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POISSON APPROXIMATION FOR CALL FUNCTION VIA STEIN-CHEN'S METHOD

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ณัฏฐ์ ย่องหิ้น : การประมาณค่าฟังก์ชันคอลด้วยการแจกแจงปัวซงโดยวิธีของสไตน์-เชน (POISSON APPROXIMATION FOR CALL FUNCTION VIA STEIN-CHEN'S METHOD) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: ศ.ดร.กฤษณะ เนียมมณี, 43 หน้า.

ฟังก์ชันคอลเป็นฟังก์ชันค่าจริงที่ไม่เป็นลบนิยามโดย $h_z(v) = (v - z)^+$ สำหรับ $z \geq 0$ เมื่อ $(v - z)^+ = \max\{v - z, 0\}$ มีการประยุกต์ฟังก์ชันคอลในด้านการเงินอย่างมากมาย ตัวอย่างเช่น การลงทุนในตราสารที่มีหนี้เป็นหลักประกัน เราให้ขอบเขตการประมาณค่าแบบปัวซงสำหรับ $h_z(V)$ โดยที่ V คือผลรวมของตัวแปรสุ่มที่มีค่าเป็นจำนวนเต็มไม่เป็นลบที่อิสระต่อกัน เทคนิคที่ใช้คือวิธีของสไตน์-เชนกับการแปลงแบบอคติศูนย์ ยิ่งไปกว่านั้นในกรณีที่ V คือผลรวมของตัวแปรสุ่มแบร์นูลลีที่อิสระต่อกัน เราปรับปรุงขอบเขตการประมาณค่าแบบปัวซงสำหรับ $h_z(V)$ โดยการเพิ่มพจน์แก้ไข

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต...

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A call function is a nonnegative real-valued function defined by $h_z(v) = (v-z)^+$ for $z \geq 0$ where $(v-z)^+ = \max\{v-z, 0\}$. There are many applications of call function in finance. For example, the standard collateralized debt obligation tranche pricing. In this work, we give bounds of Poisson approximation for $h_z(V)$ where V is a sum of independent nonnegative integer-valued random variables. The technique used is Stein-Chen's method with the zero bias transformation. Moreover, in case that V is a sum of independent Bernoulli random variables, we improve the bounds of Poisson approximation for $h_z(V)$ by adding some correction terms.

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CONTENTS

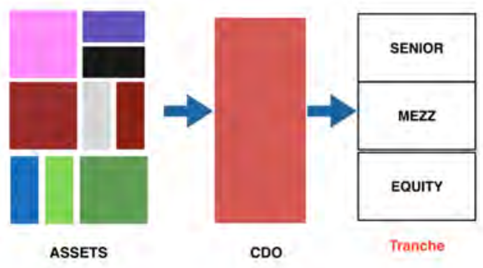
CHAPTER I

INTRODUCTION

A call function is a nonnegative real-valued function defined by

$$h_z(v) = (v - z)^+ \tag{1.1}$$

for $z \geq 0$ where $(v - z)^+ = \max\{v - z, 0\}$. There are many applications of call function in finance. The collateralized debt obligation (CDO) tranche pricing is an example of call function. The CDO is a security which is transferred into a product and sold to investors.



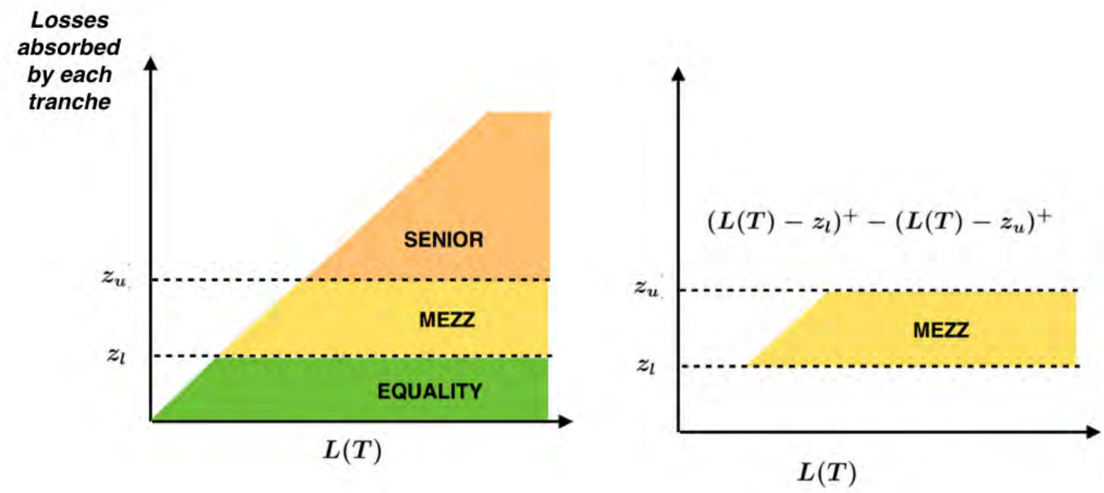
The CDO is divided into different risk classes known as tranche. The spread of a tranche is mostly determined by its credit rating, which is based on the default probability of this tranche. Each tranche is assigned a different payment priority and interest rate. Normally, the tranches primarily used in CDOs are typically known as senior, mezzanine and equity. The senior tranche includes securities with high credit ratings and tends to be low risk and therefore have lower returns. The investors can choose to invest on different tranches according to their interest.

In the standard CDO tranche pricing with underlying n portfolios, the portfolio i is assumed to have a recovery rate (the proportion of a bad debt that can be recovered) $R > 0$. Then the percentage loss at time T is defined as the total loss on the portfolio:

$$L(T) \quad = \quad \frac{(1-R)}{n}\sum_{i=1}^{n}\mathbb{I}_{\{\tau_i \le T\}}$$

where τ_i is the dafult time of the i^{th} portfolio and

$$\mathbb{I}_{\{\tau_i \le T\}}(x) = \begin{cases} 1 & \text{if } \tau_i(x) \le T \\ 0 & \text{otherwise.} \end{cases}$$



For each CDO tranche, there exist a detachment point z_u (a limit above which the tranche loss does not increase) and an attachment point z_l (a limit below which the tranche bears none of the loss). The loss on the tranche is defined as the call spread $\mathrm{E}[(L(T)-z_l)^+] - \mathrm{E}[(L(T)-z_u)^+]$. The pricing problem can be reduced to calculating the expectation of a call function, i.e., $\mathrm{E}[(L(T)-z^*)^+]$ where z^* is the attachment or the detachment point of the tranche (see [6], [10], [12] and [13] for more details). We see that

$$\mathrm{E}[(L(T)-z^*)^+] = \frac{1-R}{n}\mathrm{E}\left[(\sum_{i=1}^{n}\mathbb{I}_{\{\tau_i \le T\}} - \frac{nz^*}{1-R})^+\right].$$

and $\mathbb{I}_{\{\tau_i \le T\}}$ is a Bernoulli random variable, it suffices to compute $\mathrm{E}\left[(\sum_{i=1}^{n}\mathbb{I}_{\{\tau_i \le T\}} - z)^+\right]$ where $z = \dfrac{nz^*}{1-R} \ge 0$ is a constant.

Let $Y_1, Y_2, \dots, Y_n$ be nonnegative integer-valued independent random variables with $\mathrm{E}[Y_i] = \lambda_i < \infty$ and $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ (λ_i may depend on n). Set

$$V_n = \sum_{i=1}^{n} Y_i.$$

Let $\mathcal{P}_\lambda$ be the Poisson random variable with parameter $\lambda > 0$ i.e., $P(\mathcal{P}_\lambda = m) = \dfrac{\lambda^m e^{-\lambda}}{m!}, m = 0, 1, \dots$. It is well known that the distribution of V_n can be approximation by distribution of $\mathcal{P}_\lambda$ as $n \to \infty$ (see [9], [14], [18] for more details). That is the reason we try to approximate

$$\mathrm{E}[(V_n - z)^+] \text{ by } \mathrm{E}[(\mathcal{P}_\lambda - z)^+].$$

In 1972, Stein ([17]) proposed a general method obtaining a bound of difference between the distribution of sum of random variables and the standard normal distribution. Chen ([3]) applied the Stein's method to the Poisson approximation. In 1975, it was used to find a bound between the distribution of sum of random indicators and a Poisson distribution. This method is called the Stein-Chen's method.

In 2005, Goldstein and Reinert ([8]) introduced the zero bias transformation for Poisson approximation. Let Y be a random variable taking nonnegative integer-valued such that $\mathrm{E}[Y] < \infty$. A random variable Y^* is said to have the Y-Poisson zero biased distribution if

$$\mathrm{E}[Yf(Y)] = \lambda \mathrm{E}[f(Y^* + 1)]$$

for any function f such that $\mathrm{E}[Yf(Y)]$ exists. Jiao ([11], p.270) gave a distribution of Y^* as follow:

$$P(Y^* = y) = \frac{(y+1)P(Y = y+1)}{\mathrm{E}[Y]}$$

where $y = 0, 1, \dots$.

In 2009, Jiao and Karoui ([12]) find a uniform bound on Poisson approximation of $\mathrm{E}[(V_n - z)^+]$ by the Stein-Chen's method and zero bias transformation. In our works, we improve the results of Jiao and Karoui ([12]) in many directions.

The following theorems are our main results.

Theorem 1.1. *Let Y_i^* have the Y_i-Poisson zero biased distribution for $i = 1, 2, \ldots, n$. Then*

$$\sup_{z \geq 0} \left| \mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \right| \leq (2e^\lambda - 1)\delta_n$$

where

$$\delta_n = \sum_{i=1}^{n} \lambda_i \left| \mathrm{E}\left[Y_i^* - Y_i \right] \right| + \sum_{i=1}^{n} \lambda_i \mathrm{E}\left[Y_i^*(Y_i^* - 1) + Y_i(Y_i - 1) \right].$$

In the next theorem we improve the bound in Theorem 1.1 to a non-uniform bound.

Theorem 1.2. *Under the assumption of Theorem 1.1. and $z > 1$, we have*

$$\left| \mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \right| \leq \frac{(2e^\lambda - 1)(1 + \lambda)}{z} \delta_n.$$

The next corollaries are a special case of Theorem 1.1 and Theorem 1.2 where Y_i's are Bernoulli and Geometric random variables, respectively.

Corollary 1.3. *If $Y_i \sim Ber(p_i)$ for $i = 1, 2, \ldots, n$, then δ_n in Theorem 1.1 and Theorem 1.2 is $\sum_{i=1}^{n} p_i^2$. Furthermore, in case of $p = p_1 = p_2 = \cdots = p_n$, $\delta_n = np^2$.*

Corollary 1.4. *Let δ_n be defined as in Theorem 1.1 and Theorem 1.2. If $Y_i \sim Geo(p_i)$ for $i = 1, 2, \ldots, n$ then*

$$\delta_n = \sum_{i=1}^{n} \left[\frac{(1 - p_i)^2}{p_i^2} + 8\frac{(1 - p_i)^3}{p_i^3} \right]. \tag{1.2}$$

In 2009, Jiao and Karoui ([12]) applied the Stein-Chen's method and zero bias transformation to find a uniform bound on Poisson approximation of $\mathrm{E}[(V_n - z)^+]$ by adding correction terms. Theorem 1.5 is their result.

Theorem 1.5. *[Jiao and Karoui ([12])] Under the assumption of Theorem 1.1 and let $z \in \mathbb{N}$,*

$$\left|\mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] - C^{\mathcal{P}}_{h_z}\right| \leq (2e^\lambda - 1)\epsilon_n$$

where

$$\begin{aligned}\epsilon_n &= 2\sum_{i=1}^{n}\lambda_i \mathrm{E}\Big[|Y_i^* - Y_i|(|Y_i^* - Y_i| + 1)\Big] \\ &\quad + 2\bigg(\sum_{i=1}^{n}\lambda_i^2 \mathrm{Var}\Big[Y_i^* - Y_i\Big]\bigg)^{\frac{1}{2}} + 12\bigg(\sum_{i=1}^{n}\lambda_i \mathrm{E}\Big|Y_i^* - Y_i\Big|\bigg)^2 \end{aligned} \tag{1.3}$$

and

$$C^{\mathcal{P}}_{h_z} = \frac{\mathrm{Var}(V_n) - \lambda}{2(z-1)!} e^{-\lambda}\lambda^{z-1}.$$

Remark 1.6.

1) In case of $Y_i \sim Ber(p_i)$, by Theorem 1.5, we have

$$\epsilon_n = 4\sum_{i=1}^{n} p_i^2 + 2\bigg(\sum_{i=1}^{n} p_i^3(1-p_i)\bigg)^{\frac{1}{2}} + 12\bigg(\sum_{i=1}^{n} p_i^2\bigg)^2. \tag{1.4}$$

From the bounds in Corollary 1.3 and equality (1.4), it is easy to see that our bounds are better than the bounds in Theorem 1.5.

2) In the case that Y_i's are identically distributed with $p_i = \frac{1}{n^\delta}$ where $\delta > 1$, the order of the bounds in Theorem 1.5 and Corollary 1.3 are $O\bigg(\frac{1}{n^{\frac{3\delta-1}{2}}}\bigg)$ and $O\bigg(\frac{1}{n^{2\delta-1}}\bigg)$ respectively. Hence the rate of convergence in Corollary 1.3 is sharper than that in Theorem 1.5.

By 1) and 2), in case the $Y_i \sim Ber(p_i)$, the correction term $C^{\mathcal{P}}_{h_z}$ in Theorem 1.5 is not effective to reduce the order of the bound in (1.4).

Remark 1.7.

By Theorem 1.5 and the inequalities, $|\mathrm{E}[Y_i^* - Y_i]| \leq \mathrm{E}|Y_i^* - Y_i|$ and $|\mathrm{E}[(Y_i^* - Y_i)^2]| \leq$

$\mathrm{E}[|Y_i^* - Y_i|^2]$, *we have*

$$2\sum_{i=1}^{n}\lambda_i\left|\mathrm{E}\left[Y_i^* - Y_i\right]\right| + 2\sum_{i=1}^{n}\lambda_i\mathrm{E}\left[\left(Y_i^* - Y_i\right)^2\right] \le \epsilon_n.$$

If $Y_i \sim Geo(p_i)$ *for all* $i = 1, 2, \dots, n$ *then* $\lambda_i = \mathrm{E}[Y_i] = \frac{1-p_i}{p_i}$, $\mathrm{E}[Y_i^*] = \frac{2(1-p_i)}{p_i}$ *and*

$$\sum_{i=1}^{n}\left[\frac{2(1-p_i)^2(4-p_i)}{p_i^3} + \frac{2(1-p_i)^2}{p_i^2}\right] \le \epsilon_n. \tag{1.5}$$

It is easy to see that the first term in right hand side of (1.2) is less than the second term in left hand side of (1.5). Note that

$$8(1-p_i) \le 2(4-p_i) \text{ for } i = 1, \dots, n.$$

Then

$$8\sum_{i=1}^{n}\frac{(1-p_i)^3}{p_i^3} \le 2\sum_{i=1}^{n}\frac{(1-p_i)^2(4-p_i)}{p_i^3}.$$

Hence our bound is better than the bound in Theorem 1.5.

In standard CDO tranche pricing, we are interested in computing $\mathrm{E}[(V_n - z)^+]$ where V_n is a sum of n independent Bernoulli random variables. Next theorem, we use Stein ([18]), Neammanee and Thongtha ([16]) in order to add a correction term.

Theorem 1.8. *For* $i = 1, 2, \dots, n$, *let* $Y_i \sim Ber(p_i)$,

$$C_{call} = \sum_{j=1}^{n}\left[\mathrm{E}[(\mathcal{P}_{\lambda-p_j} - z)^+ - (\mathcal{P}_\lambda - z)^+] - p_j\mathrm{E}[(\mathcal{P}_{\lambda-p_j} - z)^+ - (\mathcal{P}_{\lambda-p_j} + 1 - z)^+]\right]$$

and $|\lambda - 1| \vee 1 = \max\{|\lambda - 1|, 1\}$. *Then*

(i) $\sup_{z\ge 0}\left|E[(V_n - z)^+] - E[(\mathcal{P}_\lambda - z)^+] - C_{call}\right| \le \frac{2(2e^\lambda - 1)}{|\lambda - 1| \vee 1}\left(\sum_{j=1}^{n}p_j^2\right)^2$, *and*

(ii) $\left|E[(V_n - z)^+] - E[(\mathcal{P}_\lambda - z)^+] - C_{call}\right| \leq \dfrac{2(2e^\lambda - 1)(1+\lambda)}{z(|\lambda - 1| \vee 1)}\left(\sum_{j=1}^{n} p_j^2\right)^2$ *where* $z \geq 1$.

Remark 1.9.

1) By Corollary 1.3, this bound tends to 0, *where* $\sum_{j=1}^{n} p_j^2 \to 0$ *as* $n \to \infty$. *In this case, the rate* $\left(\sum_{j=1}^{n} p_j^2\right)^2$ *in Theorem 1.8 is sharper than the rate* $\sum_{j=1}^{n} p_j^2$ *in Corollary 1.3.*

2) In case that Y_i*'s are identically distributed with* $p_i = \frac{1}{n^\delta}$ *where* $\delta > 1$, *the order of bound in Corollary 1.3 and Theorem 1.8 are* $O\left(\dfrac{1}{n^{2\delta-1}}\right)$ *and* $O\left(\dfrac{1}{n^{2(2\delta-1)}}\right)$ *respectively. Hence the rate of convergence of the bound in Theorem 1.8 is sharper than that of Corollary 1.3.*

This thesis is organized as follows. In Chapter II, we introduce Stein-Chen's method, zero bias transformation and some properties of Stein-Chen's solution. In Chapter III, we use Stein-Chen's method and zero bias transformation to find bounds on $|\mathrm{E}[(\mathcal{P}_\lambda - z)^+] - \mathrm{E}[(V_n - z)^+]|$ (Theorem 1.1, Theorem 1.2, Corollary 1.3 and Corollary 1.4). In Chapter IV, in case that $Y_i's$ are Bernoulli random variables, we give bounds on the difference of $\mathrm{E}[(\mathcal{P}_\lambda - z)^+]$ and $\mathrm{E}[(V_n - z)^+]$ by adding some correction terms (Theorem 1.8).

CHAPTER II
STEIN-CHEN'S METHOD AND ZERO BIAS TRANSFORMATION

In this chapter, we introduce Stein-Chen's method and zero bias transformation. We also give bounds of a solution of Stein's equation for a call function.

2.1 Stein-Chen's method

Stein ([17]) proposed a general method to obtain a bound for difference between the distribution of sum of random variables and the standard normal distribution in 1972. This method is free form Fourier transform but it relies on the differential equation,

$$f'(v) - vf(v) = h(v) - \mathrm{E}[h(\mathrm{Z})] \qquad ; v \in \mathbb{R} \tag{2.1}$$

where Z is a standard normal random variable, f is an absolutely continuous such that $\mathrm{E}|f'(\mathrm{Z})| < \infty$ and h is a real valued measurable function with $\mathrm{E}[h(\mathrm{Z})] < \infty$. The equation is called Stein's equation for the standard normal distribution.
Let Φ be the standard normal distribution function and x be a real number.
Let $\mathrm{I}_x : \mathbb{R} \to \mathbb{R}$ be defined by

$$\mathrm{I}_x(v) = \begin{cases} 1 & \text{if } v \leq x \\ 0 & \text{otherwise.} \end{cases}$$

By (2.1) with $h = \mathrm{I}_x$, we have

$$\mathrm{E}[f'(V) - Vf(V)] = P(V \le x) - \Phi(x) \tag{2.2}$$

for any random variable V. From (2.2), we can bound $\mathrm{E}[f'(V) - Vf(V)]$ instead of $P(V \le x) - \Phi(x)$ (see [1], [4], [15], [18] for more examples).

In 1975, Chen ([3]) applied the Stein's method to the Poisson approximation. It was used to find a bound of difference between the distribution of the sum of random indicators and the Poisson distribution. This method is called the Stein-Chen's method and based on a Stein's equation for Poisson distribution,

$$vf(v) - \lambda f(v+1) = h(v) - \mathcal{P}_\lambda(h) \qquad ; v = 0, 1, 2, \ldots \tag{2.3}$$

where f, h are real-valued functions on $\mathbb{N} \cup \{0\}$, $\mathcal{P}_\lambda$ is a Poisson random variable with parameter λ and $\mathcal{P}_\lambda(h) = \mathrm{E}[h(\mathcal{P}_\lambda)]$. The solution of (2.3) for a given h is

$$f_h(v) = \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{\lambda^i}{i!}(h(i) - \mathcal{P}_\lambda(h)) \qquad \text{for } v \in \mathbb{N} \tag{2.4}$$

and $f_h(0) = 0$. Many authors investigated bounds of this approximation. For example, let $Y_1, \ldots, Y_n$ be independent Bernoulli random variables with $P(Y_i = 1) = p_i = 1 - P(Y_i = 0)$. Set $V_n = Y_1 + \cdots + Y_n$ and $\lambda = p_1 + \cdots + p_n$. Stein ([17]) showed that, for any subset A of $\mathbb{N} \cup \{0\}$,

$$\Big| P(V_n \in A) - P(\mathcal{P}_\lambda \in A) \Big| \le \min\left(1, \frac{1}{\lambda}\right) \sum_{i=1}^{n} p_i^2$$

(see [3] and [14] for more examples).

In this work, let h be a call function. That is $h(v) = h_z(v) = (v - z)^+$ where $z \ge 0$ and $(v - z)^+ = \max\{v - z, 0\}$. By (2.3) in case of $h = h_z$, we have

$$vf(v) - \lambda f(v+1) = (v - z)^+ - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \qquad ; v = 0, 1, 2, \ldots \tag{2.5}$$

where

$$\mathrm{E}[(\mathcal{P}_\lambda - z)^+] = e^{-\lambda}\sum_{j=0}^{\infty}\frac{(j-z)^+\lambda^j}{j!} = e^{-\lambda}\sum_{j=\lceil z\rceil}^{\infty}\frac{(j-z)\lambda^j}{j!}$$

and $\lceil z \rceil$ is the smallest integer greater than or equal to z. The solution g_z of (2.5) is given by

$$g_z(v) = \begin{cases} 0 & \text{if } v = 0 \\ \\ \dfrac{(v-1)!}{\lambda^v}\displaystyle\sum_{i=v}^{\infty}\frac{\lambda^i}{i!}((i-z)^+ - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] & \text{if } v = 1, 2, \ldots \end{cases} \tag{2.6}$$

([12], pp. 167). Replace v by a random variable V and take expectation of (2.5), so we have

$$\mathrm{E}[Vf(V)] - \lambda\mathrm{E}[f(V+1)] = \mathrm{E}[(V-z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+]. \tag{2.7}$$

Then, we find a bound

$$\mathrm{E}[Vf(V)] - \lambda\mathrm{E}[f(V+1)] \text{ instead of } \mathrm{E}[(V-z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+].$$

In the proof of main results, we also need the following lemmas.

Lemma 2.1. *The following properties hold:*

(i) $\mathrm{E}[(\mathcal{P}_\lambda - z)^+] \le \lambda$ *for* $z \ge 0$ *and,*

(ii) $\mathrm{E}[(\mathcal{P}_\lambda - z)^+] \le \dfrac{\lambda^2}{z}$ *for* $z > 1$.

Proof. (i) Let $z \ge 0$. Since $(l-z)^+ \le l$ for $l \ge 0$,

$$\begin{aligned}\mathrm{E}[(\mathcal{P}_\lambda - z)^+] &= e^{-\lambda}\sum_{l=0}^{\infty}\frac{(l-z)^+\lambda^l}{l!} \\ &= e^{-\lambda}\sum_{l=1}^{\infty}\frac{(l-z)^+\lambda^l}{l!} \\ &\le e^{-\lambda}\sum_{l=1}^{\infty}\frac{\lambda^l}{(l-1)!}\end{aligned}$$

$$
\begin{aligned}
&= \lambda e^{-\lambda} \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda.
\end{aligned}
$$

(ii) Let $z > 1$. Hence $\lceil z \rceil \geq 2$ and

$$
\begin{aligned}
\mathrm{E}[(\mathcal{P}_\lambda - z)^+] &= e^{-\lambda} \sum_{l=\lceil z \rceil}^{\infty} \frac{(l-z)\lambda^l}{l!} \\
&\leq \frac{e^{-\lambda}}{\lceil z \rceil} \sum_{l=\lceil z \rceil}^{\infty} \frac{(l-z)\lambda^l}{(l-1)!} \\
&\leq \frac{\lambda^2 e^{-\lambda}}{\lceil z \rceil} \sum_{l=\lceil z \rceil}^{\infty} \frac{\lambda^{l-2}}{(l-2)!} \\
&\leq \frac{\lambda^2 e^{-\lambda}}{\lceil z \rceil} \sum_{l=2}^{\infty} \frac{\lambda^{l-2}}{(l-2)!} \\
&\leq \frac{\lambda^2 e^{-\lambda}}{z} \sum_{l=2}^{\infty} \frac{\lambda^{l-2}}{(l-2)!} \\
&= \frac{e^{-\lambda}\lambda^2}{z} e^{\lambda} \\
&= \frac{\lambda^2}{z}.
\end{aligned}
$$

□

Lemma 2.2. *Let $z \geq 0$ and $\Delta g_z(v) = g_z(v+1) - g_z(v)$ where g_z be defined as (2.6). Then for $v = 0, 1, 2, \ldots,$ we have*

(i) $|g_z(v)| \leq e^{\lambda}$ and,

(ii) $|\Delta g_z(v)| \leq 2e^{\lambda} - 1$.

Proof. (i) Let $v \in \mathbb{N} \cup \{0\}$. Since $g_z(0) = 0$, it suffices to prove (i) in the case of $v \geq 1$. Note that

$$
\begin{aligned}
0 < \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{\lambda^i}{i!}(i-z)^+ &\le \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{i\lambda^i}{i!} \\
&= \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{\lambda^i}{(i-1)!} \\
&= \sum_{i=v}^{\infty} \frac{(v-1)!\lambda^{i-v}}{(i-1)!} \\
&= 1 + \sum_{i=v+1}^{\infty} \frac{\lambda^{i-v}}{v\cdots(i-1)} \\
&= 1 + \sum_{i=1}^{\infty} \frac{\lambda^{i}}{v\cdots(v+i-1)} \\
&\le 1 + \sum_{i=1}^{\infty} \frac{\lambda^{i}}{i!} \\
&= e^{\lambda}, \qquad (2.8)
\end{aligned}
$$

and

$$
\begin{aligned}
0 < \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{\lambda^i}{i!} &= \sum_{i=v}^{\infty} \frac{(v-1)!\lambda^{i-v}}{i!} \\
&= \sum_{i=v}^{\infty} \frac{\lambda^{i-v}}{v\cdots i} \\
&= \frac{1}{\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{i+1}}{v\cdots(v+i)} \qquad (2.9) \\
&\le \frac{1}{\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{i+1}}{(i+1)!} \\
&= \frac{e^{\lambda}-1}{\lambda}. \qquad (2.10)
\end{aligned}
$$

By Lemma 2.1 (i) and (2.10), we have

$$
\begin{aligned}
0 < \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{\lambda^i}{i!} \mathrm{E}[(\mathcal{P}_\lambda - z)^+] &= \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{\lambda^i}{i!} \\
&\le e^{\lambda} - 1. \qquad (2.11)
\end{aligned}
$$

From (2.6), (2.8) and (2.11), we have

$$\begin{aligned}|g_z(v)| &= \left|\frac{(v-1)!}{\lambda^v}\sum_{i=v}^{\infty}\frac{\lambda^i}{i!}[(i-z)^+ - \mathrm{E}[(\mathcal{P}_\lambda - z)^+]]\right| \\ &\le e^\lambda.\end{aligned}$$

(ii) If $v = 0$ then, by (i),

$$|\Delta g_z(0)| = |g_z(1)| \le e^\lambda \le 2e^\lambda - 1. \tag{2.12}$$

Let $v \ge 1$. Note that

$$\begin{aligned}\Delta g_z(v) &= g_z(v+1) - g_z(v) \\ &= A(v) + B(v) \end{aligned}\tag{2.13}$$

where

$$\begin{aligned}A(v) &= \frac{v!}{\lambda^{v+1}}\sum_{i=v+1}^{\infty}\frac{\lambda^i}{i!}(i-z)^+ - \frac{(v-1)!}{\lambda^v}\sum_{i=v}^{\infty}\frac{\lambda^i}{i!}(i-z)^+, \\ B(v) &= \frac{(v-1)!}{\lambda^v}\sum_{i=v}^{\infty}\frac{\lambda^i}{i!}\mathrm{E}[(\mathcal{P}_\lambda - z)^+] - \frac{v!}{\lambda^{v+1}}\sum_{i=v+1}^{\infty}\frac{\lambda^i}{i!}\mathrm{E}[(\mathcal{P}_\lambda - z)^+].\end{aligned}\tag{2.14}$$

By (2.8), $|A(v)| \le e^\lambda$ and by (2.11), $|B(v)| \le e^\lambda - 1$. Hence

$$|\Delta g_z(v)| \le |A(v)| + |B(v)| \le 2e^\lambda - 1.$$

□

Lemma 2.3. *Let $z > 1$. The following properties hold:*

(i) $|\Delta g_z(v)| \le \dfrac{2e^\lambda - 1}{z}$ *for $v \in \{2, 3, 4, \ldots\}$ and,*

(ii) $|\Delta g_z(1)| \le \dfrac{1 + (2e^\lambda - 1)\lambda}{z}$.

Proof. Note that for $i \geq v$ and $i \geq \lceil z \rceil$, we have

$$\begin{aligned}
\Big| v(i+1-z) - (i-z)(i+1) \Big| &= \Big| v(i+1) - vz - i(i+1) + z(i+1) \Big| \\
&= \Big| -(i-v)(i+1) + z(i-v) + z \Big| \\
&\leq \Big| -(i-v)(i+1) + z(i-v) \Big| + z \\
&= (i+1)(i-v) - z(i-v) + z \\
&= (i+1)(i-v) - z(i-v-1).
\end{aligned}$$

Hence for $i \geq \lceil z \rceil$,

$$\Big| v(i+1-z) - (i-z)(i+1) \Big| \leq \begin{cases} (i+1)(i-v) & \text{if } i > v \\ \\ z & \text{if } i = v. \end{cases} \tag{2.15}$$

(i) Let $A(v)$ and $B(v)$ be defined as in (2.14). Note that

$$\begin{aligned}
A(v) &= \frac{v!}{\lambda^{v+1}} \sum_{i=v+1}^{\infty} \frac{\lambda^i}{i!}(i-z)^+ - \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{\lambda^i}{i!}(i-z)^+ \\
&= \frac{v!}{\lambda^{v+1}} \sum_{i=v}^{\infty} \frac{\lambda^{i+1}}{(i+1)!}(i+1-z)^+ - \frac{(v-1)!}{\lambda^v} \sum_{i=v}^{\infty} \frac{\lambda^i}{i!}(i-z)^+ \\
&= v! \sum_{i=v}^{\infty} \frac{\lambda^{i-v}}{(i+1)!}(i+1-z)^+ - (v-1)! \sum_{i=v}^{\infty} \frac{\lambda^{i-v}}{i!}(i-z)^+ \\
&= (v-1)! \sum_{i=v}^{\infty} \frac{\lambda^{i-v}}{(i+1)!} \Big[v(i+1-z)^+ - (i-z)^+(i+1) \Big].
\end{aligned} \tag{2.16}$$

If $v \geq z$ then, by (2.15) and (2.16),

$$\begin{aligned}
|A(v)| &= \left| (v-1)! \sum_{i=v}^{\infty} \frac{\lambda^{i-v}}{(i+1)!} \Big[v(i+1-z) - (i-z)(i+1) \Big] \right| \\
&\leq \frac{z}{v(v+1)} + (v-1)! \sum_{i=v+1}^{\infty} \frac{(i-v)\lambda^{i-v}}{i!}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{v} + (v-1)! \sum_{i=v+1}^{\infty} \frac{\lambda^{i-v}}{(i-1)!} \\
&= \frac{1}{v} + \sum_{i=v+1}^{\infty} \frac{\lambda^{i-v}}{v \cdots (i-1)} \\
&= \frac{1}{v} + \sum_{i=1}^{\infty} \frac{\lambda^{i}}{v \cdots (i+v-1)} \\
&= \frac{1}{v} + \frac{\lambda}{v} + \sum_{i=2}^{\infty} \frac{\lambda^{i}}{v \cdots (i+v-1)} \\
&= \frac{1}{v}\Bigg[1 + \lambda + \sum_{i=2}^{\infty} \frac{\lambda^{i}}{(1)(v+1) \cdots (i+v-1)}\Bigg] \\
&\leq \frac{1}{v}\Bigg[1 + \lambda + \sum_{i=2}^{\infty} \frac{\lambda^{i}}{1 \cdot 2 \cdots (i)}\Bigg] \\
&= \frac{1}{v}\Bigg[1 + \lambda + \sum_{i=2}^{\infty} \frac{\lambda^{i}}{i!}\Bigg] \\
&\leq \frac{e^{\lambda}}{v} \\
&\leq \frac{e^{\lambda}}{z}. \qquad (2.17)
\end{aligned}
$$

For $v < z$, by (2.15) and (2.16), we see that

$$
\begin{aligned}
|A(v)| &= \Bigg|(v-1)! \sum_{i=\lceil z\rceil-1}^{\infty} \frac{\lambda^{i-v}}{(i+1)!}\Big[v(i+1-z) - (i-z)^{+}(i+1)\Big]\Bigg| \\
&\leq v! \frac{\lambda^{\lceil z\rceil-1-v}(\lceil z\rceil - z)}{(\lceil z\rceil)!} + (v-1)! \sum_{i=\lceil z\rceil}^{\infty} \frac{\lambda^{i-v}}{(i+1)!}\Big|v(i+1-z) - (i-z)(i+1)\Big| \\
&\leq \frac{v!\lambda^{\lceil z\rceil-v-1}(\lceil z\rceil - z)}{\lceil z\rceil!} + (v-1)! \sum_{i=\lceil z\rceil}^{\infty} \frac{(i-v)\lambda^{i-v}}{i!} \\
&\leq \frac{v!\lambda^{\lceil z\rceil-v-1}(\lceil z\rceil - z)}{\lceil z\rceil!} + \frac{1}{\lceil z\rceil}(v-1)! \sum_{i=\lceil z\rceil}^{\infty} \frac{(i-v)\lambda^{i-v}}{(i-1)!} \\
&\leq \frac{v!\lambda^{\lceil z\rceil-v-1}(\lceil z\rceil - z)}{\lceil z\rceil!} + \frac{1}{\lceil z\rceil}(v-1)! \sum_{i=\lceil z\rceil}^{\infty} \frac{\lambda^{i-v}}{(i-2)!} \\
&= \frac{1}{\lceil z\rceil}\Bigg[\frac{v!\lambda^{\lceil z\rceil-v-1}(\lceil z\rceil - z)}{(\lceil z\rceil-1)!} + (v-1)! \sum_{i=\lceil z\rceil}^{\infty} \frac{\lambda^{i-v}}{(i-2)!}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\lceil z\rceil}\left[\frac{v!\lambda^{\lceil z\rceil-v-1}(\lceil z\rceil - z)}{(\lceil z\rceil-1)!} + (v-1)!\sum_{i=\lceil z\rceil - v}^{\infty}\frac{\lambda^i}{(i+v-2)!}\right] \\
&\le \frac{1}{z}\left[\frac{v!\lambda^{\lceil z\rceil-v-1}(\lceil z\rceil - z)}{(\lceil z\rceil-1)!} + (v-1)!\sum_{i=\lceil z\rceil - v}^{\infty}\frac{\lambda^i}{(i+v-2)!}\right] \qquad (2.18)\\
&\le \frac{1}{z}\left[\frac{v!\lambda^{\lceil z\rceil-v-1}}{(\lceil z\rceil-1)!} + (v-1)!\sum_{i=\lceil z\rceil - v}^{\infty}\frac{\lambda^i}{(i+v-2)!}\right]. \qquad (2.19)
\end{aligned}
$$

Let $v \geq 2$. If $v = \lceil z\rceil - 1$, then

$$
\frac{v!\lambda^{\lceil z\rceil-v-1}}{(\lceil z\rceil-1)!} = 1 \qquad (2.20)
$$

and

$$
\begin{aligned}
(v-1)!\sum_{i=\lceil z\rceil - v}^{\infty}\frac{\lambda^i}{(i+v-2)!} &= (\lceil z\rceil - 2)!\sum_{i=1}^{\infty}\frac{\lambda^i}{(i+\lceil z\rceil-3)!} \\
&= \lambda + \sum_{i=2}^{n}\frac{\lambda^i}{(\lceil z\rceil-1)\cdots(i+\lceil z\rceil-3)} \\
&\le \lambda + \sum_{i=2}^{n}\frac{\lambda^i}{2\cdots i} \\
&= \sum_{i=1}^{\infty}\frac{\lambda^i}{i!}. \qquad (2.21)
\end{aligned}
$$

If $v < \lceil z\rceil - 1$, then

$$
\frac{v!\lambda^{\lceil z\rceil-v-1}}{(\lceil z\rceil-1)!} = \frac{\lambda^{\lceil z\rceil-v-1}}{(v+1)\cdots(\lceil z\rceil-1)} \le \frac{\lambda^{\lceil z\rceil-v-1}}{(\lceil z\rceil-v-1)!} \qquad (2.22)
$$

and

$$
\begin{aligned}
(v-1)!\sum_{i=\lceil z\rceil - v}^{\infty}\frac{\lambda^i}{(i+v-2)!} &= \sum_{i=\lceil z\rceil - v}^{\infty}\frac{\lambda^i}{v\cdots(i+v-2)} \\
&\le \sum_{i=\lceil z\rceil - v}^{\infty}\frac{\lambda^i}{2\cdots i} \\
&= \sum_{i=\lceil z\rceil - v}^{\infty}\frac{\lambda^i}{i!}. \qquad (2.23)
\end{aligned}
$$

Hence by (2.19) - (2.23), for $2 \le v < z$ we have

$$\begin{aligned} |A(v)| &\le \frac{1}{z}\left[\frac{\lambda^{\lceil z\rceil - v - 1}}{(\lceil z\rceil - v - 1)!} + \sum_{i=\lceil z\rceil - v}^{\infty} \frac{\lambda^i}{i!}\right] \\ &= \frac{1}{z}\sum_{i=\lceil z\rceil - v - 1}^{\infty} \frac{\lambda^i}{i!} \\ &\le \frac{1}{z}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} \\ &= \frac{e^\lambda}{z}. \end{aligned}$$

From this fact and (2.17),

$$|A(v)| \le \frac{e^\lambda}{z} \text{ for } v \in \{2, 3, \ldots\}. \tag{2.24}$$

Next, we consider

$$\begin{aligned} |B(v)| &= \left|\frac{(v-1)!}{\lambda^v}\sum_{i=v}^{\infty}\frac{\lambda^i}{i!}\mathrm{E}[(\mathcal{P}_\lambda - z)^+] - \frac{v!}{\lambda^{v+1}}\sum_{i=v+1}^{\infty}\frac{\lambda^i}{i!}\mathrm{E}[(\mathcal{P}_\lambda - z)^+]\right| \\ &= \mathrm{E}[(\mathcal{P}_\lambda - z)^+]\left|\frac{(v-1)!}{\lambda^v}\sum_{i=v}^{\infty}\frac{\lambda^i}{i!} - \frac{v!}{\lambda^{v+1}}\sum_{i=v+1}^{\infty}\frac{\lambda^i}{i!}\right|. \end{aligned} \tag{2.25}$$

By (2.9) and $v \ge 2$, we have

$$\begin{aligned} 0 < \frac{(v-1)!}{\lambda^v}\sum_{i=v}^{\infty}\frac{\lambda^i}{i!} &\le \frac{1}{\lambda^2}\sum_{i=0}^{\infty}\frac{\lambda^{i+2}}{v\cdots(v+i)} \\ &\le \frac{1}{\lambda^2}\sum_{i=0}^{\infty}\frac{\lambda^{i+2}}{2\cdots(i+2)} \\ &= \frac{1}{\lambda^2}\sum_{i=2}^{\infty}\frac{\lambda^i}{i!} \\ &\le \frac{e^\lambda - 1}{\lambda^2}, \end{aligned} \tag{2.26}$$

and

$$0 < \frac{v!}{\lambda^{v+1}} \sum_{i=v+1}^{\infty} \frac{\lambda^i}{i!} \le \frac{1}{\lambda^2} \sum_{i=0}^{\infty} \frac{\lambda^{i+2}}{(v+1)\cdots(v+1+i)}$$
$$\le \frac{1}{\lambda^2} \sum_{i=0}^{\infty} \frac{\lambda^{i+2}}{3\cdots(i+3)}$$
$$\le \frac{1}{\lambda^2} \sum_{i=0}^{\infty} \frac{\lambda^{i+2}}{2\cdots(i+2)}$$
$$= \frac{1}{\lambda^2} \sum_{i=2}^{\infty} \frac{\lambda^i}{i!}$$
$$\le \frac{e^\lambda - 1}{\lambda^2}. \tag{2.27}$$

Then by (2.25) - (2.27) and Lemma 2.1 (ii), we have

$$|B(v)| \le \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \frac{e^\lambda - 1}{\lambda^2}$$
$$\le \frac{e^\lambda - 1}{z}. \tag{2.28}$$

Hence by (2.24) and (2.28), we have

$$\Big| g_z(v+1) - g_z(v) \Big| \le |A(v)| + |B(v)|$$
$$\le \frac{e^\lambda}{z} + \frac{e^\lambda - 1}{z}$$
$$= \frac{2e^\lambda - 1}{z}.$$

(ii) Let $v = 1$. Then by (2.18), we have

$$|A(1)| \le \frac{1}{z}\left[\frac{\lambda^{\lceil z \rceil - 2}(\lceil z \rceil - z)}{(\lceil z \rceil - 1)!} + \sum_{i=\lceil z \rceil - 1}^{\infty} \frac{\lambda^i}{(i-1)!} \right]$$
$$\le \frac{1}{z}\left[\frac{\lambda^{\lceil z \rceil - 2}}{(\lceil z \rceil - 1)!} + \sum_{i=\lceil z \rceil - 1}^{\infty} \frac{\lambda^i}{(i-1)!} \right].$$

Thus

$$|A(1)| \leq \begin{cases} \frac{1}{z}\left[1 + \lambda \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}\right] & \text{if } \lceil z \rceil = 2 \\ \frac{\lambda}{z}\left[\frac{\lambda^{\lceil z \rceil - 3}}{(\lceil z \rceil - 3)!} + \sum_{i=\lceil z \rceil - 1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}\right] & \text{if } \lceil z \rceil > 2 \end{cases}$$

$$\leq \begin{cases} \frac{1 + e^{\lambda}\lambda}{z} & \text{if } \lceil z \rceil = 2 \\ \frac{e^{\lambda}\lambda}{z} & \text{if } \lceil z \rceil > 2 \end{cases}$$

$$\leq \frac{1 + e^{\lambda}\lambda}{z}. \tag{2.29}$$

By (2.10) and Lemma 2.1 (ii), we have

$$\begin{aligned} |B(1)| &= \mathrm{E}[(\mathcal{P}_\lambda - z)^+]\left[\frac{1}{\lambda}\sum_{i=1}^{\infty}\frac{\lambda^i}{i!} - \frac{1}{\lambda^2}\sum_{i=2}^{\infty}\frac{\lambda^i}{i!}\right] \\ &\leq \mathrm{E}[(\mathcal{P}_\lambda - z)^+]\frac{e^{\lambda} - 1}{\lambda} \\ &\leq \frac{(e^{\lambda} - 1)\lambda}{z}. \end{aligned} \tag{2.30}$$

Hence, by (2.13), (2.29) and (2.30),

$$\begin{aligned} |\Delta g_z(1)| &\leq |A(1)| + |B(1)| \\ &\leq \frac{1 + (2e^{\lambda} - 1)\lambda}{z}. \end{aligned}$$

□

Next section, we introduce zero bias transformation. In the proofs of Theorem 3.1 and Theorem 3.2, we use this to rewrite the expression $\mathrm{E}[Vf(V)] - \lambda\mathrm{E}[f(V+1)]$.

2.2 Zero bias transformation

In 2005, Goldstein and Reinert ([8]) introduced the zero bias transformation for Poisson approximation. Let Y be a random variable taking nonnegative integer-values such that $\mathrm{E}[Y] = \lambda < \infty$. A random variable Y^* is said to have the Y-Poisson zero biased distribution if

$$\mathrm{E}[Yf(Y)] \quad = \quad \lambda \mathrm{E}[f(Y^*+1)]$$

for any function f such that $\mathrm{E}[Yf(Y)]$ exists. Jiao ([11], pp. 270) give a distribution of Y^* as follow:

$$P(Y^* = y) = \frac{(y+1)P(Y = y+1)}{\mathrm{E}[Y]}. \tag{2.31}$$

where $y = 0, 1, \ldots$. Note that by (2.31), we have $Y^* \geq 0$.

Proposition 2.4. *Let $Y \sim Ber(p)$ i.e. $P(Y = 1) = 1 - P(Y = 0) = p > 0$. Then $Y^* = 0$.*

Proof. By (2.31), we have

$$P(Y^* = 0) = \frac{P(Y = 1)}{E[Y]} = \frac{p}{p} = 1.$$

Hence $Y^* = 0$. □

Proposition 2.5. *Let $Y \sim Geo(p)$, i.e. $P(Y = y) = p(1-p)^y$ for $y = 0, 1, 2, \ldots$ and $0 < p \leq 1$. Then*

$$\mathrm{E}[Y^*] = \frac{2(1-p)}{p} \text{ and } \mathrm{E}[Y^{*2}] = \frac{2(1-p)(3-2p)}{p^2}.$$

Proof. By (2.31) and the fact that $\mathrm{E}[Y] = \dfrac{1-p}{p}$ (see [5], pp. 60), we have

$$P(Y^* = y) = \frac{(y+1)P(Y = y+1)}{E[Y]} = \frac{(y+1)p(1-p)^{y+1}}{E[Y]} = (y+1)p^2(1-p)^y$$

where $y \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
\mathrm{E}[Y^*] &= \sum_{y=0}^{\infty} y(y+1)p^2(1-p)^y \\
&= \sum_{y=0}^{\infty} y^2p^2(1-p)^y + \sum_{y=0}^{\infty} yp^2(1-p)^y \\
&= \sum_{y=0}^{\infty} y^2p^2(1-p)^y + p\sum_{y=0}^{\infty} yp(1-p)^y \\
&= \sum_{y=0}^{\infty} y^2p^2(1-p)^y + p\mathrm{E}(Y) \\
&= \sum_{y=0}^{\infty} y^2p^2(1-p)^y + (1-p) \\
&= p\mathrm{E}[Y^2] + (1-p)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}[Y^{*2}] &= \sum_{y=0}^{\infty} y^2(y+1)p^2(1-p)^y \\
&= \sum_{y=0}^{\infty} y^3p^2(1-p)^y + \sum_{y=0}^{\infty} y^2p^2(1-p)^y \\
&= p\sum_{y=0}^{\infty} y^3p(1-p)^y + p\sum_{y=0}^{\infty} y^2p(1-p)^y \\
&= p\mathrm{E}[Y^3] + p\mathrm{E}[Y^2].
\end{aligned}$$

Note that the moment generating function of Y is $m(t) = \dfrac{p}{1-(1-p)e^t}$ where $t = 0, 1, \dots,$ (see [5], pp. 221). Hence

$$\begin{aligned}
\mathrm{E}[Y^2] &= m''(0) = \frac{(1-p)(2-p)}{p^2}, \\
\mathrm{E}[Y^3] &= m^{(3)}(0) = \frac{(1-p)(p^2-6p+6)}{p^3}.
\end{aligned}$$

Then

$$\begin{aligned}\mathrm{E}[Y^*] &= \frac{(1-p)(2-p)}{p} + (1-p)\\ &= \frac{(1-p)(2-p)+(1-p)p}{p}\\ &= \frac{2(1-p)}{p}\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}[Y^{*2}] &= \frac{(1-p)(p^2-6p+6)}{p^2} + \frac{(1-p)(2-p)}{p}\\ &= \frac{(1-p)(p^2-6p+6)+(1-p)(2-p)p}{p^2}\\ &= \frac{(1-p)(p^2-6p+6+2p-p^2)}{p^2}\\ &= \frac{2(1-p)(3-2p)}{p^2}.\end{aligned}$$

□

Next we consider the sum of independent nonnegative integer-valued random variables. The following proposition are shown by Jiao and Karoui ([12], pp. 167).

Proposition 2.6. *For any $i = 1, \cdots, n$, assume that Y_i^* have the Y_i- Poisson zero biased distribution. Let I be a random index independent of $Y_1, \ldots, Y_n, Y_1^*, \ldots, Y_n^*$ satisfying $P(I = i) = \dfrac{\lambda_i}{\lambda}$. Then $V_n^{(I)} + Y_I^*$ has the V_n-Poisson zero biased distribution where $V_n = \sum_{i=1}^{n} Y_i$ and $V_n^{(i)} = V_n - Y_i$.*

CHAPTER III
BOUND ON POISSON APPROXIMATION FOR CALL FUNCTION

Let $Y_1, Y_2, \dots, Y_n$ be nonnegative integer-valued independent random variables with $\mathrm{E}[Y_i] = \lambda_i < \infty$ and $\mathcal{P}_\lambda$ be a Poisson random variable with parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Define

$$V_n \;=\; \sum_{i=1}^{n} Y_i.$$

The aim of this chapter is to find upper bounds of

$$\Big|\mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+]\Big|$$

by using Stein-Chen's method and zero bias transformation.

Theorem 3.1. *Let Y_i^* have the Y_i-Poisson zero biased distribution for $i = 1, \dots, n$. Then*

$$\sup_{z \ge 0}\Big|\mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+]\Big| \le (2e^\lambda - 1)\delta_n$$

where

$$\delta_n = \sum_{i=1}^{n} \lambda_i \Big|\mathrm{E}[Y_i^* - Y_i]\Big| + \sum_{i=1}^{n} \lambda_i \mathrm{E}\Big[Y_i^*(Y_i^* - 1) + Y_i(Y_i - 1)\Big].$$

Proof. Let I be a random index independent of $Y_1, \dots, Y_n, Y_1^*, \dots, Y_n^*$ satisfying $P(I = i) = \dfrac{\lambda_i}{\lambda}$. By Proposition 2.6, we know that $V_n^{(I)} + Y_I^*$ has the V_n-Poisson zero biased distribution where $V_n^{(i)} = V_n - Y_i$. Then

$$\mathrm{E}[V_n g_z(V_n)] = \lambda \mathrm{E}[g_z(V_n^{(I)} + Y_I^* + 1)] \tag{3.1}$$

where g_z is the solution of a Stein's equation for Poisson distribution and defined as in (2.6).

Let $\Delta g_z(x) = g_z(x+1) - g_z(x)$ and $\Delta^2 g_z(x) = \Delta(\Delta g_z(x)) = \Delta g_z(x+1) - \Delta g_z(x)$. By (2.7) and (3.1), we have

$$\begin{aligned}
&\mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \\
&= \mathrm{E}[V_n g_z(V_n)] - \lambda \mathrm{E}[g_z(V_n+1)] \\
&= \lambda \mathrm{E}\Big[g_z(V_n^{(I)} + Y_I^* + 1) - g_z(V_n + 1)\Big] \\
&= \lambda \mathrm{E}\Big[g_z(V_n^{(I)} + Y_I^* + 1) - g_z(V_n^{(I)} + Y_I + 1)\Big] \\
&= \lambda \mathrm{E}\Big[\sum_{i=1}^{n}(g_z(V_n^{(i)} + Y_i^* + 1) - g_z(V_n^{(i)} + Y_i + 1))\mathbb{I}(I=i)\Big] \\
&= \lambda \sum_{i=1}^{n} \mathrm{E}\Big[(g_z(V_n^{(i)} + Y_i^* + 1) - g_z(V_n^{(i)} + Y_i + 1))\mathbb{I}(I=i)\Big] \\
&= \lambda \sum_{i=1}^{n} \mathrm{E}\Big[g_z(V_n^{(i)} + Y_i^* + 1) - g_z(V_n^{(i)} + Y_i + 1)\Big]\mathrm{E}\Big[\mathbb{I}(I=i)\Big] \\
&= \lambda \sum_{i=1}^{n} \frac{\lambda_i}{\lambda}\mathrm{E}\Big[g_z(V_n^{(i)} + Y_i^* + 1) - g_z(V_n^{(i)} + Y_i + 1)\Big] \\
&= \sum_{i=1}^{n} \lambda_i \mathrm{E}\Big[g_z(V_n^{(i)} + Y_i^* + 1) - g_z(V_n^{(i)} + Y_i + 1)\Big]. \qquad (3.2)
\end{aligned}$$

Note that for $a \in \mathbb{N} \cup \{0\}$,

$$g_z(x+a) = g_z(x) + a\Delta g_z(x) + \sum_{j=0}^{(a-1)\vee 0} (((a-1)\vee 0) - j)\Delta^2 g_z(x+j)$$

where $(a-1)\vee 0 = \max\{a-1, 0\}$ ([13], pp. 17). This implies that

$$\begin{aligned}
\mathrm{E}[g_z(V_n^{(i)} + Y_i^* + 1)] = \mathrm{E}\Big[&g_z(V_n^{(i)} + 1) + Y_i^*\Delta g_z(V_n^{(i)} + 1) \\
&+ \sum_{j=0}^{(Y_i^*-1)\vee 0} (((Y_i^* - 1)\vee 0) - j)\Delta^2 g_z(V_n^{(i)} + 1 + j)\Big] \qquad (3.3)
\end{aligned}$$

and

$$\mathrm{E}[g_z(V_n^{(i)} + Y_i + 1)] = \mathrm{E}\Bigg[g_z(V_n^{(i)} + 1) + Y_i \Delta g_z(V_n^{(i)} + 1) + \sum_{j=0}^{(Y_i-1)\vee 0} (((Y_i - 1) \vee 0) - j)\Delta^2 g_z(V_n^{(i)} + 1 + j)\Bigg]. \quad (3.4)$$

Let $||\Delta g_z|| = \sup_{v\geq 1}\{|\Delta g_z(v)|\}$ and $||\Delta^2 g_z|| = \sup_{v\geq 1}\{|\Delta^2 g_z(v)|\}$. Hence, by (3.3) and (3.4),

$$\begin{aligned}
&\Big|\mathrm{E}[g_z(V_n^{(i)} + Y_i^* + 1) - g_z(V_n^{(i)} + Y_i + 1)]\Big| \\
&= \Bigg|\mathrm{E}\Big[(Y_i^* - Y_i)\Delta g_z(V_n^{(i)} + 1)\Big] + \mathrm{E}\Bigg[\sum_{j=0}^{(Y_i^*-1)\vee 0} (((Y_i^* - 1) \vee 0) - j)\Delta^2 g_z(V_n^{(i)} + 1 + j)\Bigg] \\
&\quad - \mathrm{E}\Bigg[\sum_{j=0}^{(Y_i-1)\vee 0} (((Y_i - 1) \vee 0) - j)\Delta^2 g_z(V_n^{(i)} + 1 + j)\Bigg]\Bigg| \\
&\leq \Bigg|\mathrm{E}\Big[(Y_i^* - Y_i)\Delta g_z(V_n^{(i)} + 1)\Big]\Bigg| \\
&\quad + \Bigg|\mathrm{E}\Bigg[\sum_{j=0}^{(Y_i^*-1)\vee 0} (((Y_i^* - 1) \vee 0) - j)\Delta^2 g_z(V_n^{(i)} + 1 + j)\Bigg]\Bigg| \\
&\quad + \Bigg|\mathrm{E}\Bigg[\sum_{j=0}^{(Y_i-1)\vee 0} (((Y_i - 1) \vee 0) - j)\Delta^2 g_z(V_n^{(i)} + 1 + j)\Bigg]\Bigg| \\
&\leq ||\Delta g_z||\Big|\mathrm{E}[Y_i^* - Y_i]\Big| + ||\Delta^2 g_z||\Bigg|\mathrm{E}\Bigg[\sum_{j=0}^{(Y_i^*-1)\vee 0} (((Y_i^* - 1) \vee 0) - j)\Bigg]\Bigg| \\
&\quad + ||\Delta^2 g_z||\Bigg|\mathrm{E}\Bigg[\sum_{j=0}^{(Y_i-1)\vee 0} (((Y_i - 1) \vee 0) - j)\Bigg]\Bigg|. \qquad (3.5)
\end{aligned}$$

Note that

$$\sum_{j=0}^{(Y_i^*-1)\vee 0} (((Y_i^* - 1) \vee 0) - j) = 0 \text{ for } Y_i^* = 0, 1$$

and for $Y_i^* \geq 2$,

$$\sum_{j=0}^{(Y_i^*-1)\vee 0} (((Y_i^*-1)\vee 0)-j) = \sum_{j=0}^{Y_i^*-1} (Y_i^*-1-j)$$
$$= (Y_i^*-1)+(Y_i^*-2)+\cdots+0$$
$$= \frac{Y_i^*(Y_i^*-1)}{2}.$$

Then

$$\sum_{j=0}^{(Y_i^*-1)\vee 0} (((Y_i^*-1)\vee 0)-j) = \frac{Y_i^*(Y_i^*-1)}{2} \text{ for } Y_i^* \geq 0. \tag{3.6}$$

Similarly (3.6), we can show that

$$\sum_{j=0}^{(Y_i-1)\vee 0} (((Y_i-1)\vee 0)-j) = \frac{Y_i(Y_i-1)}{2} \text{ for } Y_i^* \geq 0. \tag{3.7}$$

By (3.2) and (3.5)-(3.7), we have

$$\begin{aligned}
&\left|\mathrm{E}[(V_n-z)^+]-\mathrm{E}[(\mathcal{P}_\lambda-z)^+]\right| \\
&\leq ||\Delta g_z||\sum_{i=1}^{n}\lambda_i\left|\mathrm{E}\left[Y_i^*-Y_i\right]\right| + \frac{||\Delta^2 g_z||}{2}\sum_{i=1}^{n}\lambda_i\mathrm{E}\left[Y_i^*(Y_i^*-1)+Y_i(Y_i-1)\right] \\
&\leq ||\Delta g_z||\sum_{i=1}^{n}\lambda_i\left|\mathrm{E}\left[Y_i^*-Y_i\right]\right| + ||\Delta g_z||\sum_{i=1}^{n}\lambda_i\mathrm{E}\left[Y_i^*(Y_i^*-1)+Y_i(Y_i-1)\right] \\
&= ||\Delta g_z||\delta_n
\end{aligned} \tag{3.8}$$

where we use the fact that $||\Delta^2 g_z|| = \sup_{v\geq 1}\{|\Delta g_z(v+1)-\Delta g_z(v)|\} \leq 2||\Delta g_z||$.
By Lemma 2.2 (ii) and (3.8), we have

$$\left|\mathrm{E}[(V_n-z)^+]-\mathrm{E}[(\mathcal{P}_\lambda-z)^+]\right| \leq (2e^{\lambda}-1)\delta_n.$$

□

In the next theorem we improve a bound in Theorem 3.1 to a non-uniform bound.

Theorem 3.2. *Let Y_i^* have the Y_i-Poisson zero biased distribution for $i = 1, \ldots, n$ and $z > 1$. Then*

$$\left| \mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \right| \leq \frac{(2e^\lambda - 1)(1 + \lambda)}{z} \delta_n.$$

Proof. Let $z > 1$. By Lemma 2.3, we have

$$\begin{aligned} |\Delta g_z(v)| &\leq \max\{\frac{2e^\lambda - 1}{z}, \frac{1 + (2e^\lambda - 1)\lambda}{z}\} \\ &= \frac{2e^\lambda - 1}{z} \max\{1, \frac{1}{2e^\lambda - 1} + \lambda\} \\ &\leq \frac{2e^\lambda - 1}{z} \max\{1, 1 + \lambda\} \\ &\leq \frac{(2e^\lambda - 1)(1 + \lambda)}{z} \end{aligned} \tag{3.9}$$

for $v \in \{1, 2, \ldots\}$. Form this fact and (3.8) we have

$$\left| \mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \right| \leq \frac{(2e^\lambda - 1)(1 + \lambda)}{z} \delta_n.$$

□

We can see that for $z > 1 + \lambda$ the bound in Theorem 3.2 is smaller than the bound in Theorem 3.1.

The next corollaries are special cases of Theorem 3.1 and Theorem 3.2 where Y_i's are Bernoulli and Geometric random variables, respectively.

Corollary 3.3. *If $Y_i \sim Ber(p_i)$ for $i = 1, 2, \ldots, n$, then δ_n in Theorem 3.1 and Theorem 3.2 is $\sum_{i=1}^{n} p_i^2$. Furthermore, in case of $p = p_1 = p_2 = \cdots = p_n$, $\delta_n = np^2$.*

Proof. Since $Y_i \sim Ber(p_i)$, $\lambda_i = \mathrm{E}[Y_i] = p_i$, $\mathrm{E}[Y_i^2] = p_i$ and by Proposition 2.4,

$Y_i^* = 0$ for $i = 1, 2, \ldots n$. Hence

$$\begin{aligned}
\delta_n &= \sum_{i=1}^{n} \lambda_i \Big| \mathrm{E}[Y_i^* - Y_i] \Big| + \sum_{i=1}^{n} \lambda_i \mathrm{E}\Big[Y_i^*(Y_i^* - 1) + Y_i(Y_i - 1) \Big] \\
&= \sum_{i=1}^{n} p_i \mathrm{E}[Y_i] + \sum_{i=1}^{n} p_i \mathrm{E}\Big[Y_i(Y_i - 1) \Big] \\
&= \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \mathrm{E}\Big[Y_i^2 - Y_i \Big] \\
&= \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \Big[\mathrm{E}[Y_i^2] - \mathrm{E}[Y_i] \Big] \\
&= \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \Big[p_i - p_i \Big] \\
&= \sum_{i=1}^{n} p_i^2.
\end{aligned}$$

□

Remark 3.4.

1) In case of $Y_i \sim Ber(p_i)$, by Corollary 3.3 we have

$$\sup_{z \geq 0} \Big| \mathrm{E}[(V - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \Big| \leq (2e^\lambda - 1) \sum_{i=1}^{n} p_i^2 \tag{3.10}$$

and by Theorem 1.5, we have

$$\Big| \mathrm{E}[(V - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] - C_{h_z}^{\mathcal{P}} \Big| \leq (2e^\lambda - 1)\epsilon_n$$

where

$$\epsilon_n = 4 \sum_{i=1}^{n} p_i^2 + 2\Big(\sum_{i=1}^{n} p_i^3 (1 - p_i)\Big)^{\frac{1}{2}} + 12\Big(\sum_{i=1}^{n} p_i^2\Big)^2. \tag{3.11}$$

Form (3.10) and (3.11), it is easy to see that our bound is better than the bound in Theorem 1.5.

2) In case of Y_i's are identically distributed with $p_i = \frac{1}{n^\delta}$ where $\delta > 1$, we have

$$\epsilon_n = O\left(\frac{1}{n^{\frac{3\delta-1}{2}}}\right) \text{ and } \delta_n = O\left(\frac{1}{n^{2\delta-1}}\right).$$

Hence our bound is sharper than bound in Theorem 1.5.

By 1) and 2), in case that $Y_i \sim Ber(p_i)$, the correction term $C^{\mathcal{P}}_{h_z}$ in Theorem 1.5 is not effective to reduce the order of the bound (3.11).

Corollary 3.5. *Let δ_n be defined as in Theorem 3.1 and Theorem 3.2. If $Y_i \sim Geo(p_i)$ for $i = 1, 2, \dots, n$ then*

$$\delta_n = \sum_{i=1}^{n}\left[\frac{(1-p_i)^2}{p_i^2} + 8\frac{(1-p_i)^3}{p_i^3}\right].$$

Proof. Since $Y_i \sim Geo(p_i), \lambda_i = \mathrm{E}[Y_i] = \frac{1-p_i}{p_i}$. By Proposition 2.5, we have $\mathrm{E}[Y_i^2] = \frac{(1-p_i)(2-p_i)}{p_i^2}$, $\mathrm{E}[Y_i^*] = \frac{2(1-p_i)}{p_i}$ and $\mathrm{E}[Y_i^{*2}] = \frac{2(1-p_i)(3-2p_i)}{p_i^2}$. Then

$$\lambda_i\left|\mathrm{E}[Y_i^* - Y_i]\right| = \frac{1-p_i}{p_i}\left|\frac{2(1-p_i)}{p_i} - \frac{1-p_i}{p_i}\right| = \frac{(1-p_i)^2}{p_i^2}$$

and

$$\begin{aligned}
&\lambda_i \mathrm{E}\Big[Y_i^*(Y_i^*-1) + Y_i(Y_i-1)\Big] \\
&= \lambda_i \mathrm{E}\Big[Y_i^{*2} - Y_i^* + Y_i^2 - Y_i\Big] \\
&= \frac{1-p_i}{p_i}\left[\frac{2(1-p_i)(3-2p_i)}{p_i^2} - \frac{2(1-p_i)}{p_i} + \frac{(1-p_i)(2-p_i)}{p_i^2} - \frac{1-p_i}{p_i}\right] \\
&= \frac{1-p_i}{p_i}\left[8\frac{(1-p_i)^2}{p_i^2}\right] \\
&= 8\frac{(1-p_i)^3}{p_i^3}.
\end{aligned}$$

Hence

$$\delta_n = \sum_{i=1}^{n}\lambda_i\left|\mathrm{E}\Big[Y_i^* - Y_i\Big]\right| + \sum_{i=1}^{n}\lambda_i\mathrm{E}\Big[Y_i^*(Y_i^*-1) + Y_i(Y_i-1)\Big]$$

$$= \sum_{i=1}^{n} \left[\frac{(1-p_i)^2}{p_i^2} + 8\frac{(1-p_i)^3}{p_i^3} \right].$$

□

Remark 3.6.

If $Y_i \sim Geo(p_i)$ for all $i = 1, 2, \ldots, n$ then, by Corollary 3.5, we have

$$\sup_{z \geq 0} \left| \mathrm{E}[(V-z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] \right| \leq (2e^\lambda - 1) \sum_{i=1}^{n} \left[\frac{(1-p_i)^2}{p_i^2} + 8\frac{(1-p_i)^3}{p_i^3} \right] \tag{3.12}$$

Note that $\lambda_i = \mathrm{E}[Y_i] = \dfrac{1-p_i}{p_i}$, $\mathrm{E}[Y_i^*] = \dfrac{2(1-p_i)}{p_i}$ *and*

$$\begin{aligned} & 2(2e^\lambda - 1) \sum_{i=1}^{n} \lambda_i \left[\mathrm{E}\left[(Y_i^* - Y_i)^2 \right] + \left| \mathrm{E}\left[Y_i^* - Y_i \right] \right| \right] \\ & = (2e^\lambda - 1) \sum_{i=1}^{n} \left[\frac{2(1-p_i)^2(4-p_i)}{p_i^3} + \frac{2(1-p_i)^2}{p_i^2} \right] \end{aligned} \tag{3.13}$$

It is easy to see that the first term in right hand side of (3.12) is less than the second term in right hand side of (3.13). Note that

$$8(1-p_i) \leq 2(4-p_i) \text{ for } i = 1, \ldots, n.$$

Then

$$8 \sum_{i=1}^{n} \frac{(1-p_i)^3}{p_i^3} \leq 2 \sum_{i=1}^{n} \frac{(1-p_i)^2(4-p_i)}{p_i^3}.$$

Hence

$$\delta_n \leq 2 \sum_{i=1}^{n} \lambda_i \left[\mathrm{E}\left[(Y_i^* - Y_i)^2 \right] + \left| \mathrm{E}\left[Y_i^* - Y_i \right] \right| \right] \tag{3.14}$$

By Theorem 1.5 and the inequalities, $|\mathrm{E}[Y_i^* - Y_i]| \leq \mathrm{E}|Y_i^* - Y_i|$ *and* $|\mathrm{E}[(Y_i^* - Y_i)^2]| \leq$

$\mathrm{E}[|Y_i^* - Y_i|^2]$, *we have*

$$2\sum_{i=1}^{n}\lambda_i\Big[\mathrm{E}\Big[(Y_i^*-Y_i)^2\Big]+\Big|\mathrm{E}\Big[Y_i^*-Y_i\Big]\Big|\Big] \leq 2\sum_{i=1}^{n}\lambda_i\mathrm{E}\Big[|Y_i^*-Y_i|(|Y_i^*-Y_i|+1)\Big]$$
$$\leq \epsilon_n. \tag{3.15}$$

Hence by (3.14) and (3.15), we have our bound is better than the bound in Theorem 1.5.

CHAPTER IV

REFINEMENT OF BOUND IN CASE OF BERNOULLI RANDOM VARIABLES

In standard CDO tranche pricing, we are interested in computing $\mathrm{E}[(V_n - z^+)]$ where V_n is a sum of Bernoulli random variables ([12], [13] for more details).

Let $Y_1, Y_2, \ldots, Y_n$ be independent Bernoull random variables with $\mathrm{P}(Y_i = 1) = p_i > 0$ for $i = 1, 2, \ldots, n$ and $\mathcal{P}_\lambda$ be a Poisson random variable with parameter $\lambda = p_1 + p_2 + \cdots + p_n$. Define

$$V_n \;=\; \sum_{i=1}^{n} Y_i.$$

In this chapter we improve the result in Corollary 3.3. We use the technique from Stein ([18]), Neammanee and Thongtha ([16]) in order to add a correction term. In the proof of the main result, we also need the following lemma.

Lemma 4.1. *Let f be a real-valued function. For each $j = 1, 2, \ldots, n$, we have*

(i) $\displaystyle \frac{1}{\lambda}\sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) = \frac{e^{\lambda-p_j}}{p_j}\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})]$ *and,*

(ii) $\displaystyle \frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v+1}\frac{\lambda^i}{i!}f(i) = e^{\lambda-p_j}\left[\frac{\mathrm{E}[f(\mathcal{P}_{\lambda-p_j}+1)]}{p_j} + \frac{\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})]}{p_j^2}\right].$

Proof. i) Let $T = \displaystyle\frac{1}{\lambda}\sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i)$. Then

$$\begin{aligned}
p_jT &= \lambda T - (\lambda - p_j)T \\
&= \sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \left(1-\frac{p_j}{\lambda}\right)\sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) \\
&= \sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^{v+1}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i)
\end{aligned}$$

$$
\begin{aligned}
&= f(0) + \sum_{v=1}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \sum_{v=1}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v-1}\frac{\lambda^i}{i!}f(i)\\
&= f(0) + \sum_{v=1}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^v \left[\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \sum_{i=0}^{v-1}\frac{\lambda^i}{i!}f(i)\right]\\
&= f(0) + \sum_{v=1}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^v \frac{f(v)\lambda^v}{v!}\\
&= \sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^v \frac{f(v)\lambda^v}{v!}\\
&= \sum_{v=0}^{\infty}\frac{(\lambda-p_j)^v f(v)}{v!}\\
&= e^{\lambda-p_j}\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})].
\end{aligned}
$$

Hence $\frac{1}{\lambda}\sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) = \frac{e^{\lambda-p_j}}{p_j}\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})]$.

ii) For $j = 1, 2, \ldots, n$, let $S = \frac{1}{\lambda^2}\sum_{v=0}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i)$.

Then by (i), we have

$$
\begin{aligned}
&\frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v+1}\frac{\lambda^i}{i!}f(i)\\
&= \frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) + \frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\left(1-\frac{p_j}{\lambda}\right)^v \frac{\lambda^{v+1}}{(v+1)!}f(v+1)\\
&= S + \frac{1}{\lambda^2}\sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) + \frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\left(1-\frac{p_j}{\lambda}\right)^v \frac{\lambda^{v+1}}{(v+1)!}f(v+1)\\
&= S + \frac{1}{\lambda^2}\sum_{v=0}^{\infty}\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) + \frac{1}{\lambda}\sum_{v=0}^{\infty}\frac{(\lambda-p_j)^v}{v!}f(v+1)\\
&= S + \frac{e^{\lambda-p_j}}{\lambda p_j}\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})] + \frac{e^{\lambda-p_j}}{\lambda}\mathrm{E}[f(\mathcal{P}_{\lambda-p_j}+1)], \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
p_j S &= \lambda S - (\lambda - p_j)S\\
&= \frac{1}{\lambda}\sum_{v=0}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \frac{1}{\lambda}\left(1-\frac{p_j}{\lambda}\right)\sum_{v=0}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^v \sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\lambda}\sum_{v=1}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \frac{1}{\lambda}\sum_{v=0}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^{v+1}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) \\
&= \frac{1}{\lambda}\sum_{v=1}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \frac{1}{\lambda}\sum_{v=1}^{\infty} (v-1)\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v-1}\frac{\lambda^i}{i!}f(i) \\
&= \left[\frac{1}{\lambda}\sum_{v=1}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \frac{1}{\lambda}\sum_{v=1}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v-1}\frac{\lambda^i}{i!}f(i)\right] \\
&\quad + \frac{1}{\lambda}\sum_{v=1}^{\infty} \left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v-1}\frac{\lambda^i}{i!}f(i) \\
&= \frac{1}{\lambda}\sum_{v=1}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^{v}\left[\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) - \sum_{i=0}^{v-1}\frac{\lambda^i}{i!}f(i)\right] + \frac{1}{\lambda}\sum_{v=0}^{\infty} \left(1-\frac{p_j}{\lambda}\right)^{v+1}\sum_{i=0}^{v}\frac{\lambda^i}{i!}f(i) \\
&= \frac{1}{\lambda}\sum_{v=1}^{\infty} v\left(1-\frac{p_j}{\lambda}\right)^{v}\frac{\lambda^v}{v!}f(v) + \left(1-\frac{p_j}{\lambda}\right)\frac{e^{\lambda-p_j}}{p_j}\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})] \\
&= \frac{1}{\lambda}\sum_{v=1}^{\infty} \frac{(\lambda-p_j)^v}{(v-1)!}f(v) + e^{\lambda-p_j}\left(\frac{1}{p_j}-\frac{1}{\lambda}\right)\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})] \\
&= \frac{\lambda-p_j}{\lambda}\sum_{v=0}^{\infty} \frac{(\lambda-p_j)^v}{v!}f(v+1) + e^{\lambda-p_j}\left(\frac{1}{p_j}-\frac{1}{\lambda}\right)\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})] \\
&= e^{\lambda-p_j}\left(1-\frac{p_j}{\lambda}\right)\mathrm{E}[f(\mathcal{P}_{\lambda-p_j}+1)] + e^{\lambda-p_j}\left(\frac{1}{p_j}-\frac{1}{\lambda}\right)\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})]. \qquad (4.2)
\end{aligned}
$$

Form (4.1) and (4.2) we have

$$
S = e^{\lambda-p_j}\left[\left(\frac{1}{p_j}-\frac{1}{\lambda}\right)\mathrm{E}[f(\mathcal{P}_{\lambda-p_j}+1)] + \left(\frac{1}{p_j^2}-\frac{1}{\lambda p_j}\right)\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})]\right]
$$

and, hence

$$
\begin{aligned}
&\frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\left(1-\frac{p_j}{\lambda}\right)^{v}\sum_{i=0}^{v+1}\frac{\lambda^i}{i!}f(i) \\
&= e^{\lambda-p_j}\left[\left(\frac{1}{p_j}-\frac{1}{\lambda}\right)\mathrm{E}[f(\mathcal{P}_{\lambda-p_j}+1)] + \left(\frac{1}{p_j^2}-\frac{1}{\lambda p_j}\right)\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})] + \frac{1}{\lambda p_j}\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})]\right. \\
&\quad \left. + \frac{1}{\lambda}\mathrm{E}[f(\mathcal{P}_{\lambda-p_j}+1)]\right] \\
&= e^{\lambda-p_j}\left[\frac{\mathrm{E}[f(\mathcal{P}_{\lambda-p_j}+1)]}{p_j} + \frac{\mathrm{E}[f(\mathcal{P}_{\lambda-p_j})]}{p_j^2}\right].
\end{aligned}
$$

□

The following theorem is our main result.

Theorem 4.2. *Let*

$$C_{call} = \sum_{j=1}^{n} \Big[\mathrm{E}[(\mathcal{P}_{\lambda - p_j} - z)^+ - (\mathcal{P}_\lambda - z)^+] - p_j \mathrm{E}[(\mathcal{P}_{\lambda - p_j} - z)^+ - (\mathcal{P}_{\lambda - p_j} + 1 - z)^+] \Big]$$

and $|\lambda - 1| \vee 1 = \max\{|\lambda - 1|, 1\}$. *Then*

(i) $\sup_{z \geq 0} \Big| E[(V_n - z)^+] - E[(\mathcal{P}_\lambda - z)^+] - C_{call} \Big| \leq \frac{2(2e^\lambda - 1)}{|\lambda - 1| \vee 1} \Big(\sum_{j=1}^{n} p_j^2 \Big)^2$ *and,*

(ii) $\Big| E[(V_n - z)^+] - E[(\mathcal{P}_\lambda - z)^+] - C_{call} \Big| \leq \frac{2(2e^\lambda - 1)(1 + \lambda)}{z(|\lambda - 1| \vee 1)} \Big(\sum_{j=1}^{n} p_j^2 \Big)^2$ *where* $z \geq 1$.

For $z > 1 + \lambda$, *the non-uniform bound in (ii) is smaller than uniform bound in (i).*

Proof. For $h : \mathbb{N} \cup \{0\} \to \mathbb{R}$, Stein ([18], pp. 86) derived the equation

$$\mathrm{E}[h(V_n)] = \mathcal{P}_\lambda(h) + \sum_{j=1}^{n} p_j^2 E\Big[\tilde{g}_\lambda(h)(V_n^{(j)}) \Big] \tag{4.3}$$

where $\tilde{g}_\lambda(h)(v) = g_\lambda(h)(v+2) - g_\lambda(h)(v+1)$, $V_n^{(j)} = V_n - Y_j$, $\mathcal{P}_\lambda(h) = \mathrm{E}[h(\mathcal{P}_\lambda)] = e^{-\lambda} \sum_{i=1}^{\infty} \frac{h(i)\lambda^i}{i!}$ and

$$g_\lambda(h)(v) = \begin{cases} 0 & \text{if } v = 0 \\ \\ \frac{(v-1)!}{\lambda^v} \sum_{i=0}^{v-1} \frac{\lambda^i}{i!} (h(i) - \mathcal{P}_\lambda(h)) & \text{if } v = 1, 2, \ldots. \end{cases} \tag{4.4}$$

In this proof, let h be a call function, i.e., $h(v) = h_z(v) = (v - z)^+$ for some $z \geq 0$. Then by (4.3),

$$\mathrm{E}[h_z(V_n)] = \mathcal{P}_\lambda(h_z) + \sum_{j=1}^{n} p_j^2 \mathrm{E}[\tilde{g}_\lambda(h_z)(V_n^{(j)})].$$

That is

$$\mathrm{E}[(V_n - z)^+] - \mathrm{E}[(\mathcal{P}_\lambda - z)^+] = \sum_{j=1}^{n} p_j^2 \mathrm{E}[\tilde{g}_\lambda(h_z)(V_n^{(j)})]. \tag{4.5}$$

By $\tilde{g}_\lambda : \mathbb{N} \cup \{0\} \to \mathbb{R}$ and (4.3), we choose $h = \tilde{g}_\lambda(h_z)$. Then for $j = 1, 2, \ldots, n$,

$$\mathrm{E}[\tilde{g}_\lambda(h_z)(V_n^{(j)})] = \mathcal{P}_{\lambda - p_j}(\tilde{g}_\lambda(h_z)) + \sum_{l=1, l\neq j}^{n} p_l^2 \mathrm{E}[\tilde{g}_{\lambda - p_j}(\tilde{g}_\lambda(h_z))(V_n^{(j,l)})] \tag{4.6}$$

where $V_n^{(j,l)} = V_n - (Y_j + Y_l)$. Hence, by (4.5) and (4.6) we have

$$\begin{aligned} &E[(V_n - z)^+] - E[(\mathcal{P}_\lambda - z)^+] \\ &= \sum_{j=1}^{n} p_j^2 \mathcal{P}_{\lambda - p_j}(\tilde{g}_\lambda(h_z)) + \sum_{j=1}^{n} p_j^2 \sum_{l=1, l\neq j}^{n} p_l^2 \mathrm{E}[\tilde{g}_{\lambda - p_j}(\tilde{g}_\lambda(h_z))(V_n^{(j,l)})]. \end{aligned}$$

That is

$$\begin{aligned} &\left| E[(V_n - z)^+] - E[(\mathcal{P}_\lambda - z)^+ - \sum_{j=1}^{n} p_j^2 \mathcal{P}_{\lambda - p_j}(\tilde{g}_\lambda(h_z)) \right| \\ &= \left| \sum_{j=1}^{n} p_j^2 \sum_{l=1, l\neq j}^{n} p_l^2 \mathrm{E}[\tilde{g}_{\lambda - p_j}(\tilde{g}_\lambda(h_z))(V_n^{(j,l)})] \right|. \end{aligned} \tag{4.7}$$

First, we will show that $\sum_{j=1}^{n} p_j^2 \mathcal{P}_{\lambda - p_j}(\tilde{g}_\lambda(h_z)) = C_{call}$. Note that by (4.4),

$$\begin{aligned} \mathcal{P}_{\lambda - p_j}(\tilde{g}_\lambda(h_z)) &= \mathrm{E}[\tilde{g}_\lambda(h_z)(\mathcal{P}_{\lambda - p_j})] \\ &= e^{-(\lambda - p_j)} \sum_{v=0}^{\infty} \frac{\tilde{g}_\lambda(h_z)(v)(\lambda - p_j)^v}{v!} \\ &= e^{-(\lambda - p_j)} \sum_{v=0}^{\infty} \frac{(g_\lambda(h_z)(v+2) - g_\lambda(h_z)(v+1))(\lambda - p_j)^v}{v!} \\ &= e^{-(\lambda - p_j)} \left[\sum_{v=0}^{\infty} \frac{g_\lambda(h_z)(v+2)(\lambda - p_j)^v}{v!} + \sum_{v=0}^{\infty} \frac{g_\lambda(h_z)(v+1)(\lambda - p_j)^v}{v!} \right] \\ &= e^{-(\lambda - p_j)} \Big[-R_1 + R_2 + R_3 - R_4 \Big] \end{aligned} \tag{4.8}$$

where

$$R_1 = \frac{1}{\lambda}\sum_{v=0}^{\infty}\frac{(\lambda - p_j)^v}{\lambda^v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}(i-z)^+,$$
$$R_2 = \frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\frac{(\lambda - p_j)^v}{\lambda^v}\sum_{i=0}^{v+1}\frac{\lambda^i}{i!}(i-z)^+,$$
$$R_3 = \frac{1}{\lambda}\sum_{v=0}^{\infty}\frac{(\lambda - p_j)^v}{\lambda^v}\sum_{i=0}^{v}\frac{\lambda^i}{i!}\mathcal{P}_\lambda(h_z),$$
$$R_4 = \frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\frac{(\lambda - p_j)^v}{\lambda^v}\sum_{i=0}^{v+1}\frac{\lambda^i}{i!}\mathcal{P}_\lambda(h_z).$$

By Lemma 4.1, we choose $f = h_z$ then

$$R_1 = \frac{e^{\lambda - p_j}}{p_j}\mathrm{E}[(\mathcal{P}_{\lambda - p_j} - z)^+] \tag{4.9}$$

and

$$R_2 = e^{\lambda - p_j}\left[\frac{1}{p_j}\mathrm{E}[(\mathcal{P}_{\lambda - p_j} + 1 - z)^+] + \frac{1}{p_j^2}\mathrm{E}[(\mathcal{P}_{\lambda - p_j} - z)^+]\right]. \tag{4.10}$$

To calculate R_3 and R_4, we will apply Lemma 4.1 by letting $f = 1$. Hence

$$\begin{aligned} R_3 &= \frac{e^{\lambda - p_j}}{p_j}\mathcal{P}_\lambda(h_z) \\ &= \frac{e^{\lambda - p_j}}{p_j}\mathrm{E}[(\mathcal{P}_\lambda - z)^+] \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} R_4 &= \left[\frac{1}{\lambda^2}\sum_{v=0}^{\infty}(v+1)\frac{(\lambda - p_j)^v}{\lambda^v}\sum_{i=0}^{v+1}\frac{\lambda^i}{i!}\right]\mathcal{P}_\lambda(h_z) \\ &= e^{\lambda - p_j}\left[\frac{1}{p_j} + \frac{1}{p_j^2}\right]\mathcal{P}_\lambda(h_z) \\ &= e^{\lambda - p_j}\left[\frac{1}{p_j} + \frac{1}{p_j^2}\right]\mathrm{E}[(\mathcal{P}_\lambda - z)^+]. \end{aligned} \tag{4.12}$$

By (4.8) - (4.12), we have

$$\begin{aligned}
\mathcal{P}_{\lambda-p_j}(\tilde{g}_\lambda(h_z)) &= e^{-(\lambda-p_j)}\Big[-R_1+R_2+R_3-R_4\Big]\\
&= -\frac{\mathrm{E}[(\mathcal{P}_{\lambda-p_j}-z)^+]}{p_j}+\frac{\mathrm{E}[(\mathcal{P}_{\lambda-p_j}+1-z)^+]}{p_j}+\frac{\mathrm{E}[(\mathcal{P}_{\lambda-p_j}-z)^+]}{p_j^2}\\
&\quad+\frac{\mathrm{E}[(\mathcal{P}_{\lambda}-z)^+]}{p_j}-\frac{\mathrm{E}[(\mathcal{P}_{\lambda}-z)^+]}{p_j}-\frac{\mathrm{E}[(\mathcal{P}_{\lambda}-z)^+]}{p_j^2}.\\
&= \frac{\mathrm{E}[(\mathcal{P}_{\lambda-p_j}-z)^+]-\mathrm{E}[(\mathcal{P}_{\lambda}-z)^+]}{p_j^2}\\
&\quad-\frac{\mathrm{E}[(\mathcal{P}_{\lambda-p_j}-z)^+])-\mathrm{E}[(\mathcal{P}_{\lambda-p_j}+1-z)^+]}{p_j}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{j=1}^{n}p_j^2\mathcal{P}_{\lambda-p_j}(\tilde{g}_\lambda(h_z)) &= \sum_{j=1}^{n}\Big[\mathrm{E}[(\mathcal{P}_{\lambda-p_j}-z)^+]-\mathrm{E}[(\mathcal{P}_{\lambda}-z)^+]\Big]\\
&\quad-\sum_{j=1}^{n}p_j\Big[\mathrm{E}[(\mathcal{P}_{\lambda-p_j}-z)^+]-\mathrm{E}[(\mathcal{P}_{\lambda-p_j}+1-z)^+]\Big]\\
&= C_{call}. \qquad (4.13)
\end{aligned}$$

Next, we will show that

$$\left|\sum_{j=1}^{n}p_j^2\sum_{l=1,l\neq j}^{n}p_l^2\mathrm{E}[\tilde{g}_{\lambda-p_j}(\tilde{g}_\lambda(h_z)(V_n^{(j,l)}))]\right|\le\frac{2(2e^\lambda-1)}{(|\lambda-1|\vee 1)}\left(\sum_{j=1}^{n}p_j^2\right)^2$$

and for $z>1$,

$$\left|\sum_{j=1}^{n}p_j^2\sum_{l=1,l\neq j}^{n}p_l^2\mathrm{E}[\tilde{g}_{\lambda-p_j}(\tilde{g}_\lambda(h_z)(V_n^{(j,l)}))]\right|\le\frac{2(2e^\lambda-1)(1+\lambda)}{z(|\lambda-1|\vee 1)}\left(\sum_{j=1}^{n}p_j^2\right)^2.$$

For $h:\mathbb{N}\cup\{0\}\to\mathbb{R}$, Stein ([18], pp. 89) show that

$$||\tilde{g}_\lambda(h)||_\infty\le 2||h||_\infty\min\left(\frac{1}{\lambda},1\right)$$

where $||h||_\infty = \sup_{v\in\mathbb{N}\cup\{0\}} |h(v)|$. If $h = \tilde{g}_\lambda(h_z)$ then

$$||\tilde{g}_{\lambda-p_j}(\tilde{g}_\lambda(h_z))||_\infty \leq 2||\tilde{g}_\lambda(h_z)||_\infty \min\left(\frac{1}{\lambda-p_j},1\right).$$

Observe that for $\lambda > 1$, we have

$$\min(\frac{1}{\lambda-p_j},1) \leq \min(\frac{1}{\lambda-1},1) \leq \frac{1}{|\lambda-1|\vee 1}$$

and for $0 < \lambda \leq 1$

$$\min(\frac{1}{\lambda-p_j},1) = 1 = \frac{1}{|\lambda-1|\vee 1}.$$

Hence

$$\begin{aligned}E[\tilde{g}_{\lambda-p_j}(\tilde{g}_\lambda(h_z)(V_n^{(j,l)}))] &\leq ||\tilde{g}_{\lambda-p_j}(\tilde{g}_\lambda(h_z)||_\infty \\ &\leq \frac{2||\tilde{g}_\lambda(h_z)||_\infty}{|\lambda-1|\vee 1}.\end{aligned}$$

This implies,

$$\left|\sum_{j=1}^n p_j^2 \sum_{l=1,l\neq j}^n p_l^2 E[\tilde{g}_{\lambda-p_j}(\tilde{g}_\lambda(h_z)(V_n^{(j,l)}))]\right| \leq \frac{2||\tilde{g}_\lambda(h_z)||_\infty}{|\lambda-1|\vee 1}\left(\sum_{j=1}^n p_j^2\right)^2. \tag{4.14}$$

Note that

$$\begin{aligned}||\tilde{g}_\lambda(h_z)||_\infty &= \sup_{v\in\mathbb{N}\cup\{0\}} |g_\lambda(h_z)(v+2) - g_\lambda(h_z)(v+1)| \\ &= \sup_{v\in\mathbb{N}} |g_\lambda(h_z)(v+1) - g_\lambda(h_z)(v)| \\ &= \sup_{v\in\mathbb{N}} |g_z(v+1) - g_z(v)|, \\ &= \sup_{v\in\mathbb{N}} |\Delta g_z(v)| \end{aligned} \tag{4.15}$$

where the third equality follows form the fact that $g_\lambda(h_z)(v) = -g_z(v)$ for all $v = 0, 1, \ldots$ ([18], pp. 82).

Hence, by Lemma 2.2 (ii), (3.9), (4.14) and (4.15), we have

$$\left|\sum_{j=1}^{n}\sum_{l=1,l\neq j}^{n} p_j^2p_l^2\mathrm{E}[\tilde{g}_{\lambda-p_j}(\tilde{g}_{\lambda}(h_z)(V_n^{(j,l)}))]\right| \leq \frac{2(2e^{\lambda}-1)}{|\lambda-1|\vee 1}\left(\sum_{j=1}^{n}p_j^2\right)^2,$$

and for $z>1$,

$$\left|\sum_{j=1}^{n}\sum_{l=1,l\neq j}^{n} p_j^2p_l^2\mathrm{E}[\tilde{g}_{\lambda-p_j}(\tilde{g}_{\lambda}(h_z)(V_n^{(j,l)}))]\right| \leq \frac{2(2e^{\lambda}-1)(1+\lambda)}{z(|\lambda-1|\vee 1)}\left(\sum_{j=1}^{n}p_j^2\right)^2.$$

From this fact, (4.7) and (4.13), we have

$$\sup_{z\geq 0}\left|E[(V_n-z)^+]-E[(\mathcal{P}_\lambda-z)^+-C_{call}\right| \leq \frac{2(2e^{\lambda}-1)}{|\lambda-1|\vee 1}\left(\sum_{j=1}^{n}p_j^2\right)^2,$$

and for $z>1$

$$\left|E[(V_n-z)^+]-E[(\mathcal{P}_\lambda-z)^+-C_{call}\right| \leq \frac{2(2e^{\lambda}-1)(1+\lambda)}{z(|\lambda-1|\vee 1)}\left(\sum_{j=1}^{n}p_j^2\right)^2.$$

□

Remark 4.3.

1) By Corollary 3.3, we have

$$\sup_{z\geq 0}\left|\mathrm{E}[(V_n-z)^+]-\mathrm{E}[(\mathcal{P}_\lambda-z)^+]\right| \leq (2e^{\lambda}-1)\sum_{i=1}^{n}p_i^2.$$

If $\sum_{j=1}^{n}p_j^2 \to 0$ *as* $n\to\infty$ *then this bound tends to* 0 *as* $n\to\infty$*. In this case, the rate* $\left(\sum_{j=1}^{n}p_j^2\right)^2$ *in Theorem 4.2 is sharper than the rate* $\sum_{j=1}^{n}p_j^2$ *in Corollary 3.3.*

2) In case of Y_i*'s are identically distributed with* $p_i=\frac{1}{n^{\delta}}$ *where* $\delta>1$*, we have the order of bound in Corollary 3.3 and Theorem 4.2 are* $O\left(\frac{1}{n^{2\delta-1}}\right)$ *and* $O\left(\frac{1}{n^{2(2\delta-1)}}\right)$ *respectively. Hence the rate of convergence of the bound in Theorem 4.2 is sharper than the bound in Corollary 3.3.*

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