การไม่มีจริงของผลเฉลยวงกว้างของสมการเชิงพาราโบลาเทียมกึ่งเชิงเส้นในโดเมนรูปกรวย

นายธานินทร์ หนูแดง

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2559

NONEXISTENCE OF GLOBAL SOLUTIONS OF SEMILINEAR PSEUDOPARABOLIC EQUATION IN CONE DOMAIN

Mr. Tanin Noodaeng

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2016

Thesis Title	NONEXISTENCE OF GLOBAL SOLUTIONS OF SEMILINEAR PSEUDOPARABOLIC EQUATION IN CONE DOMAIN
By	Mr. Tanin Noodaeng
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Sujin Khomrutai, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.... Dean of the Faculty of Science

(Associate Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

.... Chairman

(Associate Professor Nataphan Kitisin, Ph.D.)

.... Thesis Advisor

(Assistant Professor Sujin Khomrutai, Ph.D.)

.... Examiner

(Assistant Professor Ratinan Boonklurb, Ph.D.)

.... External Examiner

(Assistant Professor Tawikan Treeyaprasert, Ph.D.)

ธานินทร์ หนูแดง : การไม่มีจริงของผลเฉลยวงกว้างของสมการเชิงพาราโบลาเทียมกึ่งเชิงเส้นในโดเมนรูปกรวย (NONEXISTENCE OF GLOBAL SOLUTIONS OF SEMILINEAR PSEUDOPARABOLIC EQUATION IN CONE DOMAIN)
อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผู้ช่วยศาสตราจารย์ ดร.สุจินต์ คมฤทัย, 25 หน้า.

ในวิทยานิพนธ์ฉบับนี้เราศึกษาสมการเชิงพาราโบลาเทียมกึ่งเชิงเส้น

$$\begin{cases} \partial_t u - \Delta \partial_t u = \Delta u + a(x) u^p, & x \in \mathcal{C}, t > 0 \\ u(x,0) = u_0(x), & x \in \mathcal{C}, \\ u(x,\mathrm{t}) = 0 & x \in \partial\mathcal{C}, t > 0 \end{cases}$$

โดยที่ $\mathcal{C}$ เป็นโดเมนรูปกรวย $p > 1$ และ $u_0(x)$ เป็นฟังก์ชันที่ไม่เป็นลบ และ $a(x)$ เป็นฟังก์ชันศักย์ที่สอดคล้องว่ามี $\sigma > -2$ ซึ่ง

$$a(x) \gtrsim |x|^\sigma \quad \text{สำหรับทุกค่า } x \in \mathcal{C}$$

เราพิสูจน์ได้ว่าถ้า

$$1 < p < p_c := \min\left\{1 + \frac{\sigma + 2}{n + e_*}, 1 + \frac{2}{e_*}\right\}$$

โดยที่ $e_* = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(\Omega)}$ และ $\lambda_1(\Omega)$ เป็นค่าเฉพาะตัวแรกของลาปลาซ-เบลตรามี Δ_ω บน Ω แล้วจะได้ว่า $u \equiv 0$ เป็นผลเฉลยอย่างอ่อนเพียงหนึ่งเดียวที่นิยามสำหรับทุก $t \in [0, \infty)$ ของสมการเชิงพาราโบลาเทียมกึ่งเชิงเส้นข้างต้น

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต
สาขาวิชา คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก
ปีการศึกษา 2559	

＃＃5772014723 : MAJOR MATHEMATICS

KEYWORDS : SEMILINEAR PSEUDOPARABOLIC EQUATION, CONE DOMAIN

TANIN NOODAENG : NONEXISTENCE OF GLOBAL SOLUTIONS OF SEMILINEAR PSEUDOPARABOLIC EQUATION IN CONE DOMAIN

ADVISOR : ASSISTANT PROFESSOR SUJIN KHOMRUTAI, Ph.D., 25 pp.

In this thesis, we investigate the following *semilinear pseudoparabolic* equation:

$$\begin{cases} \partial_t u - \triangle \partial_t u = \triangle u + a(x)u^p & x \in \mathcal{C}, t > 0, \\ u(x,0) = u_0(x) & x \in \mathcal{C}, \\ u(x,t) = 0 & x \in \partial\mathcal{C}, t > 0, \end{cases}$$

where $\mathcal{C}$ is a cone domain, $p > 1$, u_0 is a nonnegative function and $a(x)$ is a potential satisfying that there exist $\sigma > -2$ such that

$$a(x) \gtrsim |x|^\sigma \quad \text{for all} \quad x \in \mathcal{C}.$$

We prove that if

$$1 < p < p_c := \min\left\{1 + \frac{\sigma+2}{n+e_*}, 1 + \frac{2}{e_*}\right\},$$

where $e_* = -\frac{n-2}{2} + \sqrt{(\frac{n-2}{2})^2 + \lambda_1(\Omega)}$ and $\lambda_1(\Omega)$ is the first eigenvalue of the Laplace-Beltrami operator Δ_ω on Ω, then $u \equiv 0$ is the only weak solution which is defined for all $t \in [0, \infty)$, to the above semilinear pseudoparabolic equation.

Department: Mathematics and Computer Science Student's Signature: ....

Field of Study: ....Mathematics...... Advisor's Signature: ....

Academic Year: ....2016.........

ACKNOWLEDGEMENTS

I would like to acknowledge and extend my deepest gratitude to my advisor, Assistant Professor Dr.Sujin Khomrutai, for his supervision and motivation that encourage me to accomplish this thesis. I am also sincerely grateful to Associate Professor Dr.Nataphan Kitisin, Assistant Professor Dr.Ratinan Boonklurb and Assistant Professor Dr.Tawikan Treeyaprasert, the thesis committee, for their suggestion and valuable comments. Moreover, I would like to express my heartiest appreciation to my parents for supporting me. This thesis is funded by The Development and Promotion of Science and Technology Talents Project (DPST).

CONTENTS

CHAPTER I

INTRODUCTION

There have been many studies of nonlinear heat equations with initial and boundary conditions of the form

$$\begin{cases} \partial_t u = \triangle u + f(\bigtriangledown u, u, x, t) & x \in \Omega, t > 0, \\ u(x,t) = g(x,t) & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $u : \Omega \times [0, \infty) \to \mathbb{R}, \Omega$ is an open subset of $\mathbb{R}^n$, f and g are given functions and u_0 is the initial condition of u. The heat equations have been employed to model many phenomena in physic, chemistry, biology, population dynamic etc. In 1960's, H. Fujita [5] studied positive solutions of (1.1) in the case that $f = u^p$ and $\Omega = \mathbb{R}^n$, i.e.,

$$\begin{cases} \partial_t u = \triangle u + u^p & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x) \geq 0, \end{cases} \tag{1.2}$$

where $p > 1$ is a real constant. He proved that $p_c = 1 + \frac{2}{n}$ is a critical value for the exponent p in the sense that if $1 < p < p_c$, then the solution with nontrivial initial condition blows up in a finite time, whereas if $p > p_c$, then the solution can be global provided u_0 is sufficiently small and it blows up in a finite time provided u_0 is sufficiently large. We recall that a solution u is said to be *blow-up in a finite time* if there is $T > 0$ such that $\|u(\cdot,t)\|_{L^\infty} \to \infty$ as $t \to T^-$ and u is said to be *global* if it is defined on $\mathbb{R}^n \times [0, \infty)$. In the case that $p = p_c$, the nontrivial solution is shown to be blow-up by K. Hayakawa [6], when $n = 1, 2$ and by K. Kobayashi et al. [8] (and also F.B. Weissler [16]) for the general $n \geq 1$. The number $p_c = 1 + \frac{2}{n}$ is called the Fujita critical exponent for the semilinear heat equation (1.2).

In 2002, G.G. Laptev derived the Fujita type critical exponent for weak solutions to the semilinear heat equation on cone domains using the test function method developed by Pohozaev and Miditeri ([12], [13]). He studied the following differential inequality

$$u_t - \Delta(|u|^{m-1}u) \geq |x|^\sigma |u|^q \qquad (x,t) \in \mathcal{C} \times (0, \infty), \tag{1.3}$$

where $\mathcal{C}$ is a cone domain, $1 \leq m < q$ and $\sigma > -2$ and he proved that (1.3) has no nontrivial global weak solution when $q < q^* := m + \frac{2+\sigma}{e^*+2}$ (see in Chapter 2 for the definition of e^*).

Apart from the heat equations, there is another type of equations which are widely studied, the so-called, *pseudo parabolic equations*, which have the following form

$$\begin{cases} \partial_t u - k\triangle\partial_t u = \triangle u + a(x)u^p & x \in \mathcal{C}, t > 0, \\ u(x,t) = 0 & x \in \partial\mathcal{C}, t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.4}$$

where $p \geq 1, k > 0$ is a constant, $a(x) \geq 0$ and $u_0 \geq 0$ are given functions. It is used to model many physical systems such as lightning [1], seepage of fluids in fissured rocks [2], radiation with time delay [11], the heat conduction models [15] etc. Observe that H. Fujita studied (1.4) in the case $k = 0, a(x) = 1$ and $\mathcal{C}$ is replaced with $\mathbb{R}^n$.

In 2009, Y. Cao, J. Yin and C. P. Wang [3] studied (1.4) when $k > 0, a(x) = 1$ and $\mathcal{C}$ is replaced with $\mathbb{R}^n$ and proved that $p_c = 1 + \frac{2}{n}$ is the Fujita type critical exponent for the initial value problem. This is the same as the corresponding result for the semilinear heat equation.

Moreover, in 2015, S. Khomrutai [7] studied (1.4) in the case $k = 1$, $a(x)$ is an unbounded function and $\mathcal{C}$ is replaced with $\mathbb{R}^n$. He proved by the test function method that if $a(x) = |x|^\sigma$ where $0 \leq \sigma \leq \frac{4}{n-2}$ for $n \geq 3$ and $\sigma \in [0, \infty)$ for $n = 1, 2$, then the critical exponent of (1.4) is $p_c = 1 + \frac{\sigma+2}{n}$.

There have been many studies on nonlinear pseudo parabolic on $\mathbb{R}^n$ or a bounded domain in $\mathbb{R}^n$ such as above. To the author knowledge, there is no investigation of the problem on cone domains. The aim of this work is to fill this gap.

In my thesis, motivated by [7], [9], we consider the Cauchy problem (1.4) on a cone domain and we have the main result that if

$$1 < p < p_c := \min\left\{1 + \frac{\sigma + 2}{n + e_*}, 1 + \frac{2}{e_*}\right\},$$

where $a(x) \gtrsim |x|^\sigma$, then $u \equiv 0$ is the only solution which is defined for all $t \in [0, \infty)$.

This thesis is organized into four chapters as follows.

In Chapter II, we introduce some basic analysis, definitions and theories that are useful to study (1.4) on cone domains. Next, in Chapter III, we prove some preliminary estimates and we introduce the notion of a weak solution for (1.4). Finally, we proved our main results in Chapter IV.

CHAPTER II

PRELIMINARIES

In this chapter, we give some basic concepts in PDEs which are omitted the details of proofs. The proof can be found in common PDEs textbooks.

2.1 Bisic analysis

Lemma 2.1. *If* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then, for any* $u, v \geq 0$*,*

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q. \tag{2.1}$$

Lemma 2.2. *(Holder's inequality). Let* f, g *be measurable functions on a measure space* (X, μ)*. Let* $1 < p, q < \infty$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then,*

$$\int_X |fg| d\mu \leq \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q d\mu\right)^{\frac{1}{q}}. \tag{2.2}$$

Theorem 2.3. *(Minkowski's inequalilty). Let* $1 \leq p < \infty$ *and let* f, g *be measurable functions on a measure space* (X, μ)*. Then,*

$$\left(\int_X |f+g|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p d\mu\right)^{\frac{1}{p}}. \tag{2.3}$$

Theorem 2.4. *Let* (X, μ) *and* (Y, ν) *be* σ*-finite measure spaces and* $f : X \times Y \longrightarrow [0, \infty)$ *be an integrable function. Then,*

$$\begin{aligned} \int_{X\times Y} f(x,y)\ d(\mu\times\nu) &= \int_X \left(\int_Y f(x,y)\ d\nu(y)\right)\ d\mu(x) \\ &= \int_Y \left(\int_X f(x,y)\ d\mu(x)\right)\ d\nu(y). \end{aligned} \tag{2.4}$$

2.2 Integration on cones

Let

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + ... + x_n^2} = 1\}$$

be the unit sphere. Any $x \in \mathbb{R}^n$ $(x \neq 0)$ can be written in polar coordinations as $x = r\omega$, where

$$r := |x| \in (0, \infty) \quad \text{and} \quad \omega := \frac{x}{|x|} \in S^{n-1}.$$

For a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ which is integrable, it is well-known that

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \int_{S^{n-1}} f(r\omega) r^{n-1} d\omega \, dr, \tag{2.5}$$

where $d\omega$ is the surface measure on the unit sphere.

Definition 2.5. Let $\Omega \subseteq S^{n-1}$ be an open set and $\rho > 0$. A set of the form

$$\mathcal{C}_{\rho,\Omega} = \{x = r\omega \in \mathbb{R}^n : r > \rho, \omega \in \Omega\} \tag{2.6}$$

is called a *cone* in $\mathbb{R}^n$.

Note that the boundary of the cone $\mathcal{C}_{\rho,\Omega}$ is $\partial\mathcal{C}_{\rho,\Omega} = \{r\omega \in \mathbb{R}^n \big| r = \rho, \omega \in \Omega\} \cup \{r\omega \in \mathbb{R}^n \big| r > \rho, \omega \in \partial\Omega\} =: \partial\mathcal{C}^1 \cup \partial\mathcal{C}^2$.

Example 2.6. If $\Omega = S^{n-1}$ and $\rho = 0$, then $\mathcal{C}_{0,S^{n-1}} = \mathbb{R}^n - \{0\}$ and $\partial\mathcal{C}_{\rho,\Omega} = \{0\}$.

Example 2.7. If $\Omega = \{x = (x_1, x_2, x_3, ..., x_n) \in S^{n-1}, x_n > 0\}$ and $\rho = 0$, then $\mathcal{C}_{0,S^{n-1}}$ is the upperhalf space and $\partial\mathcal{C}_{\rho,\Omega} = \{(x_1, x_2, x_3, ..., x_{n-1}, 0)\}$.

Proposition 2.8. *Let $f : \mathcal{C}_{\rho,\Omega} \longrightarrow \mathbb{R}$ be an integrable function. Then,*

$$\int_{\mathcal{C}_{\rho,\Omega}} f(x) \, dx = \int_\rho^\infty \int_\Omega f(r\omega) r^{n-1} \, d\omega \, dr. \tag{2.7}$$

Lemma 2.9 (Integrating by parts)**.** *Let* $f : \mathcal{C}_{\rho,\Omega} \longrightarrow \mathbb{R}$ *be a differentiable function. Then, for all* $g \in C_c^\infty(\mathcal{C}_{\rho,\Omega})$(= *the space of smooth functions with compact support, defined on* $\mathcal{C}_{\rho,\Omega}$),

$$\int_{\mathcal{C}_{\rho,\Omega}} (\partial_{x_i} f) g \ dx = -\int_{\mathcal{C}_{\rho,\Omega}} (\partial_{x_i} g) f \ dx. \tag{2.8}$$

Theorem 2.10 (Green's formula)**.** *Let* $f, g \in C^2(\mathcal{C}_{\rho,\Omega}) \cap C(\overline{\mathcal{C}_{\rho,\Omega}})$*. Then,*

$$\int_{\mathcal{C}_{\rho,\Omega}} (f\Delta g - g\Delta f)\, dx = \int_{\partial\mathcal{C}_{\rho,\Omega}} \left(f\frac{\partial g}{\partial \overrightarrow{n}} - g\frac{\partial f}{\partial \overrightarrow{n}} \right) dS, \tag{2.9}$$

where $\frac{\partial g}{\partial \overrightarrow{n}}$ *is the directional derivative of* g *with respect to the outward unit normal vector* $\overrightarrow{n}$.

2.3 Helmholtz eigenvalue

For simplicity, we denote $\mathcal{C} = \mathcal{C}_{\rho,\Omega}$. We shall use the first Helmholtz eigenvalue $\lambda_\omega \equiv \lambda_1(\Omega) > 0$ of the Laplace-Beltrami operator Δ_ω on Ω and the corresponding eigenfunction $\Psi > 0$, that is, λ_1 and Ψ satisfy,

$$\begin{cases} \Delta_\omega \Psi + \lambda_1 \Psi = 0 & \text{in } \ \Omega, \\ \Psi = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{2.10}$$

The following results are true by standard elliptic theory (see for instance [4]):

1. $\Psi \in C^\infty(\Omega) \cap C(\overline{\Omega})$,
2. $\Psi(x) > 0 \quad \forall x \in \Omega$,
3. $\frac{\partial \Psi(x)}{\partial \overrightarrow{n}} \leq 0 \quad \forall x \in \partial\Omega$ by Hopf's lemma.

CHAPTER III

MAIN RESULT 1

3.1 Weak solution

Let $\rho \geq 0$ and $\Omega \subset S^{n-1}$ be an open set. We denote $\mathcal{C} = \mathcal{C}_{\rho,\Omega}$, $\partial\mathcal{C}^1 = \partial\mathcal{C}^1_{\rho,\Omega}$ and $\partial\mathcal{C}^2 = \partial\mathcal{C}^2_{\rho,\Omega}$. In this work, we consider non-negative solutions $u = u(x,t)$ for the semilinear pseudoparabolic equation

$$\begin{cases} \partial_t u - \triangle\partial_t u = \triangle u + a(x)u^p & \text{for } \ x \in \mathcal{C}, t > 0, \\ u(x,0) = u_0(x) & \text{for } \ x \in \mathcal{C}, \\ u(x,t) = 0 & \text{for } \ x \in \partial\mathcal{C}, t > 0, \end{cases} \tag{3.1}$$

where $p > 1$, $a(x)$ and u_0 are given non-negative functions.

We have the following relationship between classical and weak solutions for (3.1).

Proposition 3.1. *If u is a classical solution of (3.1), then for any $\varphi \in C_c^\infty(\mathcal{C} \times [0,\infty))$, we have*

$$\int_0^\infty \int_{\mathcal{C}} u\mathcal{A}\varphi \ dxdt + \int_0^\infty \int_{\mathcal{C}} \varphi a(x)u^p \ dxdt = \int_{\mathcal{C}} u_0(x)\Big(\triangle\varphi(x,0) - \varphi(x,0)\Big) \ dx, \tag{3.2}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\varphi = \partial_t\varphi - \triangle\partial_t\varphi + \triangle\varphi. \tag{3.3}$$

Proof. Multiplying (3.1) with φ and integrating over $\mathcal{C} \times [0, \infty)$, we have

$$\begin{aligned}\int_0^\infty \int_{\mathcal{C}} \varphi \partial_t u \; dxdt - \int_0^\infty \int_{\mathcal{C}} \varphi \triangle \partial_t u \; dxdt \\ = \int_0^\infty \int_{\mathcal{C}} \varphi \triangle u \; dxdt + \int_0^\infty \int_{\mathcal{C}} \varphi a(x) u^p \; dxdt. \qquad (3.4)\end{aligned}$$

Consider the first term on the left hand side of (3.4). Applying the Fubini's theorem (Theorem 2.4) and then integrating by parts with respect to t, we get

$$\begin{aligned}\int_0^\infty \int_{\mathcal{C}} \varphi \partial_t u \; dxdt &= \int_{\mathcal{C}} \int_0^\infty \varphi \partial_t u \; dtdx \\ &= \int_{\mathcal{C}} \left(\varphi u \Big|_0^\infty - \int_0^\infty u \partial_t \varphi \; dt \right) dx \\ &= - \int_{\mathcal{C}} \varphi(x, 0) u(x, 0) \; dx - \int_{\mathcal{C}} \int_0^\infty u \partial_t \varphi \; dtdx \\ &= - \int_{\mathcal{C}} \varphi(x, 0) u_0(x) \; dx - \int_0^\infty \int_{\mathcal{C}} u \partial_t \varphi \; dxdt, \qquad (3.5)\end{aligned}$$

where we have used that φ has compact support in the third equality.

Next, we consider the second term on the left hand side of (3.4). We get by the Green's identity (Theorem 2.10) that

$$\begin{aligned}\int_0^\infty \int_{\mathcal{C}} \varphi \triangle \partial_t u \; dxdt &= \left(\int_0^\infty \int_{\mathcal{C}} \varphi \triangle \partial_t u \; dxdt - \int_0^\infty \int_{\mathcal{C}} \partial_t u \triangle \varphi \; dxdt \right) \\ &\quad + \int_0^\infty \int_{\mathcal{C}} \partial_t u \triangle \varphi \; dxdt \\ &= \int_0^\infty \int_{\partial \mathcal{C}} \left(\varphi \frac{\partial(\partial_t u)}{\partial \overrightarrow{n}} - \partial_t u \frac{\partial \varphi}{\partial \overrightarrow{n}} \right) dSdt + \int_0^\infty \int_{\mathcal{C}} \partial_t u \triangle \varphi \; dxdt \\ &= \int_0^\infty \int_{\partial \mathcal{C}} \varphi \frac{\partial(\partial_t u)}{\partial \overrightarrow{n}} \; dSdt - \int_0^\infty \int_{\partial \mathcal{C}} \partial_t u \frac{\partial \varphi}{\partial \overrightarrow{n}} \; dSdt \\ &\quad + \int_0^\infty \int_{\mathcal{C}} \partial_t u \triangle \varphi \; dxdt.\end{aligned}$$

Since $\partial_t u = 0$ and $\varphi = 0$ on $\partial \mathcal{C}$, we get

$$\int_0^\infty \int_{\mathcal{C}} \varphi \triangle \partial_t u \; dxdt = \int_0^\infty \int_{\mathcal{C}} (\triangle \varphi) \partial_t u \; dxdt.$$

Applying the integrating by parts with respect to t, we get

$$\begin{aligned}\int_0^\infty\int_{\mathcal{C}}\varphi\triangle\partial_t u\;dxdt &= \int_0^\infty\int_{\mathcal{C}}(\triangle\varphi)\partial_t u\;dxdt\\ &= \int_{\mathcal{C}} u\triangle\varphi\Big|_0^\infty\;dx - \int_{\mathcal{C}}\int_0^\infty u\frac{\partial\triangle\varphi}{\partial t}\;dtdx\\ &= -\int_{\mathcal{C}} u_0(x)\triangle\varphi(x,0)\;dx - \int_0^\infty\int_{\mathcal{C}} u\frac{\partial\triangle\varphi}{\partial t}\;dxdt. \qquad (3.6)\end{aligned}$$

Now, we consider the first term on the right hand side of (3.4), we get

$$\begin{aligned}\int_0^\infty\int_{\mathcal{C}}\varphi\triangle u\;dxdt &= \left(\int_0^\infty\int_{\mathcal{C}}\varphi\triangle u\;dxdt - \int_0^\infty\int_{\mathcal{C}} u\triangle\varphi\;dxdt\right)\\ &\quad + \int_0^\infty\int_{\mathcal{C}} u\triangle\varphi\;dxdt\\ &= \int_0^\infty\int_{\partial\mathcal{C}}\left[\varphi\frac{\partial u}{\partial\overrightarrow{n}} - u\frac{\partial\varphi}{\partial\overrightarrow{n}}\right]dSdt + \int_0^\infty\int_{\mathcal{C}} u\triangle\varphi\;dxdt.\end{aligned}$$

Since $u = 0$ and $\varphi = 0$ on $\partial\mathcal{C}$, we get $\displaystyle\int_0^\infty\int_{\partial\mathcal{C}}\varphi\frac{\partial(\partial_t u)}{\partial\overrightarrow{n}}\;dSdt = 0$ and $\displaystyle\int_0^\infty\int_{\partial\mathcal{C}} u\frac{\partial\varphi}{\partial\overrightarrow{n}}\;dSdt = 0$. Then,

$$\int_0^\infty\int_{\mathcal{C}}\varphi\triangle u\;dxdt = \int_0^\infty\int_{\mathcal{C}} u\triangle\varphi\;dxdt. \qquad (3.7)$$

Substituting (3.5), (3.6) and (3.7) in (3.4) and using the operator $\mathcal{A}$, we have

$$\int_0^\infty\int_{\mathcal{C}} u\mathcal{A}\varphi\;dxdt + \int_0^\infty\int_{\mathcal{C}}\varphi a(x)u^p\;dxdt = \int_{\mathcal{C}} u_0(x)\Big(\triangle\varphi(x,0) - \varphi(x,0)\Big)\;dx.$$

So, we have the proposition. □

In this work we are interested in weak solutions of (3.1).

Definition 3.2. (Weak solution) A function $u \in L^1_{loc}(\mathcal{C}\times[0,\infty))$ is called a *weak solution* to (3.1) provided it satisfies (3.2) for all $\varphi \in C_c^\infty(\mathcal{C}\times[0,\infty))$.

CHAPTER IV

MAIN RESULT 2

4.1 Test function

Notation: For two functions f and g, we write $f \lesssim g$ if there is a constant C, called a *constant multiple*, such that $f \leq Cg$ at every point in the domain.

Lemma 4.1 ([9],[14])**.** *Let $q \in (1, \infty)$. There is $\phi \in C^3(\mathbb{R}), 0 \leq \phi \leq 1$ such that*

$$\begin{cases} \phi(s) = 1 & \text{if } \ s \leq 1, \ \phi(s) = 0 \ \ \text{if } \ s \geq 2, \\ \phi'(s) \leq 0 & \text{for all } \ s \in \mathbb{R} \text{and} \\ |\ \phi^{(j)}(s)| \lesssim \phi(s)^{\frac{q-1}{q}} & \text{for all } \ 1 \leq s \leq 2, j \in \{1,2,3\}. \end{cases} \tag{4.1}$$

Proof. Choose $\xi \in C^2(\mathbb{R})$ satisfying $0 \leq \xi \leq 1, \xi' \leq 0$ and

$$\begin{cases} \xi(s) = 1 & \text{for } s \leq 1, \\ \xi(s) = 0 & \text{for } s \geq 2. \end{cases}$$

Then, it is directly to verify that

$$\phi(s) = \xi(s)^{2q} \qquad (s \in \mathbb{R})$$

has the desired properties. □

Let $R, \rho > 0$ (the latter is given by (3.1)) and $T > 0$ are constants with $R > \rho$. In polar coordinates $x = r\omega$, we define

$$\varphi(x,t) = \left[\left(\frac{|x|}{\rho} \right)^{e_*} - \left(\frac{|x|}{\rho} \right)^{-e^*} \right] \phi \left(\frac{|x|}{R} \right) \Psi_s(\omega) \phi \left(\frac{t}{T} \right), \tag{4.2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is given in Lemma 4.1, $\Psi_s(\omega) \geq 0$ is the Helmholtz eigenfunction of Δ_ω on $\Omega \subset S^{n-1}$ corresponding to the first eigenvalue $\lambda_1(\Omega)$ (see Chapter 3), $e_* = -\frac{n-2}{2} + \sqrt{(\frac{n-2}{2})^2 + \lambda_1(\Omega)}$ and $e^* = \frac{n-2}{2} + \sqrt{(\frac{n-2}{2})^2 + \lambda_1(\Omega)}$. For convenience, we will use the following functions

$$\xi_\rho(|x|) = \left(\frac{|x|}{\rho}\right)^{e_*} - \left(\frac{|x|}{\rho}\right)^{-e^*},$$

$$\Phi(|x|) = \phi\left(\frac{|x|}{R}\right),$$

$$\widetilde{\Phi}(t) = \phi\left(\frac{t}{T}\right),$$

$$\psi = \psi_{\rho,R} = \xi_\rho(|x|)\Phi(|x|)\Psi_s(\omega).$$

Thus,

$$\begin{aligned}\varphi(x,t) &= \psi(x,t)\widetilde{\Phi}(t)\\ &= \xi_\rho(|x|)\Phi(|x|)\Psi_s(\omega)\widetilde{\Phi}(t)\\ &= \xi_\rho(r)\Phi(r)\Psi_s(\omega)\widetilde{\Phi}(t). \end{aligned} \tag{4.3}$$

We will need the following lemma.

Lemma 4.2. *$\Delta(\xi_\rho\Psi_s) = 0$ in $\mathcal{C}$.*

Proof. Note that Ψ_s is an eigenfunction for the Dirichlet problem of Δ_ω in Ω;

$$\begin{cases} \Delta_\omega\Psi_s + \lambda_1\Psi_s = 0 & \text{in } \Omega,\\ \Psi_s = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

By expressing of the Laplace-Beltrami operator in polar coordinates (r,ω), i.e.,

$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega$, we have

$$\begin{aligned}
\Delta(\xi_\rho \Psi_s) &= \frac{\partial^2(\xi_\rho \Psi_s)}{\partial r^2} + \frac{n-1}{r}\frac{\partial(\xi_\rho \Psi_s)}{\partial r} + \frac{1}{r^2}\Delta_\omega \xi_\rho \Psi_s \\
&= \Psi_s \frac{\partial^2 \xi_\rho}{\partial r^2} + \frac{n-1}{r}\Psi_s \frac{\partial \xi_\rho}{\partial r} + \frac{\xi_\rho}{r^2}\Delta_\omega \Psi_s \\
&= \Psi_s \frac{\partial^2 \xi_\rho}{\partial r^2} + \frac{n-1}{r}\Psi_s \frac{\partial \xi_\rho}{\partial r} + \frac{\xi_\rho}{r^2}(-\lambda_1 \Psi_s) \\
&= \Psi_s \left[\frac{\partial^2 \xi_\rho}{\partial r^2} + \frac{n-1}{r}\frac{\partial \xi_\rho}{\partial r} - \frac{\xi_\rho \lambda_1}{r^2} \right] \\
&= \Psi_s \Bigg[\frac{e_*(e_*-1)}{\rho^{e_*}} r^{e_*-2} + e^*(-e^*-1)\rho^{e^*} r^{-e^*-2} + \frac{n-1}{r}\frac{e_*}{\rho^{e_*}} r^{e_*-1} \\
&\quad + \frac{n-1}{r} e^* \rho^{e^*} r^{-e^*-1} - \frac{\lambda_1}{r^2}\left[\left(\frac{r}{\rho}\right)^{e_*} - \left(\frac{r}{\rho}\right)^{-e^*} \right] \Bigg] \\
&= \Psi_s \Bigg[\frac{1}{r^2}\left(\frac{r}{\rho}\right)^{e_*} [e_*(e_*-1) + (n-1)e_* - \lambda_1] + \frac{1}{r^2}\left(\frac{r}{\rho}\right)^{e^*} [e^*(-e^*-1) \\
&\quad + (n-1)e^* + \lambda_1] \Bigg]. \qquad (4.5)
\end{aligned}$$

Since e_* and $-e^*$ are roots of the quadratic equation $r(r-1) + (n-1)r - \lambda_1 = 0$, we have the lemma. □

4.2 Pointwise estimates of the test functions

Since

$$\begin{aligned}
\varphi(x,t) &= \xi_\rho(r)\Phi(r)\Psi_s(\omega)\widetilde{\Phi}(t) \\
&= \xi_\rho(r)\phi\left(\frac{r}{R}\right)\Psi_s(\omega)\phi\left(\frac{t}{T}\right),
\end{aligned}$$

we have,

$$\partial_t \varphi(x,t) = \frac{1}{T}\xi_\rho(r)\phi\left(\frac{r}{R}\right)\Psi_s(\omega)\phi'\left(\frac{t}{T}\right).$$

Thus, by Lemma 4.1,

$$\begin{aligned}
|\partial_t \varphi(x,t)| &= \frac{1}{T}\xi_\rho(r)\phi\left(\frac{r}{R}\right)\Psi_s(\omega)\left|\phi'\left(\frac{t}{T}\right)\right| \\
&\lesssim \frac{1}{T}\xi_\rho(r)\phi\left(\frac{r}{R}\right)\Psi_s(\omega)\phi\left(\frac{t}{T}\right)^{\frac{q-1}{q}} \chi_{\{t: T \le t \le 2T\}}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
|\partial_t \varphi|^q &\lesssim \left[\frac{1}{T}\xi_\rho(r)\phi\left(\frac{r}{R}\right)\Psi_s(\omega)\phi\left(\frac{t}{T}\right)^{\frac{q-1}{q}}\right]^q \chi_{\{t:T\le t\le 2T\}}\\
&= T^{-q}\left(\xi_\rho(r)\right)^q\left(\phi\left(\frac{r}{R}\right)\right)^q\left(\Psi_s(\omega)\right)^q\left(\phi\left(\frac{t}{T}\right)\right)^{q-1}\chi_{\{t:T\le t\le 2T\}}\\
&= T^{-q}\left(\xi_\rho(r)\phi\left(\frac{r}{R}\right)\Psi_s(\omega)\phi\left(\frac{t}{T}\right)\right)^{q-1}\xi_\rho(r)\phi\left(\frac{r}{R}\right)\Psi_s(\omega)\chi_{\{t:T\le t\le 2T\}}.
\end{aligned}
$$

Since $\xi_\rho(r)\le\left(\frac{r}{\rho}\right)^{e*}$, $\phi\left(\frac{r}{R}\right)\le 1$ and $\Psi_s(\omega)\lesssim 1$, we get that

$$
\begin{aligned}
|\partial_t \varphi|^q &\lesssim \left(\frac{r}{\rho}\right)^{e*} T^{-q}\varphi^{q-1}\chi_{\{t:T\le t\le 2T\}}\\
&\lesssim r^{e*}T^{-q}\varphi^{q-1}\chi_{\{t:T\le t\le 2T\}}.
\end{aligned}
$$

Since $r\le 2R$ on supp φ, we obtain that

$$
\begin{aligned}
|\partial_t \varphi|^q &\lesssim (2R)^{e*}T^{-q}\varphi^{q-1}\chi_{\{t:T\le t\le 2T\}}\\
&\lesssim R^{e*}T^{-q}\varphi^{q-1}\chi_{\{t:T\le t\le 2T\}}.
\end{aligned}
$$

$$
\begin{aligned}
\Delta\varphi(x,t) &= \frac{\partial^2\varphi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\triangle_\omega\varphi\\
&= \Psi_s\widetilde{\Phi}\ \frac{\partial^2\left(\xi_\rho\Phi\right)}{\partial r^2}+\frac{n-1}{r}\Psi_s\widetilde{\Phi}\ \frac{\partial\left(\xi_\rho\Phi\right)}{\partial r}+\frac{1}{r^2}\xi_\rho\Phi\widetilde{\Phi}\left(\Delta_\omega\Psi_s\right)\\
&= \Psi_s\widetilde{\Phi}\left[\Phi\frac{\partial^2\xi_\rho}{\partial r^2}+2\frac{\partial\xi_\rho}{\partial r}\frac{\partial\Phi}{\partial r}+\xi_\rho\frac{\partial^2\Phi}{\partial r^2}\right]+\frac{n-1}{r}\Psi_s\widetilde{\Phi}\left[\xi_\rho\frac{\partial\Phi}{\partial r}+\Phi\frac{\partial\xi_\rho}{\partial r}\right]\\
&\quad+\frac{1}{r^2}\xi_\rho\Phi\widetilde{\Phi}\left(\Delta_\omega\Psi_s\right)\\
&= \Psi_s\widetilde{\Phi}\Phi\xi_\rho''+\frac{2}{R}\xi_\rho'\phi'\left(\frac{r}{R}\right)\Psi_s\widetilde{\Phi}+\frac{1}{R^2}\Psi_s\widetilde{\Phi}\xi_\rho\phi''\left(\frac{r}{R}\right)\\
&\quad+\frac{n-1}{rR}\Psi_s\widetilde{\Phi}\xi_\rho\phi'\left(\frac{r}{R}\right)+\frac{n-1}{r}\Psi_s\widetilde{\Phi}\Phi\xi_\rho'+\frac{1}{r^2}\xi_\rho\Phi\widetilde{\Phi}\left(\Delta_\omega\Psi_s\right)\\
&= \widetilde{\Phi}\Phi\left[\Psi_s\xi_\rho''+\frac{n-1}{r}\Psi_s\xi_\rho'+\frac{1}{r^2}\xi_\rho\left(\Delta_w\Psi_s\right)\right]+\frac{2}{R}\xi_\rho'\phi'\left(\frac{r}{R}\right)\Psi_s\widetilde{\Phi}\\
&\quad+\frac{1}{R^2}\Psi_s\widetilde{\Phi}\xi_\rho\phi''\left(\frac{r}{R}\right)+\frac{n-1}{rR}\Psi_s\widetilde{\Phi}\xi_\rho\phi'\left(\frac{r}{R}\right).
\end{aligned}
$$

By $\Delta(\xi_\rho\Psi_s)=0$, we have

$$
\Delta\varphi(x,t)=\frac{2}{R}\xi_\rho'\phi'\left(\frac{r}{R}\right)\Psi_s\widetilde{\Phi}+\frac{1}{R^2}\Psi_s\widetilde{\Phi}\xi_\rho\phi''\left(\frac{r}{R}\right)+\frac{n-1}{rR}\Psi_s\widetilde{\Phi}\xi_\rho\phi'\left(\frac{r}{R}\right).
$$

Thus, by Lemma 4.1,

$$
\begin{aligned}
&|\Delta\varphi(x,t)|^q \\
&= \left|\frac{2}{R}\xi_\rho'\phi'\left(\frac{r}{R}\right)\Psi_s\widetilde{\Phi} + \frac{1}{R^2}\Psi_s\widetilde{\Phi}\xi_\rho\phi''\left(\frac{r}{R}\right) + \frac{n-1}{rR}\Psi_s\widetilde{\Phi}\xi_\rho\phi'\left(\frac{r}{R}\right)\right|^q \chi_{\{r:R<r<2R\}} \\
&\lesssim \{\left(\frac{2}{R}\xi_\rho'\Psi_s\widetilde{\Phi}\right)^q \left|\phi'\left(\frac{r}{R}\right)\right|^q + \left(\frac{1}{R^2}\Psi_s\widetilde{\Phi}\xi_\rho\right)^q \left|\phi''\left(\frac{r}{R}\right)\right|^q \\
&\quad + \left(\frac{n-1}{rR}\Psi_s\widetilde{\Phi}\xi_\rho\right)^q \left|\phi'\left(\frac{r}{R}\right)\right|^q\}\chi_{\{r:R<r<2R\}} \\
&\lesssim \{\left(\frac{2}{R}\xi_\rho'\Psi_s\widetilde{\Phi}\right)^q \phi\left(\frac{r}{R}\right)^{q-1} + \left(\frac{1}{R^2}\Psi_s\widetilde{\Phi}\xi_\rho\right)^q \phi\left(\frac{r}{R}\right)^{q-1} \\
&\quad + \left(\frac{n-1}{rR}\Psi_s\widetilde{\Phi}\xi_\rho\right)^q \phi\left(\frac{r}{R}\right)^{q-1}\}\chi_{\{r:R<r<2R\}} \\
&\lesssim \widetilde{\Phi}^q\phi\left(\frac{r}{R}\right)^{q-1}\left[\left(\frac{1}{R^2}\xi_\rho\Psi_s\right)^q + \left(\frac{1}{R^2}\Psi_s\xi_\rho\right)^q + \left(\frac{1}{rR}\Psi_s\xi_\rho\right)^q\right]\chi_{\{r:R<r<2R\}} \\
&\lesssim \frac{1}{R^{2q}}\widetilde{\Phi}\xi_\rho\Psi_s\varphi^{q-1} + \frac{1}{R^{2q}}\widetilde{\Phi}\xi_\rho\Psi_s\varphi^{q-1} + \frac{1}{r^qR^q}\widetilde{\Phi}\xi_\rho\Psi_s\varphi^{q-1}\chi_{\{r:R<r<2R\}} \\
&\lesssim R^{-2q}\widetilde{\Phi}\xi_\rho\Psi_s\varphi^{q-1}\chi_{\{r:R<r<2R\}} \\
&\lesssim R^{-2q+e_*}\varphi^{q-1}\chi_{\{r:R<r<2R\}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\Delta\varphi(x,t) &= \frac{2}{R}\xi_\rho'\phi'\left(\frac{r}{R}\right)\Psi_s\widetilde{\Phi} + \frac{1}{R^2}\Psi_s\widetilde{\Phi}\xi_\rho\phi''\left(\frac{r}{R}\right) + \frac{n-1}{rR}\Psi_s\widetilde{\Phi}\xi_\rho\phi'\left(\frac{r}{R}\right) \\
&= \widetilde{\Phi}\left[\frac{2}{R}\xi_\rho'\phi'\left(\frac{r}{R}\right)\Psi_s + \frac{1}{R^2}\Psi_s\xi_\rho\phi''\left(\frac{r}{R}\right) + \frac{n-1}{rR}\Psi_s\xi_\rho\phi'\left(\frac{r}{R}\right)\right],
\end{aligned}
$$

we obtain that

$$
\partial_t\Delta\varphi(x,t) = \frac{1}{T}\phi'\left(\frac{t}{T}\right)\left[\frac{2}{R}\xi_\rho'\phi'\left(\frac{r}{R}\right)\Psi_s + \frac{1}{R^2}\Psi_s\xi_\rho\phi''\left(\frac{r}{R}\right) + \frac{n-1}{rR}\Psi_s\xi_\rho\phi'\left(\frac{r}{R}\right)\right].
$$

Thus, by Lemma 4.1,

$$
\begin{aligned}
&|\partial_t \Delta\varphi(x,t)|^q \\
&= \left|\frac{1}{T}\phi'\left(\frac{t}{T}\right)\left[\frac{2}{R}\xi_\rho'\phi'\left(\frac{r}{R}\right)\Psi_s + \frac{1}{R^2}\Psi_s\xi_\rho\phi''\left(\frac{r}{R}\right) + \frac{n-1}{rR}\Psi_s\xi_\rho\phi'\left(\frac{r}{R}\right)\right]\right|^q \\
&\lesssim \frac{1}{T^q}\left[\phi'\left(\frac{t}{T}\right)\right]^q\left[\left(\frac{1}{R}\xi_\rho'\Psi_s\right)^q\left|\phi'\left(\frac{r}{R}\right)\right|^q + \left(\frac{1}{R^2}\Psi_s\xi_\rho\right)^q\left|\phi''\left(\frac{r}{R}\right)\right|^q\right] \\
&\quad + \frac{1}{T^q}\left[\phi'\left(\frac{t}{T}\right)\right]^q\left[\left(\frac{n-1}{rR}\Psi_s\xi_\rho\right)^q\left|\phi'\left(\frac{r}{R}\right)\right|^q\right] \\
&\lesssim \{\frac{1}{T^q}\left(\phi\left(\frac{t}{T}\right)\right)^{q-1}\left[\left(\frac{1}{R}\xi_\rho'\Psi_s\right)^q\phi^{q-1}\left(\frac{r}{R}\right) + \left(\frac{1}{R^2}\Psi_s\xi_\rho\right)^q\phi^{q-1}\left(\frac{r}{R}\right)\right] \\
&\quad + \frac{1}{T^q}\left(\phi\left(\frac{t}{T}\right)\right)^{q-1}\left[\left(\frac{n-1}{rR}\Psi_s\xi_\rho\right)^q\phi^{q-1}\left(\frac{r}{R}\right)\right]\}\chi_{\{r:R<r<2R\}}\chi_{\{t:T\le t\le 2T\}} \\
&\lesssim \{\frac{1}{T^q}\left(\phi\left(\frac{t}{T}\right)\right)^{q-1}\left[\left(\frac{1}{R^2}\xi_\rho\Psi_s\right)^q\phi^{q-1}\left(\frac{r}{R}\right) + \left(\frac{1}{R^2}\Psi_s\xi_\rho\right)^q\phi^{q-1}\left(\frac{r}{R}\right)\right] \\
&\quad + \frac{1}{T^q}\left(\phi\left(\frac{t}{T}\right)\right)^{q-1}\left[\left(\frac{1}{rR}\Psi_s\xi_\rho\right)^q\phi^{q-1}\left(\frac{r}{R}\right)\right]\}\chi_{\{r:R<r<2R\}}\chi_{\{t:T\le t\le 2T\}} \\
&\lesssim \frac{1}{T^q}\left[\frac{1}{R^{2q}}\xi_\rho\Psi_s\varphi^{q-1} + \frac{1}{R^{2q}}\xi_\rho\Psi_s\varphi^{q-1} + \frac{1}{r^qR^q}\xi_\rho\Psi_s\varphi^{q-1}\right]\chi_{\{r:R<r<2R\}}\chi_{\{t:T\le t\le 2T\}} \\
&\lesssim T^{-q}R^{-2q}\xi_\rho\Psi_s\varphi^{q-1}\chi_{\{r:R<r<2R\}}\chi_{\{t:T\le t\le 2T\}} \\
&\lesssim T^{-q}R^{-2q+e_*}\rho^{-e_*}\varphi^{q-1}\chi_{\{r:R<r<2R\}}\chi_{\{t:T\le t\le 2T\}}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
|\mathcal{A}\varphi|^q &= |\partial_t\varphi - \triangle\partial_t\varphi + \triangle\varphi|^q \\
&\le |\partial_t\varphi|^q + |\triangle\partial_t\varphi|^q + |\triangle\varphi|^q \\
&\lesssim \left(R^{e*}T^{-q}\chi_{K_1} + T^{-q}R^{-2q+e_*}\chi_{K_3} + R^{-2q+e_*}\chi_{K_2}\right)\varphi^{q-1},
\end{aligned}
$$

where

$$
\begin{aligned}
K_1 &= \{(x,t) : \rho \le |x| \le 2R, T \le t \le 2T\}, \\
K_2 &= \{(x,t) : R \le |x| \le 2R, 0 \le t \le 2T\}, \\
K_3 &= \{(x,t) : R \le |x| \le 2R, T \le t \le 2T\}.
\end{aligned}
$$

Hence, we obtain the following pointwise estimate for $\mathcal{A}\varphi$:

$$\frac{|\mathcal{A}\varphi|^q}{\varphi^{q-1}} \lesssim R^{e*}T^{-q}\chi_{K_1} + R^{-2q+e_*}\chi_{K_2} + T^{-q}R^{-2q+e_*}\chi_{K_3}. \tag{4.6}$$

4.3 Proof of the Main Result

For this section, we study blow-up of solutions to the semilinear pseudoparabolic equation (3.1). Assume that there exist $\sigma > -2$ such that

$$a(x) \gtrsim |x|^\sigma \quad \text{for all} \quad x \in \mathcal{C}. \tag{4.7}$$

Thus, a can be singular if $\sigma < 0$. We prove that if

$$1 < p < p_c := \min\{p_{c_1}, p_{c_2}\},$$

where $p_{c_1} := 1 + \frac{\sigma+2}{n+e_*}$ and $p_{c_2} := 1 + \frac{2}{e_*}$, then $u \equiv 0$ is the only solution which is defined for all $t \in [0, \infty)$.

Theorem 4.3. *Let a and u_0 be non-negative continuous functions, a satisfies* (4.7) *with $\sigma > -2$. If $1 < p < p_c$ and u is a global solution of* (3.1)*, then $u \equiv 0$.*

Proof. First, we carry out some integral estimates. Assume that $u \geq 0$ is a weak solution of (3.1). Then, by (3.2) and $\varphi(x, 0) \geq 0$, we get

$$\int_0^\infty \int_{\mathcal{C}} u\mathcal{A}\varphi \; dxdt + \int_0^\infty \int_{\mathcal{C}} \varphi a(x)u^p \; dxdt \leq \int_{\mathcal{C}} u_0(x)\triangle\varphi(x, 0) \; dx. \tag{4.8}$$

Observe that supp $\mathcal{A}\varphi \subset$ supp φ. Let $\mathcal{C}' = \mathcal{C} \cap \text{supp}\, \mathcal{A}\varphi$ and $q = \frac{p}{p-1}$. By Hölder and Young inequalities,

$$\begin{aligned}
\int_0^\infty \int_{\mathcal{C}} |u\mathcal{A}\varphi| dxdt &= \int_0^\infty \int_{\mathcal{C}'} |a^{\frac{1}{p}} u\varphi^{\frac{1}{p}}| \left| \frac{\mathcal{A}\varphi}{(a\varphi)^{\frac{1}{p}}} \right| dxdt \\
&\leq \left(\int_0^\infty \int_{\mathcal{C}'} |a^{\frac{1}{p}} u\varphi^{\frac{1}{p}}|^p dxdt \right)^{\frac{1}{p}} \left(\int_0^\infty \int_{\mathcal{C}'} \left| \frac{\mathcal{A}\varphi}{(a\varphi)^{\frac{1}{p}}} \right|^q dxdt \right)^{\frac{1}{q}} \\
&\leq \frac{1}{p} \int_0^\infty \int_{\mathcal{C}'} au^p\varphi dxdt + \frac{1}{q} \int_0^\infty \int_{\mathcal{C}'} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dxdt.
\end{aligned}$$

Hence, (4.8) becomes

$$\begin{aligned}\int_0^\infty \int_{\mathcal{C}} \varphi a(x) u^p \; dxdt &\le \int_0^\infty \int_{\mathcal{C}'} |u\mathcal{A}\varphi| \; dxdt + \int_{\mathcal{C}} u_0(x) \triangle\varphi(x,0) \; dx \\ &\le \frac{1}{p}\int_0^\infty \int_{\mathcal{C}} a u^p \varphi dxdt + \frac{1}{q}\int_0^\infty \int_{\mathcal{C}'} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dxdt \\ &\quad + \int_{\mathcal{C}} u_0(x) \triangle\varphi(x,0) \; dx.\end{aligned}$$

Then,

$$\left(1-\frac{1}{p}\right)\int_0^\infty \int_{\mathcal{C}} \varphi a(x) u^p \; dxdt \le \frac{1}{q}\int_0^\infty \int_{\mathcal{C}'} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dxdt + \int_{\mathcal{C}} u_0(x) \triangle\varphi(x,0) \; dx.$$

Thus, we have

$$\int_0^\infty \int_{\mathcal{C}} \varphi a(x) u^p \; dxdt \le \int_0^\infty \int_{\mathcal{C}} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dxdt + q\int_{\mathcal{C}} u_0(x) \triangle\varphi(x,0) \; dx. \tag{4.9}$$

Next, we estimate an upper bound for the first term on the right hand side of (4.9). Since $a(x) \gtrsim |x|^\sigma (\sigma > -2)$ and supp $\mathcal{A}\varphi$, supp $\varphi \subset \{(x,t) : \rho \le |x| \le 2R, 0 \le t \le 2T\}$, it follows that

$$\begin{aligned}\int_0^\infty \int_{\mathcal{C}'} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dxdt &\lesssim \int_0^\infty \int_{\mathcal{C}'} \frac{1}{|x|^{\sigma(q-1)}} \frac{|\mathcal{A}\varphi|^q}{\varphi^{q-1}} dxdt \\ &\lesssim \int_0^{2T} \int_{\rho\le|x|\le 2R} \frac{1}{|x|^{\sigma(q-1)}} \frac{|\mathcal{A}\varphi|^q}{\varphi^{q-1}} dxdt.\end{aligned}$$

By (4.6), we obtain that

$$\begin{aligned}&\int_0^\infty \int_{\mathcal{C}'} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dxdt \\ &\lesssim \int_0^{2T} \int_{\rho\le|x|\le 2R} \frac{1}{|x|^{\sigma(q-1)}} \left[R^{e*}T^{-q}\chi_{K_1} + R^{-2q+e_*}\chi_{K_2} + T^{-q}R^{-2q+e_*}\chi_{K_3}\right] dxdt \\ &= \left[\int_0^{2T} \int_{\rho\le|x|\le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e*}T^{-q}\chi_{K_1} dxdt + \int_0^{2T} \int_{\rho\le|x|\le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{-2q+e_*}\chi_{K_2} dxdt\right] \\ &\quad + \int_0^{2T} \int_{\rho\le|x|\le 2R} \frac{1}{|x|^{\sigma(q-1)}} T^{-q}R^{-2q+e_*}\chi_{K_3} dxdt \\ &=: I + II + III.\end{aligned}$$

No, we choose $T = R^2$. The integral I is estimated by

$$\begin{aligned} I &= \int_0^{2R^2} \int_{\rho \le |x| \le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e*} R^{-2q} dx dt \\ &\lesssim \int_0^{2R^2} \int_{\rho \le |x| \le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_* - 2q} dx dt \\ &\lesssim \int_{\rho \le |x| \le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_* - 2q + 2} dx. \end{aligned}$$

By integration using spherical coordinates, we get

$$\begin{aligned} I &= \int_{\rho \le |x| \le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_* - 2q + 2} dx \\ &= \int_{S^{n-1}} \int_\rho^{2R} \frac{1}{r^{\sigma(q-1)}} R^{e_* - 2q + 2} r^{n-1} dr d\omega. \end{aligned}$$

If $p = 1 + \frac{\sigma}{n}$, then $n - 1 - \sigma(q-1) = -1$ and hence, the last integral becomes

$$\begin{aligned} I &= \int_{S^{n-1}} \int_\rho^{2R} \frac{1}{r} R^{e_* - 2q + 2} dr d\omega \\ &= \omega_n (\ln r) \Big|_\rho^{2R} R^{e_* - 2q + 2} \\ &\lesssim \ln\left(\frac{2R}{\rho}\right) R^{e_* - 2q + 2} \\ &\lesssim \ln\left(\frac{R}{\rho}\right) R^{e_* - \frac{2}{p-1}}. \end{aligned}$$

If $p \neq 1 + \frac{\sigma}{n}$, then $n - \sigma(q-1) \neq 0$ and hence, it becomes

$$\begin{aligned} I &= \int_{S^{n-1}} \int_\rho^{2R} \frac{1}{r^{\sigma(q-1)}} R^{e_* - 2q + 2} r^{n-1} dr d\omega \\ &= \omega_n \left(\frac{r^{n - \sigma q + \sigma}}{n - \sigma q + \sigma}\right) \Bigg|_\rho^{2R} R^{e_* - 2q + 2} \\ &\lesssim \frac{1}{n - \sigma q + \sigma} \left((2R)^{n - \sigma q + \sigma} - \rho^{n - \sigma q + \sigma}\right) R^{e_* - 2q + 2} \\ &\lesssim \frac{1}{n - \sigma q + \sigma} \left[(R)^{n - \sigma q + \sigma + e_* - 2q + 2} - \rho^{n - \sigma q + \sigma} R^{e_* - 2q + 2}\right]. \end{aligned}$$

If $p > 1 + \frac{\sigma}{n}$, then $n - \sigma q + \sigma > 0$ and we obtain that

$$\begin{aligned} I &\lesssim \frac{1}{n - \sigma q + \sigma} R^{n - \sigma q + \sigma + e_* - 2q + 2} \\ &= R^{\frac{n}{p-1}\left(p - 1 - \frac{\sigma + 2}{n}\right) + e_*}. \end{aligned}$$

If $p < 1 + \frac{\sigma}{n}$, then $n - \sigma q + \sigma < 0$ and we obtain that

$$\begin{aligned} I &\lesssim \frac{1}{n - \sigma q + \sigma} \rho^{n-\sigma q+\sigma} R^{e_* - 2q + 2} \\ &\lesssim R^{e_* - \frac{2}{p-1}}. \end{aligned}$$

The integral II is estimated by

$$\begin{aligned} II &\lesssim \int_0^{2R^2} \int_{\rho \le |x| \le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_* - 2q} dx dt \\ &\lesssim \int_{\rho \le |x| \le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_* - 2q + 2} dx. \end{aligned}$$

By integration using spherical coordinates, we get

$$\begin{aligned} II &\lesssim \int_{\rho \le |x| \le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_* - 2q + 2} dx \\ &\lesssim \int_\rho^{2R} \frac{1}{r^{\sigma(q-1)}} R^{e_* - 2q + 2} r^{n-1} dr. \end{aligned}$$

If $p = 1 + \frac{\sigma}{n}$, then it becomes

$$\begin{aligned} II &= \int_\rho^{2R} \frac{1}{r^{\sigma(q-1)}} R^{e_* - 2q + 2} r^{n-1} dr \\ &= (\ln r)\Big|_\rho^{2R} R^{e_* - 2q + 2} \\ &\lesssim \ln\left(\frac{2R}{\rho}\right) R^{e_* - 2q + 2} \\ &\lesssim \ln\left(\frac{R}{\rho}\right) R^{e_* - \frac{2}{p-1}}. \end{aligned}$$

If $p \neq 1 + \frac{\sigma}{n}$, then it becomes

$$\begin{aligned} II &\lesssim \int_\rho^{2R} \frac{1}{r^{\sigma(q-1)}} R^{e_* - 2q + 2} r^{n-1} dr \\ &= \left(\frac{r^{n-\sigma q+\sigma}}{n - \sigma q + \sigma}\right)\Bigg|_\rho^{2R} R^{e_* - 2q + 2} \\ &\lesssim \frac{1}{n - \sigma q + \sigma} \left((2R)^{n-\sigma q+\sigma} - \rho^{n-\sigma q+\sigma}\right) R^{e_* - 2q + 2} \\ &\lesssim \frac{1}{n - \sigma q + \sigma} \left[(R)^{n-\sigma q+\sigma+e_* - 2q + 2} - \rho^{n-\sigma q+\sigma} R^{e_* - 2q + 2}\right]. \end{aligned}$$

If $p > 1 + \frac{\sigma}{n}$, then $n - \sigma q + \sigma > 0$ and we obtain that

$$\begin{aligned} II &\lesssim \frac{1}{n - \sigma q + \sigma} R^{n-\sigma q+\sigma+e_*-2q+2} \\ &\lesssim R^{\frac{n}{p-1}\left(p-1-\frac{\sigma+2}{n}\right)+e_*}. \end{aligned}$$

If $p < 1 + \frac{\sigma}{n}$, then $n - \sigma q + \sigma < 0$ and we obtain that

$$\begin{aligned} II &\lesssim R^{e_*-2q+2} \\ &\lesssim R^{e_*-\frac{2}{p-1}}. \end{aligned}$$

The integral *III* is estimated by

$$\begin{aligned} III &\lesssim \int_{R^2}^{2R^2} \int_{\rho\le|x|\le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_*-4q} dx dt \\ &\lesssim \int_{\rho\le|x|\le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_*-4q+2} dx. \end{aligned}$$

Yet again, we get

$$\begin{aligned} III &\lesssim \int_{\rho\le|x|\le 2R} \frac{1}{|x|^{\sigma(q-1)}} R^{e_*-4q+2} dx \\ &= \int_{S^{n-1}} \int_{\rho}^{2R} \frac{1}{r^{\sigma(q-1)}} R^{e_*-4q+2} r^{n-1} dr d\omega. \end{aligned}$$

If $p = 1 + \frac{\sigma}{n}$, then it becomes

$$\begin{aligned} \mathrm{III} &= \int_{S^{n-1}} \int_{\rho}^{2R} \frac{1}{r^{\sigma(q-1)}} R^{e_*-4q+2} r^{n-1} dr d\omega \\ &= \omega_n (\ln r)\Big|_{\rho}^{2R} R^{e_*-4q+2} \\ &\lesssim \ln\left(\frac{2R}{\rho}\right) R^{e_*-4q+2} \\ &\lesssim \ln\left(\frac{R}{\rho}\right) R^{e_*-2-\frac{4}{p-1}}. \end{aligned}$$

If $p \neq 1 + \frac{\sigma}{n}$, then it becomes

$$
\begin{aligned}
III &\lesssim \int_{S^{n-1}} \int_{\rho}^{2R} \frac{1}{r^{\sigma(q-1)}} R^{e_*-4q+2} r^{n-1} dr d\omega \\
&= \omega_n \left(\frac{r^{n-\sigma q+\sigma}}{n-\sigma q+\sigma} \right) \Bigg|_{\rho}^{2R} R^{e_*-4q+2} \\
&\lesssim \frac{1}{n-\sigma q+\sigma} \left((2R)^{n-\sigma q+\sigma} - \rho^{n-\sigma q+\sigma} \right) R^{e_*-4q+2} \\
&\lesssim \frac{1}{n-\sigma q+\sigma} \left[(2R)^{n-\sigma q+\sigma+e_*-4q+2} - \rho^{n-\sigma q+\sigma} R^{e_*-4q+2} \right].
\end{aligned}
$$

If $p > 1 + \frac{\sigma}{n}$, then $n - \sigma q + \sigma > 0$ and we obtain that

$$
\begin{aligned}
III &\lesssim \frac{1}{n-\sigma q+\sigma} R^{n-\sigma q+\sigma+e_*-4q+2} \\
&\lesssim R^{\frac{n}{p-1}\left(p-1-\frac{\sigma+2}{n}\right)+e_*-2q}.
\end{aligned}
$$

If $p < 1 + \frac{\sigma}{n}$, then $n - \sigma q + \sigma < 0$ and we obtain that

$$
\begin{aligned}
III &\lesssim R^{e_*-4q+2} \\
&\lesssim R^{e_*-2-\frac{2}{p-1}}.
\end{aligned}
$$

Combining these estimates with (4.6), we have

$$
\int_0^\infty \int_{\mathcal{C}} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dx dt \lesssim \begin{cases} R^{\frac{n}{p-1}\left(p-1-\frac{\sigma+2}{n}\right)+e_*} & \text{if } p > 1+\frac{\sigma}{n}, \\ R^{e_*-\frac{2}{p-1}} & \text{if } 1 < p < 1+\frac{\sigma}{n}, \\ (\ln R)\, R^{e_*-\frac{2}{p-1}} & \text{if } p = 1+\frac{\sigma}{n}. \end{cases} \tag{4.10}
$$

If $1 < p < p_{c_1} := 1 + \frac{\sigma+2}{n+e_*}$, then $\frac{n}{p-1}\left(p - 1 - \frac{\sigma+2}{n}\right) + e_* < 0$.

If $1 < p < p_{c_2} := 1 + \frac{2}{e_*}$, then $e_* - \frac{2}{p-1} < 0$.

Now, we assume that $1 < p < \min\{p_{c_1}, p_{c_2}\}$ and $R \longrightarrow \infty$, we obtain

$$
\lim_{R\to\infty} \int_0^\infty \int_{\mathcal{C}'} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dx dt = 0.
$$

Observe that $\Delta\varphi(x,0) \lesssim R^{-2}$. Then, (4.9) becomes

$$\int_0^\infty \int_{\mathcal{C}} \varphi a(x) u^p \, dxdt \le \int_0^\infty \int_{\mathcal{C}} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dxdt + q \int_{\mathcal{C}} u_0(x) \triangle\varphi(x,0) \, dx$$
$$\le \int_0^\infty \int_{\mathcal{C}} \frac{|\mathcal{A}\varphi|^q}{(a\varphi)^{q-1}} dxdt + R^{-2} \|u_0\|_{L^1}.$$

Hence, if $1 < p < \min\{p_{c_1}, p_{c_2}\}$, then passing $R \to \infty$ in the last estimate we get

$$\int_0^\infty \int_{\mathcal{C}} \varphi a(x) u^p \, dxdt \le 0,$$

it follows that

$$\int_0^\infty \int_{\mathcal{C}} a(x) u^p \, dxdt = 0.$$

This means $u \equiv 0$, we have the theorem. □

4.4 Conclusions

Now, we have the relationship between classical and weak solutions of (1.4) which was shown in Proposition 3.1. We have shown that (1.4) has no nontrivial global weak solution which is defined for all $t \in [0, \infty)$. For future works, it is interesting to investigate the local existence problems, that is to show that given any initial condition on the cone domain, the pseudoparabolic equation admits a solution defined on some time interval [0,T]. Another direction would be the investigation of blowing-up phenomena when the exponent p lies outside $(1, p_c)$. More importantly, we haven't explored whether p_c is a critical exponent of Fujita type for the system or not, i.e. we didn't study the case $p > p_c$.

REFERENCES

[1] Aslan, B.C., Hager, W.W. and Moskow, S., A generalized eigenproblem for the Laplacian which arises in lightning, *J. Math. Anal. Appl.*, **341**, 1028-104 (2008).

[2] Barenblatt, G.I., Zheltov, I.P. and Kochina, I.N., Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks(strata), *J. Appl. Math. Mech.*, **24**, 1286-1303 (1960).

[3] Cao, Y., Yin, J., Wang, C.P., Cauchy problems of semilinear pseudo-parabolic equation, *J. Differential equations*, **246**, 4568-4590 (2009).

[4] L.C. Evans, *Partial Differential Equations*, *Amer Mathematical Society*, Providence, (1998).

[5] Fujita, H., On the blowing up of solutions of the Cauchy problem for $u_t = \triangle u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo* Sect. (1) **16**, 109-124 (1966).

[6] Hayakawa K., On the nonexistence of global solutions of some semilinear parabolic equations, *Proc. Japan Acad.*, **49**, 503-525 (1973).

[7] Khomruthai, S., Global and blow-up solutions of superlinear psedoparabolic equations with unbounded coefficient, *Nonlinear Anal.*, **122**, 192–214 (2015).

[8] Kobayashi, K., Sirao, T., Tanaka, H., On the blowing up problem for semilinear heat equations,*J. Math. Soc. Japan*, **29**, 407-424 (1977).

[9] Laptev, G. G., Nonexistence of solutions for parabolic inequalities in unbounded cone-like domain via the test function method, *J. Evol. Equa*, **2**, 459–470 (2002).

[10] Laptev, G. G., Nonexistence of global positive solutions for systems of semilinear elliptic inequalities in cone, *Russian Acad. Sci. Izv. Math.*, **64**, 108–124 (2000).

[11] Milne, E., The diffusion of imprisoned radiation through a gas, *J. London Math. Soc.*, **1**, 40-51 (1962).

[12] Mitidieri, E., and Pohozaev, S. I., Nonexistence of global positive solutions to quasilinear elliptic inequalities, *Russian Acad. Sci. Dokl. Math. 359*, **359**, 456-460 (1998).

[13] Mitidieri, E., and Pohozaev, S. I., A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities, Nauka, Moscow, 2001 (Proc. Steklov Inst. Math. 234).

[14] Pohozaev, S.I., and A., Tesei, Blow-up of nonnegative solutions to quasilinear parabolic inequalities, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, **11**, 99–109 (2000).

[15] Rubinstein, L.I., On the problem of the process of propagation of heat in heterogeneous media, *IZV. Akad. Nauk SSSR, Ser. Geogr*, **1** (1984).

[16] Weissler, F.B., Existence and nonexistence of global solutions of a semi-linear heat equation, *Israel J. Math.*, **38**, 29-40 (1981).

VITA

Name Mr. Tanin Noodaeng

Date of Birth 8 December 1991

Place of Birth Phatthalung, Thailand

Education B.Sc. (First-Class Degree Honours) Mathematics,
Prince of Songkla University, 2013

Scholarship The Development and Technology Talents Project (DPST)

Publication Noodaeng, T., Khomrutai, S.: Nonexistence of Global Solutions of Semilinear Pseudoparabolic Equation in Cone Domain, Proceedings of The Annual Pure and Applied Mathematics Conference 2017, 249-265.

Conference **Speaker**
Nonexistence of Global Solutions of Semilinear Pseudoparabolic Equation in Cone Domain at The Annual Pure and Applied Mathematics Conference 2017,28-30 June 2017 at Chulalongkorn University, Bangkok.