การลดมิติของซูเปอร์กราวิตีสิบเอ็ดมิติบน SU(2) กรุปแมนิโฟลด์ และ N=4 เกจซูเปอร์กราวิตี

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DIMENSIONAL REDUCTION OF ELEVEN DIMENSIONAL SUPERGRAVITY ON SU(2) GROUP MANIFOLD AND N=4 GAUGED SUPERGRAVITY

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งานวิจัยนี้ทำการตรวจสอบความสอดคล้องที่ระดับของสมการการเคลื่อนที่จากการลดมิติที่นำไปสู่ทฤษฎี $N = 4$ เกจซูเปอร์กราวิตีในกาลอวกาศสี่มิติจากซูเปอร์กราวิตีที่มีอยู่ชนิดเดียวในสิบเอ็ดมิติอย่างละเอียด การลดมิติจะสอดคล้องที่ระดับของสมการการเคลื่อนที่ต่อเมื่อการแทนค่าวิธีการลดมิติในสมการการเคลื่อนที่ในมิติที่สูงกว่าให้ผลลัพธ์เป็นสมการการเคลื่อนที่ของทฤษฎีในมิติที่ต่ำกว่าทุกสมการ โดยทั่วไปการลดมิติคาลูซา-ไคลน์บนทรงกลมจะไม่สอดคล้อง แต่มีอีกวิธีหนึ่งที่สามารถทำให้การลดมิติสอดคล้องได้นั่นคือทำการลดมิติบนกรุปแมนิโฟลด์ของลีกรุปซึ่งได้รับการรับรองว่าเป็นการลดมิติที่สอดคล้องโดยจะเรียกการลดมิตินี้ว่าการลดมิติเชิร์ก-ชวาร์ซ จากความจริงที่ว่ากรุปแมนิโฟลด์ของลีกรุป $SU(2)$ มีทอพอโลยีเป็นทรงกลมสามมิติ S^3 การลดมิติคาลูซา-ไคลน์บน S^3 จึงสามารถสร้างขึ้นได้จากการลดมิติบนกรุปแมนิโฟลด์นี้โดยการแทนที่ S^3 ในวิธีการลดมิติด้วย $SU(2)$ กรุปแมนิโฟลด์ ทฤษฎี $N = 4$ เกจซูเปอร์กราวิตีในสี่มิติเป็นผลมาจากวิธีการลดมิติบนทรงกลมเจ็ดมิติ S^7 โดยการเขียน S^7 ให้อยู่ในรูปของ $S^3 \times S^3 \times S^1$ และแทนที่แต่ละ S^3 ด้วย $SU(2)$ กรุปแมนิโฟลด์ ซึ่งความสอดคล้องของการลดมิติบนทรงกลมเจ็ดมิติ S^7 นี้อยู่ในระดับของสมการการเคลื่อนที่ นอกจากทฤษฎีหลัก $SO(4)$ เกจซูเปอร์กราวิตีทฤษฎีฟรีดแมน-ชวาร์ซซึ่งเป็นอีกหนึ่งรูปแบบของ $N = 4$ เกจซูเปอร์กราวิตีในสี่มิติก็เป็นผลจากวิธีการลดมิตินี้ด้วยโดยใช้การแปลงทางเดียวระหว่างทฤษฎี $N = 4$ เกจซูเปอร์กราวิตีทั้งสองรูปแบบ นอกจากนี้วิธีการลดมิติที่สอดคล้องยังยินยอมให้ทำการฝังผลเฉลยของทฤษฎี $N = 4$ $SO(4)$ เกจซูเปอร์กราวิตีในสี่มิติลงในซูเปอร์กราวิตีสิบเอ็ดมิติได้อีกด้วย

ภาควิชา ....ฟิสิกส์...........	ลายมือชื่อนิสิต ....
สาขาวิชา ....ฟิสิกส์..........	ลายมือชื่อ อ.ที่ปรึกษาหลัก ....
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PATHARADANAI NUCHINO : DIMENSIONAL REDUCTION OF ELEVEN DIMENSIONAL SUPERGRAVITY ON SU(2) GROUP MANIFOLD AND N=4 GAUGED SUPERGRAVITY. ADVISOR : PARINYA KARNDUMRI, Ph.D., 95 pp.

The consistency at the level of equations of motion from the dimensional reduction giving rise to four-dimensional $N = 4$, half-maximal gauged supergravity from the unique supergravity in eleven dimensions is thoroughly verified in this study. Dimensional reductions are said to be consistent at the level of equations of motion if and only if substitutions of reduction ansatze in higher dimensional equations of motion yield all equations of motion for lower dimensional theory. Apart from some special cases, a Kaluza-Klein sphere reduction is generally not consistent. However, there exists an alternative way to obtain a consistent dimensional reduction. The guaranteed-consistent reduction, known as the Scherk-Schwarz reduction, can be found by performing the dimensional reduction on a group manifold of some particular Lie group. From the fact that the $SU(2)$ group manifold is topologically a three-dimensional sphere S^3, the Kaluza-Klein reduction involving S^3 can be obtained from a group manifold reduction via replacing the S^3 in the reduction ansatz by an $SU(2)$ group manifold. $N = 4$ gauged supergravity in four dimensions is obtained from a seven-dimensional sphere S^7 reduction ansatze by writing the S^7 as $S^3 \times S^3 \times S^1$ and replacing each S^3 by the $SU(2)$ group manifold while the consistency of seven-dimensional sphere S^7 reduction exists at the level of equations of motion. Apart from the main $SO(4)$ gauged theory, another related $N = 4$ gauged supergravity in four dimensions, the Freedman-Schwarz model, can be obtained by using a one-way map between the two versions of $N = 4$ gauged supergravity. Moreover, the consistent reduction ansatze allow solutions in four-dimensional $N = 4$ $SO(4)$ gauged supergravity theory to be embedded in eleven-dimensional supergravity.

Department ....Physics...........	Student's Signature ....
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CONTENTS

LIST OF TABLES

Page

LIST OF FIGURES

Page

CHAPTER I
INTRODUCTION

Theory of everything is a unified description of all four fundamental forces in nature: weak, strong, electromagnetic, and gravitational forces [1, 2, 3]. Finding this final theory is one of the major unsolved problems in physics. Unfortunately, the main obstruction on the way to this unified theory is an incompatibility between Einstein's general relativity, the classical theory describing curved spacetime as gravity in large scale and high-mass conditions, and quantum field theory, expressing all the rest three non-gravitational forces in small scale and low-mass limitations. In general, this contradictory between the two pillars of physics is avoidable. However, if one wants to approach the theory of everything all extremely high-mass in small scale situations such as in a black hole or the Big Bang, the beginning stage of the universe are needed to be explained through quantum gravity, the name of the developing consistent theory between general relativity and quantum field theory.

Development of quantum gravity leads to the more fundamental theory in higher dimension for example; ten-dimensional string theory and eleven-dimensional M-theory, the two best candidates for the theory of everything. To describe our four-dimensional world via these fundamental theories, a way of extracting lower dimensional theory from the higher one is required.

Dimensional reduction is a procedure to extract a gravitational theory in lower spacetime dimensions from a higher dimensional one with some compactification imposed on some of the spacetime coordinates. It originated in 1926 known as Kaluza-Klein reduction theory [4, 5, 6, 7] where the usual Einstein's general relativity was considered in five spacetime dimensions. The fifth extra dimension is compactified to a very small, in the order of Planck length, circle or one-dimensional sphere (S^1), as shown in Figure 1.1, such that the compact space is unobservable at the present energy scale. This unobservable compact space leads to the truncation to the massless sector process giving rises to the Kaluza-Klein reduction ansatz, an expression of the higher dimensional field in terms of the lower dimensional ones, that turns the five-dimensional pure gravity theory into the Einstein-scalar-Maxwell system in four spacetime dimensions. The result is well known to be the first unification theory between gravity and electromagnetism, satisfying $U(1)$ gauge symmetry that corresponds to the symmetry on S^1. Moreover, Kaluza-Klein reduction proposes the possible way to unify gravity and other forces with a more complicated symmetry by means of consideration a gravitational theory in higher dimensions with a more complicated compact space. Unfortunately, there are some issues that are in conflict with experiments, such as the requirement of an unobserved extra dimension, the presence of the massless scalar field, called the dilaton, in the resulting theory, fading this reduction out of the main research current for a long time [8].

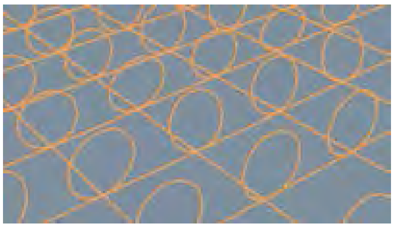

Figure 1.1: Illustration of the compact space S^1 where the rest four-dimensional spacetime are simplified to the two dimensional grid.[1]

Until 1997, the AdS/CFT correspondence, a duality relating string theory or its effective theory, supergravity, on anti-de Sitter (AdS) background to conformal field theories (CFT) on the AdS boundary [9-11] as shown in Figure 1.2 for the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, was first developed. This new duality was becoming a fascinating research topic for many theoretical physicists. Gauged supergravities, the supergravities with gauged R-symmetry group or any subgroup thereof, admit AdS vacuum solutions that have played the important role in the AdS/CFT correspondence. As mentioned before, a conformal field theory in D spacetime dimensions corresponds to a (D+1)-dimensional gauged supergravity, therefore, the derivation of gauged supergravity in some specific dimensions becomes valuable for this study. In many cases, lower dimensional gauged supergravities can be derived from higher dimensional ungauged supergravities such as the eleven-dimensional supergravity [12], the type IIA and IIB supergravities by using the faded dimensional reduction.

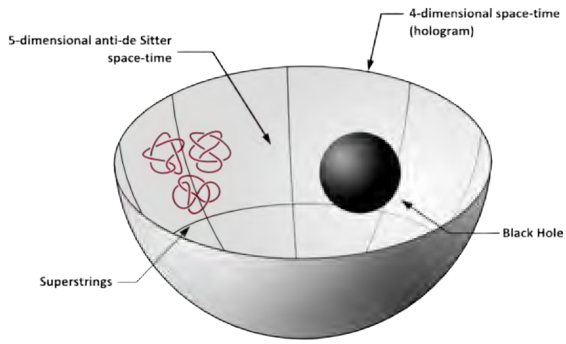

Figure 1.2: Illustration of $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence.[2]

There are two general ways to obtain gauged supergravities by dimensional reduction [13], as shown in Figure 1.3. The first way is represented by the vertical arrow. From the reduction on an n-dimensional torus (T^n), a product space of

[1] imagining-other-dimensions-merl.jpg [Online]. Available from : http://www.pbs.org/wgbh/nova/assets/img/full-size/imagining-other-dimensions-merl.jpg[2016,February]

[2] ads-cft.png [Online]. Available from : http://quantum-bits.org/wp-content/uploads/2015/09/ads-cft.png[2016,March]

n-S^1 where an example T^2 compact space is illustrated in the left panel of Figure 1.4, a lower-dimensional ungauged supergravity, which refuses an AdS vacuum, is obtained. Then, gauged supergravity in lower dimensions is achieved later by a gauging process (the horizontal arrow), selecting a subgroup, G_0 of the global symmetry, G of the ungauged supergravity and promoting it to a local gauge symmetry.

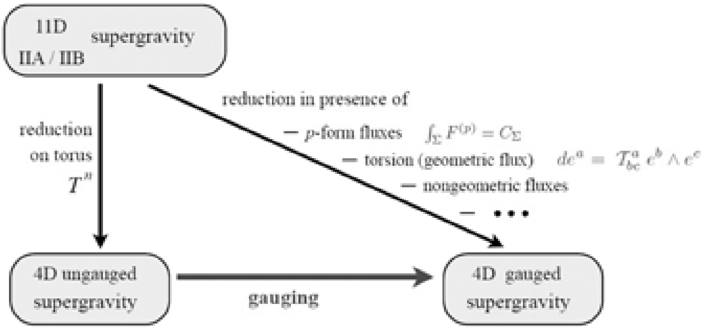

Figure 1.3: lower-dimensional gauged supergravity from the higher ungauged ones [13].

For the diagonal arrow, a lower-dimensional gauged supergravity can directly be obtained by the consistent dimensional reduction on more complicated structured compact space like an n-dimensional sphere (S^n), for example the dimensional reduction on S^2 is shown in the right panel of Figure 1.4, from the higher dimensional ungauged supergravities as demonstrated in the study of AdS black holes in [14]. Notable cases are the maximal gauged supergravities in 7 and 4 dimensions from consistent Kaluza-Klein reductions on S^4 [15, 16] and S^7 [17] of the eleven-dimensional supergravity and the five-dimensional maximal gauged supergravities from the reduction on S^5 of type IIB supergravity [18]. The reductions to maximal gauged supergravities are very complicated; the full consistent reduction of S^7 has been recently proven in [19]. On the other hand, many examples giving the half-maximal gauged supergravities in D = 7, 6, 5, and 4 have been worked out completely [20, 21, 22, 23].

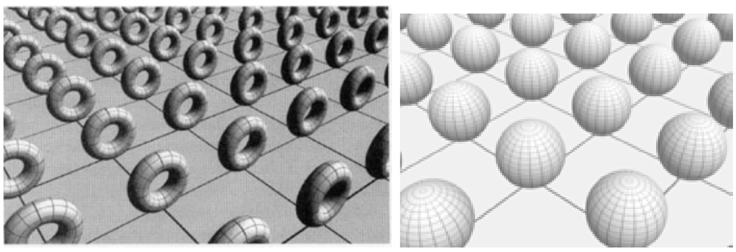

Figure 1.4: Illustration of the compact space T^2 and S^2 [3].

However, the consistencies of the Kaluza-Klein sphere reductions, mentioned above, are suspicious. They depend seriously on conspiracies between contributions from the metric and the other fields in the higher-dimensional theory and have no general understanding of their reasons unless for achieving the consistency.

In general, Kaluza-Klein reduction on S^n with an arbitrary $n > 1$ leads to inconsistencies. Fortunately, there exists an alternative way to perform a spherical reduction. This is known as the Scherk-Schwarz reduction [24] that is guaranteed to be consistent. It is a Kaluza-Klein reduction on a group manifold of a Lie group, a continuous group equipped with a Lie algebra, for example, the group manifold of $SU(2)$ is S^3, demonstrated in Appendix A. From the consistency viewpoint, Lie algebras are classified into 2 types; the Lie algebras with traceless structure constants are referred to as type A where the reduction ansatz is consistent at the level of the higher-dimensional action, while for the type B algebra, with non-vanishing trace of structure constants in Lie algebras, the reduction is consistent only at the level of the field equations. There are some previous works on this guaranteed consistent reduction. The $SU(2)$ reduction of six-dimensional (1,0) supergravity giving rise to a gauged supergravity in three dimensions was studied in [25], and in a more general case, the group reduction on an n-dimensional group manifold of the ten-dimensional heterotic supergravity is given in [26].

The main purpose of this work is to study the dimensional reduction theory for obtaining the N=4, half-maximal $SO(4)$ gauged supergravity in four-dimensional spacetime by truncating the Kaluza-Klein reduction on S^7 of the ungauged supergravity in eleven dimensions that was worked out in [23]. The compact space S^7 is described by a foliation of the two three-dimensional spheres, S^3. In group theory, S^3 is a group manifold of Lie group $SU(2)$. By replacing an S^3 with the $SU(2)$ group manifold and keeping only $SU(2)$ left invariant fields, it is guaranteed to be a consistent reduction. This could possibly lead to the more understanding in the consistency of the reduction on S^7.

The reduced theory; N=4, half-maximal $SO(4)$ gauged supergravity in four-dimensional spacetime, has some applications in the study of superconductors. By the AdS/CFT correspondence, a gauged supergravity is related to a superconformal field theory in three-dimensional spacetime, a time dimension together with the two-dimensional spatial surface, that has been recently used for study two-dimensional superconductor in [27, 28].

Besides, knowing the procedure of the dimensional reduction allows us to embed solutions in lower dimensions to higher dimensional theory through the reduction ansatz as in [29, 30]. Since the two theories are consistent, solutions in one theory have to be solutions in another one. The embedding solution in the most fundamental eleven-dimensional M-theory possibly leads to more interesting properties of the same solution in the lower dimensional description.

Thus, after the study of dimensional reduction, some solutions in this $SO(4)$ gauged supergravity in four dimensions will be reviewed, established and embedded to study in eleven-dimensional supergravity.

CHAPTER II
LITERATURE REVIEW

2.1 Einstein's Gravity Theory

Einstein's gravity theory or general relativity is one of the cornerstones of classical physics that applies special relativity to gravity. Albert Einstein spent almost ten years after created his special relativity theory to formulate this elegant theory of gravity in 1915 [31, 32]. General relativity can explain situations that Newton's gravity theory cannot figure out for a long time. The most famous example, which appears in many textbooks is the perihelion shift of Mercury. It has been long known that the point on Mercury's orbit with the nearest distance from the sun is shifted by 42.9 seconds of arc in every 100 years [33]. Calculations from Newtonian mechanics cannot describe this weird situation even though adding the effect of the neighbouring planets while Einstein's general relativity can explain it effortlessly. In this section, the brief main ideas about Einstein's gravity theory are reviewed from [33, 34, 35, 36] to introduce all the basic understandings and calculations that will be always used in this study.

There are two essences in general relativity. Firstly, gravity is not a force but a curvature of spacetime. Our four-dimensional spacetime is not only an empty flat static stage of the universe anymore but also has dynamics and can be curved. Some mathematical concepts describing the curvature of spacetime are established in the first part of this section: differential geometry. Secondly, the curvature of spacetime is caused by the existence of matter or energy. Hence, the last part will be about Einstein's field equations explaining this curvature's cause. Furthermore, some matter fields coupling to gravity will be reviewed at last.

2.1.1 Differential Geometry

2.1.1.1 Spacetime, manifold, tangent and cotangent spaces

In relativity framework, spacetime is the main character expressing all the weird but true relativistic phenomena. The idea of spacetime was emerged in special relativity by considering time as one of the universe's dimensions that can be related to different observers instead of the absolute time of the universe that is the same for all observers.

Flat space is the simplest case that satisfies Euclidean geometry, all basic geometry, such as interior angles in a triangle add up to 180 degrees. In special relativity, the four-dimensional flat spacetime $\mathbb{R}^{1,3}$ or Minkowski spacetime $\mathbb{M}^4$ has an unusual rotational symmetry: Lorentz symmetry or $SO(3,1)$. The Lorentz symmetry corresponds to Lorentz transformations, the spacetime transformations satisfying the two postulates of special relativity:

1. All physical laws are the same in all inertial frames.

2. The speed of light* is the same in all inertial frames.

The two inertial frames (O and O′) related by some relative speed v in Euclidean coordinates are shown in the Figure 2.1. All four spacetime coordinates in each frame can be related by the Lorentz transformations:

$$\begin{aligned} t' &= \gamma(t - \mathrm{v}x), \\ x' &= \gamma(x - \mathrm{v}t), \\ y' &= y, \\ z' &= z, \end{aligned} \tag{2.1.1}$$

where $\gamma = \frac{1}{\sqrt{1-\mathrm{v}^2}}$ is called Lorentz factor.

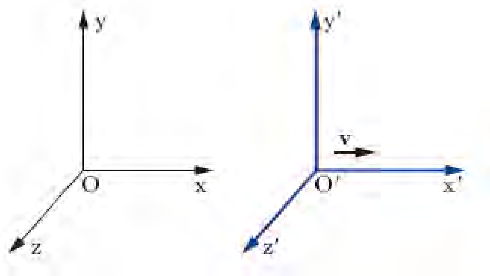

Figure 2.1: The two inertial frames O and O′ with some constant relative speed v.[1]

In the rest of this study, any vectors in spacetime coordinates can be easily described by $x^a = (t, x, y, z, ...)$ where $a = 0, 1, 2, 3, ..., D-1$ is a D-dimensional flat spacetime index where the zeroth coordinate is always the time coordinate. By this convention, Lorentz transformations in four-dimensional spacetime from (2.1.1) will become

$$\begin{aligned} x^{0'} &= \gamma(x^0 - \mathrm{v}x^1), \\ x^{1'} &= \gamma(x^1 - \mathrm{v}x^0), \\ x^{2'} &= x^2, \\ x^{3'} &= x^3, \end{aligned} \tag{2.1.2}$$

or in matrix equation,

$$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\mathrm{v} & 0 & 0 \\ -\gamma\mathrm{v} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}. \tag{2.1.3}$$

*For convenience, the speed of light is assigned to be unity, $c = 1$.

[1] relativita_04.jpg[Online]. Available from : http://images.treccani.it/enc/media/share/images/orig/system/galleries/NPT/VOL_8/IMMAGINI/relativita_04.jpg[2016,May]

By introducing Lorentz operator,

$$\Lambda = \begin{bmatrix} \gamma & -\gamma \mathrm{v} & 0 & 0 \\ -\gamma \mathrm{v} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.1.4}$$

the Lorentz transformations (2.1.2) can be written in a simple equation:

$$x^{a'} = \Lambda^{a'}{}_b x^b \tag{2.1.5}$$

where the repeated indices are summed by Einstein's summation convention $\Lambda^{a'}{}_b x^b = \sum_{b=0}^{3} \Lambda^{a'}{}_b x^b$. Moreover, if the two inertial frames (O and O′) are related by a relative speed v in arbitrary direction, (2.1.5) is still satisfied by using

$$\Lambda = \begin{bmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_x & \gamma & 0 & 0 \\ -\gamma v_y & 0 & \gamma & 0 \\ -\gamma v_z & 0 & 0 & \gamma \end{bmatrix}. \tag{2.1.6}$$

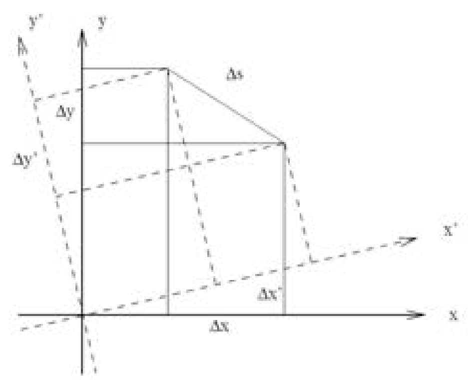

Figure 2.2: The unchanged distance Δs in two-dimensional space (x,y) due to rotational symmetry $SO(2)$ that turns (x,y) to (x′,y′) [33].

Four-dimensional Minkowski spacetime has an unusual rotational symmetry $SO(3,1)$ such that any distance in the spacetime should be unchanged or said to be invariance under these Lorentz transformations (the unusual rotation) similar to the unchanged distance in space due to rotational symmetry, for example; SO(2) in two-dimensional space illustrated in Figure 2.2. Spacetime interval is the distance between two events, points in spacetime, defined* by

$$\Delta s^2 = -(\Delta t)^2 + (\Delta \vec{x})^2 \tag{2.1.7}$$

*This definition is called the mostly plus convention that prefers to use in many studies of general relativity. However, one can use the mostly minus convention by defining spacetime interval as $\Delta s^2 = (\Delta t)^2 - (\Delta \vec{x})^2$ together with $\eta_{ab} = diag(1, -1, -1, -1)$ but their physical meanings are the same where $(\Delta\vec{x})^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$ is the distance in three-dimensional space. By introducing the Minkowski metric,

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.1.8}$$

the spacetime interval can be written in the form

$$\Delta s^2 = \eta_{ab}\Delta x^a \Delta x^b. \tag{2.1.9}$$

Hence, its infinitesimal form is

$$ds^2 = \eta_{ab} dx^a dx^b, \tag{2.1.10}$$

that is invariant under Lorentz transformations

$$\begin{aligned} ds^2 &= ds'^2, \\ \eta_{ab} dx^a dx^b &= \eta_{a'b'} dx^{a'} dx^{b'}. \end{aligned} \tag{2.1.11}$$

By using Lorentz transformations (2.1.5), Minkowski metric has to transforms by

$$\eta_{a'b'} = \Lambda^a{}_{a'}\Lambda^b{}_{b'}\eta_{ab} \tag{2.1.12}$$

where $\Lambda^a{}_{a'} = (\Lambda^{a'}{}_a)^{-1}$ is an inverse of the Lorentz operator in (2.1.5).

In vector's point of view, spacetime interval is nothing but the square magnitude of the vector dx^a. Therefore, any vector V^a in Minkowski spacetime has its square magnitude as

$$|V|^2 = \eta_{ab} V^a V^b. \tag{2.1.13}$$

A dual vector of the vector V^a in spacetime is defined by

$$V_a = \eta_{ab} V^b, \tag{2.1.14}$$

and transforms by the inverse Lorentz operator

$$V_{a'} = \Lambda^a{}_{a'} V_a. \tag{2.1.15}$$

Using this dual vector definition, the square magnitude of any vector V^a in Minkowski spacetime can be written in the form

$$V^2 = V_a V^a, \tag{2.1.16}$$

Note that the minus sign in Minkowski metric introduced in (2.1.8) distinguishes vector V^a into 3 types:

1. $V^2 > 0 : V$ is called space-like vector.

2. $V^2 = 0 : V$ is called light-like or null vector.

3. $V^2 < 0 : V$ is called time-like vector.

This dissimilarity depends on the difference between the components of time and space coordinates while Lorentz transformations cannot change their type. In group theory, Lorentz group $SO(3,1)$ is categorised as a non-compact group, see also Appendix A.

While spacetime and coordinates seem indivisible in special relativity, they become vastly different in general relativity. To yield spacetime that can be curved, the theory describes D-dimensional spacetime as a D-dimensional differentiable real manifold $\mathcal{M}$, a smooth and continuous topological real space that locally looks like flat space $\mathbb{M}^D$. A coordinate system x maps a subset of $\mathcal{M}$ to the well known $\mathbb{M}^D$. However, there is no a unique coordinate system for a manifold $\mathcal{M}$. On the other hand, one coordinate system x has a smooth map to another coordinate system x' that also maps a subset of $\mathcal{M}$ to some flat space $\mathbb{M}^D$, as shown in Figure 2.3.

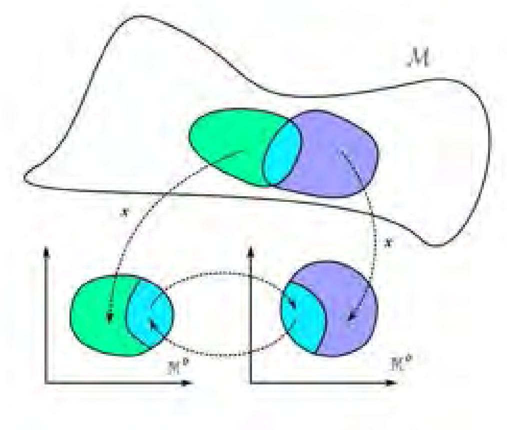

Figure 2.3: Mapping of two subsets of a manifold $\mathcal{M}$ by coordinate systems x and x' where the overlap region can be mapped smoothly between the two coordinate systems.[2]

At each point $p \in \mathcal{M}$, there exists a tangent space $T_p(\mathcal{M})$, a vector space containing tangent vectors at the point p, as displayed in Figure 2.4. $T_p(\mathcal{M})$ has the same dimensions as $\mathcal{M}$ with a particular set of basis vector $\partial/\partial x^\mu$ where x^μ is the coordinates and $\mu = 0, 1, \ldots, D-1$ is a D-dimensional curved spacetime index. Thus any vector $V \in T_p(\mathcal{M})$ can be written as

$$V = V^\mu \partial_\mu \tag{2.1.17}$$

Along with the tangent space $T_p(\mathcal{M})$, the corresponding cotangent space $T_p^*(\mathcal{M})$

[2]manifold.svg.png[Online]. Availabel from : https://upload.wikimedia.org/wikipedia/commons/thumb /0/06/Two_coordinate_charts_on_a_manifold.svg/2000px-Two_coordinate_charts_on_a_manifold.svg.png[2016,May]

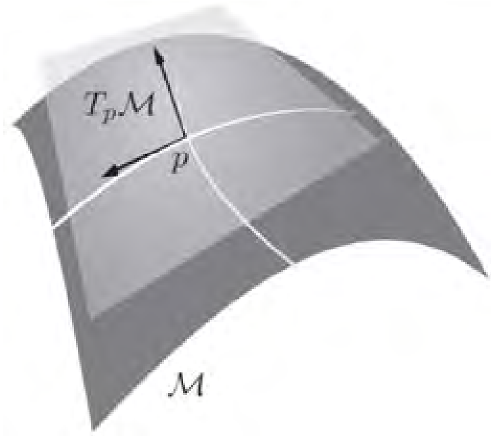

Figure 2.4: A tangent space $T_p(\mathcal{M})$ (gray plane) of a manifold $\mathcal{M}$ (dark grey) at point p [34].

with a dual basis dx^ν is also defined where $dx^\nu(\partial_\mu) = \delta^\nu_\mu$. Likewise, every vector in the cotangent space $W \in T^*_p(\mathcal{M})$ may be written as

$$W = W_\nu dx^\nu \tag{2.1.18}$$

By using the fact that physics does not depend on the choice of coordinates system, the general coordinates transformations (GCT) is defined as transformation ofs x to x' that turns basis vectors in the tangent space $T_p(\mathcal{M})$ from $\partial/\partial x^\nu$ to $\partial/\partial x^{\mu'}$ via using the chain rule,

$$\partial_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}}\, \partial_\nu. \tag{2.1.19}$$

Furthermore, any vector $V \in T_p(\mathcal{M})$ is a geometrical object that is independent of the choice of coordinate system as mentioned so V has to be invariant under GCT,

$$V = V^\nu \partial_\nu = V^{\mu'} \partial_{\mu'}. \tag{2.1.20}$$

Together with (2.1.19), this leads to the transformations of the vector components V^μ under GCT,

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} V^\nu, \tag{2.1.21}$$

where $\frac{\partial x^{\mu'}}{\partial x^\nu}$ can be claimed to be a GCT operator with its inverse $\frac{\partial x^\nu}{\partial x^{\mu'}}$. In the same fashion, the transformation properties of the cotangent vectors $W \in T^*_p(\mathcal{M})$ under GCT can be determined. Starting from transformations of the basis vectors,

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} dx^\nu. \tag{2.1.22}$$

The invariance of the cotangent vectors W leads to the transformations of the components W_μ,

$$W_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}} W_\nu. \tag{2.1.23}$$

To determine interval on curved spacetime, a multi-linear map from a product space of r cotangent space $T_p^*(\mathcal{M})$ and s tangent space $T_p(\mathcal{M})$ to the real line called (r,s) tensor $T^{(r,s)}$ is considered where (r,s) is called the rank of the tensor,

$$T^{(r,s)}: \underbrace{T_p^*(\mathcal{M}) \times ... \times T_p^*(\mathcal{M})}_{\text{r times}} \times \underbrace{T_p(\mathcal{M}) \times ... \times T_p(\mathcal{M})}_{\text{s times}} \to \mathbb{R}. \tag{2.1.24}$$

Note that $T^{(0,0)}$ is nothing but a real number or real scalar field while $T^{(1,0)}$ and $T^{(0,1)}$ are just an element of tangent and cotangent space in (2.1.17) and (2.1.18) respectively. An (r,s) tensor $T^{(r,s)}$ can be written in component form as

$$T^{(r,s)} = T^{\mu_1...\mu_r}{}_{\nu_1...\nu_s}\ \partial_{\mu_1} \otimes ... \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes ... \otimes dx^{\nu_s}, \tag{2.1.25}$$

where $\otimes$ is called a tensor product and $T^{\mu_1...\mu_r}{}_{\nu_1...\nu_s}$ are the components of any (r,s) tensor $T^{(r,s)}$ that transform under GCT as

$$T^{\mu'_1...\mu'_r}{}_{\nu'_1...\nu'_s} = \frac{\partial x^{\mu'_1}}{\partial x^{\rho_1}} ... \frac{\partial x^{\mu'_r}}{\partial x^{\rho_r}} \frac{\partial x^{\sigma_1}}{\partial x^{\nu'_1}} ... \frac{\partial x^{\sigma_s}}{\partial x^{\nu'_s}} T^{\rho_1...\rho_r}{}_{\sigma_1...\sigma_s} \tag{2.1.26}$$

For curved spacetime, any distance is now defined by a $(0,2)$ symmetric tensor called the metric g at each point $p \in \mathcal{M}$, i.e.

$$g:\ T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R}, \tag{2.1.27}$$

which can be expressed in terms of the basis vectors $dx^\mu \otimes dx^\nu$ and the components $g_{\mu\nu}(x)$ by

$$g = g_{\mu\nu}(x)\ dx^\mu \otimes dx^\nu. \tag{2.1.28}$$

By dropping $\otimes$, this equation becomes a familiar equation corresponding to the interval on curved spacetime that called the line element,

$$ds^2 = g_{\mu\nu}(x)\ dx^\mu dx^\nu. \tag{2.1.29}$$

In the case of flat spacetime, the metric components $g_{\mu\nu}$*, or called the metric tensor for simplicity, becomes the constant Minkowski metric $\eta_{\mu\nu}$ defined in (2.1.8). Moreover, $g_{\mu\nu}$ also have the same properties as $\eta_{\mu\nu}$, such as there exists inverse metric $g^{\mu\nu} = g^{-1}_{\mu\nu}$, if $g_{\mu\nu}$ are non-degenerate ($\det g_{\mu\nu} \neq 0$), which satisfy

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu. \tag{2.1.30}$$

Similarly, the metric components and their inverses can be used to raise and lower the components of tangent and cotangent vector correspondingly by mapping between tangent and cotangent space,

$$\begin{aligned} g:\ T_p(\mathcal{M}) \to T_p^*(\mathcal{M}) \Rightarrow V_\mu = g_{\mu\nu} V^\nu, \\ g^{-1}:\ T_p^*(\mathcal{M}) \to T_p(\mathcal{M}) \Rightarrow W^\mu = g^{\mu\nu} W_\nu. \end{aligned} \tag{2.1.31}$$

*For convenience, an argument (x) for the metric components $g_{\mu\nu}$ is usually omitted but they are always depend on coordinates x.

The line element in (2.1.29) is sometimes called the metric equation containing the metric tensor $g_{\mu\nu}$ encoding the curvature's information of spacetime. Therefore, metric equation (2.1.29) is the keystone for describing spacetime curvature. Before decoding the curvature's information of spacetime, the totally anti-symmetric tensor is useful to introduce.

2.1.1.2 Differential forms and volume form

Differential forms or p-form* is the totally anti-symmetric $(0,p)$ tensor $\omega^{(p)}$,

$$\omega^{(p)}:\ \underbrace{T_p(\mathcal{M})\times ...\times T_p(\mathcal{M})}_{\text{p times}} \rightarrow \mathbb{R}, \tag{2.1.32}$$

defined by

$$\omega^{(p)} = \frac{1}{p!}\,\omega_{\mu_1...\mu_p}\,dx^{\mu_1}\wedge ...\wedge dx^{\mu_p}, \tag{2.1.33}$$

where the components $\omega_{\mu_1...\mu_p}$ are anti-symmetric in all indices,

$$\omega_{\mu_1...\mu_m...\mu_n...\mu_p} = -\omega_{\mu_1...\mu_n...\mu_m...\mu_p}. \tag{2.1.34}$$

The basis operation of p-form is described by the wedge product $\wedge$ that totally anti-symmetric instead of the usual tensor product $\otimes$,

$$dx^{\mu_1}\wedge ...\wedge dx^{\mu_p} = p!\,dx^{[\mu_1}\otimes ...\otimes dx^{\mu_p]} \tag{2.1.35}$$

where anti-symmetrizing in the indices $\mu_1...\mu_p$ of any tensor component T is denoted by

$$T^{[\mu_1...\mu_p]} = \frac{1}{p!}(T^{\mu_1...\mu_p} + (-1)^P\,permutation\ of\ (\mu_1...\mu_p)), \tag{2.1.36}$$

with $P=0$ for even and $P=1$ for odd permutation.

There are some useful operations of differential forms, which are essential to the calculation performed in this study, reviewed as following:

- Wedge product: $\wedge$

$$\wedge:\ (\omega^{(p)},\omega^{(q)})\ \rightarrow \omega^{(p+q)} \tag{2.1.37}$$

The wedge product of any p-form $\omega^{(p)}$ and q-form $\omega^{(q)}$ yeilds the resulting $(p+q)$-form $\omega^{(p)}\wedge\omega^{(q)}$, for example;

$$\begin{aligned} A^{(p)} &= \frac{1}{p!}A_{\mu_1...\mu_p}\,dx^{\mu_1}\wedge ...\wedge dx^{\mu_p}, \\ B^{(q)} &= \frac{1}{q!}B_{\nu_1...\nu_q}\,dx^{\nu_1}\wedge ...\wedge dx^{\nu_q}, \\ A^{(p)}\wedge B^{(q)} &= \frac{1}{p!q!}A_{\mu_1...\mu_p}B_{\nu_1...\nu_q}dx^{\mu_1}\wedge ...\wedge dx^{\mu_p}\wedge dx^{\nu_1}\wedge ...\wedge dx^{\nu_q}. \end{aligned} \tag{2.1.38}$$

*Note that this p in differential forms is an integer number, $0\leq p\leq D$ for D dimensional spacetime, while p in $T_p\mathcal{M}$ is a point p in a manifold $\mathcal{M}$. They are confusing but fashionable notation.

Thus

$$A^{(p)} \wedge B^{(q)} = (-1)^{pq} B^{(q)} \wedge A^{(p)}. \tag{2.1.39}$$

- Exterior derivative: d

$$d: \ \omega^{(p)} \ \rightarrow \omega^{(p+1)} \tag{2.1.40}$$

The resulting object of an exterior derivative of a p-form $\omega^{(p)}$ is a $(p+1)$-form $d\omega^{(p)}$ defined as

$$d\omega^{(p)} = \frac{1}{p!} \left(\partial_\mu \omega_{\mu_1 \ldots \mu_p}\right) dx^\mu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{2.1.41}$$

The exterior derivative is a linear operation satisfying the following conditions,

$$d(\omega^{(p)} \wedge \omega^{(q)}) = d\omega^{(p)} \wedge \omega^{(q)} + (-1)^p \omega^{(p)} \wedge d\omega^{(q)}, \tag{2.1.42}$$

$$d^2 \omega^{(p)} = 0, \tag{2.1.43}$$

for any p-form $\omega^{(p)}$ and q-form $\omega^{(q)}$, where the operator d^2 in (2.1.43) is called the nilpotent operator that always gives zero as a result of anti-symmetrization.

- Hodge duality: $*$

Hodge duality is an operation with respect to the dimension of the spacetime mapping a p-form in D-dimensional spacetime to a $(D-p)$-form,

$$*: \ \omega^{(p)} \ \rightarrow \omega^{(D-p)}, \tag{2.1.44}$$

defined by

$$*\omega^{(p)} = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} * (dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}). \tag{2.1.45}$$

The hodge duality of the basis are given by

$$*(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}) = \frac{1}{(D-p)!} \epsilon_{\mu_{p+1} \ldots \mu_D}{}^{\mu_1 \ldots \mu_p} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_D}, \tag{2.1.46}$$

hence hodge duality for any p-form in D-dimensional spacetime can be written as

$$*\omega^{(p)} = \frac{1}{p!(D-p)!} \omega_{\mu_1 \ldots \mu_p} \epsilon_{\mu_{p+1} \ldots \mu_D}{}^{\mu_1 \ldots \mu_p} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_D}. \tag{2.1.47}$$

Here, $\epsilon_{\mu_1 \ldots \mu_D}$ is the totally anti-symmetric Levi-Civita tensor in D-dimensional curved spacetime given by

$$\epsilon_{\mu_1 \ldots \mu_D} = \sqrt{|g|}\, \varepsilon_{\mu_1 \ldots \mu_D}, \tag{2.1.48}$$

where $g = \det g_{\mu\nu}$ and $\varepsilon_{\mu_1 \ldots \mu_D}$ is the totally anti-symmetric Levi-Civita symbol defined by

$$\varepsilon_{\mu_1 \ldots \mu_D} = \begin{cases} +1 : (\mu_1 \ldots \mu_D) \text{ is even permutation of } (0, 1, \ldots, D-1), \\ -1 : (\mu_1 \ldots \mu_D) \text{ is odd permutation of } (0, 1, \ldots, D-1), \\ \ \ 0 : \text{otherwise.} \end{cases} \tag{2.1.49}$$

Note that the Levi-Civita tensor becomes the Levi-Civita symbol in flat Minkowski spacetime where $\det g_{\mu\nu} = \det \eta_{\mu\nu} = -1$. For any p-form $\omega^{(p)}$, applying twice Hodge dualities gives rise to the same p-form $\omega^{(p)}$ with positive or negative sign by

$$**\,\omega^{(p)} = (-1)^{p(D-p)+t}\omega^{(p)}, \tag{2.1.50}$$

where t is the number of time-like coordinate.

Volume form is a canonical volume that plays an important role for the calculus on D-dimensional curved spacetime $\mathcal{M}$ defined by the unique D-form,

$$Vol(\mathcal{M}) = \sqrt{|g|}\, d^D x = \sqrt{|g|}\, dx^0 \wedge ... \wedge dx^{D-1}. \tag{2.1.51}$$

By using the totally anti-symmetric Levi-Civita symbol defined in (2.1.49), the volume form can be written as

$$\begin{aligned} Vol(\mathcal{M}) &= \sqrt{|g|}\,\frac{1}{D!}\,\varepsilon_{\mu_1...\mu_D} dx^{\mu_1} \wedge ... \wedge dx^{\mu_D} \\ &= \frac{1}{D!}\,\epsilon_{\mu_1...\mu_D} dx^{\mu_1} \wedge ... \wedge dx^{\mu_D}, \end{aligned} \tag{2.1.52}$$

where the second line is obtained by applying the relation (2.1.48) that is obviously invariant under GCT. Moreover, the Hodge duality of the pure number 1 (a 0-form) denoted by $\epsilon_{(D)}$ in D dimensions is also the volume form,

$$\begin{aligned} *1 = \epsilon_{(D)} &= \frac{1}{D!}\,\epsilon_{\mu_1...\mu_D} dx^{\mu_1} \wedge ... \wedge dx^{\mu_D}, \\ &= \sqrt{|g|}\, d^D x = Vol(\mathcal{M}). \end{aligned} \tag{2.1.53}$$

Together with the wedge product, Hodge duality can describe the inner product between any two p-forms, A and B, as follows

$$*A \wedge B = *B \wedge A = \frac{1}{p!}|A \cdot B| * 1, \tag{2.1.54}$$

where the inner product is defined by

$$|A \cdot B| \equiv A_{\mu_1...\mu_p} B^{\mu_1...\mu_p}, \tag{2.1.55}$$

that frequently appears in the kinetic terms in many theories for A and B being derivative terms.

2.1.1.3 Spacetime curvature from vielbein formalism

Now, we are ready to describe curved spacetime. As declared before, the information about the curvature of spacetime is encoded in the metric tensor $g_{\mu\nu}(x)$ that are functions depending on coordinates x at each point $p \in \mathcal{M}$. Unfortunately, dealing with these tensor is more complicated. Especially, their inversion $g^{-1}_{\mu\nu}$ are difficult to find.

In this study, an easier way for describing curved spacetime known as the vielbein formalism is utilised. Since a D-dimensional manifold $\mathcal{M}$ is locally flat, D-dimensional flat spacetimes can be defined at each point $p \in \mathcal{M}$. Precisely there are the two Minkowski spacetimes defined before: the tangent space $T_p(\mathcal{M})$ and cotangent space $T_p^*(\mathcal{M})$. These flat spacetimes, known as Lorentz frames, are described by the non-coordinate vielbein bases e_a and e^a respectively, where a is the flat spacetime index defined previously. The relation between the metric tensor and the Minkowski metric in these Lorentz frames is defined by

$$g_{\mu\nu}(x) = \eta_{ab}\, e^a_\mu(x) e^b_\nu(x), \tag{2.1.56}$$

where* e^a_μ , called vielbein, are the components of the vielbein basis e^a with respect to the coordinates basis dx^μ, i.e. $e^a = dx^a = e^a_\mu dx^\mu$. Equation (2.1.56) describing the vielbein like the square-root of the metric $g_{\mu\nu}$ is practically used to find these vielbein components from a given metric tensor. Besides, inverse vielbein are defined as e^μ_a, which are the components of the inverse vielbein basis e_a with respect to the coordinates basis ∂_μ, i.e. $e_a = \partial_a = e^\mu_a \partial_\mu$, and they satisfy these relations:

$$e^\mu_a e^a_\nu = \delta^\mu_\nu, \quad e^\mu_a e^b_\mu = \delta^b_a. \tag{2.1.57}$$

By these vielbein basis component, any tangent vector V and cotangent vector W can be described as

$$V = V^a e_a, \;\; W = W_a e^a, \tag{2.1.58}$$

with their components,

$$V^a = e^a_\mu V^\mu, \;\; W_a = e^\mu_a W_\mu. \tag{2.1.59}$$

Therefore, any (r, s) tensor components can be transformed via these vielbein basis components to be an (r, s) Lorentz tensor components in flat spacetime,

$$T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = e^{a_1}_{\mu_1} \ldots e^{a_r}_{\mu_r} e^{\nu_1}_{b_1} \ldots e^{\nu_s}_{b_s}\, T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}. \tag{2.1.60}$$

However, the choice of the non-coordinate basis e^a is not unique. One can define another basis $e^{b'}$ satisfying (2.1.56). However, the transformations between $e^{b'}$ and e^a is not the GCT but the familiar Lorentz transformations in (2.1.5). Moreover, the Lorentz operator now depends on spacetime coordinates, $\Lambda^{b'}{}_a(x)$, such that the transformations between non-coordinate bases is called the local Lorentz transformations (LLT) where the transformations of vielbein and inverse vielbein are defined by

$$e^{b'}_\mu(x) = \Lambda^{b'}{}_a(x)\, e^a_\mu(x), \;\; e^\mu_{b'}(x) = \Lambda^a{}_{b'}(x)\, e^\mu_a(x), \tag{2.1.61}$$

*In the same case as $g_{\mu\nu}$, an argument (x) is omitted but keep in mind that both e^a_μ and e^μ_a are always depend on coordinates x.

similarly, $\Lambda^{a}{}_{b'}(x)$ is defined to be the inverse of $\Lambda^{b'}{}_{a}(x)$. Moreover, since both vielbein and their inverse contain the spacetime coordinate index μ, they also transform under the GCT,

$$e^{a}_{\mu'}(x) = \frac{\partial x^{\nu}}{\partial x^{\mu'}}\, e^{a}_{\nu}(x), \;\; e^{\mu'}_{a}(x) = \frac{\partial x^{\mu'}}{\partial x^{\nu}}\, e^{\nu}_{a}(x). \tag{2.1.62}$$

Consequently, vielbein are both $(1,0)$ Lorentz tensor components and $(0,1)$ coordinate tensor components at the same time, while their inverse are $(0,1)$ Lorentz tensor components and $(1,0)$ coordinate tensor components. Therefore, any tensor components could be both Lorentz and coordinate with respect to their flat and curve spacetime indices. Even so, they can transform to pure Lorentz or coordinate tensor components by virtue of vielbein and their inverse. The LLT of any (r,s) Lorentz tensor components can be written as

$$T^{a'_1 \ldots a'_r}{}_{b'_1 \ldots b'_s} = \Lambda^{a'_1}{}_{a_1} \ldots \Lambda^{a'}_{r}{}_{a_r} \Lambda^{b_1}{}_{b'_1} \ldots \Lambda^{b_s}{}_{b'_s}\, T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s}, \tag{2.1.63}$$

after subtracting (x) for convenience. Note that, in coordinate point of view, this (r,s) Lorentz tensor components are just a scalar, which is invariant under GCT.

Apart from scalar fields, partial derivatives of any (r,s) Lorentz tensor components do not transform as (2.1.63) anymore. For example, consider the partial derivative, $\partial_a = \partial/\partial x^a$, of a one-form or a $(0,1)$ Lorentz tensor components, i.e. $\partial_a W_b$. By the index structure, they should transform as $(0,2)$ Lorentz tensor components. However, under LLT, this partial derivative transforms as

$$\begin{aligned}\partial_{a'} W_{b'}(x) &= \Lambda^{c}{}_{a'}(x)\, \partial_c \left[\Lambda^{d}{}_{b'}(x) W_d(x)\right], \\ &= \Lambda^{c}{}_{a'}(x)\, \Lambda^{d}{}_{b'}(x) \left[\partial_c\, W_d(x)\right] + W_d(x) \Lambda^{c}{}_{a'}(x) \left[\partial_c\, \Lambda^{d}{}_{b'}(x)\right].\end{aligned} \tag{2.1.64}$$

Here the second line is obtained by using the Leibniz's product rule where the first term is the expected transformation law of $(0,2)$ Lorentz tensor components while the second term ruins it. The only one settlement is an elimination of this second term by defining the Lorentz covariant derivative D_a of any one-form as

$$D_a W_b = \partial_a W_b - \omega_a{}^{f}{}_{b} W_f, \tag{2.1.65}$$

where $\omega_a{}^{c}{}_{b}$ is the spin connection that is anti-symmetric in the last two indices $\omega_{abc} = -\omega_{acb} = -\eta_{cf}\omega_a{}^{f}{}_{b}$. To get rid of the problematic term in (2.1.64), the transformations of the spin connection contracted with a one-form under LLT is defined by

$$\omega_{a'}{}^{f}{}_{b'} W_f = W_d \Lambda^{c}{}_{a'}(x) \left[\partial_c\, \Lambda^{d}{}_{b'}(x)\right] - \Lambda^{c}{}_{a'}(x)\, \Lambda^{d}{}_{b'}(x)\, \omega_a{}^{f}{}_{b} W_f. \tag{2.1.66}$$

Together with (2.1.64), the transformations of the Lorentz covariant derivative of any one-form under LLT yields the expected $(0,2)$ Lorentz tensor components' transformations,

$$\begin{aligned}D_{a'} W_{b'} &= \Lambda^{c}{}_{a'}(x)\, \Lambda^{d}{}_{b'}(x) [\partial_c\, W_d(x)] + \Lambda^{c}{}_{a'}(x)\, \Lambda^{d}{}_{b'}(x)\, \omega_a{}^{f}{}_{b} W_f, \\ &= \Lambda^{c}{}_{a'}(x)\, \Lambda^{d}{}_{b'}(x)\, D_c W_d.\end{aligned} \tag{2.1.67}$$

This procedure is called gauging acheived by promoting a global symmetry to be local resulting in a new derivative that satisfies transformation rules of the new promoted local symmetry. As demonstrated above, the Lorentz symmetry, a global symmetry in flat spacetime, was promoted to be local, ${\Lambda^{b'}}_a \to {\Lambda^{b'}}_a(x)$. Then, the transformations of the partial derivative of non-scalar fields turn out to be inconsistent with the LLT. Here, defining a new derivative, the standard derivative added by a correction term, gives the right transformations under LLT. This gauging process will be reviewed soon in the literature review of gauge theories in which some Lie groups are gauged.

Moreover, the Lorentz covariant derivative of any tangent vector, $(1,0)$ Lorentz tensor, components can be defined, by using the fact that the Lorentz covariant derivative of any scalar fields reduces to the partial derivative, $D_a(V^b W_b) = \partial_a(V^b W_b)$ together with (2.1.65), as

$$D_a V^b = \partial_a V^b + {\omega_a}^b{}_f V^f. \tag{2.1.68}$$

Thus the Lorentz covariant derivative of a (r,s) Lorentz tensor components can be written as

$$\begin{aligned} D_a T^{a_1\ldots a_r}{}_{b_1\ldots b_s} =& \partial_a T^{a_1\ldots a_r}{}_{b_1\ldots b_s} + {\omega_a}^{a_1}{}_c T^{c\ldots a_r}{}_{b_1\ldots b_s} + \ldots + {\omega_a}^{a_r}{}_c T^{a_1\ldots c}{}_{b_1\ldots b_s} \\ &- {\omega_a}^c{}_{b_1} T^{a_1\ldots a_r}{}_{c\ldots b_s} - \ldots - {\omega_a}^c{}_{b_s} T^{a_1\ldots a_r}{}_{b_1\ldots c}. \end{aligned} \tag{2.1.69}$$

Note that the Lorentz covariant derivative D_a satisfies the following properties:

1. D_a maps any (r,s) Lorentz tensor components to $(r,s+1)$ Lorentz tensor components,
2. D_a is a linear operator, i.e. $D_a(T+S) = D_a T + D_a S$,
3. D_a satisfies the Leibniz's product rule; $D_a(T \otimes S) = D_a T \otimes S + T \otimes D_a S$,
4. for Lorentz scalar fields, D_a reduces to ∂_a.

The coordinate covariant derivative of any vector in coordinate basis $\nabla_\mu V^\rho$ is defined from the transformed Lorentz covariant derivative, $D_\mu = e^a_\mu D_a = \partial_\mu + {\omega_\mu}^b{}_c$, as following

$$\begin{aligned} \nabla_\mu V^\rho \equiv e^\rho_a D_\mu V^a &= e^\rho_a D_\mu (e^a_\nu V^\nu), \\ &= \partial_\mu V^\rho + e^\rho_a (\partial_\mu e^a_\nu + {\omega_\mu}^a{}_b e^b_\nu) V^\nu, \end{aligned} \tag{2.1.70}$$

while the last term in the first line is obtained by transforming to the vielbein basis. The second line arises from the Leibniz's product rule of the Lorentz covariant derivative while the first term is just a normal partial derivative because V^ρ is $(0,0)$ Lorentz scalar. This equation is in the same form as (2.1.65) so the coordinate connection can be defined by the second term in (2.1.70) as

$$\Gamma^\rho_{\mu\nu} = e^\rho_a (\partial_\mu e^a_\nu + {\omega_\mu}^a{}_b e^b_\nu), \tag{2.1.71}$$

where this $\Gamma^{\rho}_{\mu\nu}$ is called the affine or Christoffel's connection making the coordinate covariant derivative transforms in the right way under the GCT, in the same way as the spin connection preserving the LLT transformation rules for the Lorentz covariant derivative. Substitution this Christoffel's connection back in (2.1.70) yields the definition of the coordinate covariant derivative of any tangent vector in coordinate basis,

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^{\rho}_{\mu\nu} V^\nu. \tag{2.1.72}$$

However, the coordinate covariant derivative is usually defined to be the total covariant derivative that contains both spin and Christoffel's connections. If we use the Lorentz covariant derivative in the first term of the second line in equation (2.1.70), we will find

$$\nabla_\mu V^\nu = D_\mu V^\nu + \Gamma^{\nu}_{\mu\sigma} V^\sigma, \tag{2.1.73}$$

with Lorentz covariant derivative being an ordinary partial derivative for $(0,0)$ Lorentz tensor components. This covariant derivative becomes a partial derivative only on a scalar quantity in both coordinate and non-coordinate frames. This covariant derivative also satisfies the same rules as the Lorentz covariant derivative, such as the coordinate covariant derivative of any one-form is given by

$$\nabla_\mu W_\nu = D_\mu W_\nu - \Gamma^{\rho}_{\mu\nu} W_\rho. \tag{2.1.74}$$

This leads to the definition of a coordinate covariant derivative of any (r,s) coordinate tensor components,

$$\begin{aligned}\nabla_\mu T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} =& D_\mu T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \Gamma^{\mu_1}_{\mu\rho} T^{\rho\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \ldots + \Gamma^{\mu_r}_{\mu\rho} T^{\mu_1\ldots\rho}{}_{\nu_1\ldots\nu_s} \\ &- \Gamma^{\rho}_{\mu\nu_1} T^{\mu_1\ldots\mu_r}{}_{\rho\ldots\nu_s} - \ldots - \Gamma^{\rho}_{\mu\nu_s} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\rho}.\end{aligned} \tag{2.1.75}$$

Therefore equation (2.1.71) can be written in the form of the total covariant derivative as,

$$\nabla_\mu e^a_\nu = D_\mu e^a_\nu - \Gamma^{\rho}_{\mu\nu} e^a_\rho = 0, \tag{2.1.76}$$

that is called the vielbein postulate. After anti-symmetrizing this equation, the torsion tensor $T^a_{\mu\nu}$ can be defined by

$$\nabla_{[\mu} e^a_{\nu]} = e^a_\rho (\Gamma^{\rho}_{\nu\mu} - \Gamma^{\rho}_{\mu\nu}) \equiv \frac{1}{2} T^a_{\mu\nu}. \tag{2.1.77}$$

Note that in common spacetime, this torsion tensor usually equals to zero, called the torsion-free condition of spacetime, that allows the Christoffel's connection to be symmetric under the two lower indices, $\Gamma^{\rho}_{\nu\mu} = \Gamma^{\rho}_{\mu\nu}$ and turns the vielbein postulate into

$$\nabla_{[\mu} e^a_{\nu]} = \partial_{[\mu} e^a_{\nu]} + \omega_{[\mu}{}^a{}_b e^a_{\nu]} = 0. \tag{2.1.78}$$

Moreover, this torsion-free condition can be rewritten using the differential forms as

$$de^a = -\omega^a{}_b \wedge e^b, \tag{2.1.79}$$

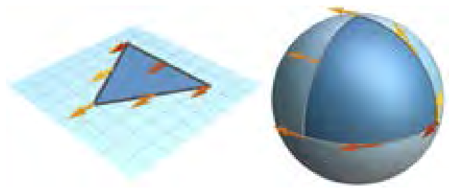

Figure 2.5: The parallel transport of a vector on a cloed path in flat and curved space.[3]

which is customarily used to find the spin connection.

To quantify the curvature of spacetime, the parallel transport of a vector $V = V^a e_a$ along a path γ parametrized by a parameter λ on a manifold $\mathcal{M}$ is defined as

$$\frac{\delta V^a}{\delta\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu V^a = 0, \tag{2.1.80}$$

which means the physical properties of a vector V are invariant along this path. By Equations (2.1.73) and (2.1.68), the variation of a vector V along this path can be described by

$$\frac{\delta V^a}{\delta\lambda} \equiv \frac{dV^a}{d\lambda} + \omega_b{}^a{}_c\frac{dx^b}{d\lambda}V^c, \tag{2.1.81}$$

where the first term describes the variation of the vector component V^a due to the change in λ and the second term expresses the variation in the non-coordinate basis e_a along γ. The parallel transport of a vector V on a closed path is directly affected by the curvature of spacetime. As shown in Figure 2.5, the parallel transport of a vector on a closed path in flat space yields the same vector at the origin, while the different vector is obtained in curved space.

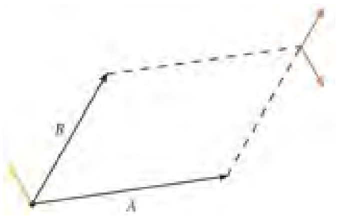

Figure 2.6: The difference between the paralell transported vectors on the paths $A \to B$ and $B \to A$ [34].

The quantity related to the curvature of spacetime can be obtained from a commutator between parallel transports of a vector V on any two different parts parametrized by λ and σ, A and B in Figure 2.6, that exhibits the difference between the parallel transported vectors that defined by

$$\begin{aligned}\delta V^a &= \frac{\delta}{\delta\sigma}\frac{\delta V^a}{\delta\lambda} - \frac{\delta}{\delta\lambda}\frac{\delta V^a}{\delta\sigma}, \\ &= \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\sigma}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^a,\end{aligned} \tag{2.1.82}$$

[3] Parallel-Transport-medium.jpg[Online] Available from : https://www.quantum-munich.de/fileadmin/media/media/Aharonov-Bohm/Parallel-Transport-medium.jpg[2016,May]

where the second line is obtained via (2.1.80). From this equation, the Riemann curvature tensor ${R_{\mu\nu}}^a{}_d$, the tensor measuring the deviation from flat spacetime, is defined by

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^a = {R_{\mu\nu}}^a{}_b V^b. \tag{2.1.83}$$

Using the definition of the covariant derivative in (2.1.73) and (2.1.68), the Riemann curvature tensor ${R_{\mu\nu}}^{ab} = \eta^{bf} {R_{\mu\nu}}^a{}_f$ can be expressed in terms of the spin connection as

$$R_{\mu\nu}{}^{ab} = 2\nabla_{[\mu}\, \omega_{\nu]}{}^{ab} = 2\partial_{[\mu}\, \omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{af}\, \omega_{\nu]f}{}^{b}, \tag{2.1.84}$$

or in differential forms,

$$R^{ab}_{(2)} = d\omega^{ab} + \omega^a{}_f \wedge \omega^{fb}, \tag{2.1.85}$$

where $R^{ab}_{(2)} = \frac{1}{2} R_{\mu\nu}{}^{ab}\, dx^\mu \wedge dx^\nu = \frac{1}{2} R_{cd}{}^{ab}\, e^c \wedge e^d$ is the curvature 2-form with the Riemann tensor being its components. Moreover, it is more comfortable to deal with the purely lower flat indices of the Riemann tensor, $R_{cdab} = e^\mu_c\, e^\nu_d\, \eta_{af}\, R_{\mu\nu}{}^f{}_b$, that has various symmetries in its indices. These symmetries are given by

$$R_{cdab} = -R_{dcab} = -R_{cdba} = R_{abcd}, \tag{2.1.86}$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{2.1.87}$$

The last equation (2.1.87) implies the Bianchi identities, where $\nabla_f = e^\mu_f\, \nabla_\mu$,

$$\nabla_{[f}\, R_{cd]ab} = 0. \tag{2.1.88}$$

By taking traces, other important quantities describing the curvature of spacetime can be derived from the Riemann tensor: the Ricci tensor R_{ab} and the Ricci scalar R defined as follow

$$R_{ab} = R_{acb}{}^c, \tag{2.1.89}$$

$$R = \eta^{ab} R_{ab}. \tag{2.1.90}$$

These two quantities play important roles in the descriptions of curved spacetime. In the next section, Einstein's field equations relating geometry and matter will be introduced through these Ricci tensor and Ricci scalar.

2.1.2 Einstein's Field Equations

After introducing all ingredients for describing curved spacetime, the elegant relation between this curvature and existences of energy and matter described by the energy-momentum tensor T_{ab} will be established in this section. For simplicity, the vacuum curved spacetime in the region containing no matter and energy will be firstly discussed to give a basic concept for a more complicated cases in which matter sources are coupled to gravity at the end of this section.

2.1.2.1 Pure gravity field equation

In the case of vacuum spacetime, there is no matter and energy occur in the region. To find an action describing the curvature of vacuum spacetime, it is more convenient if we start from the field equations expressing behaviour of the curvature. Bianchi identities in (2.1.88) are the suitable equations describing nature of the Riemann tensor, or precisely the curvature of spacetime. By taking traces, the Bianchi identities implies

$$\nabla^a(R_{ab} - \frac{1}{2}\eta_{ab}R) = \nabla^a G_{ab} = 0, \tag{2.1.91}$$

where the Einstein tensor is defined by $G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab}R$ and this equation leads to the pure gravity field equations or the vacuum Einstein's field equations,

$$R_{ab} - \frac{1}{2}\eta_{ab}R = 0, \tag{2.1.92}$$

describing the curvature of empty spacetime. Furthermore, an action giving rise to this field equation in D-dimensional spacetime is known as the Einstein-Hilbert action that is simply written in just only a term of Ricci scalar as

$$\mathcal{S}_{EH} = \int d^D x \sqrt{|g|}R. \tag{2.1.93}$$

Note that, at the level of actions, it is easier to deal with their variations with respect to the metric $g_{\mu\nu}$ so the vacuum Einstein's field equations (2.1.92) can be obtained by varying this Einstein-Hilbert action $\mathcal{S}_{EH}$ with respect to the metric $g_{\mu\nu}$,

$$\frac{\delta \mathcal{S}_{EH}}{\delta g^{\mu\nu}} = \int d^D x \ \sqrt{|g|}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R), \tag{2.1.94}$$

after applying the least action principle $\delta\mathcal{S}_{EH} = 0$ and contactions with $e^\mu_a e^\nu_b$. The Einstein-Hilbert Lagrangian density is just the integrand in the Einstein-Hilbert action (2.1.93),

$$\mathcal{L}_{EH} = \sqrt{|g|}R. \tag{2.1.95}$$

In the language of differential forms, notice that the Einstein-Hilbert action in (2.1.93) is an integration on the volume form defined in (2.1.53),

$$\mathcal{S}_{EH} = \int R * 1, \tag{2.1.96}$$

where the Einstein-Hilbert Lagrangian density takes the form

$$\mathcal{L}_{EH} = R * 1. \tag{2.1.97}$$

Note that the concept of Lagrangian density is now changed. In general, Lagrangian density is a scalar quantity whose integral over all space gives the scalar action. Henceforth, from a differential forms point of view, Lagrangian density is not a scalar quantity or 0-form anymore, but rather D-form that can be integrated over a D-dimensional manifold giving rise to a 0-form action.

2.1.2.2 Matter-coupled gravity field equation

As stated at the beginning, existences of energy and mass affact the curvature of spacetime. The energy-momentum tensor T_{ab} is defined to describe all energies, momentums, and also stresses in spacetime by relating this gravity-source-tensor to the Einstein tensor explaining the curvature of spacetime as $G_{ab} \propto T_{ab}$ or*

$$G_{ab} = \kappa^2 T_{ab} = \frac{1}{2} T_{ab}, \tag{2.1.98}$$

where the last term is obtained by using the convention $\kappa^2 = 1/2$. Using the definition of Einstein tensor, the matter coupled Einstein's field equations are obtained from (2.1.98) in the forms

$$R_{ab} - \frac{1}{2}\eta_{ab} R = \frac{1}{2} T_{ab}. \tag{2.1.99}$$

This equation is the famous Einstein's field equations describing the relation between the curvature of spacetime and matter sources. This equation can be derived from a matter coupled Einstein-Hilbert action,

$$\mathcal{S}(g, X) = \mathcal{S}_{EH}(g) + \mathcal{S}_{Matter}(g, X), \tag{2.1.100}$$

where the argument g refers to the dependence of the metric and X denotes any matter field. The total variation of this matter coupled action takes the form

$$\delta\mathcal{S} = \{\sqrt{|g|}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{\delta\mathcal{S}_{Matter}}{\delta g^{\mu\nu}}\}\delta g^{\mu\nu} + \frac{\delta\mathcal{S}_{Matter}}{\delta X}\delta X. \tag{2.1.101}$$

By applying the least action principle, $\delta\mathcal{S} = 0$, the last term corresponds to the field equation of the matter source X while the terms in the bracket will become the Einstein's field equations (2.1.99), if the energy-momentum tensor ${T_\mu}^a$ is defined by

$$T_{\mu\nu} = \frac{-2}{\sqrt{|g|}} \frac{\delta\mathcal{S}_{Matter}}{\delta g^{\mu\nu}}. \tag{2.1.102}$$

Note that, for some matter fields that will be henceforth described, this coordinate version of the energy-momentum tensor is conveniently derived from matter actions describing their behaviour on curved spacetime by the metric $g_{\mu\nu}$. However, Einstein equations are simpler in flat spacetime as in (2.1.99), since dealing with the Minkowski metric η_{ab} is easier than $g_{\mu\nu}$. Thus the energy-momentum tensor can transforms to $(0, 2)$ Lorentz tensor components by using the inverse vielbein, $T_{ab} = e^\mu_a e^\nu_b T_{\mu\nu}$.

There are two kinds of the matter fields involving in this study that will be reviewed. They are both bosonic fields with integer spin: 0 and 1. The simpler

*To obtain the Newton's gravity theory, this constant is related to the Newton's gravitational constant, $G \approx 6.674 \times 10^{-11}\, N \cdot m^2/kg^2$, by $\kappa^2 = 8\pi G$. In this study, we use the convention that $2\kappa^2$ is set to unity, $\kappa^2 = 1/2$.

case is the zero-spin real scalar field $\phi(x)$ described by an action,

$$\mathcal{S}_\phi = \int d^D x \sqrt{|g|} \{ -\frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \}, \tag{2.1.103}$$

where m corresponds to its mass. Variation of this action with respect to ϕ leads to the well-known field equation of scalar field; the Klien-Gordon equation,

$$\Box \, \phi - m^2 \phi = 0, \tag{2.1.104}$$

where the d'Alembert operator $\Box$ is defined by $\Box = \nabla^\mu \nabla_\mu$, which can be reduced to $\Box \, \phi = \nabla^\mu \partial_\mu \phi$ for any scalar field ϕ. Its energy-momentum tensor can be derived by (2.1.102) as

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \{ \partial_\rho \phi \partial^\rho \phi + m^2 \phi^2 \}. \tag{2.1.105}$$

Moreover, by using the volume form (2.1.53) and the inner product in (2.1.55), the Lagrangian density of a real scalar field can be written in form of

$$\mathcal{L}_\phi = -\frac{1}{2} * d\phi \wedge d\phi - (\frac{1}{2} m^2 \phi^2) * 1. \tag{2.1.106}$$

Another case is a vector field or gauge field A^μ corresponds to $U(1)$ gauge symmetry that will be introduced in the review of the gauge theory. An action expressing behaviour of a $U(1)$ gauge field is simply of the form

$$\mathcal{S}_A = -\frac{1}{4} \int d^D x \sqrt{|g|} \, F_{\mu\nu} F^{\mu\nu}, \tag{2.1.107}$$

where the fields strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.1.108}$$

or simpler in the form of differential form,

$$F = dA, \tag{2.1.109}$$

where $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ is the field strength 2-form. Notice that the exterior derivative of the field strength 2-form certainly equals to zero due to the nilpotent property in (2.1.43). This leads to the Bianchi identities, that can be written in component form as

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{2.1.110}$$

The variation of the action (2.1.107) with respect to the gauge field gives the field equations,

$$\nabla^\mu F_{\mu\nu} = 0. \tag{2.1.111}$$

By using (2.1.102), the energy-momentum tensor of a gauge field A^μ can be obtained as

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \tag{2.1.112}$$

Moreover, the Lagrangian density of a gauge field A^μ can be expressed by using the differential form as

$$\mathcal{L}_A = -\frac{1}{4} * F \wedge F, \tag{2.1.113}$$

where the inner product in (2.1.55) is applied.

2.2 Kaluza-Klein Reduction on S^1

Kaluza-Klein reduction, the original simplest case of dimensional reduction, which is briefly introduced in the introduction, now can be demonstrated in more mathematical details. As mentioned before, this dimensional reduction is a consideration of Einstein's general relativity in five spacetime dimensions where the fifth dimension is compactified to a very small circle or S^1, as shown in Figure 1.1. In general, this procedure can be generalized to reduce any $(D+1)$-dimensional gravitational theories to the reduced ones in D dimensions. In this section, fundamental concepts about dimensional reduction will be reviewed. Even though it is the simplest compact space S^1, these concepts and also calculational processes can be applied to a more complicated compact space such as an S^7 that we want to study.

Since the higher dimensional theory is just a pure gravity theory, a Lagrangian density describing this theory is only the Einstein-Hilbert Lagrangian density from (2.1.95),

$$\mathcal{L}_{EH} = \sqrt{|\hat{g}|}\hat{R}, \tag{2.2.1}$$

where the hat-fields are higher-dimensional fields. Equations of motion from this Lagrangian density are also the Einstein's field equations in vacuum from (2.1.92) that can be contacted and written as

$$\hat{R}_{MN} = 0, \tag{2.2.2}$$

where $M, N = 0, 1, 2..., D$ are the $(D+1)$-dimensional spacetime coordinate indices. As introduced in Section 2.1, Ricci scalar $\hat{R}$ can be derived from the metric field $\hat{g}_{MN}$ that depends on the higher dimensional spacetime coordinates y. Now suppose that one of the spatial coordinates labelled by z is compactified to a circle, S^1, with radius L, such that, the coordinates y are separated to be (x, z) where x denote the reduced D-dimensional spacetime coordinates. The coordinate z is then periodic, therefore, the metric $\hat{g}_{MN}(y)$ can be described in Fourier series of the form

$$\begin{aligned}\hat{g}_{MN}(x,z) &= \sum_n \hat{g}^{(n)}_{MN}(x)\mathrm{e}^{inz/L} \\ &= \hat{g}^{(0)}_{MN}(x) + \sum_{n\neq 0} \hat{g}^{(n)}_{MN}(x)\mathrm{e}^{inz/L},\end{aligned} \tag{2.2.3}$$

where the dependence on z is excluded. The Fourier modes n are integers associated with masses of the metric field in the lower dimensional point of view, since the twice partial derivative with respect to z will always give the real number n^2/L^2 which corresponds to the square of mass, as in (2.1.104). In the second line of (2.2.3), these massless and massive modes of the metric field are apparent separated. This process is called compactification giving rise to a stack of higher-dimensional metric field's mass states called the infinite Kaluza-Klein tower with the masses of the field in each Fourier mode n equal to $|n|/L$. Note that only compactifying procedure cannot reduce spacetime dimensions. Since in lower dimensional theory this compact space S^1 should be unobservable, the radius L should be assumed to be very small in the order of the Planck length, 10^{-35} metres. In this limit, the non-zero massive modes will have masses in order of the Planck mass, 10^{-8} kilogrammes, that is too heavy for fundamental particles. The next step of dimensional reduction is neglecting all these massive modes and keeping only the massless one called the truncation to the massless sector process. Thus the metric $\hat{g}_{MN}(x)$ is now only dependent on the D-dimensional spacetime coordinates.

In group theory, these Fourier mode functions $\mathrm{e}^{inz/L}$ in (2.2.2) are the representations of an Abelian $U(1)$ group of the circle S^1 where the Fourier mode n corresponds to a $U(1)$ charge. The $n = 0$ mode is a singlet while the $n \neq 0$ modes are all doublet where the modes n and $-n$ are complex conjugate of each other. Keeping only $n = 0$ mode makes the net charge always neutral or said to be invariant under the $U(1)$ transformation that is consistent because the rest $n \neq 0$ representations are impossibly generated from the $n = 0$ due to the charge conservation.

As a result, dimensional reductions are the compactifying some spatial coordinates together with the truncation to the massless sector that turn the higher dimensional fields to be independent of the compact space's coordinates. For the case S^1, the $(D{+}1)$-dimensional spacetime coordinate index splits to $\mu = 0, 1, 2..., D-1$ the D-dimensional spacetime coordinates and the compactified spatial coordinate z. Thus the massless mode metric field can be divided into $\hat{g}_{\mu\nu}$ symmetric $(0, 2)$ tensor components, $\hat{g}_{\mu z}$ 1-form components, and a scalar field $\hat{g}_{zz}$ in the D-dimensional point of view. Kaluza-Klein reduction ansatz is an expression of these $(D{+}1)$-dimensional metric components in the following forms

$$\hat{g}_{\mu\nu} = \mathrm{e}^{2\alpha\phi} g_{\mu\nu} + \mathrm{e}^{2\beta\phi} A_\mu A_\nu, \qquad \hat{g}_{\mu z} = \mathrm{e}^{2\beta\phi} A_\mu, \qquad \hat{g}_{zz} = \mathrm{e}^{2\beta\phi}, \tag{2.2.4}$$

where $g_{\mu\nu}$ is the metric tensor, 1-form's components A_μ, and a scalar field ϕ, which are all independent of z. The two constants α and β are chosen to be

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \qquad \beta = -(D-2)\alpha. \tag{2.2.5}$$

By using these expressions in (2.2.4), the metric equation in higher dimensions can be written as

$$d\hat{s}^2 = \mathrm{e}^{2\alpha\phi}\, ds^2 + \mathrm{e}^{2\beta\phi}(dz + A_\mu dx^\mu)^2. \tag{2.2.6}$$

Note that the reduction ansatz of the metric is always written in the form of the line element for convenience. Reduction ansatze are the key to obtain dimensional reduction describing relations between the two different dimensional worlds. As demonstrated above, the reduction ansatz of the metric describes the geometry of spacetime in which there are D non-compact spacetime dimensions together with a compact circle S^1. Translation in this compact space S^1 described by a $U(1)$ gauge transformation corresponds to the gauge transformation of the field A_μ in the ansatz. In conclusion, reduction ansatze of the metric can be deduced from the symmetry corresponding to the chosen compact space. We will use this fact to derive the reduction ansatz for the eleven-dimensional metric compactified on S^7.

To obtain the consistent dimensional reduction, substitutions of the reduction ansatze in the higher dimensional equations of motion giving rise to lower dimensional equations of motion are required. However, gravity's equations of motion, the Einstein's field equations, are only expressed by Ricci tensor. Hence, before providing the substitutions, all components of the Ricci tensor are needed to be derived from the ansatz (2.2.4). Starting from finding the higher dimensional vielbein bases, by using (2.1.56), each vielbein basis can be expressed as*

$$\hat{e}^a = \mathrm{e}^{\alpha\phi} e^a, \qquad \hat{e}^z = \mathrm{e}^{\beta\phi}(dz + A), \tag{2.2.7}$$

where $A = A_a e^a$ is a 1-form and a is a flat index in D dimensions while z is also used for the flat index of the compactified dimension. Then, $(D{+}1)$-dimensional spin connections can be obtained, in the vielbein bases through the torsion-free condition in (2.1.79),

$$\begin{aligned} \hat{\omega}^{ab} &= \omega^{ab} + \alpha \mathrm{e}^{-\alpha\phi}(\partial^b\phi \hat{e}^a - \partial^a\phi \hat{e}^b) - \frac{1}{2}F^{ab}\mathrm{e}^{(\beta-2\alpha)\phi}\hat{e}^z, \\ \hat{\omega}^{az} &= -\beta \mathrm{e}^{-\alpha\phi}\partial^a\phi \hat{e}^z - \frac{1}{2}F^a{}_b \mathrm{e}^{(\beta-2\alpha)\phi}\hat{e}^b, \end{aligned} \tag{2.2.8}$$

where F_{ab} are the components in the Lorentz frame of the field strength $F = dA$. Finally, the higher dimensional Ricci tensor components can be derived from the reduction ansatz (2.2.4) by finding the curvature 2-form via (2.1.85) and reading off their components as

$$\begin{aligned} \hat{R}_{ab} &= \mathrm{e}^{-\alpha\phi}(R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi) - \frac{1}{2}\mathrm{e}^{-2D\alpha\phi}F^2_{ab}, \\ \hat{R}_{az} &= \frac{1}{2}\mathrm{e}^{(D-3)\alpha\phi}\nabla^b\big(\mathrm{e}^{-2(D-1)\alpha\phi}F_{ab}\big), \\ \hat{R}_{zz} &= (D-2)\alpha \mathrm{e}^{-2\alpha\phi}\Box\phi + \frac{1}{4}\mathrm{e}^{-2D\alpha\phi}F^2, \end{aligned} \tag{2.2.9}$$

*Beware confusing between an exponential e and a vielbein e.

where the two type contractions of the field strengths are defined by $F^2_{ab} = F_{ac}F_b{}^c$ and $F^2 = F_{ab}F^{ab}$. After transformation to the curve spacetime indices, these $(D+1)$-dimensional Ricci tensor components can be substituted into the Einstein's field equations (2.2.2) to obtain all equations of motion of the reduced theory in D-dimensions,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{1}{2}(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\,\partial^\rho\phi) + \frac{1}{2}\mathrm{e}^{-2(D-1)\alpha\phi}(F^2_{\mu\nu} - \frac{1}{4}g_{\mu\nu}F^2), \\ \nabla^\mu(\mathrm{e}^{-2(D-1)\alpha\phi}F_{\mu\nu}) &= 0, \\ \Box\phi &= -\frac{1}{2}(D-1)\alpha\mathrm{e}^{-2(D-1)\alpha\phi}F^2. \end{aligned} \tag{2.2.10}$$

The first equation of motion is nothing but the Einstein's field equation describing the curvature of the D-dimensional spacetime corresponding to the two kinds of matter source, the scalar field ϕ and the $U(1)$ gauge field A_μ, as shown on the right-hand side of the equations. The terms in the first parenthesis are the energy-momentum tensor of a massless scalar field, as introduced in (2.1.105), and the other ones are the gauge field's energy-momentum tensor defined in (2.1.112). The rest lower dimensional equations of motion describe behaviours of the two matter fields coupled to each other. Note that the elimination of the gauge field makes the scalar field to be a harmonic function while the truncation of the scalar field, setting $\phi = 0$, turns all components of the $U(1)$ field strength to be zero. It's said to be inconsistent in the later case.

In conclusion, consistent dimensional reductions giving rise to the gravitational theory together with some additional fields in lower spacetime dimensions can be achieved if their reduction ansatze, expressions of the higher dimensional fields in terms of the lower dimensional ones, yield all equations of motion in the reduced theory via substitutions them into the higher dimensional equations of motion. By this procedure, the dimensional reductions are said to be consistent at the level of equations of motion.

Moreover, the stronger consistency of dimensional reductions can be obtained by substitutions their reduction ansatze into the higher dimensional Lagrangian density to get the lower dimensional one. In this case, dimensional reductions are said to be consistent at the level of actions. For example, this simplest Kaluza-Klein reduction on S^1 is also consistent at the level of actions. The higher dimensional Ricci tensors are contracted to be a Ricci scalar by contractions of these tensors with the $(D+1)$-dimensional Minkowski metric as follows

$$\hat{R} = \mathrm{e}^{-2\alpha\phi}(R - \frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - 2\alpha\Box\phi) - \frac{1}{4}\mathrm{e}^{-2D\alpha\phi}F^2. \tag{2.2.11}$$

The determinant of the $(D+1)$-dimensional metric can be easily calculated through the block metric form in the reduction ansatz (2.2.4),

$$\sqrt{|\hat{g}|} = \mathrm{e}^{(\beta+D\alpha)\phi}\sqrt{|g|} = \mathrm{e}^{2\alpha\phi}\sqrt{|g|}, \tag{2.2.12}$$

where the last term is obtained by the choice of β constant in (2.2.5). Then multiplication of these two quantities gives an expression for the higher dimensional Lagrangian density in lower D dimensions,

$$\mathcal{L} = \sqrt{|\hat{g}|}\,\hat{R} = \sqrt{|g|}(R - \frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - 2\alpha\Box\phi - \frac{1}{4}\mathrm{e}^{-2(D-1)\alpha\phi}F^2), \tag{2.2.13}$$

where the $\Box\phi$ term can be dropped from this Lagrangian density because it is just a total derivative that does not contribute to the equations of motion. Therefore the final form of the Lagrangian density is just a combination of the Einstein-Hilbert and the two matter fields' Lagrangian densities,

$$\mathcal{L} = \sqrt{|\hat{g}|}\,\hat{R} = \sqrt{|g|}(R - \frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{1}{4}\mathrm{e}^{-2(D-1)\alpha\phi}F^2), \tag{2.2.14}$$

which corresponds to all the lower dimensional equations of motion in (2.2.10). Note that the choices of the two constant, α and β, in (2.2.5) have their background in the following way. For α, it ensures that the kinetic term of the scalar field, $\partial_\rho\phi\,\partial^\rho\phi$, has the canonical normalisation, $\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi$, as in (2.1.103). The other thing is to ensure that the dimensionally-reduced Lagrangian density contains the usual Einstein-Hilbert form, i.e. $\sqrt{|g|}R$. It is obviously shown in the above that the choice of β makes the coefficient of the lower dimensional Ricci scalar, after multiplied by the determinant of the metric in (2.2.12), to be one.

However, consistency only at the level of equations of motion is sufficient for studying their solutions. Since all solutions must satisfy their equations of motion so consistency between equations of motion in the two theory turn solutions in lower dimensions into the higher dimensional theory's solutions, which is called lifting up or embedding solutions, and vice versa.

At this point, Kaluza-Klein reduction can be extended to produce on more complicated compact spaces a more complicated symmetry. For example, reduction on n-dimensional torus, $T^n = S^1 \times \ldots \times S^1$, reduces a D-dimensional gravitational theory to be a reduced one in (D-n)-dimensions. This dimensional reduction can be obtained by performing n-sequent of S^1 reductions. In this case, the final reduced theory will contain n-$U(1)$ gauge fields corresponding to a $U(1)^n$ symmetry.

Moreover, performing Kaluza-Klein reduction on a group manifold is also guaranteed to be consistent. Group manifold is a topological space that corresponds to a transformation satisfying group theory. For example, in this study, a group manifold of a Lie group $SU(2)$ is a three-dimensional sphere S^3, as discussed in Appendix A. The group manifold G admits a metric with the isometry group $G_L \times G_R$ corresponding to left- and right-transformations. Truncation to the set of all fields that are invariant under the left action G_L is guaranteed to be consistent since, in group theory, the retained G_L-singlet fields cannot act as sources for generating the discarded G_L-non-singlet fields, in the same way as the truncation to the massless sector process for a $U(1)$ symmetry described above. Note also that Kaluza-Klein reduction on a Lie group manifold is called Scherk-Schwarz reduction [24].

2.3 Gauge Theory

In general relativity's framework, it is more relevant to consider local symmetries than global ones. Since spacetime can be curved, so locality, or dependence on spacetime's position, becomes an essential feature of gravitational theories. Gauging is a procedure of promoting any continuous global symmetry to be a local one that depends on the spacetime coordinates x. For example, in Section 2.1, the Lorentz symmetry, which is a global symmetry on flat spacetime in special relativity, was promoted to be the local one with transformations called local Lorentz transformation or LLT. After that, the Lorentz covariant derivative was defined to preserve their transformation rules. Furthermore, this procedure can be applied to promote some continuous symmetries associated with Lie groups that are briefly introduced in Appendix A. Note that all the four fundamental forces i.e. electromagnetism, weak and strong interactions and also gravity can be formulated through this gauge procedure, therefore understanding general gauge theories is very valuable.

Classical gauge theories are summarily demonstrated in this section, starting from the simplest case; Abelian gauge theory in which a $U(1)$ gauge symmetry is promoted to be local. Then, all basic concepts from this simple gauge theory will be applied to a more involved case: non-Abelian gauge theory in which global symmetries associated with simple Lie groups, such as $SU(N)$, $SO(N)$, or $USp(N)$ are local. Finally, an $SU(2)$ gauge theory involving to this study is given at last.

2.3.1 Abelian Gauge Theory

Gauging of a $U(1)$ symmetry corresponding to a vector field or gauge field presented in Section 2.1.2.2 is mainly described in this section. To explain this simplest gauge theory, we consider a complex scalar field $\phi(x)$ with mass m in D-dimensional flat spacetime with the action given by

$$\mathcal{S}_{\phi_{\text{complex}}} = -\frac{1}{2}\int d^D x\big\{\partial_\mu\phi\,\partial^\mu\phi^* + m^2\phi^*\phi\big\}. \tag{2.3.1}$$

It is straightforward to see that this action is invariant under a global $U(1)$ transformation that can be defined as a constant phase shift on the scalar field,

$$\begin{aligned} \text{global } U(1): \phi(x) &\to \phi'(x) = \mathrm{e}^{i\alpha}\phi(x), \\ \phi^*(x) &\to (\phi^*)'(x) = \mathrm{e}^{-i\alpha}\phi^*(x), \end{aligned} \tag{2.3.2}$$

where α is a real phase parameter and $\phi^*(x)$ is a complex conjugate of the scalar field $\phi(x)$ whose global $U(1)$ transformation in the second line can be obtained from the complex conjugation of the first one.

Then, promoting this phase parameter α to depend on the spacetime coordinates, $\alpha(x)$, turns the global transformations in (2.3.2) into a local $U(1)$ transformations,

$$\begin{aligned} \text{local } U(1): \phi(x) &\to \phi'(x) = \mathrm{e}^{i\alpha(x)}\phi(x), \\ \phi^*(x) &\to (\phi^*)'(x) = \mathrm{e}^{-i\alpha(x)}\phi^*(x). \end{aligned} \tag{2.3.3}$$

However, the action in (2.3.2) is no longer invariant under this local transformation since there is an undesirable term arising when the partial derivatives of the scalar field in this action, $\partial_\mu\phi(x)$ and $\partial^\mu\phi(x)$, transform. This term makes the transformation of $\partial_\mu\phi(x)$ differ from the transformation rule in (2.3.3) such that

$$\begin{aligned} \text{local } U(1): \ \partial_\mu\phi(x) \to \partial_\mu\phi'(x) &= \partial_\mu(\mathrm{e}^{i\alpha(x)}\phi(x)), \\ &= \mathrm{e}^{i\alpha(x)}\partial_\mu\phi(x) + i\partial_\mu\alpha(x)\mathrm{e}^{i\alpha(x)}\phi(x), \end{aligned} \tag{2.3.4}$$

where the last term in the second line is the undesirable term that spoils the expected transformation rule. To achieve an action that is invariant under this local $U(1)$ transformation, the partial derivatives in (2.3.1) need to be replaced by covariant derivatives defined by

$$D_\mu\phi(x) \equiv \partial_\mu\phi(x) - iA_\mu(x)\phi(x), \tag{2.3.5}$$

where $A_\mu(x)$ is a $U(1)$ gauge field that transforms under the local $U(1)$ transformation as

$$\text{local } U(1): \ A_\mu(x) \to A'_\mu = A_\mu - \partial_\mu\alpha(x). \tag{2.3.6}$$

Therefore, this new covariant derivative satisfies the local $U(1)$ transformation rule (2.3.3) in the way that

$$\begin{aligned} D_\mu\phi(x) \to (D_\mu\phi)'(x) &= \partial_\mu\phi'(x) - iA'_\mu(x)\phi'(x), \\ &= \mathrm{e}^{i\alpha(x)}\partial_\mu\phi(x) + i\partial_\mu\alpha(x)\phi'(x) - iA_\mu\phi'(x) - i\partial_\mu\alpha(x)\phi'(x), \\ &= \mathrm{e}^{i\alpha(x)}D_\mu\phi(x). \end{aligned} \tag{2.3.7}$$

Replacing partial derivatives in the ungauged action (2.3.1) by covariant derivatives, known as the minimal coupling procedure, gives rise to the $U(1)$ gauged complex scalar field theory described by an action of the form

$$\mathcal{S}_{\phi_{\text{gauged}}} = -\frac{1}{2}\int d^D x\big\{(D_\mu\phi)^* D^\mu\phi + m^2\phi^*\phi\big\}, \tag{2.3.8}$$

that is obviously invariant under the local $U(1)$ transformation, where $(D_\mu\phi)^*$ is a complex conjugation of the covariant derivative of the complex scalar field defined in (2.3.5) whose phase shift from local $U(1)$ transformation cancels another one from (2.3.7).

However, there is still no dynamical term for the gauge field A_μ itself in (2.3.8). Adding the action expressing dynamics of the gauge field, which is introduced in (2.1.107), gives the well-known action describing interactions between a complex scalar field $\phi(x)$ and the $U(1)$ gauge field $A_\mu(x)$ in D-dimensional flat spacetime, called the scalar electrodynamics theory,

$$\mathcal{S} = -\frac{1}{2}\int d^D x\big\{(D_\mu\phi)^* D^\mu\phi + m^2\phi^*\phi + \frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu}\big\}, \tag{2.3.9}$$

where g is a coupling constant. Here, the $U(1)$ field strength is defined by

$$F_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.3.10}$$

which is also invariant under local $U(1)$ transformations,

$$\begin{aligned}\text{local } U(1): \;\; F_{\mu\nu} \to F'_{\mu\nu} &= \partial_\mu A_\nu - \partial_\mu\partial_\nu\alpha - \partial_\nu A_\mu + \partial_\nu\partial_\mu\alpha, \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ &= F_{\mu\nu},\end{aligned} \tag{2.3.11}$$

such that this scalar electrodynamics action is invariant under $U(1)$ transformations.

2.3.2 Non-Abelian Gauge Theory

As mentioned in the previous section, $U(1)$ gauge symmetry is a symmetry describing one of the four fundamental forces, electromagnetism. Apart form this Abelian gauge symmetry, in the standard model of elementary particle interactions, weak and strong interactions can be expressed through the two simple Lie groups, $SU(2)$ and $SU(3)$ that are non-Abelian, respectively. Therefore, to explain all of the four fundamental forces the study of non-Abelian gauge theory is required.

A more complicated local transformation associated to an N-dimensional compact simple Lie group $\mathcal{G}$ is now considered. In adjoint representations, see also in Appendix A, a set of scalar fields $\phi(x)^A$, where $A = 1, 2, ...N$, can be written as a scalar matrix $\Phi(x)$ that transforms by elements of the Lie group $\mathcal{G}$, $U(x)$, as

$$\Phi(x) \to \Phi'(x) = U(x)\Phi(x)U^{-1}(x) \tag{2.3.12}$$

where $\Phi(x) = T_A\phi(x)^A$. These group elements can be written in an expoenetial form,

$$U(x) \equiv \mathrm{e}^{-\alpha^A(x)T_A}, \tag{2.3.13}$$

where $\alpha^A(x)$ are real parameters depending on the spacetime coordinates x, which indicate that the above transformation (2.3.12) is local, and T_A are the group generators satisfying the Lie algebra introduced in (A.10),

$$[T_A, T_B] = f_{AB}{}^C T_C, \tag{2.3.14}$$

where $f_{AB}{}^C$ are real numbers called structure constants. A, B, and C label the number of the group generators.

As in the Abelian case, a partial derivative of the scalar matrix $\Phi(x)$ does not transforms in the same way as (2.3.12). To preserve the local transformation rule, the non-Abelian covariant derivative of a scalar matrix $\Phi(x)$ is introduced by the definition

$$D_\mu \Phi(x) \equiv \partial_\mu \Phi(x) + [\mathbf{A}_\mu(x), \Phi(x)], \tag{2.3.15}$$

where $\mathbf{A}_\mu(x) \equiv A_\mu^A(x) T_A$ are the gauge field matrices whose transformation by the group elements $U(x)$ can be imposed to be

$$\begin{aligned} \mathbf{A}_\mu(x) \to \mathbf{A}'_\mu(x) &\equiv U(x)\mathbf{A}_\mu(x)U(x)^{-1} + U(x)\partial_\mu U(x)^{-1}, \\ &= U(x)\mathbf{A}_\mu(x)U(x)^{-1} - (\partial_\mu U(x))U(x)^{-1}, \end{aligned} \tag{2.3.16}$$

where the second line is obtained through applying the Leibniz's product rule to the last term in the first line. By using these definitions, the non-Abelian covariant derivative of a scalar matrix $\Phi(x)$ now satisfies the same transformation rules as $\Phi(x)$,

$$\begin{aligned} (D_\mu \Phi)'(x) &= \partial_\mu (U \Phi\, U^{-1}) + [\, (U \mathbf{A}_\mu\, U^{-1} - \partial_\mu U\, U^{-1}), U \Phi\, U^{-1}], \\ &= U\big\{\partial_\mu \Phi + [\mathbf{A}_\mu, \Phi]\big\}U^{-1} + (\partial_\mu U)\Phi U^{-1} - (\partial_\mu U)\Phi U^{-1} \\ &\quad - U\Phi\partial_\mu U^{-1} + U\Phi\partial_\mu U^{-1}, \\ &= U(x) D_\mu \Phi(x)\, U^{-1}(x) \end{aligned} \tag{2.3.17}$$

Furthermore, by using the Lie algebra (2.3.14), the covariant derivative of the scalar fields $\phi(x)^A$ can be written as

$$D_\mu \phi^A(x) T_A \equiv \partial_\mu \phi^A(x) T_A + f_{BC}{}^A A_\mu^B(x) \phi^C(x) T_A, \tag{2.3.18}$$

where $A_\mu^B(x)$ are the $\mathcal{G}$ gauge fields. Eliminating the generators T_A gives the definition of the gauged covariant derivative of scalar fields $\phi^A(x)$,

$$D_\mu \phi^A(x) = \partial_\mu \phi^A(x) + f_{BC}{}^A A_\mu^B(x) \phi^C(x). \tag{2.3.19}$$

The non-Abelian field strength matrices are defined by $\mathbf{F}_{\mu\nu}(x) = [D_\mu, D_\nu]$ that can be expressed in terms of the gauge fields by

$$\begin{aligned} \mathbf{F}_{\mu\nu}(x) \equiv F_{\mu\nu}^A(x) T_A &\equiv \partial_\mu \mathbf{A}_\nu(x) - \partial_\nu \mathbf{A}_\mu(x) + [\mathbf{A}_\mu(x), \mathbf{A}_\nu(x)], \\ &= (\partial_\mu A_\nu^A(x) - \partial_\nu A_\mu^A(x) + f_{BC}{}^A A_\mu^B(x) A_\nu^C(x)) T_A. \end{aligned} \tag{2.3.20}$$

Thus the field strengths $F^A_{\mu\nu}(x)$ are defined in terms of the $\mathcal{G}$ gauge fields A^A_μ as

$$F^A_{\mu\nu}(x) = \partial_\mu A^A_\nu(x) - \partial_\nu A^A_\mu(x) + f_{BC}{}^A A^B_\mu(x) A^C_\nu(x). \tag{2.3.21}$$

A gauge invariant action describing the dynamics of these gauge fields in D-dimensional flat spacetime can be written in the similar form as (2.1.107) by

$$\mathcal{S}[A] = -\frac{1}{4g^2}\int d^D x F^A_{\mu\nu} F^{A\mu\nu}, \tag{2.3.22}$$

where g is a gauge coupling constant that can be absorbed into the definition of the gauge fields such that the covariant derivative of the scalar metrices and the field strength metrices can be written as

$$\begin{aligned} D_\mu \Phi(x) &= \partial_\mu \Phi(x) + g[\mathbf{A}_\mu(x), \phi(x)], \\ \mathbf{F}_{\mu\nu}(x) &= \frac{1}{g}[D_\mu, D_\nu]. \end{aligned} \tag{2.3.23}$$

Hence, the covariant derivative of the scalar fields and the field strenghts become

$$\begin{aligned} D_\mu \phi^A(x) &= \partial_\mu \phi^A(x) + g f_{BC}{}^A A^B_\mu(x) \phi^C(x), \\ F^A_{\mu\nu}(x) &= \partial_\mu A^A_\nu(x) - \partial_\nu A^A_\mu(x) + g f_{BC}{}^A A^B_\mu(x) A^C_\nu(x). \end{aligned} \tag{2.3.24}$$

The field equations and Bianchi identities describing the nature of the gauged fields A^A_μ derived from the action (2.3.22) are of the same forms as given in Section 2.1.2.2 but the partial derivatives ∂_μ are needed to be replaced by the non-Abelian covariant derivative D_μ,

$$D^\mu F^A_{\mu\nu}(x) = 0, \tag{2.3.25}$$

$$D_\mu F^A_{\nu\rho}(x) + D_\nu F^A_{\rho\mu}(x) + D_\rho F^A_{\mu\nu}(x) = 0. \tag{2.3.26}$$

Note that it is easier to write the non-Abelian gauged covariant exterior derivative of any k-form and the definition of the field strength 2-forms by using diferential forms,

$$\begin{aligned} D\omega^A_{(k)} &\equiv d\omega^A_{(k)} + g f_{BC}{}^A A^B_{(1)} \wedge \omega^A_{(k)}, \\ F^A_{(2)} &\equiv dA^A_{(1)} + \frac{1}{2} g f_{BC}{}^A A^B_{(1)} \wedge A^C_{(1)}, \end{aligned} \tag{2.3.27}$$

where $A^A_{(1)} = A^A_\mu(x) dx^\mu$ are the gauge 1-forms.

For $SU(2)$ gauge theory, the structure constant is nothing but a three-dimensional Levi-Civita symbol ε_{ijk} from (A.21), where the gauge indices become $i, j, k = 1, 2, 3$. The non-Abelian gauged covariant exterior derivative of any k-form and the definition of the field strength 2-forms defined above are now written as

$$D\omega^i_{(k)} = d\omega^i_{(k)} + g\varepsilon_{ijk} A^j_{(1)} \wedge \omega^k_{(k)}, \tag{2.3.28}$$

$$F^i_{(2)} = dA^i_{(1)} + \frac{1}{2} g\varepsilon_{ijk} A^j_{(1)} \wedge A^k_{(1)}. \tag{2.3.29}$$

By using the isomorphism $SO(4) \sim SU(2) \times SU(2)$, an $SO(4)$ gauge theory can be obtained from the two sets of commuting $SU(2)$ gauge groups that will be exhibited later when our lower dimensional theory; $N = 4$ $SO(4)$ gauged supergravity is discussed in Section 2.4.2.

2.4 Extended Supergravity

Apart from Einstein's general relativity, dimensional reductions can also be applied to another gravitational theory combined with supersymmetry, the symmetry between bosons and fermions, which is called supergravity. As mentioned in the introduction, if gravity is quantized, it will give a notorious divergence. Supersymmetry might cure this problem. In fact, supersymmetry is an essential ingredient in superstring theory, a finite quantum theory of gravity in ten spacetime dimensions. It is also strongly believed that $N = 8$ supergravity in four dimensions might give a finite quantum theory.

Supergravity is a general relativity theory whose symmetries are extended by local supersymmetry corresponding to supercharge operators Q [35, 37]. In the simplest case with only one supersymmetry operator, there exists a spin 3/2 vector-spinor field called the gravitino $\Psi_\mu(x)$ that is a superpartner of the bosonic gravitational field $e^a_\mu(x)$ relating to each other through supersymmetry transformations. Moreover, supersymmetry can be extended by adding more supercharge operators up to the number N. With more supercharge, there are more fields contained in supergravity multiplet, a set of various spin fields transforming to each other through supersymmetry transformations. For four-dimensional supergravity, $N = 8$ is the maximal case that has all fields with spin ≤ 2 in its supergravity multiplet. Furthermore, supergravity can be considered in higher dimensional spacetime in which their component fields have more degrees of freedom. The eleven-dimensional supergravity is the unique maximal case for the dimensional extension. Note that beyond these maximal limits, local supersymmetry will be ruined and their equations of motion are all inconsistent [35].

The simplest linear $N = 1$ supergravity in four-dimensional spacetime is first introduced in this section to familiarize with the universal part of supergravity, starting from gauging the global supersymmetry. After that, extended supergravities involved in this study, both higher and lower dimensional theory, will be considered.

First of all, the structure of global supersymmetry is required. The global supersymmetry transformations that change a bosonic field denoted by B into a fermionic spinor field, F, and vice versa, are schematically written in the following forms

$$\delta B \propto \bar{\epsilon} F, \qquad \delta F \propto \epsilon \gamma^\mu \partial_\mu B, \tag{2.4.1}$$

where ϵ is called the infinitesimal supersymmetry spinor parameter and $\gamma^\mu(x)$ are the coordinate gamma matrices, which are related to the Dirac gamma matrices defined in Appendix B through inverse vielbein's contractions, $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$. As shown in Appendix B, $\gamma^{\hat{0}}$ is anti-hermitian and the rest $\gamma^{\hat{i}}$ with $\hat{i} = 1, 2, 3$ are hermitian*. Using this $\gamma^{\hat{0}}$, the Dirac adjoint can be defined in the same way as (B.11) by $\bar{\epsilon} = \epsilon^\dagger i \gamma^{\hat{0}}$. The commutator of two supersymmetry transformations acting on a bosonic field leads to an operator that is proportional to the spacetime-derivative,

$$[\delta_1, \delta_2] B \propto (\bar{\epsilon}_1 \gamma^\mu \epsilon_2) \partial_\mu B. \tag{2.4.2}$$

This implies the important fact that the commutator of two supersymmetry transformations is an infinitesimal spacetime translation with a parameter $\bar{\epsilon}_1 \gamma^\mu \epsilon_2$ that transforms under GCT in the same way as a $(1, 0)$ coordinate tensor demonstrated in (2.1.21). Promoting the infinitesimal supersymmetry spinor parameter to be dependent on spacetime coordinates turns the global transformations to be local such that the supersymmetry transformation in (2.4.1) can be written as

$$\delta B \propto \bar{\epsilon}(x) F, \qquad \delta F \propto \epsilon(x) \gamma^\mu \partial_\mu B. \tag{2.4.3}$$

Thus, the commutator of two supersymmetry transformations becomes

$$[\delta_1, \delta_2] B \propto (\bar{\epsilon}_1 \gamma^\mu \epsilon_2)(x) \partial_\mu B. \tag{2.4.4}$$

For local supersymmetry, this commutator yields a vector field $(\bar{\epsilon}_1 \gamma^\mu \epsilon_2)(x)$ corresponding to an element of spacetime (local) translation, which is called diffeomorphism. Therefore, the local supersymmetry requires Einstein's general relativity describing spacetime metric as a dynamical object to assure the diffeomorphism invariance.

Four-dimensional $N = 1$ supergravity is the simplest supersymmetric theory describing gravity as a vielbein field $e^a_\mu(x)$ together with its superpartner, a Majorana gravitino $\Psi_\mu(x)$ that is a vector-spinor field containing both vector and spinor indices. The supersymmetry transformations relating these fields are given in [35] by,

$$\begin{aligned} \delta e^a_\mu &= \frac{1}{2} \bar{\epsilon}(x) \gamma^a \Psi_\mu, \\ \delta \Psi_\mu &= D_\mu \epsilon(x) \equiv \partial_\mu \epsilon + \frac{1}{4} \omega_{\mu a b} \gamma^{ab} \epsilon, \end{aligned} \tag{2.4.5}$$

where $\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$. Here, $D_\mu = e^a_\mu D_a = \partial_\mu + \frac{1}{4}\omega_{\mu a b}\gamma^{ab}$ is defined to be the Lorentz covariant derivative for a spinor. At this moment, the gravitino

*For avoiding confusion, the hat-indices are now used to separate flat spacetime indices from the curved ones.

behaves as a gauge field of the local supersymmetric theory. Therefore the linear supersymmetric Lagrangian density for this theory, which is invariant under the local supersymmetry transformation (2.4.5), is just a combination of the ordinary Einstein-Hilbert Lagrangian density, introduced in (2.1.95), and a kinetic action expressing the dynamics of the gauge field gravitino given by a Rarita-Schwinger Lagrangian density in [38], of the form

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{RS}, = \sqrt{|g|}\,[R + \bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho]. \tag{2.4.6}$$

Here $\gamma^{\mu\nu\rho} = \frac{1}{2}\{\gamma^\mu, \gamma^{\nu\rho}\}$ and the gravitino covariant derivative is given by $D_\nu \Psi_\rho \equiv \partial_\nu \Psi_\rho + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\Psi_\rho$. Note that the equation of motion for this gravitino gauge field can be obtained through the variation of this Lagrangian density with respect $\bar{\Psi}_\mu$ as

$$\gamma^{\mu\nu\rho} D_\nu \Psi_\rho = 0. \tag{2.4.7}$$

helicity	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$N=1$								1	1
$N=2$							1	2	1
$N=4$					1	4	6	4	1
$N=8$	1	8	28	56	70	56	28	8	1

Table 2.1: The various supergravity multiplets for $N = 1, 2, 4, 8$. [37]

Adding more supercharge operators extends supergravity to the $N > 1$ case whose supergravity multiplet contains various fields, as shown in Table 2.1. Here helicity is the projection of the spin onto the direction of the linear momentum. This quantity can be both positive or negative while the spin is just a positive integer or half-integer. Apart from the maximal $N = 8$ case, supergravity multiplets in Table 2.1 are not invariant under CPT discrete transformations that all physical theory must be invariant under these transformations

- Charge conjugation : the transformation between paritcle and anti-particle,
- Parity : the spatial inversion ($\vec{x} \leftrightarrow -\vec{x}$),
- Time reversal : the time inversion ($t \leftrightarrow -t$).

CPT transformations just flip the helicity of each field in the supergravity multiplet to the opposite sign. Therefore CPT invariant supergravity multiplets can be obtained by adding their CPT conjugate states to the multiplets. For example, in the N=1 case, the supergravity multiplet invariant under CPT transformations contains $(2, \frac{3}{2})$ and $(-2, -\frac{3}{2})$ states corresponding to a graviton e^a_μ and a gravitino Ψ_μ. Note also that the number of bosonic and fermionic states are always equal in each supergravity multiplet [35, 37], for example in the maximal $N = 8$ supergravity, the CPT invariant supergravity multiplet has 256 degrees of freedom, which are divided into 128 bosonic and also 128 fermionic states [37].

2.4.1 Eleven-dimensional Supergravity

The higher dimensional theory for this study is the effective theory of M-theory i.e. the unique eleven-dimensional supergravity firstly constructed by Cremmer, Julia, and Scherk in [12]. In eleven dimensions, one may expect that this $N = 1$ supergravity will contains only a graviton $\hat{e}_M^{\hat{M}}(x)$ and a vector-spinor gravitino $\Psi_M(x)$ in the same way as the simplest $N = 1$ four-dimensional supergravity introduced previously, where $M = 0, 1, ...10$ is a curved eleven-dimensional spacetime index, $\hat{M} = 0, 1, ...10$ is an eleven-dimensional flat space index, and hat-fields are eleven-dimensional bosonic fields.

However, fermionic degrees of freedom from the Majorana vector-spinor is 128 in eleven dimensions, while degrees of freedom from an eleven-dimensional graviton is just 44 [35, 37]. Thus 84 bosonic degrees of freedom are missing due to the fact that the number of bosonic and fermionic states are always identical in each supergravity multiplet. An antisymmetric 3-form potential $\hat{A}_{MNP}(x)$ corresponding to the missing 84 bosonic degrees of freedom is added to the supergravity multiplet to fix this problem. Consequently, the unique eleven-dimensional supergravity contains a graviton field $\hat{e}_M^{\hat{M}}(x)$, a 3-form potential $\hat{A}_{MNP}(x)$, and a Majorana vector-spinor gravitino $\Psi_M(x)$ where their supersymmetry transformations are given in [35] by,

$$\begin{aligned} \delta\hat{e}_M^{\hat{M}} &= \frac{1}{2}\bar{\epsilon}\gamma^{\hat{M}}\Psi_M, \\ \delta\Psi_M &= D_M(\tilde{\omega})\epsilon + \frac{\sqrt{2}}{288}(\gamma^{\hat{M}\hat{N}\hat{O}\hat{P}}{}_M - 8\gamma^{\hat{N}\hat{O}\hat{P}}\delta_M^{\hat{M}})\tilde{F}_{\hat{M}\hat{N}\hat{O}\hat{P}}\epsilon, \\ \delta\hat{A}_{MNP} &= -\frac{3\sqrt{2}}{4}\bar{\epsilon}\gamma_{[MN}\Psi_{P]}, \end{aligned} \tag{2.4.8}$$

in which

$$\begin{aligned} D_M(\tilde{\omega})\epsilon &= \partial_M\epsilon + \frac{1}{4}\tilde{\omega}_{M\hat{N}\hat{P}}\gamma^{\hat{N}\hat{P}}\epsilon, \\ \tilde{\omega}_{M\hat{N}\hat{P}} &= \hat{\omega}_{M\hat{N}\hat{P}} - \frac{1}{4}(\bar{\Psi}_M\gamma_{\hat{P}}\Psi_{\hat{N}} - \bar{\Psi}_{\hat{N}}\gamma_M\Psi_{\hat{P}} + \bar{\Psi}_{\hat{P}}\gamma_{\hat{N}}\Psi_M), \\ \tilde{F}_{MNPQ} &= 4\partial_{[M}\hat{A}_{NPQ]} + \frac{3}{2}\sqrt{2}\bar{\Psi}_{[M}\gamma_{NP}\Psi_{Q]}. \end{aligned} \tag{2.4.9}$$

The tillded spin connection and field strength are called supercovariants that are the same hat-quantity added by gravitino's interaction terms from the following 3-form potential's Lagrangian density [35, 37],

$$\begin{aligned} \mathcal{L}_A = \sqrt{|g|}\Big[&- \frac{1}{48}\hat{F}_{MNPQ}\hat{F}^{MNPQ} + \frac{1}{20736}\varepsilon^{M_1...M_{11}}\hat{F}_{M_1...M_4}\hat{F}_{M_5...M_8}\hat{A}_{M_9...M_{11}} \\ &- \frac{\sqrt{2}}{192}\bar{\Psi}_Q(\gamma^{MNOPQR} + 12\gamma^{MN}g^{OQ}g^{PR})\Psi_R(\hat{F}_{MNOP} + \tilde{F}_{MNOP})\Big], \end{aligned} \tag{2.4.10}$$

where the components of the 4-form field strength are defined by $\hat{F}_{MNPQ} = 4\partial_{[M}\hat{A}_{NPQ]}$, while the second term is called the Chern-Simon term. Together with the naive eleven-dimensional supergravity Lagrangian density, the analogue of (2.4.6),

$$\mathcal{L}_0 = \sqrt{|g|}[\hat{R} + \bar{\Psi}_M \gamma^{MNP} D_N \Psi_P], \tag{2.4.11}$$

the full Lagrangian density of the eleven-dimensional supergravity can be obtained,

$$\begin{aligned}\mathcal{L}_{11}^{full} = \sqrt{|g|}\Big[\hat{R} + \bar{\Psi}_M \gamma^{MNP} D_N \Psi_P - \frac{1}{48}\hat{F}_{MNPQ}\hat{F}^{MNPQ} \\ - \frac{\sqrt{2}}{192}\bar{\Psi}_Q(\gamma^{MNOPQR} + 12\gamma^{MN}g^{OQ}g^{PR})\Psi_R(\hat{F}_{MNOP} + \tilde{F}_{MNOP}) \\ + \frac{1}{20736}\varepsilon^{M_1 \ldots M_{11}}\hat{F}_{M_1\ldots M_4}\hat{F}_{M_5 \ldots M_8}\hat{A}_{M_9\ldots M_{11}}\Big].\end{aligned} \tag{2.4.12}$$

Nevertheless, in most applications, only bosonic parts are sufficient to perform the dimensional reduction while all fermionic fields can be obtained from supersymmetry transformations (2.4.8). Therefore, the bosonic Lagrangian density for eleven-dimensional supergravity consists only of the usual Einstein-Hilbert Lagrangian density along with kinetic and Chern-Simons terms of the 3-form potential $\hat{A}_{MNP}(x)$ as

$$\mathcal{L}_{11} = \sqrt{|g|}[\hat{R} - \frac{1}{48}\hat{F}_{MNPQ}\hat{F}^{MNPQ}] + \frac{1}{20736}\epsilon^{M_1\ldots M_{11}}\hat{F}_{M_1\ldots M_4}\hat{F}_{M_5\ldots M_8}\hat{A}_{M_9\ldots M_{11}}, \tag{2.4.13}$$

or in the more compact format using differential forms,

$$\mathcal{L}_{11} = \hat{R}\hat{*}\mathbf{1} - \frac{1}{2}\hat{*}\hat{F}_{(4)} \wedge \hat{F}_{(4)} + \frac{1}{6}\hat{F}_{(4)} \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)}, \tag{2.4.14}$$

where the 4-form field strength is defined by $\hat{F}_{(4)} \equiv d\hat{A}_{(3)}$. There are three bosonic equations of motion in this theory,

$$\hat{R}_{MN} = \frac{1}{12}(\hat{F}^2_{MN} - \frac{1}{12}\hat{g}_{MN}\hat{F}^2_{(4)}), \tag{2.4.15}$$

$$d\hat{*}\hat{F}_{(4)} = -\frac{1}{2}\hat{F}_{(4)} \wedge \hat{F}_{(4)}, \tag{2.4.16}$$

$$d\hat{F}_{(4)} = 0, \tag{2.4.17}$$

where the two contractions of the field strength's components are given by $\hat{F}^2_{MN} = \hat{F}_{MPQR}\hat{F}_N{}^{PQR}$ and $\hat{F}^2_{(4)} = \hat{F}_{MNPQ}\hat{F}^{MNPQ}$. While the first two equations describing the dynamics of both bosonic fields, the graviton and the 3-form potential, can be directly derived from the bosonic Lagrangian density (2.4.14), the last equation is just the Bianchi identity arisen from the definition of the field strength 4-form.

The clues to the dimensional reduction on S^7 of this theory can be investigated from the eleven-dimensional Einstein's field equations (2.4.15) describing the curvature of eleven-dimensional spacetime coupled by the gauge matter fields. This is obtained by assuming that eleven-dimensional spacetime is splitted such that the spacetime index can be written as $M = (\mu, m)$, where μ is a four-dimensional spacetime index and m is running over the remaining seven-dimensional space, along with setting all components of the 4-form field strength to be zero except

$$\hat{F}_{\mu\nu\rho\sigma} = -3\sqrt{2}g\epsilon_{\mu\nu\rho\sigma}, \tag{2.4.18}$$

where g is a constant. Note that this field strength's assumption satisfies both equations of motion in (2.4.16) and (2.4.17). Thus, all contractions of the two field strengths can be obtained by

$$\hat{F}^2_{\mu\nu} = -108g^2 g_{\mu\nu}, \qquad \hat{F}^2_{mn} = 0, \qquad \hat{F}^2_{(4)} = -432g^2. \tag{2.4.19}$$

Substitutions of these contractions into the eleven-dimensional Einstein's field equations (2.4.15) lead to two separated Ricci tensors corresponding to four-dimensional spacetime and seven-dimensional compact space,

$$\hat{R}_{\mu\nu} = -6g^2 g_{\mu\nu}, \qquad \hat{R}_{mn} = 3g^2 g_{mn}. \tag{2.4.20}$$

The opposite-sign between the two Ricci tensors indicate that the eleven-dimensional theory admits $AdS_4 \times S^7$ solutions since these Ricci tensors can be directly obtained from the four-dimensional metric $g_{\mu\nu}$ describing AdS_4 and the seven-dimensional metric g_{mn} on S^7 with radius $2g^{-1}$ respectively. Therefore if the dimensional reduction on S^7 is applied to the eleven-dimensional supergravity, four-dimensional supergravity will be obtained together with non-vanishing cosmological constant related to the seven-dimensional sphere's radius.

In this study, we will use this fact to perform dimensional reduction of the eleven-dimensional supergravity to obtain $N = 4$ gauged supergravity in four dimensions. However, the compact space is not the full S^7 but the truncated foliation $\times S^3 \times S^3$. Note that these eleven-dimensional equations of motion, (2.4.15) to (2.4.17), are needed for the dimensional reduction such that their substitutions by given reduction ansatze, expressions of the two eleven-dimensional bosonic fields in terms of the lower dimensional ones, have to yield all equations of motion in the reduced theory to achieve the consistent dimensional reduction.

2.4.2 $N = 4$ Gauged Supergravity in Four Dimensions

There are two discovered versions for four-dimensional $N = 4$ supergravity [39]; the first one is a theory that has a global $SO(4)$ symmetry [40, 41], and another version with global $SU(4)$ symmetry [42]. $SO(4)$ gauged supergravity is the main dimensionally reduced theory obtained from the dimensional reduction of the eleven-dimensional supergravity in this study. Besides, there is a one-way map from this standard $SO(4)$ gauged theory to another gauged supergravity, known as Freedman-Schwarz model [43], in which the four-dimensional supergravity is gauged by $SU(2)\times SU(2)$ subgroup of the global $SU(4)$ symmetry. Therefore, after introducing the main reduced theory, $N=4$ $SO(4)$ gauged supergravity, this related Freedman-Schwarz $N=4$ $SU(2)\times SU(2)$ gauged supergravity will be given together with the one-way map from the main $SO(4)$ gauged theory.

As shown in Table 2.1, $N=4$ supergravity contains the following fields in its CPT invariant field content, i.e.

- 1 spin-2 graviton e^a_μ,
- 4 Majorana vector-spinor gravitinos, Ψ^α_μ,
- 6 gauge fields $A^{\alpha\beta}_\mu$, where the two upper indices are anti-symmetrized,
- 4 Majorana spinor fields λ^α,
- 2 real scalar fields, A and B, corresponding to the complex scalar field $W\equiv -A+iB$ [39],

where $\alpha,\beta=1,2,3,4$. For the first kind of this $N=4$ theory that is invariant under a global $SO(4)$ transformation, the local supersymmetry transformations relating the above bosonic and fermionic fields are given in [40] by

$$\begin{aligned}
\delta e^a_\mu =& -i\bar{\epsilon}^\alpha\gamma^a\Psi^\alpha_\mu,\\
\delta\bar{\Psi}^\alpha_\mu =& \bar{\epsilon}^\alpha\overleftarrow{\nabla}_\mu-\frac{1}{4}\bar{\epsilon}^\beta\gamma_\mu\gamma^{\nu\rho}F^{\alpha\beta}_{\nu\rho},\\
\delta A^{\alpha\beta}_\mu =& \frac{i}{2\sqrt{2}}\Big[\varepsilon^{\alpha\beta\gamma\delta}\bar{\epsilon}^\gamma\gamma_\mu\lambda^\delta+\bar{\epsilon}^\alpha\Psi^\beta_\mu-\bar{\epsilon}^\beta\Psi^\alpha_\mu\Big],\\
\delta\bar{\lambda}^\alpha =& \frac{i}{\sqrt{2}}\bar{\epsilon}^\alpha(\partial_\mu A+i\gamma_5\partial B)\gamma^\mu-\frac{1}{4\sqrt{2}}\varepsilon^{\alpha\beta\gamma\delta}\bar{\epsilon}^\beta\gamma^{\mu\nu}F^{\gamma\delta}_{\mu\nu},\\
\delta A =& \frac{1}{\sqrt{2}}\bar{\epsilon}^\alpha\lambda^\alpha,\\
\delta B =& \frac{i}{\sqrt{2}}\bar{\epsilon}^\alpha\gamma_5\lambda^\alpha,
\end{aligned} \tag{2.4.21}$$

where ϵ^α are four spacetime dependent majorana spinor in which right covariant dericatives of their dirac adjoint are given by $\bar{\epsilon}^\alpha\overleftarrow{\nabla}_\mu=\partial_\mu\bar{\epsilon}^\alpha+\frac{1}{4}\omega_{\mu ab}\bar{\epsilon}^\alpha\gamma^{ab}$ and the field strengths $F^{\alpha\beta}_{\mu\nu}$ are defined by $F^{\alpha\beta}_{\mu\nu}\equiv\partial_\mu A^{\alpha\beta}_\nu-\partial_\mu A^{\alpha\beta}_\nu$.

To obtain the $SO(4)$ gauged supergravity, the isomorphism $SO(4)\sim SU(2)\times SU(2)$ is used. Thus the six gauge fields in the above $N=4$ supergravity multiplet are divided into the two sets associated with each $SU(2)$ gauge group defined by

$$A^i_\mu=\mathbf{a}^i_{\alpha\beta}A^{\alpha\beta}_\mu,\qquad \tilde{A}^i_\mu=\mathbf{b}^i_{\alpha\beta}A^{\alpha\beta}_\mu, \tag{2.4.22}$$

where $i=1,2,3$. Here $\mathbf{a}^i$ and $\mathbf{b}^i$ are the six real anti-symmetric 4×4 matrices represented by the Pauli matrices from (A.19) in [43] as

$$\mathbf{a}^1=\frac{1}{2}\begin{bmatrix}0 & \sigma_3\\ -\sigma_3 & 0\end{bmatrix},\quad \mathbf{a}^2=\frac{1}{2}\begin{bmatrix}-i\sigma_2 & 0\\ 0 & -i\sigma_2\end{bmatrix},\quad \mathbf{a}^3=\frac{1}{2}\begin{bmatrix}0 & -\sigma_1\\ \sigma_1 & 0\end{bmatrix},$$
$$\mathbf{b}^1=\frac{1}{2}\begin{bmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{bmatrix},\quad \mathbf{b}^2=\frac{1}{2}\begin{bmatrix}-i\sigma_2 & 0\\ 0 & i\sigma_2\end{bmatrix},\quad \mathbf{b}^3=\frac{1}{2}\begin{bmatrix}0 & i\sigma_2\\ i\sigma_2 & 0\end{bmatrix}. \tag{2.4.23}$$

These matrices generate the Lie algebra $SU(2)\times SU(2)$,

$$[\mathbf{a}^i,\mathbf{a}^j]=\varepsilon^{ijk}\mathbf{a}^k,\qquad [\mathbf{b}^i,\mathbf{b}^j]=\varepsilon^{ijk}\mathbf{b}^k,\qquad [\mathbf{a}^i,\mathbf{b}^j]=0, \tag{2.4.24}$$

which are two commuting $SU(2)$ Lie algebras from (A.21). Therefore, the $SO(4)$ gauged supergravity can be described as a theory that is gauged by two commuting $SU(2)$ gauge groups. The $SU(2)$ Yang-Mills field strength 2-forms are given by

$$\begin{aligned} F^i_{(2)} &= dA^i_{(1)}+\frac{1}{2}g\varepsilon_{ijk}A^j_{(1)}\wedge A^k_{(1)},\\ \tilde{F}^i_{(2)} &= d\tilde{A}^i_{(1)}+\frac{1}{2}g\varepsilon_{ijk}\tilde{A}^j_{(1)}\wedge \tilde{A}^k_{(1)}, \end{aligned} \tag{2.4.25}$$

where g is a gauge coupling constant. The couplings for the two factors of $SU(2)$ can be chosen to be equal in this case without losing of generality. The bosonic Lagrangian density of the four-dimensional $N=4$, $SO(4)$ gauged supergravity can be written in differential form as [23],

$$\begin{aligned} \mathcal{L}_4^{SO(4)} = {} & R*\mathbf{1}-\frac{1}{2}*d\phi\wedge d\phi-\frac{1}{2}X^4*d\chi\wedge d\chi - V*\mathbf{1}-\frac{1}{2}X^{-2}*F^i_{(2)}\wedge F^i_{(2)}\\ & -\frac{1}{2}\tilde{X}^{-2}*\tilde{F}^i_{(2)}\wedge\tilde{F}^i_{(2)}-\frac{1}{2}\chi F^i_{(2)}\wedge F^i_{(2)}+\frac{1}{2}\chi X^2\tilde{X}^{-2}\tilde{F}^i_{(2)}\wedge\tilde{F}^i_{(2)}, \end{aligned} \tag{2.4.26}$$

where the potential V is given by

$$V=-2g^2(4+X^2+\tilde{X}^2), \tag{2.4.27}$$

and the two scalar fields, called the dilaton ϕ and the axion χ, are written as

$$X\equiv \mathrm{e}^{\frac{1}{2}\phi},\qquad \tilde{X}\equiv X^{-1}q\qquad \text{where}\ \ q^2\equiv 1+\chi^2X^4. \tag{2.4.28}$$

These two real scalar fields are related to the complex scalar field W defined by $W\equiv -A+iB$ or $W=\mathrm{e}^{i\sigma}\tanh\frac{1}{2}\lambda$ through the following parametrisation

$$\begin{aligned} \cosh\lambda &= \cosh\phi+\frac{1}{2}\chi^2\mathrm{e}^{\phi},\\ \cos\sigma\sinh\lambda &= \sinh\phi-\frac{1}{2}\chi^2\mathrm{e}^{\phi},\\ \sin\sigma\sinh\lambda &= \chi\mathrm{e}^{\phi}, \end{aligned} \tag{2.4.29}$$

which is a map from a two-dimensional metric $ds_2^2 = d\lambda^2 + \sinh^2\lambda d\sigma^2$ to $ds_2^2 = d\phi^2 + \mathrm{e}^{2\phi}d\chi^2$. This $SO(4)$ gauged Lagrangian density is invariant under the ungauged supersymmetry transformations in (2.4.21) added by some extra terms for fermionic fields due to gauging process,

$$\begin{aligned}\delta'\bar{\Psi}^{\alpha}_{\mu} &= \frac{i}{\sqrt{2}}\bar{\epsilon}^{\alpha}\gamma_{\mu}\frac{g}{(1-|W|^2)^{1/2}},\\ \delta'\bar{\lambda}^{\alpha} &= \frac{1}{2}\bar{\epsilon}^{\alpha}\frac{g(A-i\gamma_5 B)}{(1-|W|^2)^{1/2}}.\end{aligned} \tag{2.4.30}$$

All equations of motion describing each bosonic field can be directly obtained from variations of the above bosonic Lagrangian density (2.4.26). Firstly, the equations of motion for the two scalar fields; the dilaton ϕ, which is now described by X, and the axion χ, are

$$\begin{aligned}d(X^{-1}*dX) = &-\frac{1}{2}X^4 * d\chi\wedge d\chi + g^2(X^2 - X^{-2} + \chi^2X^2)\epsilon_{(4)} + \frac{1}{4}X^{-2}*F^i_{(2)}\wedge F^i_{(2)}\\ &+\frac{1}{2}\chi\tilde{X}^{-4}\tilde{F}^i_{(2)}\wedge\tilde{F}^i_{(2)} - \frac{1}{4}(1-\chi^2X^4)X^2q^{-4}*\tilde{F}^i_{(2)}\wedge\tilde{F}^i_{(2)},\end{aligned} \tag{2.4.31}$$

$$\begin{aligned}d(X^4*d\chi) = &\,4g^2\chi X^2\epsilon_{(4)} + \frac{1}{2}(1-\chi^2X^4)\tilde{X}^{-4}\tilde{F}^i_{(2)}\wedge\tilde{F}^i_{(2)} - \frac{1}{2}F^i_{(2)}\wedge F^i_{(2)}\\ &+\chi X^6q^{-4}*\tilde{F}^i_{(2)}\wedge\tilde{F}^i_{(2)},\end{aligned} \tag{2.4.32}$$

where $\epsilon_{(4)} = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}$ is the four-dimensional volume form defined in (2.1.53). The two Yang-Mills equations, the equations of motion describing each $SU(2)$ gauge fields, are given by

$$D(X^{-2}*F^i_{(2)}) = -d\chi\wedge F^i_{(2)}, \tag{2.4.33}$$

$$\tilde{D}(\tilde{X}^{-2}*\tilde{F}^i_{(2)}) = d(\chi X^2\tilde{X}^{-2})\wedge\tilde{F}^i_{(2)}, \tag{2.4.34}$$

where both D and $\tilde{D}$ are the $SU(2)$ gauge covariant exterior derivatives for each $SU(2)$ gauge fields, expressed in (2.3.28). Moreover, there are two Bianchi identities automatically followed from the definitions of both $SU(2)$ field strength in (2.4.25),

$$DF^i_{(2)} \equiv dF^i_{(2)} + g\varepsilon_{ijk}A^j_{(1)}\wedge F^k_{(2)} = 0, \tag{2.4.35}$$

$$\tilde{D}\tilde{F}^i_{(2)} \equiv d\tilde{F}^i_{(2)} + g\varepsilon_{ijk}\tilde{A}^j_{(1)}\wedge\tilde{F}^k_{(2)} = 0. \tag{2.4.36}$$

Finally, the last four-dimensional equations of motion are the usual contracted Einstein's field equations with the remaining fields being gravitational sources together with the scalar potential V. These can be written in the following typical component form,

$$
\begin{aligned}
R_{\mu\nu} = & \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{2}X^4\partial_\mu\chi\,\partial_\nu\chi + \frac{1}{2}g_{\mu\nu}V + \frac{1}{2}X^{-2}(F^i_{\mu\rho}F^{i\rho}_\nu - \frac{1}{4}g_{\mu\nu}(F^i_{(2)})^2) \\
& + \frac{1}{2}\tilde{X}^{-2}(\tilde{F}^i_{\mu\rho}\tilde{F}^{i\rho}_\nu - \frac{1}{4}g_{\mu\nu}(\tilde{F}^i_{(2)})^2)
\end{aligned}
\tag{2.4.37}
$$

where $R_{\mu\nu}$ is the four-dimensional Ricci tensor and $\mu,\nu = 0,1,2,3$ are the four-dimensional curved spacetime indices. It is easy to see that this gauged theory admits an AdS vacuum solution through these Einstein's field equations. By setting all the six gauge fields and the two scalar fields* to be zero such that there are only the spacetime's equations of motion,

$$
R_{\mu\nu} = -6g_{\mu\nu}g^2, \tag{2.4.38}
$$

which is the Ricci tensor describing vacuum AdS_4 spacetime's curvature.

The Freedman-Schwarz $N = 4$ $SU(2)\times SU(2)$ gauged supergravity contains the same bosonic fields as the $SO(4)$ gauged theory while its bosonic Lagrangian density is given in [23] by

$$
\begin{aligned}
\mathcal{L}^{FS}_4 = & R*\mathbf{1} - \frac{1}{2}*d\phi\wedge d\phi - \frac{1}{2}\mathrm{e}^{2\phi}*d\chi\wedge d\chi - V_{FS}*\mathbf{1} \\
& - \frac{1}{2}\mathrm{e}^{\phi}*F^i_{(2)}\wedge F^i_{(2)} - \frac{1}{2}\mathrm{e}^{-\phi}*\tilde{F}^i_{(2)}\wedge\tilde{F}^i_{(2)} - \frac{1}{2}\chi F^i_{(2)}\wedge F^i_{(2)} - \frac{1}{2}\chi\tilde{F}^i_{(2)}\wedge\tilde{F}^i_{(2)}.
\end{aligned}
\tag{2.4.39}
$$

Here the Freedman-Schwarz potential is

$$
V_{FS} = -2(g^2 + \tilde{g}^2)\mathrm{e}^{\phi}, \tag{2.4.40}
$$

where the two coupling constants g and $\tilde{g}$ are independent and correspond to each $SU(2)$ gauge group. The two $SU(2)$ field strength 2-forms are defined in the same way as (2.4.25) but using different coupling constants,

$$
\begin{aligned}
F^i_{(2)} &= dA^i_{(1)} + \frac{1}{2}g\varepsilon_{ijk}A^j_{(1)}\wedge A^k_{(1)}, \\
\tilde{F}^i_{(2)} &= d\tilde{A}^i_{(1)} + \frac{1}{2}\tilde{g}\varepsilon_{ijk}\tilde{A}^j_{(1)}\wedge\tilde{A}^k_{(1)}.
\end{aligned}
\tag{2.4.41}
$$

This Freedman-Schwarz Lagrangian density is invariant under the ungauged supersymmetry transformations of the second version of four-dimensional $N = 4$ supergravity containing a global $SU(4)$ symmetry given in [43] together with the following extra terms for fermionic fields,

$$
\begin{aligned}
\delta'\bar{\Psi}^\alpha_\mu &= \frac{i}{2\sqrt{2}}\mathrm{e}^{\phi/2}\bar{\epsilon}^\alpha(g + i\gamma_5\tilde{g})\gamma_5, \\
\delta'\bar{\lambda}^\alpha &= \frac{1}{2}\mathrm{e}^{\phi/2}\bar{\epsilon}^\alpha(g + i\gamma_5\tilde{g}).
\end{aligned}
\tag{2.4.42}
$$

*In general vacuum solutions, scalar fields are the critical points of the scalar potential and need not be zero.

As mentioned before, there exists a one-way map to this Freedman-Schwarz supergravity from the standard $SO(4)$ gauged theory [23]. The distinction between the two equal $SU(2)$ gauge coupling constants in the standard theory can be obtained by making the fields and coupling constant redefinitions as follow

$$\begin{aligned} \phi &= \phi' + k, & \chi &= \chi' \mathrm{e}^{-k}, \\ A^i_{(1)} &= A'^i_{(1)} \mathrm{e}^{k/2}, & \tilde{A}^i_{(1)} &= \tilde{A}'^i_{(1)} \mathrm{e}^{-k/2}, \\ g' &= g\, \mathrm{e}^{k/2}, & \tilde{g}' &= g\, \mathrm{e}^{-k/2}, \end{aligned} \tag{2.4.43}$$

where k is a constant. After that, dropping the primes makes the $SO(4)$ bosonic Lagrangian density in (2.4.26) unchanged under these redefinitions except for the definitions of the two $SU(2)$ field strength 2-forms, which are now defined by (2.4.41), and the potential V that can be written in the following forms

$$\begin{aligned} V &= -8g\tilde{g} - 2g^2 X^2 - 2\tilde{g}^2 \tilde{X}^2, \\ &= -8g\tilde{g} - 2(g^2 + \tilde{g}^2)\cosh\lambda - 2(g^2 - \tilde{g}^2)\cos\sigma \sinh\lambda, \\ &= -\frac{1}{1 - |W|^2}(g_+^2(3 - |W|^2) - g_-^2(1 - 3|W|^2) - 4g_+ g_- A). \end{aligned} \tag{2.4.44}$$

In the second line, the two scalar fields are parametrized by (2.4.29), while in the last line, the potential V is written in terms of the complex scalar field W, and two coupling constants $g_\pm \equiv g \pm \tilde{g}$. Besides, the redefinitions (2.4.43) also turn the extra terms in the supersymmetric transformations for fermionic fields (2.4.30) into [23, 39]

$$\begin{aligned} \delta' \bar{\Psi}^\alpha_\mu &= \frac{i}{\sqrt{2}} \bar{\epsilon}^\alpha \gamma_\mu \frac{[g_+ + g_-(-A + i\gamma_5 B)]}{(1 - |W|^2)^{1/2}}, \\ \delta' \bar{\lambda}^\alpha &= \frac{1}{2} \bar{\epsilon}^\alpha \frac{[g_+(A - i\gamma_5 B) - g_-]}{(1 - |W|^2)^{1/2}}. \end{aligned} \tag{2.4.45}$$

From this $g \neq \tilde{g}$ $SO(4)$ gauged supergravity, the Freedman-Schwarz model can be obtained by applying the following redefinitions of the fields and coupling constants,

$$\chi = \chi' + b, \qquad \tilde{A}^i_{(1)} = b\tilde{A}'^i_{(1)}, \qquad \tilde{g} = \tilde{g}' b^{-1}, \tag{2.4.46}$$

where b is also a constant. Then, after taking the limit $b \to \infty$ and dropping the primes, the bosonic Lagrangian density for the Freedman-Schwarz supergravity in (2.4.39) is finally obtained together with the extra terms in fermionic supersymmetry transformations (2.4.42), while the ungauged supersymmetry transformations (2.4.21) map to the transformation rules in [43] appropriately.

In conclusion, the Freedman-Schwarz model can be derived from the $SO(4)$ gauged supergravity. However, this map is irreversible since there is no analogue scaling of fields and coupling constants turning the Freedman-Schwarz gauged theory into the $SO(4)$ gauged supergravity [23]. The scaling (2.4.43) means there is no distinction between the situations where the gauge coupling constants g and $\tilde{g}$ are equal or unequal because they leave the $SO(4)$ gauged Lagrangian density invariant. Whereas, the loss of this scaling in the Freedman-Schwarz model indicate that the ratio between these two $SU(2)$ gauge coupling constants is a genuine parameter of the theory. Afterwards, the redefinitions of these fields and coupling constants mapping the standard $SO(4)$ gauged supergravity to the Freedman-Schwarz model will be reconsidered in the investigation of the effects of this one-way map on the Kaluza-Klein reduction ansatze giving N=4, half-maximal SO(4) gauged supergravity from eleven-dimensional supergravity in Section 3.1.

CHAPTER III
DIMENSIONAL REDUCTION

The dimensional reduction of eleven-dimensional supergravity giving rise to half-maximal $N = 4$ $SO(4)$ gauged theory in four dimensions will be demonstrated in this chapter. The reduction ansatze for the bosonic eleven-dimensional fields, i.e. the metric and the 4-form field strength, are first discussed together with their geometry and symmetry. After that these ansatze will be substituted in all eleven-dimensional equations of motion in Section 3.2 to verify the consistency at the level of equations of motion for the dimensional reduction.

3.1 Reduction Ansatze

The consistent reduction ansatz for the eleven-dimensional metric can be deduced from the maximal Abelian case $U(1)^4$ and the full S^7 reduction ansatze in [14] and [17] respectively, as explicated in Appendix C. This metric ansatz describes eleven-dimensional spacetime as a product space between four-dimensional spacetime and a distorted seven-dimensional sphere where its geometry can be described as a foliation of $S^3 \times S^3$. By the fact that three-dimensional sphere S^3 is a group manifold of Lie group $SU(2)$, parts of the metric ansatz involving S^3 can be obtained from the Scherk-Schwarz reduction [24]. Therefore the whole metric reduction ansatz is given by

$$d\hat{s}_{11}^2 = \Delta^{\frac{2}{3}} ds_4^2 + 2g^{-2}\Delta^{\frac{2}{3}} d\xi^2 + \frac{1}{2} g^{-2}\Delta^{\frac{2}{3}} \Big[\frac{c^2}{c^2X^2+s^2}\sum_i (h^i)^2 + \frac{s^2}{s^2\tilde{X}^2+c^2}\sum_i (\tilde{h}^i)^2\Big], \tag{3.1.1}$$

where

$$\begin{aligned} \Delta &\equiv \left[(c^2X^2+s^2)(s^2\tilde{X}^2+c^2)\right]^{\frac{1}{2}}, \\ c &\equiv \cos\xi, \qquad s \equiv \sin\xi, \\ h^i &\equiv \sigma_i - gA^i_{(1)}, \quad \tilde{h}^i \equiv \tilde{\sigma}_i - g\tilde{A}^i_{(1)}. \end{aligned} \tag{3.1.2}$$

The six quantities, σ_i and $\tilde{\sigma}_i$, are left-invariant 1-forms on each S^3 [44] satisfying the Maurer-Cartan algebras:

$$d\sigma_i = -\frac{1}{2}\varepsilon_{ijk}\sigma_j \wedge \sigma_k, \qquad d\tilde{\sigma}_i = -\frac{1}{2}\varepsilon_{ijk}\tilde{\sigma}_j \wedge \tilde{\sigma}_k. \tag{3.1.3}$$

It is more explicit to consider the geometry of eleven-dimensional spacetime in "unexcited" state where the SU(2) gauge fields, axion, and dilaton vanish such that

$$\begin{aligned} X &= \tilde{X} = 1, \\ \Delta &= [(c^2+s^2)(s^2+c^2)]^{\frac{1}{2}} = 1, \\ h^i &= \sigma_i, \quad \tilde{h}^i = \tilde{\sigma}_i. \end{aligned} \tag{3.1.4}$$

Therefore the metric reduction ansatz (3.1.1) becomes

$$d\hat{s}_{11}^2 = ds_4^2 + 2g^{-2}\Big[d\xi^2 + c^2\frac{1}{4}\sum_i (\sigma_i)^2 + s^2\frac{1}{4}\sum_i (\tilde{\sigma}_i)^2\Big], \tag{3.1.5}$$

where $\frac{1}{4}\sum_i(\sigma_i)^2 = d\Omega_3^2$ and $\frac{1}{4}\sum_i(\tilde{\sigma}_i)^2 = d\tilde{\Omega}_3^2$ are the metrics on the two round unit three-dimensional spheres S^3 [38]. The terms in the bracket represent the metric on the round unit seven-dimensional sphere S^7,

$$d\Omega_7^2 = d\xi^2 + c^2 d\Omega_3^2 + s^2 d\tilde{\Omega}_3^2, \tag{3.1.6}$$

where ξ is called the "latitude" coordinate running between the limits $0 \leq \xi \leq \pi/2$, at which one of the two S^3 shrinks to zero radii. In this unexcited state, the metric ansatz in (3.1.5) clearly describes the compact space as a round S^7 with radius $\sqrt{2}/g$. However, the existence of other fields distorts the shape of the round S^7.

While the consistent reduction ansatz for the metric can be deduced by using the $SU(2)$ group reduction, the 4-form field strength ansatz are determined by a trial and error process introducing additional terms from the already known reduction ansatz in the Abelian case [17] until the dimensional reduction are finally consistent. The reduction ansatz for the 4-form field strength is given in [23] by

$$\hat{F}_{(4)} = -g\sqrt{2}U\epsilon_{(4)} - \frac{4sc}{g\sqrt{2}}X^{-1} * dX \wedge d\xi + \frac{\sqrt{2}sc}{g}\chi X^4 * d\chi \wedge d\xi + \hat{F}'_{(4)} + \hat{F}''_{(4)}, \tag{3.1.7}$$

where

$$U = (X^2c^2 + \tilde{X}^2s^2 + 2). \tag{3.1.8}$$

The primed terms can be described as an exterior derivation of the primed 3-form potential ansatz, $\hat{F}'_{(4)} = d\hat{A}'_{(3)}$, that is purely given in terms of the two S^3 of the form

$$\hat{A}'_{(3)} = f\epsilon_{(3)} + \tilde{f}\tilde{\epsilon}_{(3)}, \tag{3.1.9}$$

where the two volume forms on each S^3 are defined by

$$\begin{aligned} \epsilon_{(3)} &= \frac{1}{3!}\varepsilon_{ijk}h^i \wedge h^j \wedge h^k, \\ \tilde{\epsilon}_{(3)} &= \frac{1}{3!}\varepsilon_{ijk}\tilde{h}^i \wedge \tilde{h}^j \wedge \tilde{h}^k. \end{aligned} \tag{3.1.10}$$

The two functions, f and $\tilde{f}$, are given by

$$\begin{aligned} f &= \frac{1}{2\sqrt{2}}g^{-3}c^4\chi X^2(c^2X^2 + s^2)^{-1}, \\ \tilde{f} &= -\frac{1}{2\sqrt{2}}g^{-3}s^4\chi X^2(s^2\tilde{X}^2 + c^2)^{-1}. \end{aligned} \tag{3.1.11}$$

Hence, the primed 4-form field strength ansatz can be written as

$$\begin{aligned} \hat{F}'_{(4)} = &\frac{\partial f}{\partial \chi}d\chi \wedge \epsilon_{(3)} + \frac{\partial f}{\partial X}dX \wedge \epsilon_{(3)} + \frac{\partial f}{\partial \xi}d\xi \wedge \epsilon_{(3)} \\ &+ \frac{\partial \tilde{f}}{\partial \chi}d\chi \wedge \tilde{\epsilon}_{(3)} + \frac{\partial \tilde{f}}{\partial X}dX \wedge \tilde{\epsilon}_{(3)} + \frac{\partial \tilde{f}}{\partial \xi}d\xi \wedge \tilde{\epsilon}_{(3)} \\ &- \frac{1}{2}fg\varepsilon_{ijk}\, h^i \wedge h^j \wedge F^k_{(2)} - \frac{1}{2}\tilde{f}g\varepsilon_{ijk}\, \tilde{h}^i \wedge \tilde{h}^j \wedge \tilde{F}^k_{(2)}. \end{aligned} \tag{3.1.12}$$

The remaining terms contained in $\hat{F}''_{(4)}$ involving to the two $SU(2)$ field strength 2-forms, $F^i_{(2)}$ and $\tilde{F}^i_{(2)}$, are given by

$$\begin{aligned}\hat{F}''_{(4)} =& \frac{g^{-2}}{\sqrt{2}} scX^{-2} d\xi \wedge h^i \wedge *F^i_{(2)} + \frac{g^{-2}}{4\sqrt{2}} c^2 X^{-2} \varepsilon_{ijk} h^i \wedge h^j \wedge *F^k_{(2)} \\ &- \frac{g^{-2}}{\sqrt{2}} sc\tilde{X}^{-2} d\xi \wedge \tilde{h}^i \wedge *\tilde{F}^i_{(2)} + \frac{g^{-2}}{4\sqrt{2}} s^2 \tilde{X}^{-2} \varepsilon_{ijk} \tilde{h}^i \wedge \tilde{h}^j \wedge *\tilde{F}^k_{(2)} \\ &+ \frac{g^{-2}}{\sqrt{2}} sc\chi d\xi \wedge h^i \wedge F^i_{(2)} + \frac{g^{-2}}{4\sqrt{2}} c^2 \chi \varepsilon_{ijk} h^i \wedge h^j \wedge F^k_{(2)} \\ &+ \frac{g^{-2}}{\sqrt{2}} sc\chi X^2 \tilde{X}^{-2} d\xi \wedge \tilde{h}^i \wedge \tilde{F}^i_{(2)} - \frac{g^{-2}}{4\sqrt{2}} s^2 \chi X^2 \tilde{X}^{-2} \varepsilon_{ijk} \tilde{h}^i \wedge \tilde{h}^j \wedge \tilde{F}^k_{(2)}.\end{aligned} \tag{3.1.13}$$

Note that these reduction ansatze for both eleven-dimensional bosonic fields still depend on the internal space coordinate ξ through its sinusoidal functions, s and c. In order to obtain all four-dimensional equations of motions in $SO(4)$ gauged supergravity, (2.4.31)-(2.4.34) and (2.4.37), every single ξ-dependent term arising from substitutions of the eleven-dimensional equations of motion, (2.4.15)-(2.4.17), by these reduction ansatze need to be canceled. Therefore the consistent dimensional reduction between the two supergravities is achieved.

The reduction ansatze, (3.1.1) and (3.1.7), are invariant under a residual Z_2 subgroup of the original global SL(2,R) symmetry of the ungauged four-dimensional $N = 4$ supergravity that transforms the various quantities to their primed image by

$$\begin{aligned} X' = \tilde{X}', &\quad \chi' X'^2 = -\chi X^2, \\ A'^i_{(1)} = \tilde{A}^i_{(1)}, &\quad \tilde{A}'^i_{(1)} = A^i_{(1)}, \\ c' = s, &\quad s' = -c, \\ h'^i = \tilde{h}^i, &\quad \tilde{h}'^i = h^i, \\ \epsilon'_{(3)} = \tilde{\epsilon}_{(3)}, &\quad \tilde{\epsilon}'_{(3)} = \epsilon_{(3)}. \end{aligned} \tag{3.1.14}$$

This residual Z_2 corresponds to an interchange of the two S^3 in the foliation of $S^3 \times S^3$. In the four-dimensional theory, this associates with an interchange of the two $SU(2)$ gauge fields.

Moreover, it is of interest to see what happens to the reduction ansatze, if the fields and coupling constants redefinitions in (2.4.43) and (2.4.46) are taken. For the metric ansatz, these redefinitions turn the metric (3.1.1) into

$$d\hat{s}^2_{11} = (\frac{1}{2}bX)^{\frac{2}{3}} \Big[ds^2_4 + \frac{2}{g\tilde{g}} d\tilde{\xi}^2 + \frac{1}{2} g^{-2} X^{-2} \sum_i (h^i)^2 + \frac{1}{2} \tilde{g}^{-2} X^{-2} \sum_i (\tilde{h}^i)^2 \Big], \tag{3.1.15}$$

where the new latitude coordinate is defined by $\xi \equiv b^{-\frac{1}{2}} \tilde{\xi} + \frac{1}{4}\pi$. The reduction ansatz for the 4-form field strength (3.1.7) reduces to

$$\begin{aligned}\hat{F}_{(4)} = \frac{b}{\sqrt{2g\tilde{g}}}\Big(& X^4 * d\chi \wedge d\tilde{\xi} - \frac{1}{2}g^{-2}d\tilde{\xi}\wedge\epsilon_{(3)} - \frac{1}{2}\tilde{g}^{-2}d\tilde{\xi}\wedge\tilde{\epsilon}_{(3)} \\ & + \frac{1}{2}g^{-1}d\tilde{\xi}\wedge F^i_{(2)}\wedge h^i + \frac{1}{2}\tilde{g}^{-1}d\tilde{\xi}\wedge\tilde{F}^i_{(2)}\wedge\tilde{h}^i\Big)\end{aligned} \tag{3.1.16}$$

through these fields and coupling constants redefinitions. Note that even though b is sent to infinity, these reduction ansatze are still consistent. In eleven-dimensional theory, there is a scaling symmetry under

$$\hat{g}_{MN} \to e^{2k}\hat{g}_{MN}, \qquad \hat{A}_{MNP} \to e^{3k}\hat{A}_{MNP}, \tag{3.1.17}$$

where k is a constant, that leave all equations of motion unchanged. Substitutions of these dimensional reduction ansatze, which contain overall constant factors $b^{\frac{2}{3}}$ and b for the metric and the 4-form ansatze respectively, into the eleven-dimensional equations of motion cancel these factors out from the equations of motion. Therefore, the factor b is effectively set to be any value under these scaling (3.1.17) and it is conveniently set to be $b = 2$ [23].

The reduction ansatze (3.1.15) and (3.1.16) can be interpreted as ansatze for two-step dimensional reduction obtaining another $N = 4$ gauged theory in four dimensions. The first step is the simplest Kaluza-Klein reduction of eleven-dimensional supergravity on an internal space S^1 described by the new latitude coordinate $\tilde{\xi}$ giving rise to ten-dimensional type IIA supergravity. In this case, the reduction ansatze for the metric can be written in the same form as (2.2.6), while the 4-form ansatz for the Kaluza-Klein reduction on S^1 is given in [7] by the following form

$$\begin{aligned} d\hat{s}^2_{11} &= e^{-\frac{1}{6}\varphi}ds^2_{10} + e^{\frac{4}{3}\varphi}(d\tilde{\xi}+\mathcal{A}_{(1)})^2, \\ \hat{F}_{(4)} &= F_{(4)} + F_{(3)}\wedge(d\tilde{\xi}+\mathcal{A}_{(1)}), \end{aligned} \tag{3.1.18}$$

where φ is a ten-dimensional dilaton scalar field and $\mathcal{A}_{(1)}$ is called the Kaluza-Klein $U(1)$ potential 1-form obtained from a dimensional reduction of the metric S^1, while the lower-dimensional 4-from and 3-form field strengths are defined by $F_{(4)} \equiv dA_{(3)} - dA_{(2)}\wedge\mathcal{A}_{(1)}$ and $F_{(3)} \equiv dA_{(2)}$ respectively. Comparing these ansatze with (3.1.15) and (3.1.16) leads to expressions of all bosonic fields in the type IIA supergravity in terms of the four-dimensional ones,

$$\begin{aligned} ds^2_{10} &= (\frac{2}{g\tilde{g}})^{\frac{1}{8}}\Big[e^{\frac{3}{4}\phi}ds^2_4 + e^{-\frac{1}{4}\phi}\big(g^{-2}\sum_i(h^i)^2 + \tilde{g}^{-2}\sum_i(\tilde{h}^i)^2\big)\Big], \\ F_{(3)} &= \frac{1}{\sqrt{2g\tilde{g}}}\Big[2e^{2\phi} * d\chi + g^{-2}\epsilon_{(3)} + \tilde{g}^{-2}\tilde{\epsilon}_{(3)} - g^{-1}F^i_{(2)}\wedge h^i - \tilde{g}^{-1}\tilde{F}^i_{(2)}\wedge\tilde{h}^i\Big], \\ \varphi &= \frac{1}{2}\phi - \frac{3}{4}\log\Big(\frac{g\tilde{g}}{2}\Big), \\ F_{(4)} &= 0, \qquad \mathcal{A}_{(1)} = 0. \end{aligned} \tag{3.1.19}$$

These expressions can be viewed as the reduction ansatze for the second step dimensional reduction of the ten-dimensional heterotic supergravities on an $S^3 \times S^3$ internal space giving rise to the Freedman-Schwarz model demonstrated in [45]. Therefore, the reduction ansatze (3.1.1) and (3.1.7) can also reduce eleven-dimensional supergravity to another version of four-dimensional gauged supergravity, the Freedman-Schwarz model, through the two-step dimensional reduction described above when the field and coupling constant redefinitions in (2.4.43) and (2.4.46) are taken.

3.2 Dimensional Reduction Procedures

All procedures of examining the consistency at the level of equations of motion for the dimensional reduction are established in this section through substitutions of eleven-dimensional equations of motion, (2.4.15)-(2.4.17), by the two reduction ansatze given in the previous section. If these ansatze lead to the consistent dimensional reduction of eleven-dimensional supergravity giving rise to the $N = 4$ $SO(4)$ gauged supergravity in four dimensions, these substitutions have to yield all four-dimensional equations of motion that are introduced in Section 2.4.2.

Fortunately, there are two equations of motion in the four-dimensional $SO(4)$ gauged supergravity directly arisen from the definitions of the $SU(2)$ Yang-Mills field strengths in (2.4.25) without dimensional reduction procedure. These two equations of motion are the Bianchi identities (2.4.35) and (2.4.36) for each set of $SU(2)$ Yang-Mills field strengths that can be written in component forms as

$$\begin{aligned} D_a F^i_{bc} + D_b F^i_{ca} + D_c F^i_{ab} &= 0, \\ \tilde{D}_a \tilde{F}^i_{bc} + \tilde{D}_b \tilde{F}^i_{ca} + \tilde{D}_c \tilde{F}^i_{ab} &= 0. \end{aligned} \tag{3.2.1}$$

Here D_a and $\tilde{D}_a$ are the Lorentz $SU(2)$-gauged covariant derivatives defined by

$$\begin{aligned} D_a P^i_{b_1 \dots b_p} &= \partial_a P^i_{b_1 \dots b_p} - \omega_a{}^c{}_{b_1} P^i_{c \dots b_p} - \dots - \omega_a{}^c{}_{b_p} P^i_{b_1 \dots c} + g \varepsilon_{ijk} A^j_a P^k_{b_1 \dots b_p}, \\ \tilde{D}_a \tilde{P}^i_{b_1 \dots b_p} &= \partial_a \tilde{P}^i_{b_1 \dots b_p} - \omega_a{}^c{}_{b_1} \tilde{P}^i_{c \dots b_p} - \dots - \omega_a{}^c{}_{b_p} \tilde{P}^i_{b_1 \dots c} + g \varepsilon_{ijk} \tilde{A}^j_a \tilde{P}^k_{b_1 \dots b_p}, \end{aligned} \tag{3.2.2}$$

in which $P^i_{b_1 \dots b_p}$ and $\tilde{P}^i_{b_1 \dots b_p}$ are components of the two p-forms separately charged by each $SU(2)$. These Bianchi identities are very helpful for substitutions involving the equations of motion for the $SU(2)$ field strengths. Note that it is simpler to deal with flat spacetime so all substitutions of the eleven-dimensional equations of motion will be more conveniently performed in the Lorentz frame described by vielbein non-coordinates bases.

Our dimensional reduction procedures take the first step from the simplest equation of motion in eleven dimensions, the Bianchi identity (2.4.17),

$$d\hat{F}_{(4)} = 0.$$

If the reduction ansatz for the eleven-dimensional anti-symmetric tensor is constructed on the fundamental 3-form potential $\hat{A}_{(3)}$, this equation will obviously satisfy due to the property of the nilpotent operator in (2.1.43), $d^2\hat{A}_{(3)} = 0$. However, there is no way to express an explicit form of the potential from our field strength ansatz (3.1.7) [23], so the Bianchi identity is not a true identity. Substitution of the 4-form reduction ansatz in (3.1.7) in this equation of motion leads to some of the four-dimensional equations of motion. However, an exterior derivative of the primed 4-form field strength ansatz vanish due to the property of the nilpotent operator, $d\hat{F}'_{(4)} = d^2\hat{A}'_{(3)} = 0$. Thus the remaining terms in an exterior derivative of the 4-form ansatz (3.1.7) are

$$\begin{aligned}
d\hat{F}_{(4)} =& - d\Big(g\sqrt{2}U\epsilon_{(4)}\Big) - d\Big(\frac{4sc}{g\sqrt{2}}X^{-1} * dX \wedge d\xi\Big) + d\Big(\frac{\sqrt{2}sc}{g}\chi X^4 * d\chi \wedge d\xi\Big)\\
&+ d\Big(\frac{g^{-2}}{\sqrt{2}}scX^{-2}d\xi \wedge h^i \wedge *F^i_{(2)}\Big) + d\Big(\frac{g^{-2}}{4\sqrt{2}}c^2X^{-2}\varepsilon_{ijk}h^i \wedge h^j \wedge *F^k_{(2)}\Big)\\
&- d\Big(\frac{g^{-2}}{\sqrt{2}}sc\tilde{X}^{-2}d\xi \wedge \tilde{h}^i \wedge *\tilde{F}^i_{(2)}\Big) + d\Big(\frac{g^{-2}}{4\sqrt{2}}s^2\tilde{X}^{-2}\varepsilon_{ijk}\tilde{h}^i \wedge \tilde{h}^j \wedge *\tilde{F}^k_{(2)}\Big)\\
&+ d\Big(\frac{g^{-2}}{\sqrt{2}}sc\chi d\xi \wedge h^i \wedge F^i_{(2)}\Big) + d\Big(\frac{g^{-2}}{4\sqrt{2}}c^2\chi\varepsilon_{ijk}h^i \wedge h^j \wedge F^k_{(2)}\Big)\\
&+ d\Big(\frac{g^{-2}}{\sqrt{2}}sc\chi X^2\tilde{X}^{-2}d\xi \wedge \tilde{h}^i \wedge \tilde{F}^i_{(2)}\Big) - d\Big(\frac{g^{-2}}{4\sqrt{2}}s^2\chi X^2\tilde{X}^{-2}\varepsilon_{ijk}\tilde{h}^i \wedge \tilde{h}^j \wedge \tilde{F}^k_{(2)}\Big).
\end{aligned} \tag{3.2.3}$$

There are five non-zero components survive from this simplest substitution corresponding to the following 5-form's bases i.e.

- $(d\xi \wedge \epsilon_{(4)})$ basis
  This basis is obtained from the first line in (3.2.3) and the terms containing $d\xi$ in their bases. The component corresponding to this basis equals to a particular combination of components of the two equations of motion (3.2.10) and (3.2.11) for the dilaton and the axion respectively,

$$\begin{aligned}
\Box\phi - X^4\chi\Box\chi =& X^4\partial^a\chi\partial_a\chi + 2\chi X^4\partial^a\phi\partial_a\chi + 2g^2[X^2 - X^2 + \chi^2X^2]\\
&- \frac{1}{4}X^{-2}F^i_{ab}F^{iab} - \frac{1}{8}\chi\varepsilon^{abcd}F^i_{ab}F^i_{cd}\\
&+ \frac{1}{4}\tilde{X}^{-2}\tilde{F}^i_{ab}\tilde{F}^{iab} - \frac{1}{8}\chi X^2\tilde{X}^{-2}\varepsilon^{abcd}\tilde{F}^i_{ab}\tilde{F}^i_{cd}.
\end{aligned} \tag{3.2.4}$$

- <u>$(d\xi \wedge h^i \wedge e^a \wedge e^b \wedge e^f)$ and $(h^i \wedge h^j \wedge e^a \wedge e^b \wedge e^f)$ bases</u>
  These two bases can be obtained from the second and the fourth lines in (3.2.3). While the first basis is obtained from the two left terms containing $d\xi$, the second basis is obtained from the right terms. By using the four-dimensional Bianchi identity of the untilded $SU(2)$ field strength given in (2.4.35), the two components imply the same equation of motion for the first $SU(2)$ Yang-Mills field strengths,

$$D_a F^{iab} = \partial_a \phi F^{iab} + \frac{1}{2} X^2 \varepsilon^{abcd} \partial_a \chi F^i_{cd}, \tag{3.2.5}$$

  that is a component form of the first Yang-Mills equation of motion (2.4.33).

- <u>$(d\xi \wedge \tilde{h}^i \wedge e^a \wedge e^b \wedge e^f)$ and $(\tilde{h}^i \wedge \tilde{h}^j \wedge e^a \wedge e^b \wedge e^f)$ bases</u>
  These two bases are parallel to the previous ones and can be obtained from the third and the fifth lines in (3.2.3). These two components equal to the component form of the equation of motion for the tilded $SU(2)$ Yang-Mills field strength $\tilde{F}^i_{(2)}$ in (2.4.34),

$$\begin{aligned} \tilde{D}_a \tilde{F}^{iab} =& (\chi^2 X^4 - 1) q^{-2} \partial_a \phi \tilde{F}^{iab} + 2\chi \tilde{X}^{-2} X^2 \partial_a \chi \tilde{F}^{iab} \\ & - \frac{1}{2} \varepsilon^{abcd} \tilde{X}^{-2} \left[2\chi \partial_a \phi + (1 - \chi^2 X^4) \partial_a \chi \right] \tilde{F}^i_{cd}, \end{aligned} \tag{3.2.6}$$

  when another four-dimensional Bianchi identity is applied.

Therefore, the substitution in this simplest higher-dimensional equation of motion of the 4-form reduction ansatz implies the two $SU(2)$ Yang-Mills equations and a particular combination of the equations of motion for the dilaton and the axion in four-dimensional spacetime.

Then, the next substitution in the equation of motion for the 4-form field strength (2.4.16) with the 4-form reduction ansatz is considered. As shown in (2.4.16),

$$d\hat{*}\hat{F}_{(4)} = -\frac{1}{2} \hat{F}_{(4)} \wedge \hat{F}_{(4)},$$

this equation requires the eleven-dimensional Hodge duality of $\hat{F}_{(4)}$ that turns the 4-form field strength into a 7-form expressed through (3.1.7) by

$$\begin{aligned} \hat{*}\hat{F}_{(4)} =& \frac{1}{4} g^{-6} s^3 c^3 \Delta^{-2} U \, d\xi \wedge \epsilon_{(3)} \wedge \tilde{\epsilon}_{(3)} - \frac{1}{4} g^{-6} s^4 c^4 \Delta^{-2} X^{-1} dX \wedge \epsilon_{(3)} \wedge \tilde{\epsilon}_{(3)} \\ & + \frac{1}{8} g^{-6} s^4 c^4 \Delta^{-2} X^4 \chi d\chi \wedge \epsilon_{(3)} \wedge \tilde{\epsilon}_{(3)} + \hat{*}\hat{F}'_{(4)} + \hat{*}\hat{F}''_{(4)}. \end{aligned} \tag{3.2.7}$$

The eleven-dimensional Hodge duality of $\hat{F}'_{(4)}$ is given by

$$
\begin{aligned}
\hat{*}\hat{F}'_{(4)} =& -\sqrt{2}g^{-1}s^3c^{-3}\Delta^{-2}\Omega^3\frac{\partial f}{\partial\chi} * d\chi\wedge d\xi\wedge\tilde{\epsilon}_{(3)}\\
&-\sqrt{2}g^{-1}s^3c^{-3}\Delta^{-2}\Omega^3\frac{\partial f}{\partial X} * dX\wedge d\xi\wedge\tilde{\epsilon}_{(3)}\\
&+\frac{1}{\sqrt{2}}gs^3c^{-3}\Delta^{-2}\Omega^3\frac{\partial f}{\partial\xi}\,\epsilon_{(4)}\wedge\tilde{\epsilon}_{(3)}\\
&+\sqrt{2}g^{-1}s^{-3}c^3\Delta^{-2}\tilde{\Omega}^3\frac{\partial\tilde{f}}{\partial\chi} * d\chi\wedge d\xi\wedge\epsilon_{(3)}\\
&+\sqrt{2}g^{-1}s^{-3}c^3\Delta^{-2}\tilde{\Omega}^3\frac{\partial\tilde{f}}{\partial X} * dX\wedge d\xi\wedge\epsilon_{(3)}\\
&-\frac{1}{\sqrt{2}}gs^{-3}c^3\Delta^{-2}\tilde{\Omega}^3\frac{\partial\tilde{f}}{\partial\xi}\,\epsilon_{(4)}\wedge\epsilon_{(3)}\\
&-\frac{1}{\sqrt{2}}fg^{-2}s^3c^{-1}\Omega\tilde{\Omega}^{-1}d\xi\wedge h^i\wedge *F^i_{(2)}\wedge\tilde{\epsilon}_{(3)}\\
&+\frac{1}{\sqrt{2}}\tilde{f}g^{-2}s^{-1}c^3\Omega^{-1}\tilde{\Omega}d\xi\wedge\tilde{h}^i\wedge *\tilde{F}^i_{(2)}\wedge\epsilon_{(3)},
\end{aligned}
\tag{3.2.8}
$$

where the two scalar quantities $\Omega = c^2X^2s^2$ and $\tilde{\Omega} = s^2\tilde{X}^2 + c^2$ are defined in (D.2), and the last term $\hat{*}\hat{F}''_{(4)}$ is given by

$$
\begin{aligned}
\hat{*}\hat{F}''_{(4)} =& -\frac{1}{16}g^{-5}s^4c^2\tilde{\Omega}^{-1}X^{-2}\varepsilon_{ijk}\,F^i_{(2)}\wedge h^j\wedge h^k\wedge\tilde{\epsilon}_{(3)}\\
&-\frac{1}{4}g^{-5}s^3c\Omega\tilde{\Omega}^{-1}X^{-2}d\xi\wedge h^i\wedge F^i_{(2)}\wedge\tilde{\epsilon}_{(3)}\\
&-\frac{1}{16}g^{-5}s^2c^4\Omega^{-1}\tilde{X}^{-2}\varepsilon_{ijk}\,\tilde{F}^i_{(2)}\wedge\tilde{h}^j\wedge\tilde{h}^k\wedge\epsilon_{(3)}\\
&+\frac{1}{4}g^{-5}sc^3\Omega^{-1}\tilde{\Omega}\tilde{X}^{-2}d\xi\wedge\tilde{h}^i\wedge\tilde{F}^i_{(2)}\wedge\epsilon_{(3)}\\
&+\frac{1}{16}g^{-5}s^4c^2\tilde{\Omega}^{-1}\chi\varepsilon_{ijk} * F^i_{(2)}\wedge h^j\wedge h^k\wedge\tilde{\epsilon}_{(3)}\\
&+\frac{1}{4}g^{-5}s^3c\Omega\tilde{\Omega}^{-1}\chi d\xi\wedge h^i\wedge *F^i_{(2)}\wedge\tilde{\epsilon}_{(3)}\\
&-\frac{1}{16}g^{-5}s^2c^4\Omega^{-1}\chi X^2\tilde{X}^{-2}\varepsilon_{ijk} * \tilde{F}^i_{(2)}\wedge\tilde{h}^j\wedge\tilde{h}^k\wedge\epsilon_{(3)}\\
&+\frac{1}{4}g^{-5}sc^3\Omega^{-1}\tilde{\Omega}\chi X^2\tilde{X}^{-2}d\xi\wedge\tilde{h}^i\wedge *\tilde{F}^i_{(2)}\wedge\epsilon_{(3)}.
\end{aligned}
\tag{3.2.9}
$$

We now take an exterior derivative on this 7-form and substitute this into the left-handed side of the equation of motion for the 4-form field strength in (2.4.16) together with a wedge product of the two $\hat{F}_{(4)}$ on the right-handed side. This equation of motion now describes an 8-form in which there are 13 differential bases in which each component equals to some four-dimensional equations of motions or zero, as exhibited in the following:

- $(\epsilon_{(4)} \wedge d\xi \wedge \epsilon_{(3)}) \leftrightarrow$ the two scalar fields' equations of motion,
- $(\epsilon_{(4)} \wedge d\xi \wedge \tilde{\epsilon}_{(3)}) \leftrightarrow$ the two scalar fields' equations of motion,
- $(\epsilon_{(4)} \wedge d\xi \wedge h^i \wedge h^j \wedge \tilde{h}^k) \leftrightarrow 0$,
- $(\epsilon_{(4)} \wedge d\xi \wedge h^i \wedge \tilde{h}^j \wedge \tilde{h}^k) \leftrightarrow 0$,
- $(\epsilon_{(4)} \wedge h^i \wedge h^j \wedge \tilde{h}^m \wedge \tilde{h}^n) \leftrightarrow 0$,
- $(e^a \wedge e^b \wedge e^f \wedge \epsilon_{(3)} \wedge \tilde{h}^i \wedge \tilde{h}^j) \leftrightarrow$ the second Yang-Mills equation,
- $(e^a \wedge e^b \wedge e^f \wedge h^i \wedge h^j \wedge \tilde{\epsilon}_{(3)}) \leftrightarrow$ the first Yang-Mills equation,
- $(e^a \wedge e^b \wedge e^f \wedge d\xi \wedge \epsilon_{(3)} \wedge \tilde{h}^i) \leftrightarrow$ the second Yang-Mills equation,
- $(e^a \wedge e^b \wedge e^f \wedge d\xi \wedge h^i \wedge \tilde{\epsilon}_{(3)}) \leftrightarrow$ the first Yang-Mills equation,
- $(e^a \wedge e^b \wedge d\xi \wedge \epsilon_{(3)} \wedge \tilde{h}^i \wedge \tilde{h}^j) \leftrightarrow 0$,
- $(e^a \wedge e^b \wedge d\xi \wedge h^i \wedge h^j \wedge \tilde{\epsilon}_{(3)}) \leftrightarrow 0$,
- $(e^a \wedge e^b \wedge \epsilon_{(3)} \wedge \tilde{\epsilon}_{(3)}) \leftrightarrow 0$,
- $(e^a \wedge d\xi \wedge \epsilon_{(3)} \wedge \tilde{\epsilon}_{(3)}) \leftrightarrow 0$.

There are 6 differential bases that give non-zero components equal to some four-dimensional equations of motion. For the first two 8-form's bases, their components contain the component form of the equation of motion for the dilaton,

$$\begin{aligned}\Box\phi =& X^4\partial^a\chi\partial_a\chi - 2g^{-2}(X^2 - X^{-2} + \chi^2X^2) - \frac{1}{4}X^{-2}F^i_{ab}F^{iab} \\ &+ \frac{1}{4}(1-\chi^2X^4)q^{-4}X^2\tilde{F}^i_{ab}\tilde{F}^{iab} + \frac{1}{4}\chi\tilde{X}^{-4}\varepsilon^{abcd}\tilde{F}^i_{ab}\tilde{F}^i_{cd},\end{aligned} \tag{3.2.10}$$

together with the component of the axion's equation of motion,

$$\begin{aligned}\Box\chi =& -2\partial^a\phi\partial_a\chi - 4g^{-2}\chi X^{-2} - \frac{1}{2}\chi X^2q^{-4}\tilde{F}^i_{ab}\tilde{F}^{iab} \\ &- \frac{1}{8}X^{-4}\varepsilon^{abcd}F^i_{ab}F^i_{cd} + \frac{1}{8}(1-\chi^2X^4)q^{-4}\varepsilon^{abcd}\tilde{F}^i_{ab}\tilde{F}^i_{cd},\end{aligned} \tag{3.2.11}$$

in some particular combinations in which the two scalar fields' equations of motion, (3.2.10) and (3.2.11) are each multiplied by some different coefficient proportional to $\cos^2\xi$ or $\sin^2\xi$. By using the fact that these coefficients are orthogonal to each other, the components of the first two 8-form's bases therefore contain the two independent scalar fields' equations of motion.

In addition, the non-zero components of the 8-form's bases equal to the component of the $SU(2)$ Yang-Mills equations, (3.2.5) and (3.2.6), in the same way as the previous substitution. Therefore, the substitution of the reduction ansatz for $\hat{F}_{(4)}$ (3.1.7) in the eleven-dimensional 4-form's equation of motion gives rise to four-dimensional equations of motion for the dilaton, the axion, and the $SU(2)$ gauge fields.

Finally, substitution in the eleven-dimensional Einstein's field equation given in (2.4.15) is performed,

$$\hat{R}_{\hat{M}\hat{N}} = \frac{1}{12}(\hat{F}^2_{\hat{M}\hat{N}} - \frac{1}{12}\hat{\eta}_{\hat{M}\hat{N}}\hat{F}^2_{(4)}).$$

Note that this is the most significant equation of motion for our dimensional reduction in which the reduction ansatze for both bosonic fields are required for this substitution. In Appendix D, all components of the eleven-dimensional Ricci tensor will be derived from the reduction ansatz for the metric given in (3.1.1). The contractions of the 4-form's components on the right-hand side of the equation can be obtained from the reduction ansatz for the 4-form field strength. Starting from all non-zero components of the 4-form field strength that can be read off from (3.1.7) where the eleven-dimensional flat spacetime indices split to $\hat{M}, \hat{N} = (a, 0, i, \tilde{i})$ where a is a four-dimensional flat spacetime index, 0 is a flat space index corresponding to the spatial coordinate ξ, i and $\tilde{i}$ are the flat coordinate index on each S^3,

$$\hat{F}_{abcd} = -\sqrt{2}g\Delta^{-\frac{4}{3}}(c^2x^2 + s^2\tilde{X}^2 + 2)\varepsilon_{abcd}, \tag{3.2.12}$$

$$\hat{F}_{abc0} = sc\Delta^{-\frac{4}{3}}(\chi X^4\partial_f\chi - \partial_f\phi)\varepsilon_{abc}{}^f, \tag{3.2.13}$$

$$\hat{F}_{aijk} = c\,\Omega^{\frac{1}{2}}\Delta^{-\frac{4}{3}}\Big(\frac{s^2\chi X^2\partial_a\phi}{\Omega} + X^2\partial_a\chi\Big)\varepsilon_{ijk}, \tag{3.2.14}$$

$$\hat{F}_{a\tilde{i}\tilde{j}\tilde{k}} = -s\,\tilde{\Omega}^{-\frac{1}{2}}\Delta^{-\frac{4}{3}}\Big[\chi X^2(2s^2X^{-2} + c^2)\partial_a\phi + (s^2 - s^2\chi^2X^4 + c^2X^2)\partial_a\chi\Big]\varepsilon_{ijk}, \tag{3.2.15}$$

$$\hat{F}_{0ijk} = -\sqrt{2}gs\,\Omega^{-\frac{1}{2}}\Delta^{-\frac{4}{3}}\chi X^2(1+\Omega)\varepsilon_{ijk}, \tag{3.2.16}$$

$$\hat{F}_{0\tilde{i}\tilde{j}\tilde{k}} = -\sqrt{2}gc\,\tilde{\Omega}^{-\frac{1}{2}}\Delta^{-\frac{4}{3}}\chi X^2(1+\tilde{\Omega})\varepsilon_{ijk}, \tag{3.2.17}$$

$$\hat{F}_{ab0i} = \frac{1}{\sqrt{2}}s\,\Omega^{\frac{1}{2}}\Delta^{-\frac{4}{3}}\Big(\chi F^i_{ab} + \frac{X^{-2}}{2}F^i_{cd}\varepsilon_{ab}{}^{cd}\Big), \tag{3.2.18}$$

$$\hat{F}_{ab0\tilde{i}} = \frac{1}{\sqrt{2}}c\,\tilde{\Omega}^{\frac{1}{2}}\Delta^{-\frac{4}{3}}\tilde{X}^{-2}\Big(\chi X^2\tilde{F}^i_{ab} - \frac{1}{2}\tilde{F}^i_{cd}\varepsilon_{ab}{}^{cd}\Big), \tag{3.2.19}$$

$$\hat{F}_{abij} = \frac{1}{\sqrt{2}}\Omega\Delta^{-\frac{4}{3}}\varepsilon_{ijk}\Big(\frac{\chi s^2}{\Omega}F^k_{ab} + \frac{X^{-2}}{2}F^k_{cd}\varepsilon_{ab}{}^{cd}\Big), \tag{3.2.20}$$

$$\hat{F}_{ab\tilde{i}\tilde{j}} = \frac{1}{\sqrt{2}}\tilde{\Omega}\Delta^{-\frac{4}{3}}\varepsilon_{ijk}\tilde{X}^{-2}\Big(-\frac{\chi X^2c^2}{\tilde{\Omega}}\tilde{F}^k_{ab} + \frac{1}{2}\tilde{F}^k_{cd}\varepsilon_{ab}{}^{cd}\Big), \tag{3.2.21}$$

the contractions of these components in $\hat{F}^2_{\hat{M}\hat{N}} = \hat{F}_{\hat{M}\hat{P}\hat{Q}\hat{R}}\hat{F}_{\hat{N}}{}^{\hat{P}\hat{Q}\hat{R}}$ can be obtained as follow

$$\begin{aligned}
\hat{F}^2_{00} = 3!\Delta^{-\frac{8}{3}}s^2c^2\Big\{&2\chi X^4\partial^a\chi\partial_a\phi - \partial^a\phi\partial_a\phi - \chi^2X^8\partial^a\chi\partial_a\chi \\
&+ 2g^2\big[\frac{c^{-2}\chi^2X^4}{\Omega}(1+2\Omega+\Omega^2) + \frac{s^{-2}\chi^2X^4}{\tilde{\Omega}}(1+2\tilde{\Omega}+\tilde{\Omega}^2)\big] \\
&+ \frac{c^{-2}X^{-2}}{4}\Omega\big[\tilde{X}^2F^i_{ab}F^{iab} + \chi\varepsilon^{abcd}F^i_{ab}F^i_{cd}\big] \\
&+ \frac{s^{-2}\tilde{X}^{-4}}{4}\tilde{\Omega}\big[(\chi^2X^4-1)\tilde{F}^i_{ab}\tilde{F}^{iab} + \chi X^2\varepsilon^{abcd}\tilde{F}^i_{ab}\tilde{F}^i_{cd}\big]\Big\},
\end{aligned} \tag{3.2.22}$$

$$\begin{aligned}
\hat{F}^2_{0a} = 3!\sqrt{2}g\Delta^{-\frac{8}{3}}sc \times& \\
\Big\{\partial_a\phi\big[&\frac{1+\tilde{\Omega}}{\tilde{\Omega}}\chi^2X^4(2s^2X^{-2}+c^2) - \frac{s^2}{\Omega}\chi^2X^4(1+\Omega) - (c^2x^2+s^2\tilde{X}^2+2)\big] \\
+ \chi X^4\partial_a\chi&\big[(c^2x^2+s^2\tilde{X}^2+2) - (1+\Omega) + \frac{(1+\tilde{\Omega})}{\tilde{\Omega}}X^{-2}(\Omega - s^2\chi^2X^4)\big]\Big\},
\end{aligned} \tag{3.2.23}$$

$$\hat{F}^2_{0i} = 0, \qquad\qquad \hat{F}^2_{0\tilde{i}} = 0, \tag{3.2.24}$$

$$\begin{aligned}
\hat{F}^2_{ab} = 3!\ \Delta^{-\frac{8}{4}}\Big\{&s^2c^2(2\chi X^4\partial_c\chi\partial^c\phi - \partial_c\phi\partial^c\phi - \chi^2X^8\partial_c\chi\partial^c\chi)\eta_{ab} \\
&- 2g^2(\Omega+\tilde{\Omega}+1)^2\eta_{ab} \\
&+ s^2\partial_a\phi\partial_b\phi\big[c^2 + \chi^2X^4(\frac{s^2c^2}{\Omega} + \frac{(2s^2X^{-2}+c^2)^2}{\tilde{\Omega}})\big] \\
&+ \partial_a\chi\partial_b\chi\big[s^2c^2\chi^2X^8 + c^2X^4\Omega + \frac{s^2}{\Omega}(\Omega - s^2\chi^2X^4)^2\big] \\
&+ (\partial_a\chi\partial_b\phi + \partial_a\phi\partial_b\chi)\frac{s^2\chi X^2}{\tilde{\Omega}}(\Omega - s^2\chi^2X^4)(2s^2X^{-2}+c^2) \\
&+ \frac{1}{2}\big[c^2\tilde{\Omega}X^2\tilde{X}^{-2} + \tilde{X}^{-4}(\tilde{\Omega}^2 + c^4\chi^2X^4)\big]\tilde{F}^i_{ac}\tilde{F}^{ic}_b \\
&+ \frac{1}{2}(s^2\Omega X^{-2}\tilde{X}^2 + X^{-4}\Omega^2 + s^4\chi^2)F^i_{ac}F^{ic}_b \\
&- \frac{\Omega X^{-4}}{4}(s^2+\Omega)F^i_{cd}F^{icd}\eta_{ab} - \frac{\tilde{\Omega}\tilde{X}^{-4}}{4}(c^2+\tilde{\Omega})\tilde{F}^i_{cd}\tilde{F}^{icd}\eta_{ab} \\
&+ s^2\Omega\chi X^{-2}(F^i_{ad}F^i_{cf}\varepsilon_b{}^{dcf} + F^i_{cf}\varepsilon_a{}^{dcf}F^i_{bd}) \\
&- c^2\tilde{\Omega}\chi X^2\tilde{X}^{-4}(\tilde{F}^i_{ad}\tilde{F}^i_{cf}\varepsilon_b{}^{dcf} + \tilde{F}^i_{cf}\varepsilon_a{}^{dcf}\tilde{F}^i_{bd})\Big\},
\end{aligned} \tag{3.2.25}$$

$$\hat{F}^2_{ai} = -\frac{3}{\sqrt{2}}c\,\Omega^{\frac{1}{2}}\Delta^{-\frac{8}{3}}\Big\{2s^2X^{-2}(1+\frac{s^2\chi^2x^4}{\Omega}\partial_b\phi F^ib_a) + \varepsilon_a{}^{bcd}(s^2\chi^2X^4+\Omega)\partial_b\chi F^i_{cd}\Big\}, \tag{3.2.26}$$

$$\hat{F}^2_{a\tilde{i}} =\frac{3}{\sqrt{2}}s\,\tilde{\Omega}^{\frac{1}{2}}\Delta^{-\frac{8}{3}}\tilde{X}^{-2}\times$$
$$\Big\{2c^2\big[(1-\frac{\chi^2X^4(2s^2X^{-2})}{\tilde{\Omega}})\partial_b\phi-\chi X^4(1+\frac{\Omega X^{-2}-s^2\chi^2X^2}{\tilde{\Omega}})\partial_b\chi\big]\tilde{F}^{ib}_a$$
$$+\Omega\varepsilon_a{}^{bcd}\big[2\chi\partial_b\phi+(1-\chi^2X^4)\partial_b\chi\big]\tilde{F}^i_{cd}\Big\}, \tag{3.2.27}$$
$$\hat{F}^2_{ij} = 3!\Delta^{-\frac{8}{3}}\Big\{c^2X^4\delta_{ij}(\frac{s^4\chi^2}{\Omega}\partial_a\phi\partial^a\phi+\Omega\partial_a\chi\partial^a\chi+2s^2\chi\partial_a\chi\partial^a\phi)$$
$$+2g^2\frac{s^2\chi^2X^4}{\Omega}(1+\Omega)^2\delta_{ij}$$
$$+\frac{1}{4}\Omega\big[s^2(\chi^2-X^{-4})-\Omega(\frac{\chi^2s^4}{\Omega^2}-X^{-4})\big]F^i_{ab}F^{jab}$$
$$+\frac{1}{4}\Omega^2\delta_{ij}\big[(\frac{\chi^2s^4}{\Omega^2}-x^{-4})F^k_{ab}F^{kab}+\frac{\chi X^{-2}s^2}{\Omega}\varepsilon^{abcd}F^k_{ab}F^k_{cd}\big]\Big\}, \tag{3.2.28}$$
$$\hat{F}^2_{\tilde{i}\tilde{j}} = 3!\Delta^{-\frac{8}{3}}\Big\{\frac{s^2}{\tilde{\Omega}}\delta_{ij}[\chi^2X^4(2s^2x^{-2}+c^2)^2\partial_a\phi\partial^a\phi+(\Omega-s^2\chi^2X^4)^2\partial_a\chi\partial^a\chi$$
$$+2\chi X^2(2s^2x^{-2}+c^2)(\Omega-s^2\chi^2X^4)\partial_a\chi\partial^a\phi]$$
$$+2g^2\frac{c^2\chi^2X^4}{\tilde{\Omega}}(1+\tilde{\Omega})^2\delta_{ij}$$
$$+\frac{1}{4}\tilde{\Omega}\tilde{X}^{-4}\big[c^2(\chi^2x^4-1)-\tilde{\Omega}(\frac{c^4\chi^2X^4}{\tilde{\Omega}^2}-1)\big]\tilde{F}^i_{ab}\tilde{F}^{jab}$$
$$+\frac{1}{4}\tilde{\Omega}^2\tilde{X}^{-4}\delta_{ij}\big[(\frac{c^4\chi^2x^4}{\tilde{\Omega}^2}-1)\tilde{F}^k_{ab}\tilde{F}^{kab}-\frac{c^2\chi X^2}{\tilde{\Omega}}\varepsilon^{abcd}\tilde{F}^k_{ab}\tilde{F}^k_{cd}\big]\Big\}, \tag{3.2.29}$$
$$\hat{F}^2_{i\tilde{j}} =\frac{3}{2}sc\Delta^{-\frac{5}{3}}F^i_{ab}\tilde{F}^{jab}. \tag{3.2.30}$$

At this point, all off-diagonal-block components of the Einstein's field equation can be substituted. By using the fact that all off-diagonal-block components of the eleven-dimensional Minkowski metric tensor are zeros (D.6),

$$\hat{\eta}_{0a}=\hat{\eta}_{0i}=\hat{\eta}_{0\tilde{i}}=\hat{\eta}_{ai}=\hat{\eta}_{a\tilde{i}}=\hat{\eta}_{i\tilde{j}}=0, \tag{3.2.31}$$

the second term on the right-hand side of the eleven-dimensional Einstein's field equation vanishes for off-diagonal-block components. While most of substitutions with these contractions of the 4-form ansatz together with the Ricci tensor's components from (D.9) into off-diagonal-block components of the eleven-dimensional Einstein's equations lead to zero, there are two components, i.e. (a,i) and $(a,\tilde{i})$, in which their substitutions give rise to the components of the four-dimensional $SU(2)$ Yang-Mills equations (3.2.5) and (3.2.6) respectively.

In order to obtain the substitutions for the diagonal-block components of the eleven-dimensional Einstein's equation, the full contraction of the 4-form field strength ansatz is needed. Since our calculations are performed in the eleven-dimensional flat spacetime, the full contraction $\hat{F}^2_{(4)}$ can be comfortably obtained from above contractions through

$$\begin{aligned}\hat{F}^2_{(4)} &= \hat{\eta}^{\hat{M}\hat{N}}\hat{F}^2_{\hat{M}\hat{N}},\\ &= \eta^{ab}\hat{F}^2_{ab} + \hat{F}^2_{00} + \delta^{ij}\hat{F}^2_{ij} + \delta^{\tilde{i}\tilde{j}}\hat{F}^2_{\tilde{i}\tilde{j}}.\end{aligned} \tag{3.2.32}$$

Therefore the full contraction of the 4-form field strength components is then given by

$$\begin{aligned}\hat{F}^2_{(4)} = \; 4!\Delta^{-\frac{8}{3}}\Big\{ & s^2\partial_a\phi\partial^a\phi\big[\frac{s^2c^2\chi^2x^4}{\Omega} + \frac{\chi^2(2s^2+c^2x^2)^2}{\tilde{\Omega}} - c^2\big]\\ & + \partial_a\chi\partial^a\chi(\Omega - s^2\chi^2X^4)\big[c^2x^4 + \frac{s^2}{\tilde{\Omega}}(\Omega - s^2\chi^2X^4)\big]\\ & + 2s^2\chi X^2\partial_a\chi\partial^a\phi\big[2c^2x^2 + \frac{1}{\tilde{\Omega}}(\Omega - s^2\chi^2X^4)(2s^2X^{-2}+c^2)\big]\\ & - 2g^2\big[(\Omega+\tilde{\Omega}+1)^2 - \chi^2X^4\big(\frac{s^2}{\Omega}(1+\Omega)^2 + \frac{c^2}{\tilde{\Omega}}(1+\tilde{\Omega})^2\big)\big]\\ & + \frac{1}{4}\big[s^2\Omega(\chi^2 - X^{-4}) + s^4\chi^2 - \Omega^2X^{-4}\big]F^i_{ab}F^{iab}\\ & + \frac{\tilde{X}^{-4}}{4}\big[c^2\tilde{\Omega}(\chi^2x^4 - 1) + c^4\chi^2x^4 - \tilde{\Omega}^2\big]\tilde{F}^i_{ab}\tilde{F}^{iab}\\ & + \varepsilon^{abcd}\big[\frac{s^2\chi X^{-2}\Omega}{2}F^i_{ab}F^i_{cd} - \frac{c^2\chi X^2\tilde{X}^{-4}\tilde{\Omega}}{2}\tilde{F}^i_{ab}\tilde{F}^i_{cd}\big]\Big\}.\end{aligned} \tag{3.2.33}$$

Substitutions of these components in the diagonal-block components of the eleven-dimensional Einstein's field equation give some particular combinations of the two scalar fields' equations of motion, (3.2.10) and (3.2.11), in the same way as the previous substitution in the 4-form's equation of motion. However, the (a,b) component gives the last four-dimensional equation of motion that never been obtained from the previous substitutions, the four-dimensional Einstein's field equation in (2.4.37) described in flat spacetime indices by

$$\begin{aligned}R_{ab} = &\frac{1}{2}\partial_a\phi\,\partial_b\phi + \frac{1}{2}X^4\partial_a\chi\,\partial_b\chi + \frac{1}{2}\eta_{ab}V + \frac{1}{2}X^{-2}(F^i_{ac}F^{ic}_b - \frac{1}{4}\eta_{ab}(F^i_{(2)})^2)\\ &+ \frac{1}{2}\tilde{X}^{-2}(\tilde{F}^i_{ac}\tilde{F}^{ic}_b - \frac{1}{4}\eta_{ab}(\tilde{F}^i_{(2)})^2).\end{aligned} \tag{3.2.34}$$

Hence, substitution of the reduction ansatze for the both metric (3.1.1) and 4-form (3.1.7) in the last eleven-dimensional equations of motion leads to all equations of motion in the four-dimensional $SO(4)$ gauged supergravity.

In conclusion, substitutions of the reduction ansatze (3.1.1) and (3.1.7) into the equations of motion of the eleven-dimensional supergravity (2.4.15)-(2.4.17) yield all four-dimensional equations of motion for N=4, SO(4) gauged supergravity (2.4.31)-(2.4.37). Consequently, this dimensional reduction is said to be consistent only at the level of equations of motion. Note that the consistency at the level of equations of motion for this reduction is satisfactory for studying embedding of the four-dimensional solutions in eleven dimensions. Since solutions must satisfy equations of motion in each theory, solutions of one theory are also the solutions of the other one through these reduction ansatze. In the next chapter, some interesting solutions in the lower dimensional $SO(4)$ gauged supergravity will be examined together with their embedding in eleven dimensions obtained by substitutions of the four-dimensional solutions into the reduction ansatze.

CHAPTER IV
APPLICATIONS

The most interesting and very useful application of consistent dimensional reductions that we want to present in this chapter is the embedding of lower dimensional solutions in higher dimensions. As seen in (3.1.1) and (3.1.7), the eleven-dimensional bosonic fields are expressed in terms of four-dimensional ones in the $SO(4)$ gauged supergravity. Embedding is a procedure of lifting lower dimensional bosonic solutions to the higher dimensional theory through expressions from dimensional reduction ansatze. Note that things are very simpler and clearer in eleven dimensions so embedding solutions of the four-dimensional $N = 4$ $SO(4)$ gauged supergravity into eleven-dimensional fundamental framework give a very beneficial way to learn about the four-dimensional solutions.

In this chapter, two static solutions in four-dimensional $N = 4$ $SO(4)$ gauged supergravity are established together with their supersymmetries. After that, the embedding of these solutions will be given. Finally, the way to embed the maximal $N = 8$ solutions using the $N = 4$ consistent reduction ansatze when all $SU(2)$ Yang-Mills gauge fields vanish will be given.

4.1 The Simplest Static Four-dimensional Solutions

We will begin with the simplest static vacuum solution containing nothing in four-dimensional spacetime. By setting all matter fields to zero, the four-dimensional Lagrangian density of the $SO(4)$ gauged supergravity given in (2.4.26) becomes*

$$\mathcal{L}_4^{\text{vac}} = \sqrt{|g|}\ (R + 12\,\alpha^2). \tag{4.1.1}$$

It can also be checked that setting the gauge fields and scalars to zero satisfies their field equations. Note that, in this chapter, Lagrangian densities are conveniently considered in component form since our example solutions are some particular truncated theories of the $SO(4)$ gauged supergravity in which most of the matter fields vanish. The only one equation of motion from this Lagrangian density is just the vacuum Einstein's field equation containing a negative cosmological constant,

$$R_{\mu\nu} = -6\,\alpha^2 g_{\mu\nu}. \tag{4.1.2}$$

It is well known that the Einstein's field equation containing a negative cosmological constant Λ in any D dimensions can be written as

$$R_{\mu\nu} = \frac{\Lambda}{(D-2)} g_{\mu\nu}, \tag{4.1.3}$$

where $\Lambda \equiv -(D-1)(D-2)/L^2$ and L is a constant. The solution for metric describes AdS_D spacetime [35, 36] of the form

$$ds^2_{AdS_D} = \frac{L^2}{r^2}(dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu). \tag{4.1.4}$$

*To avoid confusion between the gauge coupling constant g and the determinant of the metric tensor $g = \det g_{\mu\nu}$, the gauge coupling constant will be denoted by α in this chapter.

Here $\mu, \nu = 0, 1, ..., D-1$, $x^0 = t$, and $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, ..., +1)$. It is easily to see that $L^2 = 1/2\alpha^2$ for (4.1.2). Therefore, the vacuum solution for the above Vacuum Einstein's field equation (4.1.2) is a four-dimensional metric describing AdS_4 spacetime given by

$$ds^2_{AdS_4} = \frac{1}{2\alpha^2 r^2}(-dt^2 + dr^2 + dx^2 + dy^2). \tag{4.1.5}$$

The second example of this application is a more complicated static solution, the Einstein-Yang-Mills theory. For the standard $SO(4)$ gauged supergravity in four dimensions, setting the two scalar fields to zero and $A^i_\mu = \tilde{A}^i_\mu$ turns the Lagrangian density in (2.4.26) into

$$\mathcal{L}_4^{\mathrm{EYM}} = \sqrt{|g|}\,(R + 12\,\alpha^2 - \frac{1}{2}(F^i)^2). \tag{4.1.6}$$

where the $SU(2)$ Yang-Mills field strengths are defined by $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \alpha\varepsilon_{ijk}A^j_\mu A^k_\nu$, which is the component of the 2-form (2.3.29), and $(F^i)^2 = F^i_{\mu\nu}F^{i\mu\nu}$. Note here that this truncated theory is still consistent after truncating the scalar fields, unlike the dimensionally reduced theory obtained from the simplest Kaluza-Klein reduction in Section 2.2 that the vanishing of the dilaton makes the $U(1)$ field strength equals to zero. This Einstein-Yang-Mills Lagrangian density leads to the following two equations of motion. The Einstein's field equation is given from the variation of (4.1.6) with respect to the inverse metric $g^{\mu\nu}$ by

$$R_{\mu\nu} = -6\,\alpha^2 g_{\mu\nu} + \big(F^i_{\mu\rho}F^{i\rho}_\nu - \frac{1}{4}g_{\mu\nu}(F^i)^2\big), \tag{4.1.7}$$

that is the Einstein's field equation containing a negative cosmological constant together with the energy-momentum tensor of the $SU(2)$ Yang-Mills gauge fields. For the $SU(2)$ gauge fields, the variation of the Einstein-Yang-Mills Lagrangian density with respect to these vector fields leads to the component form of the source-free Yang-Mills equation in (2.4.33),

$$D_\mu F^{i\mu\nu} = 0, \tag{4.1.8}$$

where D_μ is the $SU(2)$-gauged covariant derivative defined in the first equation of (3.2.2). To obtain the solution that preserve some fraction of supersymmetry, the four-dimensional metric ansatz is taken to be a product space between two-dimensional anti-de Sitter spacetime and a two-dimensional hyperbolic space, $AdS_2 \times H_2$, given in [29] by

$$ds^2_{AdS_2\times H_2} = \mathrm{e}^{2f(r)}(-dt^2 + dr^2) + \frac{\mathrm{e}^{2h(r)}}{y^2}(dx^2 + dy^2), \tag{4.1.9}$$

where r is the radial coordinate of AdS_2 and $f(r)$, $h(r)$ are functions that will be determined later. All non-zero components of the spin connections for this metric ansatz can be computed as follow

$$\begin{aligned}
\omega_{t\hat{t}\hat{r}} &= -f', \\
\omega_{x\hat{r}\hat{x}} = \omega_{y\hat{r}\hat{y}} &= -\mathrm{e}^{h-f}\frac{h'}{y}, \\
\omega_{x\hat{x}\hat{y}} &= -\frac{1}{y},
\end{aligned} \tag{4.1.10}$$

where the hat-indices refer to flat spacetime indices and the prime denotes derivatives with respect to the AdS_2's radial coordinate r. From these spin connection's components, all components of the Ricci tensor can be computed through (2.1.85) by

$$\begin{aligned}
R_{tt} &= f'' + 2h'f', \\
R_{rr} &= -f'' - 2(h')^2 - 2h'' + 2h'f', \\
R_{xx} = R_{yy} &= -\frac{1}{y^2}\Big[\mathrm{e}^{2(h-f)}(h'' + 2(h')^2) + 1\Big].
\end{aligned} \tag{4.1.11}$$

The gauge fields' solution given in [29] in order to preserve supersymmetry for the Yang-Mills equation (4.1.8) is chosen to be

$$A^3_x = \frac{k}{y}, \tag{4.1.12}$$

where k is a constant. Therefore, the only one non-zero component of the $SU(2)$ Yang-Mills field strength is

$$F^3_{xy} = \frac{k}{y^2}. \tag{4.1.13}$$

Here, x and y are the two spatial coordinates describing two-dimensional hyperbolic space H_2. Now, we consider a fixed point solution in which $h(r)$ becomes constant. Therefore, the remaining non-zero spin connection components are

$$\begin{aligned}
\omega_{t\hat{t}\hat{r}} &= -f' = \frac{1}{r}, \\
\omega_{x\hat{x}\hat{y}} &= -\frac{1}{y},
\end{aligned} \tag{4.1.14}$$

where the last term in the first line is obtained by taking the AdS_2 factor to the same form in (4.1.4) as $\mathrm{e}^{2f(r)} \approx L^2/r^2$. The Ricci tensor's components in (4.1.11) reduce to

$$\begin{aligned}
R_{tt} &= f'' = \frac{1}{r^2}, \\
R_{rr} &= -f'' = -\frac{1}{r^2}, \\
R_{xx} = R_{yy} &= -\frac{1}{y^2}.
\end{aligned} \tag{4.1.15}$$

Then substitution of all non-zero components of the Ricci tensor together with the field strengths' solution (4.1.13) into the Einstein's field equations (4.1.7) leads to the near horizon solution, (4.1.13) and (4.1.9) where the two exponential functions are given by

$$\begin{aligned} \mathrm{e}^{2f(r)} &= \frac{2}{(F^2+12\alpha^2)r^2}, \\ \mathrm{e}^{2h} &= \frac{(2k^2+y^2)y^2}{k^2+6\alpha^2y^4}, \end{aligned} \tag{4.1.16}$$

where $F^2 = 2\alpha^2/y^4$. Note that the AdS_4 and the near horizon $AdS_2 \times H_2$ solutions are supersymmetric and it is very useful to discuss their supersymmetries.

Many bosonic solutions in which all fermionic fields vanish can be interpreted as supersymmetric backgrounds whose fluctuation can be treated quantum mechanically [35]. These supersymmetric background solutions preserve some supersymmetry such that their variations due to local supersymmetry transformations vanish when the solution is substituted,

$$\delta(\epsilon)\ \text{boson} = \bar{\epsilon}\ \text{fermion} = 0, \qquad \delta(\epsilon)\ \text{fermion} = \epsilon\ \text{boson} = 0. \tag{4.1.17}$$

Since all fermionic fields are absent in supersymmetric backgrounds, the left-hand side of the first equation obviously satisfy. The remaining variation is called the supersymmetry condition giving an explicit form of the independent infinitesimal supersymmetry spinor $\epsilon(x)$ called the Killing spinor. A supersymmetric solution is called maximally supersymmetric if and only if the Killing spinor $\epsilon(x)$ can be expressed for the maximal number of Q supercharges, which are local real components of the Killing spinor, for example, $Q = 4$ for the simplest $N = 1$ in four dimensions. However, there can be some projections for the Killing spinor $\epsilon(x)$ given from the supersymmetry condition when evaluated on background solutions. These projections reduce the maximal Q local real components of $\epsilon(x)$ to Q_0 such that the solution is said to preserve a fraction Q_0/Q of the original supersymmetry. The solution with Q_0/Q unbroken supersymmetry is called a Q_0/Q Bogomol'nyi, Prasad and Sommerfield (BPS) solution [46, 47] that is invariant under a subalgebra of the supersymmetry algebra of the action. Some interesting examples of BPS solutions include multiple charged and multi-center black holes and intersecting D-branes.

For our four-dimensional $SO(4)$ gauged supergravity, supersymmetry conditions can be determined from the ungauged local supersymmetry transformation in (2.4.21) and the two extra terms in (2.4.30). Afer setting all fermionic fields and the two scalar fields to zero, the remaining local supersymmetry conditions are given by

$$\delta\bar{\lambda}^\alpha = -\frac{1}{4\sqrt{2}}\varepsilon^{\alpha\beta\gamma\delta}\bar{\epsilon}^\beta\gamma^{\mu\nu}F^{\gamma\delta}_{\mu\nu} = 0, \tag{4.1.18}$$

$$\delta\bar{\Psi}^\alpha_\mu = \bar{\epsilon}^\alpha\overleftarrow{\mathcal{D}}_\mu - \frac{1}{4}\bar{\epsilon}^\beta\gamma_\mu\gamma^{\nu\rho}F^{\alpha\beta}_{\nu\rho} + \frac{i}{\sqrt{2}}\alpha\bar{\epsilon}^\alpha\gamma_\mu = 0, \tag{4.1.19}$$

where the supercovariant derivative is $\bar{\epsilon}^\alpha\overleftarrow{\mathcal{D}}_\mu = \partial_\mu\bar{\epsilon}^\alpha + \frac{1}{4}\omega_{\mu\hat{\nu}\hat{\rho}}\bar{\epsilon}^\alpha\gamma^{\hat{\nu}\hat{\rho}} + 2\alpha\bar{\epsilon}^\beta A^{\alpha\beta}_\mu$. The above Yang-Mills field strength is now gauged under $SO(4)$ gauge group defined in [39] by

$$F^{\alpha\beta}_{\mu\nu} = \nabla_\mu A^{\alpha\beta}_\nu - \nabla_\nu A^{\alpha\beta}_\mu + 8g\mathbf{a}^i_{\alpha\beta}\varepsilon_{ijk}A^j_\mu A^k_\nu + 8g\mathbf{b}^i_{\alpha\beta}\varepsilon_{ijk}\tilde{A}^j_\mu\tilde{A}^k_\nu. \tag{4.1.20}$$

Here $\mathbf{a}^i$ and $\mathbf{b}^i$ are the six anti-symmetric 4×4 matrices expressed in (2.4.23) generating $SO(4)\sim SU(2)\times SU(2)$ algebra.

In vacuum AdS_4 background, the $SU(2)$ Yang-Mills gauge fields also vanish such that there is only one supersymmetry condition given from (4.1.19) by

$$\partial_\mu\bar{\epsilon}^\alpha + \frac{1}{4}\omega_{\mu ab}\bar{\epsilon}^\alpha\gamma^{ab} + \frac{i}{\sqrt{2}}\alpha\bar{\epsilon}^\alpha\gamma_\mu = 0, \tag{4.1.21}$$

or

$$\partial_\mu\epsilon^\alpha + \frac{1}{4}\omega_{\mu ab}\epsilon^\alpha\gamma^{ab} + \frac{i}{\sqrt{2}}\alpha\epsilon^\alpha\gamma_\mu = 0. \tag{4.1.22}$$

By a transformation of the radial coordinates $r = \frac{1}{\sqrt{2}\alpha}\mathrm{e}^{-\sqrt{2}\alpha z}$, the AdS_4 metric solution (4.1.5) turns into

$$ds_4^2 = \mathrm{e}^{2\sqrt{2}\alpha z}\eta_{\tilde{\mu}\tilde{\nu}}dx^{\tilde{\mu}}dx^{\tilde{\nu}} + dz^2, \tag{4.1.23}$$

where $\tilde{\mu},\tilde{\nu} = 0,1,2$ are the indices of the transverse coordinates t, x, and y respectively while the three-dimensional Minkowski metric is defined by $\eta_{\tilde{\mu}\tilde{\nu}} = \mathrm{diag}(-1,+1,+1)$. Thus, the spin connection 1-forms are

$$\omega^{\tilde{\mu}z} = \sqrt{2}\alpha dx^{\tilde{\mu}}, \qquad \omega^{\tilde{\mu}\tilde{\nu}} = 0. \tag{4.1.24}$$

After substituting the above spin connection's components, the supersymmetry condition (4.1.22) now splits into radial and transverse components,

$$\begin{aligned} \partial_z\epsilon^\alpha + \frac{i}{\sqrt{2}}\alpha\gamma_z\epsilon^\alpha &= 0, \\ \partial_{\tilde{\mu}}\epsilon^\alpha + \frac{\alpha}{\sqrt{2}}\gamma_{\tilde{\mu}}(\gamma_z + i)\epsilon^\alpha &= 0. \end{aligned} \tag{4.1.25}$$

Therefore the Killing spinors solution of these equations is given in [48] by

$$\epsilon^\alpha = \mathrm{e}^{(-i/\sqrt{2})\alpha z\gamma_z}\Big(1 + \frac{i}{\sqrt{2}}\alpha x^{\tilde{\mu}}\gamma_{\hat{\mu}}(1 - i\gamma_z)\Big)\epsilon^\alpha_0, \tag{4.1.26}$$

where $\gamma_{\hat{\mu}}$ are the constant Dirac gamma matrices in three-dimensional spacetime. Here, ϵ^α_0 are four independent real constant spinors containing 4 real-components for each one. Hence, the $N = 4$ AdS_4 solution is said to be maximally supersymmetric containing 16 supercharges.

For the $AdS_2 \times H_2$ near horizon solution, the only one component of the $SU(2)$ gauge field solution (4.1.12) can be written in the $\alpha\beta$ indices through the linear combinations in (2.4.22) of the form

$$A_x^{23} = -A_x^{32} = -\frac{k}{y}. \tag{4.1.27}$$

Thus, the Yang-Mills field strength in (4.1.20) can be easily obtained

$$F_{xy}^{23} = F_{xy}^{32} = -\frac{k}{y^2}. \tag{4.1.28}$$

Then, substitution of this Yang-Mills solution in the first supersymmetry condition (4.1.18) clearly implies $\epsilon^1 = \epsilon^4 = 0$. By using the non-zero components of the spin connections in (4.1.14) together with the above Yang-Mills gauge field and field strength, the remaining Killing spinors can be obtained from the other supersymmetry condition (4.1.19) in component forms,

$$\partial_t\bar{\epsilon}^2 + \bar{\epsilon}^2(\frac{1}{2r}\gamma^{\hat{t}}\gamma^{\hat{r}} - \frac{i\alpha}{\sqrt{2}}\mathrm{e}^f\gamma^{\hat{t}}) - \frac{k}{2}\mathrm{e}^{f-2h}\bar{\epsilon}^3\gamma^{\hat{t}}\gamma^{\hat{x}}\gamma^{\hat{y}} = 0, \tag{4.1.29}$$

$$\partial_r\bar{\epsilon}^2 + \frac{i\alpha}{\sqrt{2}}\mathrm{e}^f\bar{\epsilon}^2\gamma^{\hat{r}} + \frac{k}{2}\mathrm{e}^{f-2h}\bar{\epsilon}^3\gamma^{\hat{r}}\gamma^{\hat{x}}\gamma^{\hat{y}} = 0, \tag{4.1.30}$$

$$\partial_x\bar{\epsilon}^2 + \bar{\epsilon}^2(\frac{1}{2y}\gamma^{\hat{y}}\gamma^{\hat{x}} + \frac{i\alpha}{\sqrt{2}}\frac{\mathrm{e}^h}{y}\gamma^{\hat{x}}) + \bar{\epsilon}^3(\frac{2\alpha k}{y} + \frac{k}{2y\mathrm{e}^h}\gamma^{\hat{y}}) = 0, \tag{4.1.31}$$

$$\partial_y\bar{\epsilon}^2 + \frac{i\alpha}{\sqrt{2}}\frac{\mathrm{e}^h}{y}\bar{\epsilon}^2\gamma^{\hat{y}} - \frac{k}{2y\mathrm{e}^h}\bar{\epsilon}^3\gamma^{\hat{x}} = 0. \tag{4.1.32}$$

Assuming these killing spinors to depend only on the AdS_2 radial coordinate as in [49] by imposing $\partial_t\epsilon^i = \partial_x\epsilon^i = \partial_x\epsilon^i = 0$ where $i = 2, 3$, the first two equations can be solved by

$$\bar{\epsilon}^2(r) = \sqrt{r}\bar{\epsilon}_0, \tag{4.1.33}$$

where ϵ_0 is a real constant spinor containing 4 real-components. Afterwards, the relation between this expression and $\bar{\epsilon}^3$ can be found from (4.1.31) and (4.1.32) of the form

$$\bar{\epsilon}^2 = i\bar{\epsilon}^3, \tag{4.1.34}$$

after applying the twisting projection from [49]

$$\bar{\epsilon}^i\gamma^{\hat{x}}\gamma^{\hat{y}} = -i\bar{\epsilon}^i. \tag{4.1.35}$$

This projection reduces the 4 independent real components of ϵ_0 to 2. Therefore, the $AdS_2 \times H_2$ near horizon solution is said to preserve 1/8 supersymmetry.

4.2 Embedding Solutions in Eleven-dimensional Spacetime

As declared in the beginning of this chapter, eleven-dimensional solutions can be obtained from four-dimensional ones by using the reduction ansatze (3.1.1) and (3.1.7). the embedding of the two simplest solutions in Section 4.1 will be expressed in this section through appropriate ansatze.

By using the "unexcited" state of the metric ansatz in (3.1.5), the vacuum AdS_4 solution can be embedded in eleven-dimensional spacetime as

$$d\hat{s}_{11}^2 = \frac{1}{2\alpha^2 r^2}(-dt^2 + dr^2 + dx^2 + dy^2) + \frac{2}{\alpha^2}\Big[d\xi^2 + c^2\frac{1}{4}\sum_i(\sigma_i)^2 + s^2\frac{1}{4}\sum_i(\tilde{\sigma}_i)^2\Big], \tag{4.2.1}$$

which is an eleven-dimensional metric describing spacetime as a $AdS_4 \times S^7$ product space. Then the embedded solution of the eleven-dimensional 4-form field strength can be easily obtained from (3.1.7) by

$$\hat{F}_{(4)} = -3\sqrt{2}\,\alpha\epsilon_{(4)}, \tag{4.2.2}$$

which is the well known solution for the 4-form field strength giving rise to the product spacetime solution $AdS_4 \times S^7$ in (2.4.18).

For the $AdS_2 \times H_2$ near horizon metric solution (4.1.9), the metric ansatz (3.1.1) describes eleven-dimensional spacetime as a product space $AdS_2 \times H_2 \times \tilde{S}^7$,

$$\begin{aligned} d\hat{s}_{11}^2 =& \frac{2}{(F^2+12\alpha^2)r^2}(-dt^2 + dr^2) + \frac{e^{2h}}{y^2}(dx^2+dy^2) + \frac{2}{\alpha^2}d\xi^2 \\ &+ \frac{1}{2\alpha^2}\Big[c^2(\sigma_1^2 + \sigma_1^2 + [\sigma_3 - \frac{k}{y}dx]^2) + s^2(\tilde{\sigma}_1^2 + \tilde{\sigma}_1^2 + [\tilde{\sigma}_3 - \frac{k}{y}dx]^2)\Big], \end{aligned} \tag{4.2.3}$$

where e^{2h} is given in (4.1.16) and tilde refers to the squashing of S^7 containing two copies of $SU(2)$ gauge fields pointing in one of the three S^3 directions. Moreover, the eleven-dimensional 4-form field strength can be obtained via (3.1.7) by

$$\begin{aligned} \hat{F}_{(4)} =& -3\sqrt{2}\alpha\epsilon_{(4)} + \hat{F}'_{(4)}, \\ \sqrt{2}\alpha^2\hat{F}'_{(4)} =& \, s^2c^2 d\xi \wedge \Big(h^i \wedge F^i_{(2)} - \tilde{h}^i \wedge \tilde{F}^i_{(2)}\Big) + \\ &+ \frac{1}{4}\epsilon_{ijk}\Big(c^2\ h^i \wedge h^j \wedge *F^k_{(2)} + s^2\ \tilde{h}^i \wedge \tilde{h}^j \wedge *\tilde{F}^k_{(2)}\Big), \end{aligned} \tag{4.2.4}$$

where the 1-form gauge fields and the 2-form field strengths are given by

$$\begin{aligned} A^i_{(1)} &= \tilde{A}^i_{(1)} = \frac{k}{y}dx\delta^i_3, \\ F^i_{(2)} &= \tilde{F}^i_{(2)} = \frac{k}{y^2}dx \wedge dy\delta^i_3. \end{aligned} \tag{4.2.5}$$

The vacuum AdS_4 and the $AdS_2 \times H_2$ near horizon solutions are the simplest ones demonstrated in this chapter for giving some examples of the embedded solutions. Apart from these static solutions, our dimensional reduction ansatze, (3.1.1) and (3.1.7), can be used to embed more complicated solutions of the four-dimensional $SO(4)$ gauged supergravity. For example in [50], a time-dependent solution describing a decaying white hole that settles down to the final state as a static charged black hole has been embedded in eleven dimensions. The embedded solution describes decaying, rotating M2-branes, fundamental objects in eleven-dimensional supergravity.

Furthermore, not only solutions in the $N = 4$ $SO(4)$ gauged supergravity but in the absence of gauge fields also $N = 8$ four-dimensional solutions, such as dielectric flow and Janus solutions in [51] and [52] respectively, can be embedded in eleven-dimensional spacetime through approppriately truncated reduction ansatze. Since the isometry on a three-dimensional sphere S^3 is a Lie group $SO(4)$, setting all $SU(2)$ Yang-Mills gauged fields to zero turns our reduction ansatze into (C.23) and

$$\hat{F}_{(4)} = -g\sqrt{2}U\epsilon_{(4)} - \frac{4sc}{g\sqrt{2}}X^{-1} * dX \wedge d\xi + \frac{\sqrt{2}sc}{g}\chi X^4 * d\chi \wedge d\xi + \hat{F}'_{(4)}, \quad (4.2.6)$$

which is the 4-form ansatz in (3.1.7) without $\hat{F}''_{(4)}$ terms. These ansatze containing only metric and scalar fields have $SO(4) \times SO(4)$ symmetry, a subgroup of the $N = 8$ gauge group $SO(8)$, as shown in [51].

CHAPTER V
CONCLUSIONS AND DISCUSSIONS

The consistent dimensional reduction giving rise to four-dimensional $N = 4$ $SO(4)$ gauged supergravity from the unique supergravity in eleven dimensions has been achieved through the reduction ansatze, expressions of eleven-dimensional metric and 4-form field strength in terms of all bosonic fields in four-dimensional $N = 4$ $SO(4)$ gauged supergravity given in (3.1.1) and (3.1.7) respectively. Apart from the main dimensional reduction of interest, another related $N = 4$ gauged supergravity in four dimensions, the Freedman-Schwarz model, can also be obtained from these reduction ansatze when the one-way mapping between the two versions of $N = 4$ gauged supergravity is applied. However, this alternative dimensional reduction giving rise to another $N = 4$ gauged supergravity is not the main interest in this study because the Freedman-Schwarz model is not directly obtained from a dimensional reduction of eleven-dimensional supergravity but a particular type of ten-dimensional one obtained from a Kaluza-Klein reduction of eleven-dimensional supergravity on S^1.

As shown in Section 3.1 and also Appendix D, the metric ansatz (3.1.1) is conveniently described by symmetry of the $SO(4) \sim SU(2) \times SU(2)$ gauge group in which each term involving $SU(2)$ is obtained by the Scherk-Schwartz reduction that is guaranteed to be consistent. Unfortunately, the reduction ansatz for the eleven-dimensional anti-symmetric tensor is constructed on the 4-form field strength $\hat{F}_{(4)}$ by a trial and error process adding terms into the Abelian ansatz in [17] to obtain a consistent reduction without another reason. Moreover, the explicit form of the fundamental 3-form potential $\hat{A}_{(3)}$ is impossible to express from the 4-form ansatz (3.1.7), so the dimensional reduction is consistent only at the level of equations of motion where this consistency has been thoroughly verified in Section 3.2.

Nevertheless, the consistency at the level of the equations of motion of the dimensional reduction allows us to embed any bosonic solutions of the four-dimensional $N = 4$ $SO(4)$ gauged supergravity into a more fundamental theory, the supergravity in eleven dimensions. Some examples of embedded solutions have been demonstrated in Chapter 4. Embedding gives a new way to study four-dimensional solutions in eleven-dimensional theory that has a precise geometrical interpretation in terms of various M-brane configurations. Therefore, interesting solutions in four-dimensional $N = 4$ $SO(4)$ gauged supergravity, such as in the study of holographic renormalization group (RG) flows and holographic superconductors, can be embedded in the eleven-dimensional world via our reduction ansatz in the future.

For further research effort, these reduction ansatze may be further developed to embed four-dimensional solutions of the $N = 4$ gauged supergravity coupled to a number of vector multiplet. As mentioned at the end of Section 4.2, our reduction ansatze can be used to embed $N = 8$ solutions with $SO(4) \times SO(4)$ symmetry into eleven-dimensional spacetime in the absence of gauge fields since the dimensional reduction ansatze in this case already exhibit a symmetry $SO(4)\times SO(4)$ subgroup of $SO(8)$. It might be possible to construct dimensional reduction ansatze of the eleven-dimensional theory giving rise to $N = 8$ $SO(4) \times SO(4)$ gauged supergravity base on these reduction ansatze. In order to enlarge $SU(2)$ gauge groups on each S^3, the standard $SO(4)$ gauged supergravity needs to couple with some particular vector multiplets for a larger number of gauge fields.

Finally, the complete truncation of the $N = 8$ gauged supergravity has been recently given in [19]. Truncating this reduction to the half-maximal theory coupled to six vector multiplets would be interesting and may give some insights to the relation between the maximal and half-maximal gauged supergravities in four dimensions. This will eventually be useful for the studies of the AdS/CFT holography and string dualities.

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APPENDICES

APPENDIX A INTRODUCTION TO LIE GROUPS

In physics, the concept of Lie groups is significant to express continuous symmetries in which their generators of an infinitesimal symmetry transformation form a Lie algebra. Since Lie groups have played many important roles in this study, this appendix is provided to give some brief introductions to Lie groups including their identities, representations, and classifications. Furthermore, some examples of Lie groups involving to this study are discussed in the end of this appendix.

From group theory, group is an abstract mathematical concept defined as a set $\mathcal{G}$ of elements g that satisfy the following properties:

1. Binary operations between any two elements are also elements of $\mathcal{G}$,
$$g_i \otimes g_j \in \mathcal{G},\ \forall g_i, g_j \in \mathcal{G}. \tag{A.1}$$
2. Binary operations of group elements $g_i, g_j, g_k \in \mathcal{G}$ are associative,
$$g_i \otimes (g_j \otimes g_k) = (g_i \otimes g_j) \otimes g_k. \tag{A.2}$$
3. There exists an identity element, $e \in \mathcal{G}$ such that,
$$e \otimes g_i = g_i \otimes e = g_i, \forall g_i \in \mathcal{G}. \tag{A.3}$$
4. For every group elements g_i, there exists an inverse $g_i^{-1} \in \mathcal{G}$ such that,
$$g_i^{-1} \otimes g_i = g_i \otimes g_i^{-1} = e. \tag{A.4}$$

When $\mathcal{G}$ contains a finite number r of elements, it is called a finite group, and r is called the order of the group.

Lie group is a continuous group whose elements can be described by a finite number of parameters, $g = g(\lambda_1, ..., \lambda_n)$, where the integer n is called the dimension of a Lie group $\mathcal{G}$. Since Lie group is continuous, it can be viewed as a manifold called group manifold on which each point corresponds to a group element g_i. By the closure identity (A.1), binary operations of any g_i by g_j on left- or right-hand side provide translations on the group manifold to the new point $g_k = g_j \otimes g_i$ or $\tilde{g}_k = g_i \otimes g_j$ respectively. Note that, in general, g_k and $\tilde{g}_k$ are unidentical so there are two types of translations on group manifolds that are the generally given by non-commuting left- and right-handed transformations. In the same concept with general relativity, due to the fact that a manifold is locally flat, any group elements can be considered as expansions around the identity element e of the infinitesimal forms,

$$g(\lambda_1, ..., \lambda_n) \approx e - \partial_A g\, \delta\lambda_A + ..., \tag{A.5}$$

where $\partial_A = \partial/\partial\lambda_A$ and $A = 1, 2, ..., n$. By defining generators $T_A \equiv \partial_A g$ and parameters $\epsilon^A \equiv \delta\lambda_A$, group elements g in (A.5) can be written in exponential forms,

$$g = \mathrm{e}^{-\epsilon^A T_A}. \tag{A.6}$$

These generators T_A form basis vectors in the tangent space $T_e(\mathcal{G})$ at the identity element e of the Lie group $\mathcal{G}$ and form a Lie algebra $Lie(\mathcal{G})$, a vector space with a Lie bracket operation $[\cdot,\cdot] : Lie(\mathcal{G}) \otimes Lie(\mathcal{G}) \to Lie(\mathcal{G})$, which is

- bilinear,

$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z], \quad \text{for } \alpha, \beta \in \mathbb{R} \text{ and } X, Y, Z \in Lie(\mathcal{G}), \tag{A.7}$$

- antisymmetric,

$$[X, Y] = -[Y, X], \quad \text{for } X, Y \in Lie(\mathcal{G}), \tag{A.8}$$

- and consistent the Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad \text{for } X, Y, Z \in Lie(\mathcal{G}). \tag{A.9}$$

Lie bracket of the two basis vectors gives a linear combination of other generators with structure constants ${f_{AB}}^C$,

$$[T_A, T_B] = {f_{AB}}^C T_C, \tag{A.10}$$

where this equation is called the Lie algebra. Moreover any finite dimensional Lie algebra can be represented in terms of matrices through the homomorphic map D preserving their algebraic structures, i.e.

$$D([X, Y]) = [D(X), D(Y)], \quad \text{for any } X, Y \in Lie(\mathcal{G}), \tag{A.11}$$

such that the Lie algebra (A.10) is the same. There are some important representations introduced in the following.

1. The trivial or singlet representation mapping all elements to the (1×1) matrix 0,

$$D(X) = 0, \quad \text{for all } X \in Lie(\mathcal{G}). \tag{A.12}$$

   Therefore this trivial representation is one-dimensional.

2. The fundamental representation is the smallest irreducible finite-dimensional representation of Lie algebra $Lie(\mathcal{G})$ such that any finite-dimensional representations of Lie algebras can be constructed from this elementary representation.

3. The adjoint representation is another important representation maps the Lie algebra to the general linear group of the vector space $Lie(\mathcal{G})$,

$$D : Lie(\mathcal{G}) \to GL(Lie(\mathcal{G})), \tag{A.13}$$

Using (A.10), the adjoint representation is given in terms of its generators as $(\dim(Lie(\mathcal{G}))\times \dim(Lie(\mathcal{G})))$ matrices by

$$(T_A^{adj})^B{}_C = f_A{}^B{}_C. \tag{A.14}$$

A Lie group is called Abelian if all the structure constants vanish, such that all generators commute with each other. For non-Abelian groups, there are two interesting classes; semi–simple and simple Lie group. A semi-simple Lie group G_0 is a direct product of simple Lie groups G_i that are non-Abelian,

$$G_0 = G_1 \times G_2 \times ... \times G_k. \tag{A.15}$$

The simple Lie groups are completely classified in terms of four infinite series A_n, B_n, C_n, and D_n, where n is an integer called rank of Lie groups, and the exceptional cases G_2, F_4, E_6, E_7, and E_8, as show in Table A.1.

Cartan	Lie group	Name	Dimensions	Rank
A_n	$SU(n+1)$	Special Unitary	$(n+1)^2-1$	n
B_n	$SO(2n+1)$	Odd Special Orthogonal	$n(2n+1)$	n
C_n	$Sp(2n)$	Symplectic	$n(2n+1)$	n
D_n	$SO(2n)$	Even Special Orthogonal	$n(2n-1)$	n
G_2	G_2	Exceptional	14	2
F_4	F_4	Exceptional	52	4
E_6	E_6	Exceptional	78	6
E_7	E_7	Exceptional	133	7
E_8	E_8	Exceptional	248	8

Table A.1: Catan Classification of simple Lie groups.

For example, consider the main Lie group of this study, an $SU(2)$. Starting from the simple Lie group $SU(N)$, its group elements can be represented as $(N \times N)$ matrices g satisfy the following two conditions:

1. <u>Special</u> : $\det g = 1$.
   By (A.6), $\det g = \mathrm{e}^{\mathrm{Tr}\left(-\epsilon^A T_A\right)} = 1$, which implies

$$T_A \text{ are traceless.} \tag{A.16}$$

2. <u>Unitary</u> : $g^\dagger = g^{-1}$
   By (A.6), this condition implies the generators T_A to be anti-hermitian matrices,

$$(T_A)^\dagger = -T_A. \tag{A.17}$$

For a three-dimensional Lie group $SU(2)$, the three genereators T_A can be represented by (2×2) Pauli matrices σ_A satisfying (A.16) and (A.17) of the form

$$T_A = -\frac{i}{2}\sigma_A, \quad \text{where } A = 1, 2, 3. \tag{A.18}$$

The three Pauli matrices are tracless and hermitian defined by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{A.19}$$

that satisfy the commutaion relations,

$$[\sigma_A, \sigma_B] = 2i\varepsilon_{ABC}\sigma_C. \tag{A.20}$$

These commutation relations of the Pauli matrices turn the Lie algebra in (A.10) to be

$$[T_A, T_B] = \varepsilon_{ABC}T_C, \tag{A.21}$$

called the $SU(2)$ Lie algebra where the structure constants equal to the Levi-Civita symbols with the three upper and lowwer indices are identical. Substitution of the generators from (A.18) turns the $SU(2)$'s group elements to be unitary operators of the forms

$$\mathcal{U}(\vec{\theta}) = \exp(\frac{i}{2}\theta^A\sigma_A), \tag{A.22}$$

where $\vec{\theta} = \theta^A\hat{\theta^A}$ is a parametrized three-dimensional vector described by three continuous paremeters θ^A on the basis $\hat{\theta^A}$. Since these parameters are continuous, the group element can be expanded in the form of Tylaor's series,

$$\mathcal{U}(\vec{\theta}) = 1 + i(\frac{\theta}{2})\sigma_A\hat{\theta}^A - \frac{1}{2}(\frac{\theta}{2})^2 - i\frac{1}{6}(\frac{\theta}{2})^3\sigma_A\hat{\theta}^A + ..., \tag{A.23}$$

by using $\hat{\theta}^A\hat{\theta}^A = 1$, $\theta = \sqrt{\theta^A\theta^A}$, and $\sigma_A\sigma_B = i\varepsilon_{ABC}\sigma_C + \delta_{AB}\mathbf{1}$. This series can be divided into two series; odd and even orders of $\hat{\theta}^A$ and σ_A, then written as

$$\mathcal{U}(\vec{\theta}) = \cos\left(\frac{\theta}{2}\right) + i\sigma_A\hat{\theta}^A \sin\left(\frac{\theta}{2}\right). \tag{A.24}$$

Due to the imaginary i in the last term, these operators corespond to rotations in a two-dimensional complex space that leaves the quadratic form,

$$|z_1|^2 + |z_2|^2, \tag{A.25}$$

invariant. In general, $SU(2)$'s group elements can be written in the (2×2) unitary matrices,

$$\begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}, \tag{A.26}$$

satisfying the special conditions

$$|\alpha|^2 + |\beta|^2 = 1. \tag{A.27}$$

Obviously, these conditions are preserved under group transformation, multiplications by the operators $\mathcal{U}(\vec{\theta})$.The conditions above indicate that group elements can be interpreted as points on the surface of a unit three-dimensional sphere, S^3. Therefore, a group manifold of a Lie group $SU(2)$ is an S^3,

Another example is the neighbouring $SO(3)$ Lie group. Starting from the simple Lie groups $SO(N)$ that their elements can be represented as a $(N \times N)$ real orthogonal matrices M satisfying $M^T M = \mathbf{1}$, By (A.6), The $SO(N)$ generators T_A have to be traceless, due to the special condition (A.16), and anti-symmetric (3×3) matrices. Thus the $SO(3)$ generators can be written as

$$(T_A)_{BC} = -\,\varepsilon_{ABC}, \tag{A.28}$$

where ε_{ABC} are the three-dimensional totally antisymmetric Levi-Civita symbols, which are defined as $\varepsilon_{123} = 1$. These representations of the $SO(3)$ generators satisfy the same Lie algebra of $SU(2)$ in (A.21), therefore $SU(2)$ and $SO(3)$ are said to be isomorphic, in the sense that there exists a homomorphic map preserving Lie algebra between these two Lie group. However, $SO(3)$ group manifold is not S^3, $SU(2)$ group manifold. If the representations in (A.26) substitute in (A.6) the $SO(3)$ group transformations will be corresponding to rotations by an angle $\theta \in [-\pi, \pi]$ about some three-dimensional axis $\hat{\theta}^A$,

$$\mathcal{R}_{BC}(\vec{\theta}) = \delta_{BC}\cos(\theta) + \varepsilon_{BCA}\hat{\theta}_A \sin(\theta) + \hat{\theta}_B\hat{\theta}_C(1 - \cos(\theta)), \tag{A.29}$$

Hence a group manifold of a $SO(3)$ Lie group is the inside and the surface of a two-dimensional sphere S^2 with radius π,

$$(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = (\frac{\theta}{\pi})^2, \tag{A.30}$$

where ξ_i with $i = 1, 2, 3$ are the three-dimensional coordinates of any $SO(3)$ group element, with antipodal identification on the surface $|\theta| = \pi$ that means both $\theta = \pi$ and $\theta = -\pi$ refer to the same group element as seen in (A.29).

Moreover, there is another isomorphism of the Lie groups that will be used for gauging the $N = 4$ supergravity in four dimensions. A six-dimensional Lie group $SO(4)$ is isomorphic to a direct product of two $SU(2)$ Lie groups; $SO(4) \sim SU(2) \times SU(2)$, such that a theory will be gauged by $SO(4)$, if it is gauged under the two commuting $SU(2)$ Lie groups. In addition, by the isomorphism of $SU(2)$ and $SO(3)$, it is also $SO(4) \sim SU(2) \times SU(2) \sim SO(3) \times SO(3)$.

All the simple Lie groups, introduced above, are classified as the compact Lie groups since their group manifolds have finite volumes. Moreover, these Lie groups can be turned to non-compact Lie groups by replacing the unity matrix in their defining conditions by the matrix η defined by,

$$\eta = \begin{bmatrix} -\mathbf{1}_q & 0 \\ 0 & \mathbf{1}_p \end{bmatrix}, \tag{A.31}$$

where $\mathbf{1}_m$ is a $(m \times m)$ unity matrix. For example an $SO(4)$ can be turned to be non-compact $SO(3,1)$, the Lorentz group, by using the new orthogonal condition

$$M^T \eta M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{A.32}$$

which is the same Minkovski metric introduced in (2.1.8). Note that the Lie groups $SO(N)$ and $SO(p,q)$ with $p+q=N$ have different properties. While compact Lie groups can be represented by finite-dimensional unitary matrices, non-compact ones can not.

Appendix B Spinor Representation

Another irreducible representation of the Lorentz group called a spinor representation is introduced in this appendix to understand spinor quantities describing fermionic fields in supergravity multiplets and also infinitesimal spinor parameters in supersymmetry transformations. Starting from a brief introduction of the non-compact Lorentz group represented by the more familiar vector representation, then basic concepts about the spinor representation in general dimension are given. Finally, irreducible spinor representations in both four- and eleven-dimensional spacetime will be discussed.

As explained in Appendix A, D-dimensional Lorentz group $SO(D-1,1)$ is a non-compact Lie group corresponding to an infinite-volume group manifold. By introducing the Lorentz generators M^{ab}, which are $D \times D$ matrices, elements of the Lorentz group can be written in exponential form as

$$\Lambda(\omega) = \exp\left(-\frac{1}{2}\omega_{ab}M^{ab}\right), \tag{B.1}$$

where the real parameters ω_{ab} and also the Lorentz generators M^{ab} are anti-symmetric under interchanging of the two D-dimensional flat spacetime indices a and b. In vector representation, these group elements can be written in infinitesimal form by

$$\Lambda(\omega)^c{}_d = \delta^c{}_d - \frac{1}{2}\omega_{ab}(M^{ab})^c{}_d. \tag{B.2}$$

Here, any components of the Lorentz generators can be represented by

$$(M^{ab})^c{}_d = \eta^{bc}\delta^a_d - \eta^{ac}\delta^b_d, \tag{B.3}$$

where η^{ab} is a D-dimensional inverse Minkowski metric which is defined by $\eta^{ab} \equiv \mathrm{diag}(-1,1,1,...,1,1)$. Substitution of the Lie algebra in (A.10) by this vector representation of the Lorentz generators (B.3) yields the well known Lorentz algebra,

$$[M^{ab}, M^{cd}] = \eta^{bc}M^{ad} + \eta^{ad}M^{bc} - \eta^{ac}M^{bd} - \eta^{bd}M^{ac}. \tag{B.4}$$

Therefore the infinitesimal form of the group elements in (B.2) is given by

$$\Lambda(\omega)^c{}_d = \delta^c{}_d + \eta^{ca}\omega_{ad}. \tag{B.5}$$

The D-dimensional Minkowski metric η_{ab} transforms in the same way as the 4-diemensional one in (2.1.12). This representation therefore leaves the D-dimensional spacetime interval ds^2 invariant under the Lorentz transformation. Note that the generators M^{jk}, in which $j,k = 1,...,D-1$, correspond to the (D-1)-dimensional spatial rotation, an $SO(D-1)$ subgroup of the Lorentz group $SO(D-1,1)$, whereas M^{0j} are called the boost generators. Moreover, other tensor representations can be constructed from the Lorentz generators in (B.3).

Spinor representation is another class of irreducible representations of the Lorentz group $SO(D-1,1)$ constructed from the D-dimensional Clifford algebra defined by

$$\gamma^a\gamma^b + \gamma^b\gamma^a \equiv \{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}, \tag{B.6}$$

where γ^a are the Dirac gamma matrices labeled by a D-dimensional flat spacetime index. This anti-commutation relations imply that $(\gamma^0)^2 = -\mathbf{1}$ and $(\gamma^j)^2 = \mathbf{1}$ for any $j = 1, ..., D-1$. Thus the eigenvalues of each gamma matrix are $\pm i$ for γ^0 and ± 1 for γ^j such that γ^0 is anti-hermitian while the rest γ^j are hermitian,

$$(\gamma^0)^\dagger = -\gamma^0, \qquad (\gamma^j)^\dagger = \gamma^j. \tag{B.7}$$

Note that these Dirac gamma matrices transform under the Lorentz transformation in the same way as $(1,0)$ Lorentz tensors demonstrated in (2.1.5). The Lorentz generators M^{ab} can be represented via these Dirac gamma matrices in the form

$$M^{ab} = \frac{1}{4}[\gamma^a, \gamma^b], \tag{B.8}$$

that is called the spinor representation and also satisfying the same Lorentz algebra in (B.4).

A Dirac spinor $\Psi(x)$ is an elementary field, whose quantization corresponds to a fermionic particle, represented by a D-dimensional complex column matrix that transforms under the Lorentz transformation given by the generators in (B.8),

$$\Lambda : \Psi \to \Psi' = \exp\big(-\frac{1}{8}\omega_{ab}[\gamma^a, \gamma^b]\big)\Psi. \tag{B.9}$$

To obtain a Lagrangian density describing the Dirac spinor, its Lorentz invariant bilinear form is required. However, a multiplication between a Dirac spinor and its hermitian conjugation is not invariant under the Lorentz transformation,

$$\begin{aligned}
\Lambda : \Psi^\dagger\Psi \to \Psi'^\dagger\Psi' &= \Psi^\dagger \exp\big(\frac{1}{8}\omega_{ab}[\gamma^{a\dagger}, \gamma^{b\dagger}]\big)\exp\big(-\frac{1}{8}\omega_{ab}[\gamma^a, \gamma^b]\big)\Psi, \\
&= \Psi^\dagger\Big[\exp\big(-\frac{1}{8}\omega_{0i}[\gamma^0, \gamma^i] + \frac{1}{8}\omega_{ij}[\gamma^i, \gamma^j]\big) \\
&\qquad \times \exp\big(-\frac{1}{8}\omega_{0i}[\gamma^0, \gamma^i] - \frac{1}{8}\omega_{ij}[\gamma^i, \gamma^j]\big)\Big]\Psi, \\
&\neq \Psi^\dagger\Psi,
\end{aligned} \tag{B.10}$$

where the second line is obtained by the hermitian and anti-hermitian properties of γ^a from (B.7). Here, the obvious obstacle is the bracketed term that is not equal to the unity matrix $\mathbf{1}$ since the two exponential functions do not cancel each other. To obtain the Lorentz invariant bilinear form, the Dirac adjoint is defined by

$$\bar{\Psi} \equiv \Psi^\dagger i\gamma^0. \tag{B.11}$$

Whereupon using the Clifford algebra (B.6) together with an important property of the γ^0 i.e. $[\gamma^0, \gamma^i]\gamma^0 = -\gamma^0[\gamma^0, \gamma^i]$ and $[\gamma^i, \gamma^j]\gamma^0 = \gamma^0[\gamma^i, \gamma^j]$, a multiplication between a Dirac spinor and its Dirac adjoint is now invariant under Lorentz transformation,

$$\begin{aligned}\Lambda : \bar{\Psi}\Psi \rightarrow \bar{\Psi}'\Psi' &= \Psi^\dagger \exp\big(\frac{1}{8}\omega_{ab}[\gamma^{a\dagger},\gamma^{b\dagger}]\big) i\gamma^0 \exp\big(-\frac{1}{8}\omega_{ab}[\gamma^a,\gamma^b]\big)\Psi, \\ &= \Psi^\dagger i\gamma^0\Big[\exp\big(\frac{1}{8}\omega_{0i}[\gamma^0,\gamma^i]+\frac{1}{8}\omega_{ij}[\gamma^i,\gamma^j]\big) \\ &\qquad \times \exp\big(-\frac{1}{8}\omega_{0i}[\gamma^0,\gamma^i]-\frac{1}{8}\omega_{ij}[\gamma^i,\gamma^j]\big)\Big]\Psi, \\ &= \Psi^\dagger i\gamma^0\Psi, \\ &= \bar{\Psi}\Psi.\end{aligned} \tag{B.12}$$

Note that, in particular, Dirac spinor is reducible. Besides, there are two irreducible spinor representations called Weyl and Majorana spinors satisfying different projection conditions, some notable similarity transformations for the Dirac gamma matrices.

The first transformation is simply implied by the definition of Clifford algebra (B.6) in which γ^0 can be used to turn γ^a into their hermitian conjugations $\gamma^{a\dagger}$ through the following similarity transformation,

$$\gamma^0\gamma^a\gamma^0 = \gamma^{a\dagger}. \tag{B.13}$$

In even dimension, the special gamma matrix can be defined as a multiplication of all gamma matrices,

$$\gamma_* \equiv (-i)^{D/2-1}\gamma^0\gamma^1...\gamma^{D-1}, \tag{B.14}$$

which satisfies the following properties

$$\gamma_*^2 = \mathbf{1}, \quad \mathrm{Tr}(\gamma_*) = 0, \quad \{\gamma_*,\gamma^a\} = 0, \quad [\gamma_*, M^{ab}] = 0. \tag{B.15}$$

Its eigenvalues are ± 1 due to the first property in (B.15). This special gamma matrix is used to flip the sign of γ^a by

$$\gamma_*\gamma^a\gamma_* = -\gamma^a. \tag{B.16}$$

However, in odd dimensions, γ_* is not special anymore but behaves as one of the Dirac gamma matrices satisfying the Clifford algebra (B.6).

The last important similarity transformation is given by

$$\mathcal{C}\gamma^a\mathcal{C}^{-1} = -(\gamma^a)^{\mathrm{T}}, \tag{B.17}$$

where the matrix $\mathcal{C}$ is known as the charge conjugation matrix satisfying to the following properties

$$\mathcal{C}\mathcal{C}^\dagger = \mathbf{1}, \quad \mathcal{C} = -\mathcal{C}^{\mathrm{T}}. \tag{B.18}$$

By using this transformation together with (B.13), the relation between γ^a and their complex conjugations can be obtained of the form

$$(\gamma^a)^* = -(\gamma^0\mathcal{C})\gamma^a(\gamma^0\mathcal{C})^{-1}. \tag{B.19}$$

These similarity transformations are used to identify the two irreducible spinor representations:

- Weyl spinors
  Since the eigenvalues of γ_* are ± 1 and the γ_* is also traceless, these eigenvalues can be equally divided by choosing a basis that makes γ_* diagonal, for example the γ_5 in four-dimensional spacetime expressed in (B.24). Since γ_* commutes with the Lorentz generators M^{ab}, this particularly chosen basis describes the Lorentz generators as block diagonal matrices such that a Dirac spinor Ψ in even dimensions can be projected on the complex components of the two left- and right-handed Weyl spinors, ψ_L and ψ_R, defined by
  $$\Psi_L = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix} = \mathcal{P}_+ \Psi, \qquad \Psi_R = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix} = \mathcal{P}_- \Psi, \qquad \text{where } \mathcal{P}_\pm \equiv \frac{1}{2}(\mathbf{1} \pm \gamma_*). \tag{B.20}$$
  Therefore an even dimensional Dirac spinor is a reducible representation comprised by the two complex Weyl spinors, ψ_L and ψ_R, which are inequivalent and can transform to each other through complex conjugation.

- Majorana spinor
  By using the similarity transformation relating γ^a to their complex conjugation $(\gamma^a)^*$ in (B.19), the complex conjugation of the Lorentz generators is given by
  $$(\gamma^0 \mathcal{C}) M^{ab} (\gamma^0 \mathcal{C})^{-1} = -(M^{ab})^*, \tag{B.21}$$
  which leads to the reality condition of the Dirac spinor,
  $$\Psi^* = i\gamma^0 \mathcal{C} \Psi. \tag{B.22}$$
  This projection can be used to define the Majorana spinor that is real and satisfies the condition $(\gamma^0\mathcal{C})^* \gamma^0 \mathcal{C} = \mathbf{1}$.

In particular, these two irreducible spinor representations are not equivalent. While Weyl spinors exist only in even dimensions, Majorana spinor exists if and only if the condition $(\gamma^0\mathcal{C})^* \gamma^0 \mathcal{C} = \mathbf{1}$ is satisfied. Some possible types of irreducible spinors in D-dimensional flat spacetime together with their real dimensions are given in Table B.2.

There exist both Weyl and Majorana spinors in which their real dimensions are 4 in four-dimensional spactime, as shown in Table B.2. To define Weyl spinors, Dirac matrices can be expressed by (4×4) matrices, namely the Weyl representation, containing the two (2×2) matrices; $\sigma^a = (\mathbf{1}_{2\times 2}, \sigma_i)$ and $\bar{\sigma}^a = (-\mathbf{1}_{2\times 2}, \sigma_i)$, where σ_i are the usual Pauli matrices introduced in (A.19), of the form
$$\gamma^a = \begin{bmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{bmatrix}. \tag{B.23}$$

D	spinor's dimensions (real)	Weyl	Majorana
2	1	•	•
3	2		•
4	4	•	•
5	8		
6	8	•	
7	16		
8	16	•	•
9	16		•
10	16	•	•
11	32		•

Table B.2: Some types of spinor in D-dimensional Minkowkowski spacetime [34, 37].

Thus the special gamma matrix, which is now called γ_5, is given by

$$\gamma_* \equiv \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} \mathbf{1}_{2\times 2} & 0 \\ 0 & -\mathbf{1}_{2\times 2} \end{bmatrix}, \tag{B.24}$$

such that the Dirac and Weyl spinors can be related to each other as

$$\Psi = \begin{bmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{bmatrix}. \tag{B.25}$$

While ψ_α is the left-handed Weyl spinor containing an undotted spinor index, $\alpha = 1, 2$, the right-handed Weyl spinor is $\bar{\psi}^{\dot{\alpha}}$ carrying a dotted spinor index $\dot{\alpha} = 1, 2$. Note that these spinor indices are always omitted for convienent.

Besides, a Majorana spinor in four dimensional spacetime can be defined by setting the charge conjugation matrix to be

$$\mathcal{C} = i\gamma^0, \tag{B.26}$$

such that the condition $(\gamma^0\mathcal{C})^*\gamma^0\mathcal{C} = \mathbf{1}$ satisfies. The similarity transformation (B.19) is now given by

$$(\gamma^a)^* = \gamma^a, \tag{B.27}$$

which indicates that the gamma matrices are explicitly real. This representation is called the truely real representation in which all real gamma matrices are given by

$$\begin{aligned} \gamma^0 &= \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix}, \qquad & \gamma^1 &= \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix}, \\ \gamma^2 &= \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \qquad & \gamma^3 &= \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}. \end{aligned} \tag{B.28}$$

Now, γ^0 is anti-symmetric while γ^i are symmetric corresponding to the hermitian and anti-hermitian properties of γ^a from (B.7). Therefore the reality condition

(B.22) becomes

$$\Psi^* = \Psi, \tag{B.29}$$

that implies Ψ is a real spinor called the Majorana spinor consisting of four real components,

$$\Psi = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}, \qquad \alpha, \beta, \gamma, \delta \in \mathbf{R}. \tag{B.30}$$

In eleven dimensions, the real representations can be obtained in the same way as in four-dimensional spacetime. Starting from setting the charge conjugation matrix $\mathcal{C} = i\gamma^0$ such that the reality condition in (B.22) takes the same form as (B.29), $\Psi^* = \Psi$. Here, the Majorana spinor Ψ consisting of 32 real components is the minimal spinor representations in eleven-dimensional spacetime.

Appendix C Derivation of The Metric Reduction Ansatz

In this appendix, the way to reach the metric reduction ansatz in (3.1.1) is demonstrated step by step. The first step is the deduction of the two $SU(2)$ gauge fields in the absence of the axion by the truncation of the previous result in [14]. Then the axion scalar field will be activated through the truncation of the full S^7 reduction in [17]. Finally, these two steps are combined to obtain the full metric reduction ansatz in (3.1.1).

From [14], the general Kaluza-Klein reduction ansatz for dimensional reductions of the D-dimensional metric on a unit odd-dimensional sphere S^{2k-1} are expressed of the form

$$d\hat{s}_D^2 = \tilde{\Delta}^a ds_d^2 + \tilde{\Delta}^{-b} \sum_{i=1}^{k} X_i^{-1} \big(d\mu_i^2 + \mu_i^2 (d\phi_i + A_{(1)}^i)^2\big), \tag{C.1}$$

where μ_i are the directions of cosine satisfying the constraint $\Sigma_{i=1}^k \mu_i^2 = 1$, ϕ_i are the azimuthal rotation angles, $A_{(1)}^i$ are k commuting $U(1)$ gauge 1-forms. The scalar quantities X_i satisfy the following constraint $\tilde{\Delta} = \Sigma_{i=1}^k X_i \mu_i^2$, and $\Pi_{i=1}^k X_i = 1$.

By setting $D = 11$, $d = 4$, $a = 2/3$, $b = 1/3$, and $k = 4$ the axion-free $U(1)^4$ reduction ansatz of the eleven-dimensional metric on a seven-dimensional sphere S^7 with radius $\sqrt{2}/g$ is given by

$$d\hat{s}_{11}^2 = \tilde{\Delta}^{\frac{2}{3}} ds_4^2 + 2g^{-2} \tilde{\Delta}^{-\frac{1}{3}} \sum_{i=1}^{4} X_i^{-1} \big(d\mu_i^2 + \mu_i^2 (d\phi_i + gA_{(1)}^i)^2\big). \tag{C.2}$$

Here the scalar quantities X_i are parameterised by three dilaton scalar fields, which are described as a 3-vector $\vec{\phi}$, of the form $X_i = \exp\Big(-\frac{1}{2}\vec{\mathrm{a}}_i \cdot \vec{\phi}\Big)$ where $\vec{\mathrm{a}}_i$ are four constant 3-vectors. The truncation to $U(1)^2$ can be obtained by setting two dilatons to be zero. The X_i's constraint, $X_1 X_2 X_3 X_4 = 1$, now becomes

$$\mathrm{a}_1 + \mathrm{a}_2 + \mathrm{a}_3 + \mathrm{a}_4 = 0. \tag{C.3}$$

The four scalar constants are set to be $\mathrm{a}_1 = \mathrm{a}_2 = -1$, and $\mathrm{a}_3 = \mathrm{a}_4 = 1$ such that the scalar quantities X_i are set pairwise equal, $X_1 = X_2 \equiv X$ and $X_3 = X_4 \equiv 1/X$, where the scalar field X is defined in (2.4.28). Together with setting the four $U(1)$ gauge 1-forms as $A_{(1)}^1 = A_{(1)}^2 \equiv A_{(1)}$ and $A_{(1)}^3 = A_{(1)}^4 \equiv \tilde{A}_{(1)}$, the metric ansatz (C.2) now reduces to

$$\begin{aligned} d\hat{s}_{11}^2 =& \bar{\Delta}^{\frac{2}{3}} ds_2^2 + 2g^{-2} \bar{\Delta}^{\frac{2}{3}} d\xi^2 \\ &+ \frac{1}{2} g^{-2} \bar{\Delta}^{-\frac{1}{3}} X^{-1} c^2 \Big(d\theta^2 + \sin^2\theta d\varphi^2 + (d\psi + \cos\theta d\varphi - gA_{(1)})^2\Big) \\ &+ \frac{1}{2} g^{-2} \bar{\Delta}^{-\frac{1}{3}} X s^2 \Big(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\varphi}^2 + (d\tilde{\psi} + \cos\tilde{\theta} d\tilde{\varphi} - g\tilde{A}_{(1)})^2\Big). \end{aligned} \tag{C.4}$$

Here, $\bar{\Delta} \equiv \left[(c^2X^2+s^2)(s^2X^{-2}+c^2)\right]^{\frac{1}{2}}$ where $c \equiv \cos\xi$ and $s \equiv \sin\xi$, while the four directions of cosine are parameterised by

$$\mu_1 = c\,\cos\frac{\theta}{2}, \qquad \mu_2 = c\,\sin\frac{\theta}{2}, \qquad \mu_3 = s\,\cos\frac{\tilde{\theta}}{2}, \qquad \mu_4 = s\,\sin\frac{\tilde{\theta}}{2}, \tag{C.5}$$

and the four azimuthal rotation angles ϕ_i are given by

$$\phi_1 = \frac{1}{2}(\psi+\varphi), \qquad \phi_2 = \frac{1}{2}(\psi-\varphi), \qquad \phi_3 = \frac{1}{2}(\tilde{\psi}+\tilde{\varphi}), \qquad \phi_4 = \frac{1}{2}(\tilde{\psi}-\tilde{\varphi}). \tag{C.6}$$

If the unexcited state of the spacetime is considered by setting $\phi = 0$ and $A_{(1)} = \tilde{A}_{(1)} = 0$, the above metric ansatz will be turned into

$$\begin{aligned} d\hat{s}_{11}^2 = ds_4^2 + 2g^{-2}\Big\{ d\xi^2 + c^2\big[\frac{d\theta^2}{4} + \frac{d\varphi^2}{4} + \frac{d\psi^2}{4} + \frac{1}{2}\cos\theta d\psi d\varphi\big] \\ + s^2\big[\frac{d\tilde{\theta}^2}{4} + \frac{d\tilde{\varphi}^2}{4} + \frac{d\tilde{\psi}^2}{4} + \frac{1}{2}\cos\tilde{\theta} d\tilde{\psi} d\tilde{\varphi}\big]\Big\}. \end{aligned} \tag{C.7}$$

Here, the last two terms are the metrics on unit three-dimensional spheres S^3, $d\Omega_3^2$ and $d\tilde{\Omega}_3^2$, written in terms of the Euler angles (θ,φ,ψ) and ($\tilde{\theta}$,φ,ψ) respectively [44]. Hence, the unexcited state of the metric ansatz in (3.1.5) is obtained with the two S^3 metrics expressed by the two sets of left-invariant 1-froms, $\frac{1}{4}\sum_i(\sigma_i)^2 = d\Omega_3^2$ and $\frac{1}{4}\sum_i(\tilde{\sigma}_i)^2 = d\tilde{\Omega}_3^2$. Therefore, the axion-free $U(1)^2$ metric ansatz describes the geometry of eleven-dimensional spacetime as a product space between four-dimensional spacetime and a foliation of $S^3 \times S^3$ that contains a $U(1)$ gauge field in each S^3. Furthermore, by using the fact that S^3 is a group manifold of $SU(2)$, the $U(1)$ gauge field in each S^3 can be enlarged to $SU(2)$ by turning off the $U(1)$ gauge fields and replacing all left-invariant 1-froms by the two sets of $SU(2)$-valued forms defined in [25, 26] as

$$h^i \equiv \sigma_i - gA^i_{(1)}, \qquad \tilde{h}^i \equiv \tilde{\sigma}_i - g\tilde{A}^i_{(1)}, \tag{C.8}$$

where $A^i_{(1)}$ and $\tilde{A}^i_{(1)}$ are the two sets of the $SU(2)$ Yang-Mills potential 1-forms with $i = 1, 2, 3$. This enlargment turns the $U(1)^2$ metric ansatz (C.4) into the axion-free version of the metric reduction ansatz (3.1.1),

$$d\hat{s}_{11}^2 = \bar{\Delta}^{\frac{2}{3}} ds_4^2 + 2g^{-2}\bar{\Delta}^{\frac{2}{3}} d\xi^2 + \frac{1}{2}g^{-2}\bar{\Delta}^{\frac{2}{3}}\Big[\frac{c^2}{c^2X^2+s^2}\sum_{i=1}^{3}(h^i)^2 + \frac{s^2}{s^2X^{-2}+c^2}\sum_{i=1}^{3}(\tilde{h}^i)^2\Big]. \tag{C.9}$$

To obtain the missing axion scalar field, the dimensional reduction on full seven-dimensional sphere S^7 of eleven-dimensional supergravity giving rise to the maximal $N = 8$ $SO(8)$ gauged supergravity in [17] is considered. It is shown in [17] that the internal space S^7's metric reduction ansatz is given in term of the inverse metric tensor by

$$\hat{\Delta}^{-1}(x,y)g^{mn}(x,y) = \frac{1}{2}(K^{mIJ}K^{nKL} + K^{nIJ}K^{mKL})(u_{ij}{}^{IJ} + v_{ijIJ})(u^{ij}{}_{KL} + v^{ijKL}), \tag{C.10}$$

where

$$\hat{\Delta}^2(x,y) = \frac{\det(g_{mn}(x,y))}{\det(\bar{g}_{mn}(y))}. \tag{C.11}$$

Here $\bar{g}_{mn}(y)$ denotes the metric tensor of the undistorted S^7 where x and y are spacetime and internal space coordinates respectively, and $m,n = 1,2,...,7$ are internal space indices. Moreover, K^{mIJ} are 28 Killing vectors in this internal space metric, and the tensors $u_{ij}{}^{IJ}$ and v_{ijIJ} are determined in the definition of the scalar matrix $\mathcal{V}$ and its inverse,

$$\mathcal{V} = \begin{bmatrix} u_{ij}{}^{IJ} & v_{ijIJ} \\ v^{klIJ} & u^{kl}{}_{KL} \end{bmatrix}, \qquad \mathcal{V}^{-1} = \begin{bmatrix} u^{ij}{}_{IJ} & -v_{klIJ} \\ -v^{ijKL} & u_{kl}{}^{KL} \end{bmatrix}, \tag{C.12}$$

where $i,j = 1,2,...,8$ and $I,J = 1,2,...,8$.

In the $N = 4$ $SU(2)\times SU(2)$ gauged theory, the full $N = 8$ $SO(8)$ gauged one is truncated by splitting the indices i and I into $i = (a,\bar{a})$ and $I = (a,\bar{a})$ where $a = 1,2,3,4$ and $\bar{a} = 5,6,7,8$ such that these tensors are given by

$$\begin{aligned} u_{ab}{}^{cd} &= 2\cosh\frac{\lambda}{2}\delta^{cd}_{ab}, \qquad u_{\bar{a}\bar{b}}{}^{\bar{c}\bar{d}} = 2\cosh\frac{\lambda}{2}\delta^{\bar{c}\bar{d}}_{\bar{a}\bar{b}}, \qquad u_{a\bar{b}}{}^{c\bar{d}} = 2\delta^c_a\delta^{\bar{d}}_{\bar{b}}, \\ v_{abcd} &= \sinh\frac{\lambda}{2}\mathrm{e}^{i\sigma}\varepsilon_{abcd}, \qquad v_{\bar{a}\bar{b}\bar{c}\bar{d}} = \sinh\frac{\lambda}{2}\mathrm{e}^{-i\sigma}\varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}}, \end{aligned} \tag{C.13}$$

where the two scalar fields λ and σ are related to the dilation ϕ and the axion χ through

$$\begin{aligned} \cosh\lambda &= \cosh\phi + \frac{1}{2}\chi^2\mathrm{e}^{\phi}, \\ \cos\sigma\sinh\lambda &= \sinh\phi - \frac{1}{2}\chi^2\mathrm{e}^{\phi}, \\ \sin\sigma\sinh\lambda &= \chi\mathrm{e}^{\phi}. \end{aligned} \tag{C.14}$$

Substituting these expressions (C.13) into (C.10), the reduction ansatz for the inverse metric of the internal space in the reduced $N = 4$ gauged theory are given by

$$\begin{aligned} \hat{\Delta}^{-1}(x,y)g^{mn}(x,y) = &\sum_{ij} K^{mij}K^{nij} + \frac{1}{2}(X^2-1)\sum_{\alpha=1}^{3}\Big[(J^{\alpha}_{ab}K^{mab})^2 + (J^{\alpha}_{\bar{a}\bar{b}}K^{m\bar{a}\bar{b}})^2\Big] \\ &+ \frac{1}{2}(\tilde{X}^2-1)\sum_{\alpha=1}^{3}\Big[(\tilde{J}^{\alpha}_{ab}K^{mab})^2 + (\tilde{J}^{\alpha}_{\bar{a}\bar{b}}K^{m\bar{a}\bar{b}})^2\Big], \end{aligned} \tag{C.15}$$

where

$$\begin{aligned} J^1_{12} = J^1_{34} = J^2_{13} = -J^2_{24} = J^3_{14} = J^3_{23} = 1, \\ J^1_{56} = J^1_{78} = J^2_{57} = -J^2_{68} = J^3_{58} = J^3_{67} = 1, \\ \tilde{J}^1_{12} = -\tilde{J}^1_{34} = \tilde{J}^2_{13} = \tilde{J}^2_{24} = \tilde{J}^3_{14} = -\tilde{J}^3_{23} = 1, \\ \tilde{J}^1_{56} = -\tilde{J}^1_{78} = \tilde{J}^2_{57} = \tilde{J}^2_{68} = \tilde{J}^3_{58} = -\tilde{J}^3_{67} = 1. \end{aligned} \tag{C.16}$$

By using the fact that a sum of the squares of Killing vectors yields the bi-invariant inverse metric, it is obviously seen that the first summation term on the right-hand side of (C.15) just equals to $\bar{g}^{mn}(y)$, the inverse metric tensor of the undistorted S^7. The 3 Killing vector combinations $K^{m\alpha} \equiv J^{\alpha}_{ab}K^{mab}$ and $\bar{K}^{m\alpha} \equiv J^{\alpha}_{\bar{a}\bar{b}}K^{m\bar{a}\bar{b}}$ each close on $SU(2)$ and commute with each other such that these two sets are the left- and right-translation Killing vectors on the first S^3. Likewise, both commuting Killing vector combinations $\tilde{K}^{m\alpha} \equiv \tilde{J}^{\alpha}_{ab}K^{mab}$ and $\bar{\tilde{K}}^{m\alpha} \equiv \tilde{J}^{\alpha}_{\bar{a}\bar{b}}K^{m\bar{a}\bar{b}}$ are the left- and right-translation Killing vectors on the second S^3 since each also close on $SU(2)$. Moreover, the sum of the squares of each S^3 translation Killing vectors yields the bi-invariant inverse metric g^{mn}_3 on each S^3. Therefore, (C.15) becomes

$$\hat{\Delta}^{-1}(x,y)g^{mn}(x,y) = \bar{g}^{mn}(y) + (X^2-1)g^{mn}_3 + (\tilde{X}^2-1)\tilde{g}^{mn}_3, \tag{C.17}$$

which expresses the distortion of the round S^7 due to the existance of the scalar fields by the two addition terms on the right hand side together with the scaling factor $\hat{\Delta}$. However, the bi-invariant inverse metrics on each S^3 can be written in terms of the inverse metrics on the round unit three-dimensional spheres, $d\Omega^2_3 = g_{3,ij}dx^i dx^j$ and $d\tilde{\Omega}^2_3 = g_{3,\tilde{i}\tilde{j}}dx^{\tilde{i}}dx^{\tilde{j}}$, of the forms $g^{mn}_3 = g^{ij}_3\delta^m_i\delta^n_j$ and $\tilde{g}^{mn}_3 = g^{\tilde{i}\tilde{j}}_3\delta^m_{\tilde{i}}\delta^n_{\tilde{j}}$ where the internal space index is splited into $m = (\xi, i, \tilde{i})$ where $i, j = 1, 2, 3$ and also $\tilde{i}, \tilde{j} = 1, 2, 3$.

In order to find the expression of the metric ansatz for the distorted S^7 internal space, all components of the undistorted S^7 metric tensor, $\bar{g}_{mn}(y)$, are needed. From (3.1.5), it is obvious to see that all non-zero components of this metric tensor are

$$\begin{aligned}\bar{g}_{\xi\xi} &= 1,\\ \bar{g}_{ij} &= c^2 g_{3,ij},\\ \bar{g}_{\tilde{i}\tilde{j}} &= s^2 g_{3,\tilde{i}\tilde{j}},\end{aligned} \tag{C.18}$$

where their inversions are given by

$$\begin{aligned}\bar{g}^{\xi\xi} &= 1,\\ \bar{g}^{ij} &= \frac{1}{c^2}g^{ij}_3,\\ \bar{g}^{\tilde{i}\tilde{j}} &= \frac{1}{s^2}g^{\tilde{i}\tilde{j}}_3.\end{aligned} \tag{C.19}$$

Substituting these inverse metric's components into (C.17) gives all non-zero components of the metric tensor for the distorted S^7 internal space,

$$\begin{aligned}g^{\xi\xi}(x,y) &= \hat{\Delta},\\ g^{ij}(x,y) &= \hat{\Delta}\Big(\frac{c^2X^2+s^2}{c^2}\Big)g^{ij}_3,\\ g^{\tilde{i}\tilde{j}}(x,y) &= \hat{\Delta}\Big(\frac{s^2X^2+c^2}{s^2}\Big)g^{\tilde{i}\tilde{j}}_3,\end{aligned} \tag{C.20}$$

that are also easily inverted,

$$\begin{aligned} g_{\xi\xi}(x,y) &= \hat{\Delta}^{-1}, \\ g_{ij}(x,y) &= \hat{\Delta}^{-1}\frac{c^2}{c^2X^2+s^2}g_{3,ij}, \\ g_{\tilde{i}\tilde{j}}(x,y) &= \hat{\Delta}^{-1}\frac{s^2}{s^2X^2+c^2}g_{3,\tilde{i}\tilde{j}}. \end{aligned} \tag{C.21}$$

Thus the metric ansatz for the distorted S^7 internal space can be written as

$$ds_7^2 = g_{mn}(x,y)dx^m dx^n = \hat{\Delta}^{-1}\Big(d\xi^2 + \frac{c^2}{c^2X^2+s^2}d\Omega_3^2 + \frac{s^2}{s^2\tilde{X}^2+c^2}d\tilde{\Omega}_i^2\Big). \tag{C.22}$$

Replacing the round S^7 metric $d\Omega_7^2$ in (3.1.5) by this ds_7^2 yields the metric reduction ansatz giving rise to the $N=4$ $SO(4)$ gauged theory with vanished gauge fields of the form

$$ds_{11}^2 = \Delta^{\frac{2}{3}}ds_4^2 + 2g^{-2}\Delta^{\frac{2}{3}}\Big(d\xi^2 + \frac{c^2}{c^2X^2+s^2}\frac{1}{4}\sum_{i=1}^{3}\sigma_i^2 + \frac{s^2}{s^2\tilde{X}^2+c^2}\frac{1}{4}\sum_{i=1}^{3}\tilde{\sigma}_i^2\Big) \tag{C.23}$$

where $\Delta = \hat{\Delta}^{-3/2}$ [17]. Here the two round S^3 metrics are expressed by $\frac{1}{4}\sum_i(\sigma_i)^2 = d\Omega_3^2$ and $\frac{1}{4}\sum_i(\tilde{\sigma}_i)^2 = d\tilde{\Omega}_3^2$.

Finally, turning all $SU(2)$ gauge fields on via replacing all σ_i and $\tilde{\sigma}_i$ by the two sets of $SU(2)$-valued forms defined in (C.8) turns the metric ansatz (C.23) into the fully decorated metric ansatz in (3.1.1),

$$d\hat{s}_{11}^2 = \Delta^{\frac{2}{3}}ds_4^2 + 2g^{-2}\Delta^{\frac{2}{3}}d\xi^2 + \frac{1}{2}g^{-2}\Delta^{\frac{2}{3}}\big[\frac{c^2}{c^2X^2+s^2}\sum_i(h^i)^2 + \frac{s^2}{s^2\tilde{X}^2+c^2}\sum_i(\tilde{h}^i)^2\big]. \tag{C.24}$$

Appendix D Derivation of 11D Ricci Tensor

In order to check the consistency of the dimensional reduction, components of the eleven-dimensional Ricci tensor are needed for the substitution in Einstein's field equation in (2.4.15). In this appendix, all non-zero components of the Ricci tensor are thoroughly calculated from the eleven-dimensional metric ansatz (3.1.1) through the vielbein formalism demonstrated in Section 2.1.1.

For convenience, all coefficients in the reduction ansatz for the eleven-dimensional metric will be denoted by

$$\mathrm{e}^{\beta} \equiv \Delta^{\frac{1}{3}}, \quad \mathrm{e}^{\gamma} \equiv (\sqrt{2}g)^{-1} c \Delta^{\frac{1}{3}} \Omega^{-\frac{1}{2}}, \quad \mathrm{e}^{\tilde{\gamma}} \equiv (\sqrt{2}g)^{-1} s \Delta^{\frac{1}{3}} \tilde{\Omega}^{-\frac{1}{2}}, \tag{D.1}$$

where

$$\Omega \equiv c^2 X^2 + s^2, \qquad \tilde{\Omega} \equiv s^2 \tilde{X}^2 + c^2. \tag{D.2}$$

Thus, the metric ansatz (3.1.1) becomes

$$\begin{aligned} d\hat{s}_{11}^2 &= \mathrm{e}^{2\beta} ds_4^2 + 2g^{-2} \mathrm{e}^{2\beta} d\xi^2 + \mathrm{e}^{2\gamma} \sum_i (h^i)^2 + \mathrm{e}^{2\tilde{\gamma}} \sum_i (\tilde{h}^i)^2, \\ &= \left[\mathrm{e}^{2\beta} g_{\mu\nu} + \mathrm{e}^{2\gamma} g^2 A^i_\mu A^i_\nu + \mathrm{e}^{2\tilde{\gamma}} g^2 \tilde{A}^i_\mu \tilde{A}^i_\nu \right] dx^\mu dx^\nu + 2g^{-2} \mathrm{e}^{2\beta} d\xi^2 \\ &\quad - 2\mathrm{e}^{2\gamma} g A^i_\mu \sigma_i dx^\mu - 2\mathrm{e}^{2\tilde{\gamma}} g \tilde{A}^i_\mu \tilde{\sigma}_i dx^\mu + \mathrm{e}^{2\gamma} \sigma_i^2 + \mathrm{e}^{2\tilde{\gamma}} \tilde{\sigma}_i^2. \end{aligned} \tag{D.3}$$

Here, the second line can be obtained by expressing of the four-dimensional metric as $ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu$, the four-dimensional line element equation in (2.1.29), and using the definitions of two $SU(2)$-valued forms in (C.8). Making a comparison between this form of the metric ansatz in (D.3) and the general eleven-dimensional line element equation, $d\hat{s}_{11}^2 = \hat{g}_{MN} dx^M dx^N$, in which the eleven-dimensional spacetime indices are splitted to be $M, N = (\mu, \xi, i, \tilde{i})$ with μ a four-dimensional spacetime index, ξ a ξ-coordinate index, i and $\tilde{i}$ coordinate indices on each S^3, gives all non vanishing components of the metric tensor,

$$\begin{aligned} \hat{g}_{\mu\nu} &= \mathrm{e}^{2\beta} g_{\mu\nu} + \mathrm{e}^{2\gamma} g^2 A^i_\mu A^i_\nu + \mathrm{e}^{2\tilde{\gamma}} g^2 \tilde{A}^i_\mu \tilde{A}^i_\nu, \\ \hat{g}_{\xi\xi} &= 2g^{-2} \mathrm{e}^{2\beta}, \\ \hat{g}_{i\mu} &= -\mathrm{e}^{2\gamma} g A^i_\mu, \qquad \hat{g}_{\tilde{i}\mu} = -2\mathrm{e}^{2\tilde{\gamma}} g \tilde{A}^i_\mu, \\ \hat{g}_{ij} &= \mathrm{e}^{2\gamma} \delta_{ij}, \qquad \hat{g}_{\tilde{i}\tilde{j}} = \mathrm{e}^{2\tilde{\gamma}} \delta_{\tilde{i}\tilde{j}}. \end{aligned} \tag{D.4}$$

By using the relation between metric tensor and the Minkowski metric in the Lorentz frames defined in (2.1.56),

$$\hat{g}_{MN} = e^{\hat{M}}_{M} \, \hat{\eta}_{\hat{M}\hat{N}} \, e^{\hat{N}}_{N}, \tag{D.5}$$

where $\hat{\eta}_{\hat{M}\hat{N}}$ is the eleven-dimensional Minkowski metric defined in the same way as (2.1.8) and can be written in the block-diagonal form as

$$\hat{\eta}_{\hat{M}\hat{N}} = \begin{bmatrix} \eta_{ab} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta_{ij} & 0 \\ 0 & 0 & 0 & \delta_{\tilde{i}\tilde{j}} \end{bmatrix}, \tag{D.6}$$

the eleven-dimensional non-coordinate orthogonal bases can be obtained*,

$$\hat{e}^a = \mathrm{e}^{\beta} e^a, \quad \hat{e}^0 = \sqrt{2} g^{-1} \mathrm{e}^{\beta} d\xi, \quad \hat{e}^i = \mathrm{e}^{\gamma} h^i, \quad \hat{e}^{\tilde{i}} = \mathrm{e}^{\tilde{\gamma}} \tilde{h}^i, \tag{D.7}$$

where e^a refers to the vielbein 1-form in four-dimensional spacetime. Then, all non-zero higher-dimensional spin connections can be determined through the vielbein postulate in (2.1.79) as follow

$$\begin{aligned}
\hat{\omega}^{ab} &= \omega^{ab} + (\partial^b \beta e^a - \partial^a \beta e^b) + \frac{g}{2} \mathrm{e}^{-2\beta} (\mathrm{e}^{2\gamma} F^{iab} h^i + \mathrm{e}^{2\tilde{\gamma}} \tilde{F}^{iab} \tilde{h}^i), \\
\hat{\omega}^{a0} &= \frac{g}{\sqrt{2}} (\beta' e^a - 2g^{-2} \partial^a \beta d\xi), \\
\hat{\omega}^{ai} &= -\mathrm{e}^{(\gamma - \beta)} (\partial^a \gamma h^i - \frac{g}{2} F^{ia}{}_b e^b), \\
\hat{\omega}^{a\tilde{i}} &= -\mathrm{e}^{(\tilde{\gamma} - \beta)} (\partial^a \tilde{\gamma} \tilde{h}^i - \frac{g}{2} \tilde{F}^{ia}_b e^b), \\
\hat{\omega}^{0i} &= -g \gamma' \mathrm{e}^{(\gamma - \beta)} h^i, \\
\hat{\omega}^{0\tilde{i}} &= -g \tilde{\gamma}' \mathrm{e}^{(\tilde{\gamma} - \beta)} \tilde{h}^i, \\
\hat{\omega}^{ij} &= -\frac{1}{2} \varepsilon_{ijk} (h^k + 2g A^k_{(1)}), \\
\hat{\omega}^{\tilde{i}\tilde{j}} &= -\frac{1}{2} \varepsilon_{ijk} (\tilde{h}^k + 2g \tilde{A}^k_{(1)}),
\end{aligned} \tag{D.8}$$

where ω^{ab} is a four-dimensional spin connection. Here, β', γ', and $\tilde{\gamma}'$ denote their derivatives with respect to ξ.

Some useful identities of these β, γ, and $\tilde{\gamma}$ functions are needed for calculation of the eleven-dimensional Ricci tensor. From (D.1), it is easy to exhibit the derivatives of each function with respect to the vielbein components of four-dimensional spacetime and the coordinate ξ in the following forms

$$\begin{aligned}
\partial_a \beta &= \frac{1}{6} \Big(\frac{c^2 \partial_a X^2}{\Omega} + \frac{s^2 \partial_a \tilde{X}^2}{\tilde{\Omega}} \Big), \\
\partial_a \gamma &= \partial_a \beta - \frac{c^2 \partial_a X^2}{2\Omega}, \\
\partial_a \tilde{\gamma} &= \partial_a \beta - \frac{s^2 \partial_a \tilde{X}^2}{2\tilde{\Omega}},
\end{aligned} \tag{D.10}$$

*This 0 index, which is always used for a timelike coordinate index, now refers to the flat space index corresponding to the spatial coordinate ξ and all the four-dimensional equations of motion should be independent of this ξ for consistency of the dimensional reduction.

$$\begin{aligned}
\beta' &= \frac{sc}{3}\Big[\frac{(1-X^2)}{\Omega} + \frac{(\tilde{X}^2-1)}{\tilde{\Omega}}\Big],\\
\gamma' &= \beta' - \frac{sc(1-X^2)}{\Omega} - \tan\xi,\\
\tilde{\gamma}' &= \beta' - \frac{sc(\tilde{X}^2-1)}{\tilde{\Omega}} - \cot\xi.
\end{aligned} \tag{D.11}$$

Two useful identities are obviously obtained from these derivatives,

$$\partial_a\beta + \partial_a\gamma + \partial_a\tilde{\gamma} = 0, \tag{D.12}$$

$$\beta' + \gamma' + \tilde{\gamma}' = 2\cot 2\xi. \tag{D.13}$$

Finally, all components of the eleven-dimensional Ricci tensor can be computed using (2.1.85),

$$\begin{aligned}
\hat{R}_{00} &= \Delta^{-\frac{2}{3}}\Big[-\Box\beta + \frac{1}{2}g^2(-4\beta'' - 3\gamma'' - 3\tilde{\gamma}'' + 3\beta'\gamma' + 3\beta'\tilde{\gamma}' - 3\gamma'^2 - 3\tilde{\gamma}'^2)\Big],\\
\hat{R}_{0a} &= \frac{3}{\sqrt{2}}g\Delta^{-\frac{2}{3}}\Big[(\partial_a\beta - \partial_a\gamma)\gamma' + \partial_a\beta - \partial_a\tilde{\gamma})\tilde{\gamma}'\Big],\\
\hat{R}_{0i} &= 0, \qquad\qquad \hat{R}_{0\tilde{i}} = 0,\\
\hat{R}_{ab} &= \Delta^{-\frac{2}{3}}\Big[R_{ab} - 3(\partial_a\beta\partial_b\beta + \partial_a\gamma\partial_b\gamma + \partial_a\tilde{\gamma}\partial_b\tilde{\gamma}) - \Box\beta\eta_{ab}\\
&\qquad - \frac{1}{4}c^2\Omega^{-1}F^i_{ac}F^{ic}_b - \frac{1}{4}s^2\tilde{\Omega}^{-1}\tilde{F}^i_{ac}\tilde{F}^{ic}_b - \frac{1}{2}g^2(\beta'' + 6\beta'\cot 2\xi)\eta_{ab}\Big],\\
\hat{R}_{ai} &= -\frac{1}{2\sqrt{2}}c\Delta^{-\frac{2}{3}}\Omega^{-\frac{1}{2}}\Big[D_bF^{ib}_a - 2(\partial_b\beta - \partial_b\gamma)F^{ib}_a\Big],\\
\hat{R}_{a\tilde{i}} &= -\frac{1}{2\sqrt{2}}s\Delta^{-\frac{2}{3}}\tilde{\Omega}^{-\frac{1}{2}}\Big[\tilde{D}_b\tilde{F}^{ib}_a - 2(\partial_b\beta - \partial_b\tilde{\gamma})\tilde{F}^{ib}_a\Big],\\
\hat{R}_{ij} &= \Delta^{-\frac{2}{3}}\Big[\frac{1}{2}g^2(-\gamma'' - 6\gamma'\cot 2\xi + 2\Omega c^{-2})\delta_{ij} - \Box\gamma\delta_{ij} + \frac{1}{8}c^2\Omega^{-1}F^i_{ab}F^{jab}\Big],\\
\hat{R}_{\tilde{i}\tilde{j}} &= \Delta^{-\frac{2}{3}}\Big[\frac{1}{2}g^2(-\tilde{\gamma}'' - 6\tilde{\gamma}'\cot 2\xi + 2\tilde{\Omega}s^{-2})\delta_{ij} - \Box\tilde{\gamma}\delta_{ij} + \frac{1}{8}s^2\tilde{\Omega}^{-1}\tilde{F}^i_{ab}\tilde{F}^{jab}\Big],\\
\hat{R}_{i\tilde{j}} &= \frac{1}{8}sc\Delta^{-\frac{5}{3}}F^i_{ab}\tilde{F}^{jab},
\end{aligned} \tag{D.9}$$

where R_{ab} is the Ricci tensor for the four-dimensional spacetime.

BIOGRAPHY

Mr. Patharadanai Nuchino was born on July 28, 1991, in Mae Hong Son Province, Thailand. He received his Bachelor Degree in Physics from Chiang Mai University, Thailand in March 2014. He has received Science Achievement Scholarship of Thailand (SAST) since 2010. After graduation, he continued his graduate studies in the Degree of Master of Science Program in Physics, Department of Physics, Faculty of Science, Chulalongkorn University.