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PURE IDEALS IN Γ-SEMIRINGS

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ในวิทยานิพนธ์ฉบับนี้ เรานิยามและแสดงลักษณะไอดีลบริสุทธิ์ขวา ไอดีลบริสุทธิ์ซ้าย ไอดีลบริสุทธิ์อย่างอ่อนขวา และไอดีลบริสุทธิ์อย่างอ่อนซ้ายในแกมมา-กึ่งริง เราสามารถนำสมบัติของไอดีลบริสุทธิ์ขวามาอธิบายลักษณะของแกมมา-กึ่งริงปกติอย่างอ่อนขวาได้ ถัดไปเราได้นิยามไอดีลบริสุทธิ์เฉพาะ ไอดีลบริสุทธิ์กึ่งเฉพาะ ไอดีลบริสุทธิ์ลดทอนไม่ได้ ไอดีลบริสุทธิ์ลดทอนไม่ได้อย่างเข้ม และไอดีลบริสุทธิ์ใหญ่สุดในแกมมา-กึ่งริงและตรวจสอบสมบัติของพวกมัน ยิ่งไปกว่านั้นเราสามารถหาความสัมพันธ์ระหว่างไอดีลบริสุทธิ์ขวาและไอดีลบริสุทธิ์ลดทอนไม่ได้ในแกมมา-กึ่งริง ในส่วนสุดท้ายเราตรวจสอบสมบัติของไอดีลบริสุทธิ์ขวาและไอดีลบริสุทธิ์ซ้ายบนแกมมา-กึ่งริงสาทิสสัณฐาน และได้ตรวจสอบสมบัติของมันในแกมมา-กึ่งริงผลหารและผลคูณของแกมมา-กึ่งริง

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In this thesis, we define and characterize right pure ideals, left pure ideals, right weakly pure ideals and left weakly pure ideals in Γ-semirings. We also characterize right weakly regular Γ-semirings by the properties of right pure ideals. Next, we introduce purely prime, purely semiprime, purely irreducible, strongly irreducible pure and purely maximal ideals in Γ-semirings and examine their relations. Moreover, we can find relationships between right pure ideals and purely irreducible ideals in Γ-semirings. Finally, we investigate the properties of right pure ideals and left pure ideals on Γ-semirings homomorphisms. We also investigate the properties of them in quotient Γ-semirings and products of Γ-semirings.

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CONTENTS

NOTATION

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_n$	the set of integers modulo n where $n \in \mathbb{N}$
$n\mathbb{Z}$	$\{nz \mid z \in \mathbb{Z}\}$ where $n \in \mathbb{N}$
$M_{m \times n}(S)$	the set of $m \times n$ matrices over a semiring S where $m, n \in \mathbb{N}$
$X\Gamma Y$	$\left\{ \sum_{i=1}^{l} x_i \alpha_i y_i \mid l \in \mathbb{N}, x_i \in X, \alpha_i \in \Gamma \text{ and } y_i \in Y \text{ for all } i \right\}$ where X and Y are nonempty subsets of a Γ-semiring
$\mathbb{N}X$	$\left\{ \sum_{i=1}^{l} n_i x_i \mid l \in \mathbb{N}, n_i \in \mathbb{N} \text{ and } x_i \in X \text{ for all } i \right\}$ where X is a nonempty subset of a Γ-semiring
$\mathrm{Hom}(X, Y)$	the set of all homomorphisms from X to Y where X and Y are commutative semigroups

CHAPTER I
INTRODUCTION

The notion of Γ-ring was first introduced by N. Nobusawa in 1964 [11]. J. Luh introduced the concept of left operator ring and right operator ring of Γ-ring in 1969 [9]. M. M. K. Rao introduced the concept of Γ-semiring as a generalization of semiring and Γ-ring in 1995 [12]. S. K. Sardar and T. K. Dutta modified the definition of Γ-semiring of Rao and then they defined the left operator semiring and right operator semiring of a Γ-semiring and obtained a few interesting properties. In addition, S. K. Sardar and T. K. Dutta gave the definitions of prime, semiprime irreducible and strongly irreducible ideals in Γ-semirings and also investigated some properties of them. H. Hedayati and K. P. Shum are researchers that studied Γ-semiring. In 2011, they introduced a Γ-semiring homomorphism and methods of constructing new Γ-semirings, namely a quotient Γ-semiring and the products of Γ-semirings and then they created some fundamental isomorphism theorems and the commutativity of some diagrams of Γ-semirings.

In 1989, the concept of pure and purely prime ideals in semigroups was introduced by J. Ahsan and M. Takahashi have brought forward [1]. Then M. Shabir and S. Bashir extended the concept of pure ideals in semigroups to pure ideals in ternary semigroups, in 2009. Moreover, they also defined and studied pure ideals, weakly pure ideals and purely prime ideals in ternary semigroups. Furthermore, they proved that the space of purely prime two-sided ideals is topologized [3].

In this research, we study some properties of ideals of a Γ-semiring. Later on, we define and characterize right pure ideals and left pure ideals in Γ-semirings. We characterize right weakly regular Γ-semirings by using the properties of right pure ideals. Next, we introduce purely prime, purely semiprime, purely irreducible and strongly irreducible pure ideals in Γ-semirings and examine their properties

such as relationships between right pure ideals and purely irreducible ideals in Γ-semirings. From characterization of right pure ideals and left pure ideals, we reduce the condition of right pure ideals and left pure ideals to construct right weakly pure ideals and left weakly pure ideals in Γ-semirings. We also characterize right weakly pure ideals and left weakly pure ideals in Γ-semirings. Finally, we investigate the properties of right pure ideals and left pure ideals on a Γ-semiring homomorphism. We also investigate their properties in the quotient Γ-semirings and the products of Γ-semirings.

CHAPTER II
PRELIMINARIES

In this chapter, we review some definitions and properties of Γ-semirings, which are a generalization of Γ-rings and semirings. We first introduce the concepts of semigroups.

Definition 2.1. [5] A **semigroup** $(S, *)$ is an ordered pair of a nonempty set S and an associative binary operation $*$ on S. We may write xy for $x * y$ where $x, y \in S$.

If S is a semigroup such that $x * y = y * x$ for all $x, y \in S$, we shall say that S is **commutative**.

Definition 2.2. [5] A **subsemigroup** T of a semigroup $(S, *)$ is a nonempty subset T of S such that $xy \in T$ for all $x, y \in T$.

Definition 2.3. [5] An element 1 of a semigroup S is called an **identity element** of S if $x1 = x = 1x$ for all $x \in S$.

Definition 2.4. [5] An element e of a semigroup S is called an **idempotent** if $e^2 = ee = e$. Let $E(S)$ denote the set of all idempotents of S.

Example 2.1. From Example 2 in [8], $(\mathbb{N}, max)$ is a commutative semigroup. Then every nonempty subset of $\mathbb{N}$ is a subsemigroup of $(\mathbb{N}, max)$.

Example 2.2. For each $n \in \mathbb{N}$, $(\mathbb{Z}_n, +)$ and $(\mathbb{Z}_n, \cdot)$ are commutative semigroups.

Example 2.3. For $n, m \in \mathbb{N}$, if S is a commutative semigroup, then $M_{n\times m}(S)$ is a commutative semigroup under the usual addition of matrices. Moreover, if A is a subsemigroup of S, then $M_{n\times m}(A)$ is a subsemigroup of $M_{n\times m}(S)$.

Example 2.4. Let X and Y be commutative semigroups with identity elements. If $M = \text{Hom}(X, Y)$, then M is a commutative semigroup under the usual addition of functions.

Another basic structure in this thesis is semirings. It was introduced by M. K. Sen and M. R. Adhikari [13] in 1993.

Definition 2.5. [13] A **semiring** is a nonempty set S with two binary operations $+$ and $\cdot$ satisfying the following conditions: for all $x, y, z \in S$,

(i) $(S, +)$ is a commutative semigroup;

(ii) $(S, \cdot)$ is a semigroup;

(iii) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$.

In addition, S is a **commutative semiring** if $(S, \cdot)$ is commutative.

Definition 2.6. [13] An element 0 of a semiring S is said to be an **absorbing zero** if for all $a \in S$,

$$0 \cdot a = 0 = a \cdot 0 \text{ and } a + 0 = a.$$

Definition 2.7. [13] An element 1 of a semiring S is said to be the **identity element** if $1 \cdot a = a = a \cdot 1$ for all $a \in S$.

Definition 2.8. [13] A nonempty subset A of a semiring S is called an **ideal** of S if A is a subsemigroup of $(S, +)$ and $a \cdot s, s \cdot a \in A$ for all $a \in A$ and $s \in S$.

Example 2.5. For each $n \in \mathbb{N}$, $(\mathbb{Z}_n, +, \cdot)$ is a commutative semiring with zero $[0]_n$ and identity $[1]_n$.

In 1966, W. E. Barnes introduced the definition of Γ-rings.

Definition 2.9. [2] Let $(S, +)$ and $(\Gamma, +')$ be abelian groups. S is called a **Γ-ring** if there exists a function $S \times \Gamma \times S \to S$, called a **$\Gamma$-multiplication**, whose image of (a, γ, b) is denoted by $a\gamma b$, satisfying the following conditions:

(i) **right distributive :** $(a + b)\gamma c = a\gamma c + b\gamma c$;

(ii) **left distributive :** $a\gamma(b+c) = a\gamma b + a\gamma c$;

(iii) **lateral distributive :** $a(\gamma +' \beta)b = a\gamma b + a\beta b$; and

(iv) **associative :** $a\gamma(b\beta c) = (a\gamma b)\beta c$

for all $a, b, c \in S$ and $\gamma, \beta \in \Gamma$.

The notions of Γ-semirings introduced by T. K. Dutta and S. K. Sardar [4] is a main structure for studying this thesis.

Definition 2.10. [4] Let $(S, +)$ and $(\Gamma, +')$ be commutative semigroups. S is called a **Γ-semiring** if there exists a mapping $S \times \Gamma \times S \to S$, whose image of (a, α, b) is denoted by $a\alpha b$, satisfying the following conditions: for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$,

(i) $a\alpha(b+c) = a\alpha b + a\alpha c$;

(ii) $(b+c)\alpha a = b\alpha a + c\alpha a$;

(iii) $a(\alpha +' \beta)c = a\alpha c + a\beta c$; and

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

Definition 2.11. [4] A Γ-semiring S is said to be **commutative** if $a\alpha b = b\alpha a$ for any $a, b \in S$ and for any $\alpha \in \Gamma$.

Every semiring S is a Γ-semiring where Γ is a subsemiring of S. Here is an example.

Example 2.6. Let $(S, +, \cdot)$ be a semiring and Γ a subsemiring of S. Next, define the mapping $S \times \Gamma \times S \to S$ by $x\alpha y = x \cdot \alpha \cdot y$ for all $x, y \in S$ and $\alpha \in \Gamma$. Then S is a Γ-semiring. Therefore, every semiring S is a Γ-semiring where Γ is a subsemiring of S. Moreover, if S is a commutative semiring, then S is a commutative Γ-semiring under this construction.

Similarly, we can construct a semiring from a Γ-semiring by the method as shown in the following example.

Example 2.7. Let S be a Γ-semiring. Then S is an additive commutative semigroup. Fix $\alpha \in \Gamma$ and define the binary operation $\cdot : S \times S \to S$ by $x \cdot y = x\alpha y$ for all $x, y \in S$. Therefore, S is a semiring. Furthermore, if S is a commutative Γ-semiring, then S is a commutative semiring under this construction.

The other examples of Γ-semirings are as follows.

Example 2.8. For $n, m \in \mathbb{N}$, let $S = M_{n\times m}(R)$ and $\Gamma = M_{m\times n}(R)$ where R is a semiring. By Example 2.3, $(S, +)$ and $(\Gamma, +)$ are commutative semigroups where $+$ is the usual addition of matrices. Define the mapping $S \times \Gamma \times S \to S$ by $A\alpha B$ which is the usual multiplication of matrices for all $A, B \in S$ and $\alpha \in \Gamma$. Then S is a Γ-semiring but not a commutative Γ-semiring.

Example 2.9. [8] Let $\Gamma = \{1, 2, 3\}$. By Example 2.1, $(\mathbb{N}, max)$ and (Γ, max) are commutative semigroups. Define the mapping $\mathbb{N} \times \Gamma \times \mathbb{N} \to \mathbb{N}$ by $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in \mathbb{N}$ and $\alpha \in \Gamma$. Then $\mathbb{N}$ is a commutative Γ-semiring.

Example 2.10. [12] Let $M = \mathrm{Hom}(X, Y)$ and $\Gamma = \mathrm{Hom}(Y, X)$ where X and Y are commutative semigroups with identity elements. Then M and Γ are additive commutative semigroups. Define the mapping $M \times \Gamma \times M \to M$ by $f\alpha h$ being the usual composition map for all $f, h \in M$ and $\alpha \in \Gamma$. Then M is a Γ-semiring.

We need to introduce the following notations used throughout this thesis. For nonempty subsets X, Y of a Γ-semiring S and a nonempty subset Δ of Γ,

$$X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$$

$$X\Delta Y = \left\{ \sum_{i=1}^{m} x_i\alpha_i y_i \mid m \in \mathbb{N}, x_i \in X, \alpha_i \in \Delta \text{ and } y_i \in Y \text{ for all } i \right\}$$

$$\mathbb{N}X = \left\{ \sum_{i=1}^{m} n_i x_i \mid m \in \mathbb{N}, n_i \in \mathbb{N} \text{ and } x_i \in X \text{ for all } i \right\}.$$

For convenience, we write $x\Delta Y$, $X\Delta y$ and $X\alpha Y$ instead of $\{x\}\Delta Y$, $X\Delta\{y\}$ and $X\{\alpha\}Y$, respectively, for all $x, y \in S$ and $\alpha \in \Delta$. Moreover, we simply write $\sum x_i\alpha_i y_i$ and $\sum n_i x_i$ instead of $\sum_{i=1}^{m} x_i\alpha_i y_i$ and $\sum_{i=1}^{m} n_i x_i$ where $m \in \mathbb{N}$, $x_i, y_i \in S$, $\alpha_i \in \Gamma$ and $n_i \in \mathbb{N}$ for all i, respectively.

Proposition 2.12. *Let S be a commutative Γ-semiring. Then for nonempty subsets X and Y of S, $X\Gamma Y = Y\Gamma X$.*

Proof. This is clear. □

Proposition 2.13. *Let S be a Γ-semiring. Then for nonempty subsets X, Y and Z of S and a nonempty subset Δ of Γ, the following hold:*

(i) $(X+Y)\Delta Z \subseteq X\Delta Z + Y\Delta Z$*;*

(ii) $X\Delta(Y+Z) \subseteq X\Delta Y + X\Delta Z$*;*

(iii) $(X\Delta Y)\Delta Z = X\Delta(Y\Delta Z)$*; and*

(iv) $(\mathbb{N}X)\Delta Y = \mathbb{N}(X\Delta Y) = X\Delta(\mathbb{N}Y)$.

Proof. (i) Let $\sum(x_i+y_i)\alpha_i z_i \in (X+Y)\Delta Z$ where $x_i \in X, y_i \in Y, \alpha_i \in \Delta$ and $z_i \in Z$ for all i. Since $(x_i+y_i)\alpha_i z_i = x_i\alpha_i z_i + y_i\alpha_i z_i$ for all i, we obtain

$$\sum(x_i+y_i)\alpha_i z_i = \sum x_i\alpha_i z_i + \sum y_i\alpha_i z_i \in X\Delta Z + Y\Delta Z$$

so that $(X+Y)\Delta Z \subseteq X\Delta Z + Y\Delta Z$.
(iv) $(\subseteq)$ Let $\sum z_i\alpha_i y_i \in (\mathbb{N}X)\Delta Y$ where $z_i \in \mathbb{N}X, \alpha_i \in \Delta$ and $y_i \in Y$ for all i. For each i, $z_i = \sum n_{i_j}x_{i_j}$ where $n_{i_j} \in \mathbb{N}$ and $x_{i_j} \in X$ for all i_j. It follows that

$$\sum z_i\alpha_i y_i = \sum(\sum n_{i_j}x_{i_j})\alpha_i y_i = \sum_{i,i_j}(n_{i_j}x_{i_j})\alpha_i y_i = \sum_{i,i_j} n_{i_j}(x_{i_j}\alpha_i y_i) \in \mathbb{N}(X\Delta Y).$$

Now we conclude that $(\mathbb{N}X)\Delta Y \subseteq \mathbb{N}(X\Delta Y)$.
$(\supseteq)$ Let $\sum n_i z_i \in \mathbb{N}(X\Delta Y)$ where $n_i \in \mathbb{N}$ and $z_i \in X\Delta Y$. For each i, we obtain that $z_i = \sum x_{i_j}\alpha_{i_j}y_{i_j}$ where $x_{i_j} \in X$, $\alpha_{i_j} \in \Delta$ and $y_{i_j} \in Y$. Thus $\sum n_i z_i = \sum n_i\big(\sum x_{i_j}\alpha_{i_j}y_{i_j}\big) = \sum_{i,i_j} n_i(x_{i_j}\alpha_{i_j}y_{i_j}) = \sum_{i,i_j}(n_i x_{i_j})\alpha_{i_j}y_{i_j} \in (\mathbb{N}X)\Delta Y$. Hence $\mathbb{N}(X\Delta Y) \subseteq (\mathbb{N}X)\Delta Y$.

We prove $\mathbb{N}(X\Delta Y) = X\Delta(\mathbb{N}Y)$ as same as $(\mathbb{N}X)\Delta Y = \mathbb{N}(X\Delta Y)$.

The proofs of (ii) and (iii) are obtained similarly to the proof of (i). □

Conventionally, we write $X\Delta Y\Delta Z$ instead of $(X\Delta Y)\Delta Z$ or $X\Delta(Y\Delta Z)$, for all nonempty subsets X, Y and Z of S and a nonempty subset Δ of Γ.

Proposition 2.14. *Let S be a Γ-semiring. Then for nonempty subsets X, Y and Z of S and nonempty subsets Δ and Θ of Γ the following statements hold:*

(i) if $X \subseteq Y$, then $X\Delta Z \subseteq Y\Delta Z$;

(ii) if $Y \subseteq Z$, then $X\Delta Y \subseteq X\Delta Z$;

(iii) if $\Delta \subseteq \Theta$, then $X\Delta Y \subseteq X\Theta Y$.

Proof. (i) Let $\sum x_i\alpha_i z_i \in X\Delta Z$ where $x_i \in X$, $\alpha_i \in \Delta$ and $z_i \in Z$ for all i. Since $x_i \in X \subseteq Y$, $\sum x_i\alpha_i z_i \in Y\Delta Z$ for all i. Therefore, $X\Delta Z \subseteq Y\Delta Z$.

The proofs of (ii) and (iii) are similar to the proof of (i). □

Definition 2.15. [4] Let S be a Γ-semiring. An element $0 \in S$ is said to be a **zero** if for all $a \in S$ and $\alpha \in \Gamma$,

$$0\alpha a = 0 = a\alpha 0 \text{ and } a + 0 = a.$$

Example 2.11. Let $(S, +, \cdot)$ be a semiring. By Example 2.6, S is a Γ-semiring where Γ is a subsemiring of S. If 0 is an absorbing zero of a semiring S, then $0\alpha a = 0 = a\alpha 0$ and $a + 0 = a$ for all $a \in S$ and $\alpha \in \Gamma$. Hence if 0 is an absorbing zero of a semiring S, then 0 is a zero of a Γ-semiring S.

Example 2.12. Let $S = \mathbb{N}$ and $\Gamma = \{1, 2, 3\}$. By Example 2.9, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$. Then 1 is a zero of S.

Example 2.13. For $n, m \in \mathbb{N}$, let $S = M_{n\times m}(R)$ and $\Gamma = M_{m\times n}(R)$ where R is a semiring with absorbing zero 0. By Example 2.8, S forms a Γ-semiring with $A\alpha B$ which is the usual multiplication of matrices for all $A, B \in S$ and $\alpha \in \Gamma$. Then $[0]_{n\times m}$ is a zero of S.

In semiring theory, the properties of their ideals play an important role in their structure theory. Similarly, in Γ-semiring theory, T. K. Dutta and S. K. Sardar gave the definition of ideals of a Γ-semiring in 2000.

Definition 2.16. [4] A nonempty subset I of a Γ-semiring S is called a **right ideal (left ideal)** of S if I is a subsemigroup of $(S,+)$ and $a\alpha x \in I$ $(x\alpha a \in I)$ for all $a \in I$, $x \in S$ and for all $\alpha \in \Gamma$.

Definition 2.17. [4] If I is both right and left ideal of a Γ-semiring S, then we say that I is an **ideal** of S.

Remark 1. A Γ-semiring S is a right ideal and a left ideal of S. On the other hand, every right ideal (left ideal) of a Γ-semiring is a Γ-semiring.

Remark 2. If S is a commutative Γ-semiring, then right ideals (left ideals) of S are ideals of S.

Remark 3. For each right ideal (left ideal) I of a Γ-semiring S with zero 0, $0 \in I$ because $0 = x\alpha 0 \in I$ $(0 = 0\alpha x \in I)$ for all $x \in I$ and $\alpha \in \Gamma$.

Example 2.14. Let $(S,+,\cdot)$ be a semiring and Γ a subsemiring of S. By Example 2.6, S is a Γ-semiring with $x\alpha y = x\cdot\alpha\cdot y$ for all $x, y \in S$ and $\alpha \in \Gamma$. If I is an ideal of a semiring S, then $a+b \in I$ and $x\cdot\alpha\cdot a, a\cdot\alpha\cdot x \in I$ for all $x \in S, a, b \in I$ and $\alpha \in \Gamma$. Hence I is a subsemigroup of $(S,+)$ and $x\alpha a, a\alpha x \in I$ for all $x \in S, a \in I$ and $\alpha \in \Gamma$. Now we conclude that any ideals of a semiring S are ideals of a Γ-semiring S. Conversely, let S be a semiring with identity 1 and I an ideal of a Γ-semiring S where $\Gamma = S$. So I is a subsemigroup of $(S,+)$ and for $a \in I$, $x \in S$, $a\cdot x = a\cdot 1\cdot x \in I$ and $x\cdot a = x\cdot 1\cdot a \in I$. Therefore, I is an ideal of S.

Example 2.15. For $k \in \mathbb{N}$, $n = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ where $p_1, p_2, \ldots, p_k$ are pairwise distinct primes and $n_i \in \mathbb{N}$ for all i. Let $S = \mathbb{Z}_n$ and $\Gamma = S$. By Example 2.6, S is a commutative Γ-semiring with $[x]_n[\alpha]_n[y]_n = [x\alpha y]_n$ for all $[x]_n, [y]_n \in S$ and $[\alpha]_n \in \Gamma$. Let $I = \{m[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n \mid m \in \mathbb{Z}\}$ where $0 \leq l_i \leq n_i$. For $m[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n, t[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n \in I$, $m[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n + t[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n = (m+t)[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n \in I$. Thus I is a subsemigroup of S. For $[x]_n \in S, m[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n \in I$ and $[\alpha]_n \in \Gamma$, $[x]_n[\alpha]_n m[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n = x\alpha m[p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}]_n \in I$. Therefore, I is an ideal of S.

Example 2.16. Let $S = \mathbb{N}$ and $\Gamma = \{1, 2, 3\}$. By Example 2.9, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$. Then $\{1\}, \{1, 2\}$ and $\{1, 2, 3\}$ are ideals of S. Moreover, I is an ideal of S if $\{1, 2, 3\} \subseteq I$.

Example 2.17. For $n, m \in \mathbb{N}$, let $S = M_{n\times m}(R)$ and $\Gamma = M_{m\times n}(R)$ where R is a semiring. By Example 2.8, S forms a Γ-semiring with the usual multiplication of matrices $A\alpha B$ for all $A, B \in S$ and $\alpha \in \Gamma$. Let I be an ideal of R. Then I is a subsemigroup of $(R, +)$. By Example 2.3, $M_{n\times m}(I)$ is a subsemigroup of S under the usual addition of matrices. Next, let $A = [a_{ij}]_{n\times m} \in S$, $B = [\alpha_{ij}]_{m\times n} \in \Gamma$ and $C = [c_{ij}]_{n\times m} \in M_{n\times m}(I)$. Then $ABC = \left[\sum_{k=1}^{n}\sum_{l=1}^{m} a_{il}\alpha_{lk}c_{kj}\right]_{n\times m}$ where $a_{ik}\alpha_{kl}c_{lj} \in I$ because each c_{lj} is an element of the ideal I. We now obtain that $ABC \in M_{n\times m}(I)$. Therefore, if I is an ideal of R, then $M_{n\times m}(I)$ is an ideal of S. On the other hand, let R^* be a semiring with absorbing zero 0 and identity element 1 and $M_{n\times m}(J)$ be an ideal of $M_{n\times m}(R^*)$. By Remark 3, $[0]_{n\times m} \in M_{n\times m}(J)$. Thus $0 \in J$. For $x \in R^*$, we define $[x]_{n\times m}$ is an $n \times m$ matrix with x on $(1, 1)$-entry and absorbing zeros elsewhere. Then for all $a, b \in J$, $[a + b]_{n\times m} = [a]_{n\times m} + [b]_{n\times m} \in M_{n\times m}(J)$ implies that $a + b \in J$. Next, let $x \in R^*$. We obtain from $[x]_{n\times m} \in M_{n\times m}(R^*), [a]_{n\times m} \in M_{n\times m}(J)$ and $[1]_{m\times n} \in M_{m\times n}(R^*)$ that $[ax]_{n\times m} = [a]_{n\times m}[1]_{m\times n}[x]_{n\times m} \in M_{n\times m}(J)$ and $[xa]_{n\times m} = [x]_{n\times m}[1]_{m\times n}[a]_{n\times m} \in M_{n\times m}(J)$. Hence $ax, xa \in J$. Therefore, if $M_{n\times m}(J)$ is an ideal of the $M_{m\times n}(R^*)$-semiring $M_{n\times m}(R^*)$, then J is an ideal of R^*. Now we conclude that for a semiring R^* with absorbing zero 0 and identity element 1, $M_{n\times m}(J)$ is an ideal of $M_{n\times m}(R^*)$ if and only if J is an ideal of R^*.

Example 2.18. Let $M = \text{Hom}(X, Y)$ and $\Gamma = \text{Hom}(Y, X)$ where X and Y are commutative semigroups with identity elements and $Y = E(Y)$. By Example 2.10, M forms a Γ-semiring with the usual composition map $f\alpha h$ for all $f, h \in M$ and $\alpha \in \Gamma$. Let $I = \{f \in M \mid f \text{ is a constant function}\}$. If $f, h \in I$, then there exist $y, z \in Y$, $f(x) = y$ and $h(x) = z$ for all $x \in X$. We obtain $(f + h)(x) = f(x) + h(x) = y + z$ so that $f + h \in I$. Thus I is a subsemigroup of M. Next, let $g \in M$ and $\alpha \in \Gamma$. Then $g(X) = Y'$ and $\alpha(Y) = X'$ for some $X' \subseteq X$ and $Y' \subseteq Y$. It follows that $(f\alpha g)(X) = f\alpha g(X) = f\alpha(Y') \subseteq f(X') = \{y\}$ and

$(g\alpha f)(X) = g\alpha f(X) = g\alpha(\{y\}) = g(\{t\}) = \{s\}$ for some $t \in X$ and $s \in Y$. Hence $f\alpha g \in I$ and $g\alpha f \in I$. Therefore, I is an ideal of M.

Next, some properties of ideals of Γ-semirings are given.

Proposition 2.18. *Let S be a Γ-semiring and I a subsemigroup of $(S, +)$. Then the following hold:*

(i) I is a right ideal of S if and only if $I\Gamma S \subseteq I$;

(ii) I is a left ideal of S if and only if $S\Gamma I \subseteq I$;

(iii) I is an ideal of S if and only if $S\Gamma I \subseteq I$ and $I\Gamma S \subseteq I$.

Proof. (i) $(\rightarrow)$ Let I be a right ideal of a Γ-semiring S and $y \in I\Gamma S$. Then $y = \sum a_i\alpha_i x_i$ where $a_i \in I$, $x_i \in S$ and $\alpha_i \in \Gamma$ for all i. By assumption, each $a_i\alpha_i x_i \in I$. Since I is a subsemigroup of $(S, +)$, $y = \sum a_i\alpha_i x_i \in I$.
$(\leftarrow)$ Let $I\Gamma S \subseteq I$. For all $a \in I$, $x \in S$ and $\alpha \in \Gamma$, $a\alpha x \in I\Gamma S \subseteq I$. By Definition 2.16, I is a right ideal of S.

(ii) This is similar to the proof of (i).

(iii) The proof follows from (i) and (ii). □

Proposition 2.19. *Let X be a nonempty subset of a Γ-semiring S and Δ a nonempty subset of Γ. Then the following statements hold:*

(i) $X\Delta S$ is a right ideal of S;

(ii) $S\Delta X$ is a left ideal of S;

(iii) $S\Delta X\Delta S$ is an ideal of S.

Proof. (i) We show that $X\Delta S$ is a right ideal of S. First, let $a, b \in X\Delta S$. Then $a = \sum x_i\alpha_i a_i$ and $b = \sum y_j\beta_j b_j$ where $x_i, y_j \in X$, $a_i, b_j \in S$ and $\alpha_i, \beta_j \in \Delta$ for all i, j. We obtain from $a + b = \sum x_i\alpha_i a_i + \sum y_j\beta_j b_j$ that $a + b \in X\Delta S$. Hence $X\Delta S$ is a subsemigroup of $(S, +)$. Next, let $s \in S$ and $\alpha \in \Gamma$. Then

$$a\alpha s = \left(\sum x_i\beta_i a_i\right)\alpha s = \left(\sum x_i\beta_i a_i\alpha s\right) = \sum x_i\beta_i\,(a_i\alpha s) \in X\Delta S.$$

Therefore, $X\Delta S$ is a right ideal of S.
(ii) Similarly, we can show that $S\Delta X$ is a left ideal of S.
(iii) From (i) and (ii), $(S\Delta X)\Delta S$ is a right ideal and $S\Delta(X\Delta S)$ is a left ideal of S, respectively. Hence $S\Delta X\Delta S$ is an ideal of S. □

Proposition 2.20. *Let S be a Γ-semiring with zero 0. The following statements hold:*

(i) $\{0\}$ is an ideal of S;

(ii) any finite intersection of ideals of S is an ideal of S.

Proof. (i) It is obvious that $S\Gamma\{0\} = \{0\}$ and $\{0\}\Gamma S = \{0\}$. By Proposition 2.18, $\{0\}$ is an ideal of S.
(ii) For $n \in \mathbb{N}$, let $A = \bigcap\limits_{i=1}^{n} \{J_i \mid J_i \text{ is an ideal of } S\}$. Then $A \neq \varnothing$ because $\{0\} \in A$. Let $a, b \in A$. Thus $a, b \in J_i$ for all i. For each i, J_i is a subsemigroup of $(S, +)$ so $a + b \in J_i$ for all i. It follows that $a + b \in A$. Now we obtain that A is a subsemigroup of $(S, +)$. Next, let $x \in S$ and $\alpha \in \Gamma$. Each i, J_i is an ideal of S. Thus $x\alpha a, a\alpha x \in J_i$ for all i. So $x\alpha a, a\alpha x \in A$. Hence A is an ideal of S. □

Moreover, T. K. Dutta and S. K. Sardar introduced the definition of prime, semiprime, irreducible and strongly irreducible ideals of Γ-semirings.

Definition 2.21. [4] A proper ideal P of a Γ-semiring S is said to be **prime** if for all ideals H and K of S, $H\Gamma K \subseteq P$ implies that either $H \subseteq P$ or $K \subseteq P$.

Definition 2.22. [4] A proper ideal P of a Γ-semiring S is said to be **semiprime** if for every ideal A of S, $A\Gamma A \subseteq P$ implies that $A \subseteq P$.

By virtue of Definitions 2.21 and 2.22, we can conclude that every prime ideal is semiprime. It follows that if I is not a semiprime ideal of S, then I is not a prime ideal of S. Here is an example.

Example 2.19. Let $S = \mathbb{N}$ and $\Gamma = \{1, 2, 3\}$. By Example 2.16, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$ and $\{1, 2, 3, 4\}$ is an ideal of S.

But $\{1,2,3,4\}$ is not semiprime of S because there exists an ideal $I_1 = \{1,2,3,4,5\}$ of S such that $I_1 \Gamma I_1 \subseteq \{1,2,3,4\}$. It follows that $\{1,2,3,4\}$ is not a prime ideal of S.

Definition 2.23. [4] A proper ideal I of a Γ-semiring S is said to be **irreducible** if for all ideals H and K of S, $H \cap K = I$ implies that $H = I$ or $K = I$.

Definition 2.24. [4] A proper ideal I of a Γ-semiring S is said to be **strongly irreducible** if for all ideals H and K of S, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$.

From Definitions 2.23 and 2.24, we can conclude that every strongly irreducible ideal is irreducible. It follows that if I is not an irreducible ideal of a Γ-semiring S, then I is not a strongly irreducible ideal of S.

Example 2.20. Let $S = M_{n\times 1}(\mathbb{Z}_{12})$ and $\Gamma = M_{1\times n}(\mathbb{Z}_{12})$. We have S forms a Γ-semiring with $A\alpha B$, which is the usual multiplication of matrices for all $A, B \in S$ and $\alpha \in \Gamma$. By Example 2.17, we conclude that all ideals of S are S, $I_0 = \left\{[[0]_{12}]_{n\times 1}\right\}$, $I_1 = \left\{[x_{i1}]_{n\times 1} | x_{i1} \in \{[0]_{12}, [6]_{12}\}\right\}$, $I_2 = \left\{[x_{i1}]_{n\times 1} | x_{i1} \in \{[0]_{12}, [4]_{12}, [8]_{12}\right\}$, $I_3 = \left\{[x_{i1}]_{n\times 1} | x_{i1} \in \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}\right\}$ and $I_4 = \left\{[x_{i1}]_{n\times 1} | x_{i1} \in \{[0]_{12}, [2]_{12}, [4]_{12}, [6]_{12}, [8]_{12}, [10]_{12}\right\}$. Consider sets $I_l \cap I_k$ where I_l and I_k are ideals of S,

$$
\begin{array}{llll}
I_1 \cap I_1 = I_1 & I_1 \cap I_2 = I_0 & I_1 \cap I_3 = I_1 & I_1 \cap I_4 = I_1 \\
I_2 \cap I_2 = I_2 & I_2 \cap I_3 = I_0 & I_2 \cap I_4 = I_2 & I_3 \cap I_4 = I_0 \\
I_2 \cap I_2 = I_2 & I_2 \cap I_3 = I_0 & I_2 \cap I_4 = I_2 & I_3 \cap I_3 = I_3 \\
I_3 \cap I_4 = I_1 & I_4 \cap I_4 = I_4 & I_0 \cap I_k = I_0 & I_k \cap I_0 = I_0 \\
S \cap I_k = I_k & I_k \cap S = I_k. & &
\end{array}
$$

We obtain that

- all strongly irreducible ideals of S are I_2, I_3 and I_4, and
- all irreducible ideals of S are I_2, I_3 and I_4.

Example 2.21. Let $S = \mathbb{N}$ and $\Gamma = \{1,2,3\}$. By Example 2.16, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$ and $\{1,2,3,4\}$ is an ideal of S.

But $\{1,2,3,4\}$ is not irreducible of S because there exist ideals $I_1 = \{1,2,3,4,5\}$ and $I_2 = \{1,2,3,4,6\}$ such that $I_1 \cap I_2 = \{1,2,3,4\}$. It follows that $\{1,2,3,4\}$ is not a strongly irreducible ideal of S.

In semirings, some interesting properties of semirings are true when that semirings have the identity. Likewise, in Γ-semirings, the identity element was introduced by K. Hila, I. Vardhami and K. Gjino in 2013.

Definition 2.25. [7] An element e of a Γ-semiring S is called an **identity element** of S if $a\alpha e = a = e\alpha a$, for all $a \in S$ and $\alpha \in \Gamma$.

Example 2.22. Let $S = \{1,2,3,4,5\}$ and $\Gamma = \{5\}$. We see that (S, max, min) is a semiring and Γ is a subsemiring of S. By Example 2.6, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$. Then 5 is an identity element of S.

For a Γ-semiring, the definition of the ideal generated by set is given by R. D. Jagatap and Y. S. Pawar.

Definition 2.26. [8] Let X be a nonempty subset of a Γ-semiring S. By $(X)_r$ we mean the **right ideal of S generated by X** (that is the intersection of all right ideals of S containing X).

Similarly, $(X)_l$ and $(X)_t$ denote the **left ideal and two-sided ideal generated by X**, respectively.

The following useful theorem based on a Γ-semiring with identity element is required.

Proposition 2.27. [8] *For nonempty subset X of a Γ-semiring S with identity element, we have*

(i) $(X)_r = X\Gamma S$;

(ii) $(X)_l = S\Gamma X$;

(iii) $(X)_t = S\Gamma X\Gamma S$.

CHAPTER III
PURE IDEALS

In this chapter, we separate our work into two parts. Firstly, we introduce the definitions of right pure ideals and left pure ideals in Γ-semirings and investigate their properties such as characterization of right pure ideals and left pure ideals and relationships between pure ideals and purely irreducible ideals in Γ-semirings. We also characterize right weakly regular Γ-semirings. Secondly, we include conditions into right pure ideals to define purely prime, purely semiprime, purely irreducible, strongly irreducible pure and purely maximal ideals in Γ-semirings and and investigate their relationships. Finally, We reduce conditions of right pure ideals and left pure ideals to define weakly right pure ideals and weakly left pure ideals in Γ-semirings and investigate thier properties.

Definition 3.1. An ideal I of a Γ-semiring S is called **right pure** (**left pure**) if for each $x \in I$ there exist $a \in I$, $\alpha \in \Gamma$ such that $x\alpha a = x$ ($a\alpha x = x$).

It is straightforward to show that for each commutative Γ-semiring S, right pure ideals and left pure ideals of S are coincide.

Example 3.1. Let $M = \text{Hom}(X, Y)$ and $\Gamma = \text{Hom}(Y, X)$ where X and Y are commutative semigroups with identity elements and $Y = E(Y)$. By Example 2.18, M forms a Γ-semiring with the usual composition map $f\alpha h$ for all $f, h \in M$ and $\alpha \in \Gamma$ and $I = \{f \in M \mid \ f \text{ is a constant function}\}$ is an ideal of M. Since for $f \in I$ there exists $\alpha \in \Gamma$ such that $f = f\alpha f$, I is a right pure ideal and a left pure ideal of M.

Example 3.2. Let $S = \mathbb{N}$ and $\Gamma = \{1, 2, 3\}$. By Example 2.16, S is a Γ-semiring where $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$. Then $\{1\}, \{1, 2\}$ and $\{1, 2, 3\}$ are right pure ideals of S. But $I = \{1, 2, 3, 4\}$ is not a right pure ideal of S because

$4 \neq 4\alpha x$ for all $x \in I$ and $\alpha \in \Gamma$. Since S is a commutative Γ-semiring, $\{1\}, \{1,2\}$ and $\{1,2,3\}$ are left pure ideals of S.

Example 3.3. For $n \in \mathbb{N}$, let $S = M_{n\times 1}(\mathbb{Z}_6)$ and $\Gamma = M_{1\times n}(\mathbb{Z}_6)$. Then S forms a Γ-semiring with $A\alpha B$ being the usual multiplication of matrices for all $A, B \in S$ and $\alpha \in \Gamma$. By Example 2.17, all ideals of S are S,$\left\{[x_{i1}]_{n\times 1} \,|x_{i1} \in \{[0]_6\}\right\}$, $\left\{[x_{i1}]_{n\times 1} \,|x_{i1} \in \{[0]_6, [3]_6\}\right\}$ and $\left\{[x_{i1}]_{n\times 1} \,|x_{i1} \in \{[0]_6, [2]_6, [4]_6\}\right\}$. It's easy to see that S and $\left\{[x_{i1}]_{n\times 1} \,|x_{i1} \in \{[0]_6\}\right\}$ are right pure ideals and left pure ideals of S. Next, to show that $I = \left\{[x_{i1}]_{n\times 1} \,|x_{i1} \in \{[0]_6, [3]_6\}\right\}$ is a right pure ideal of S, let $[x_{i1}]_{n\times 1} \in I$. Then there exist $[[3]_6]_{n\times 1} \in I$ and $[[1]_6]_{1\times n} \in \Gamma$ such that

$$\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} \begin{pmatrix} [1]_6 & [0]_6 & \cdots & [0]_6 \end{pmatrix} \begin{pmatrix} [3]_6 \\ [0]_6 \\ \vdots \\ [0]_6 \end{pmatrix} = \begin{pmatrix} [3]_6 x_{11} \\ [3]_6 x_{21} \\ \vdots \\ [3]_6 x_{n1} \end{pmatrix}.$$

If $x_{i1} = [0]_6$, then $[3]_6 x_{i1} = [0]_6$. If $x_{i1} = [3]_6$, then $[3]_6 x_{i1} = [3]_6$. Hence

$$\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} \begin{pmatrix} [1]_6 & [0]_6 & \cdots & [0]_6 \end{pmatrix} \begin{pmatrix} [3]_6 \\ [0]_6 \\ \vdots \\ [0]_6 \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}.$$

Therefore, I is a right pure ideal of S. Moreover, if $[x_{i1}]_{n\times 1} = [[0]_6]_{n\times 1}$, then there exist $[a_{i1}]_{n\times 1} \in I$ and $[\alpha_{1i}]_{1\times n} \in \Gamma$ such that $[x_{i1}]_{n\times 1} = [a_{i1}]_{n\times 1} [\alpha_{1i}]_{1\times n} [x_{i1}]_{n\times 1}$. But if $[x_{i1}]_{n\times 1} \neq [[0]_6]_{n\times 1}$, then there exists $k \in \{1, 2, ..., n\}$ such that $x_{k1} = [3]_6$. We choose $[\alpha_{1i}]_{1\times n} \in \Gamma$ by

$$\alpha_{1i} = \begin{cases} [1]_6 & \text{if } i = k; \\ [0]_6 & \text{if } i \neq k. \end{cases}$$

Thus

$$\begin{pmatrix} x_{11} \\ \vdots \\ x_{k1} \\ \vdots \\ x_{n1} \end{pmatrix} \begin{pmatrix} [0]_6 & \cdots & [1]_6 & \cdots & [0]_6 \end{pmatrix} \begin{pmatrix} x_{11} \\ \vdots \\ x_{k1} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} [3]_6 x_{11} \\ \vdots \\ [3]_6 x_{k1} \\ \vdots \\ [3]_6 x_{n1} \end{pmatrix} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{k1} \\ \vdots \\ x_{n1} \end{pmatrix}.$$

Therefore, I is a left pure ideal of S. Moreover, $\left\{[x_{i1}]_{n\times 1} \,|x_{i1} \in \{\bar{0}, \bar{2}, \bar{4}\}\right\}$ is a right pure ideal and left pure ideal of S which can be proved in a similar way.

Theorem 3.2. *If a and b are integers such that at least one of them is non-zero and $c = gcd(a, b)$, then there exist integers x and y such that $c = ax + by$.*

We use the above theorem to characterize right pure ideals and left pure ideals in the $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$ in the following result.

Proposition 3.3. *For $k \in \mathbb{N}$, $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where $p_1, p_2, \ldots, p_k$ are pairwise distinct primes and $n_i \in \mathbb{N}$. Let $S = \mathbb{Z}_n = \Gamma$ and $I = \{m[p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}]_n \mid m \in \mathbb{Z}\}$ where $0 \leq l_i \leq n_i$ for all i. Then I is a right pure ideal of S if and only if for each i, $l_i = 0$ or $l_i = n_i$.*

Proof. ($\leftarrow$) Let $l_i = 0$ or $l_i = n_i$ for each i. By re-arrangement, we assume that $l_1 = n_1, l_2 = n_2, \ldots, l_t = n_t$ and $l_{t+1} = 0, l_{t+2} = 0, \ldots, l_k = 0$. We will show that $I = \{m\,[p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}]_n \mid m \in \mathbb{Z}\}$ is a right pure ideal of S. First, we write $a = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$. We obtain that $gcd(a, p_{t+1}^{n_{t+1}} p_{t+2}^{n_{t+2}} \cdots p_k^{n_k}) = 1$. By Theorem 3.2, $xa + yp_{t+1}^{n_{t+1}} p_{t+2}^{n_{t+2}} \cdots p_k^{n_k} = 1$ for some $x, y \in \mathbb{Z}$. Now we conclude that there exists $x \in \mathbb{Z}$ such that $p_{t+1}^{n_{t+1}} p_{t+2}^{n_{t+2}} \cdots p_k^{n_k} | xa - 1$. Let $[z]_n \in I$. Then $[z]_n = c[a]_n$. We obtain from $xa - 1 = wp_{t+1}^{n_{t+1}} p_{t+2}^{n_{t+2}} \cdots p_k^{n_k}$ for some $w \in \mathbb{Z}$ that

$$\begin{aligned} (ca)(xa - 1) &= (ca)(wp_{t+1}^{n_{t+1}} p_{t+2}^{n_{t+2}} \cdots p_k^{n_k}) \\ (ca)(xa) - ca &= cwn. \end{aligned}$$

Since $n|(ca)(xa) - ca$, $[(ca)(xa)]_n = [ca]_n$. Thus $[z]_n[xa]_n = [z]_n$. Hence for $[z]_n \in I$ there exist $[xa]_n \in I$ and $[1]_n \in \Gamma$ such that $[z]_n[1]_n[xa]_n = [z]_n$. Therefore, I is a right pure ideal of S.

($\rightarrow$) Let $I = \{m[p_1^{l_1} \cdots p_k^{l_k}]_n \mid m \in \mathbb{Z}\}$ be a right pure ideal of S. Without loss of generality, suppose that there exists $i = 1$ such that $0 < l_1 < n_1$. Since $l_1 < n_1$, $[p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}]_n \in I - \{0\}$. Thus there exist $t[p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}]_n \in I$ and $[\alpha]_n \in \Gamma$ such that

$$\begin{aligned}
[p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}]_n &= [p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}]_n [\alpha]_n t [p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}]_n \\
&= [\alpha t p_1^{2l_1} p_2^{2l_2} \cdots p_k^{2l_k}]_n.
\end{aligned}$$

We obtain that $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \mid \Big((\alpha t p_1^{2l_1} p_2^{2l_2} \cdots p_k^{2l_k}) - (p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k})\Big)$ and therefore, $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \mid \Big((\alpha t p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k} - 1) p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}\Big)$. Since $l_1 < n_1$, we have $p_1 \mid (\alpha t p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k} - 1) p_2^{l_2} \cdots p_k^{l_k}$. Then $p_1 \mid (\alpha t p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k} - 1)$ because $p_1 \nmid p_2^{l_2} \cdots p_k^{l_k}$. Since $l_1 > 0$, $p_1 \mid (p_1(\alpha t p_1^{l_1 - 1} p_2^{l_2} \cdots p_k^{l_k}) - 1)$. We obtain that

$$\begin{aligned}
p_1(\alpha t p_1^{l_1 - 1} p_2^{l_2} \cdots p_k^{l_k}) - 1 &= c p_1 \\
p_1(\alpha t p_1^{l_1 - 1} p_2^{l_2} \cdots p_k^{l_k} - c) &= 1
\end{aligned}$$

for some $c \in \mathbb{Z}$. It is a contradiction because $\alpha t p_1^{l_1 - 1} p_2^{l_2} \cdots p_k^{l_k} - c \in \mathbb{Z}$. Therefore, $l_i = 0$ or $l_i = n_i$ for all i. □

By commutative property, this proposition holds for left pure ideals in the $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$.

Example 3.4. For the $\mathbb{Z}_{48}$-semiring $\mathbb{Z}_{48}$, we consider $I = \{m[2^4]_{48} \mid m \in \mathbb{Z}\} = \{[0]_{48}, [16]_{48}, [32]_{48}\}$. We know that $48 = (2^4)(3)$ and there exists $[1]_{48} \in \mathbb{Z}_{48}$ such that $3 \mid (1)(2^4) - 1$. We obtain that for $[x]_{48} \in I$, $[x]_{48}[1]_{48}[16]_{48} = [x]_{48}$. Thus I is a right pure ideal of the $\mathbb{Z}_{48}$-semiring $\mathbb{Z}_{48}$. Next, we consider $J = \{m[(2^2)]_{48} \mid m \in \mathbb{Z}_{48}\}$. There exists $3[2^2]_{48} = [12]_{48} \in J - \{0\}$ such that $3[2^2]_{48}[\alpha]_{48} m[2^2]_{48} = m[(3)(2^4)]_{48}[\alpha]_{48} = [0]_{48} \neq 3[2^2]_{48}$ for all $m[2^2]_{48} \in J$ and $\alpha \in \Gamma$. Hence J is not a right pure ideal in the $\mathbb{Z}_{48}$-semiring $\mathbb{Z}_{48}$.

Furthermore, we get right pure ideals in the $M_{m \times k}(\mathbb{Z}_n)$-semiring $M_{k \times m}(\mathbb{Z}_n)$ where $m \leq k$ in the following proposition.

Proposition 3.4. *Let $m \leq k$. Then I is a right pure ideal of the $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$ if and only if $M_{k\times m}(I)$ is a right pure ideal of the $M_{m\times k}(\mathbb{Z}_n)$-semiring $M_{k\times m}(\mathbb{Z}_n)$.*

Proof. ($\rightarrow$) Suppose that I is a right pure ideal of the $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$. From the proof of the above proposition, we obtain that there exist $y \in I$ such that $z = zy$ for all $z \in I$. Since $\mathbb{Z}_n$ is a semiring with identity, I is an ideal of the semiring $\mathbb{Z}_n$. By Example 2.17, $M_{k\times m}(I)$ is an ideal of the $M_{m\times k}(\mathbb{Z}_n)$-semiring $M_{k\times m}(\mathbb{Z}_n)$. To show that $M_{k\times m}(I)$ is a right pure ideal of the $M_{m\times k}(\mathbb{Z}_n)$-semiring $M_{k\times m}(\mathbb{Z}_n)$, let $[z_{ij}]_{k\times m} \in M_{k\times m}(I)$. We choose $[b_{ij}]_{k\times m} \in M_{k\times m}(I)$ by

$$b_{ij} = \begin{cases} y & \text{if } i = j; \\ [0]_n & \text{if } i \neq j \end{cases}$$

and $[\alpha_{ij}]_{m\times k} \in M_{m\times k}(\mathbb{Z}_n)$ by

$$\alpha_{ij} = \begin{cases} [1]_n & \text{if } i = j; \\ [0]_n & \text{if } i \neq j. \end{cases}$$

Then

$$[a_{ij}]_{k\times m}[\alpha_{ij}]_{m\times k}[b_{ij}]_{k\times m} = [a_{ij}]_{k\times m} y I_m = [y a_{ij}]_{k\times m} = [a_{ij}]_{k\times m}.$$

Therefore, $M_{k\times m}(I)$ is a right pure ideal of the $M_{m\times k}(\mathbb{Z}_n)$-semiring $M_{k\times m}(\mathbb{Z}_n)$.
($\leftarrow$) Let $M_{k\times m}(I)$ be a right pure ideal of the $M_{m\times k}(\mathbb{Z}_n)$-semiring $M_{k\times m}(\mathbb{Z}_n)$. By Example 2.17, I is an ideal of the semiring $\mathbb{Z}_n$. By Example 2.14, I is an ideal of the $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$. To show that I is a right pure ideal of the $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$, let $a \in I$. We define $[x]_{k\times m}$ to be the $k \times m$ matrix with x on $(1,1)$-entry and absorbing zeros elsewhere. Then $[a]_{k\times m} \in M_{k\times m}(I)$. We obtain that there exist $[b_{ij}]_{k\times m} \in M_{k\times m}(I)$ and $[\beta_{ij}]_{m\times k} \in \Gamma$ such that

$$[a]_{k\times m} = [a]_{k\times m}[\beta_{ij}]_{m\times k}[b_{ij}]_{k\times m}.$$

Then

$$[a]_{k\times m} = \begin{pmatrix} a\sum_{l=1}^{k}\beta_{1l}b_{l1} & a\sum_{l=1}^{k}\beta_{1l}b_{l2} & \cdots & a\sum_{l=1}^{k}\beta_{1l}b_{lm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence $a = a\sum_{l=1}^{k}\beta_{1l}b_{l1}$. Since I is an ideal of the semiring $\mathbb{Z}_n$ and $b_{lj} \in I$, $\sum_{l=1}^{k}\beta_{1l}b_{l1} \in I$. So there exist $\sum_{l=1}^{k}\beta_{1l}b_{l1} \in I$ and $[1]_n \in \mathbb{Z}_n$ such that $a = a[1]_n\sum_{l=1}^{k}\beta_{1l}b_{l1}$. Therefore, I is a right pure ideal of the $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$. □

A characterization of a right pure ideal in a Γ-semiring with identity element is furnished in the following theorem.

Theorem 3.5. *Let S be a Γ-semiring with identity element e. Then an ideal I of S is right pure if and only if $J \cap I = J\Gamma I$ for all right ideals J of S.*

Proof. ($\rightarrow$) Suppose that I is a right pure ideal of S and J is a right ideal of S. Then $J\Gamma I \subseteq J \cap I$. If $a \in J \cap I$, then there exist $x \in I$, $\alpha \in \Gamma$ such that $a = a\alpha x \in J\Gamma I$. This implies $J \cap I = J\Gamma I$.
($\leftarrow$) Suppose that I is an ideal of S such that $J \cap I = J\Gamma I$ for all right ideals J of S. Let $x \in I$ and $\alpha \in \Gamma$. By Proposition 2.19, $x\alpha S$ is a right ideal of S. Then

$$(x\alpha S) \cap I = (x\alpha S)\Gamma I = x\alpha(S\Gamma I) \subseteq x\alpha I.$$

We obtain from $x = x\alpha e \in x\alpha S$ that $x \in x\alpha S \cap I \subseteq x\alpha I$. This implies $x = \sum x\alpha a_i = x\alpha\sum a_i$, $a_i \in I$ all i. Since I is a subsemigroup of $(S, +)$, $\sum a_i \in I$. Thus there exist $b \in I$, $\alpha \in \Gamma$, such that $x = x\alpha b$. Hence I is a right pure ideal of S. □

Similarly, we can show that an ideal I of S is left pure if and only if $I \cap J = I\Gamma J$ for all left ideals J of S.

Next, we define and characterize right weakly regular Γ-semirings.

Definition 3.6. A Γ-semiring S is said to be **right weakly regular** if for each $x \in S$, $x \in (x\Gamma S)^2 = (x\Gamma S)\Gamma(x\Gamma S)$.

Example 3.5. Let $S = \{1, 2, 3, 4, 5\}$ and $\Gamma = \{5\}$. By Example 2.22, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$. Since $x = (x\alpha x)\alpha(x\alpha x)$ for all $x \in S$, S is right weakly regular.

Theorem 3.7. *Let S be a Γ-semiring with identity element e. Then the following assertions are equivalent:*

(i) S is right weakly regular;

(ii) $J^2 = J\Gamma J = J$ for all right ideals J of S;

(iii) $J \cap I = J\Gamma I$ for all right ideal J and ideal I of S.

Proof. $(i) \to (ii)$ Let S be right weakly regular and J a right ideal of S. We obtain that $J\Gamma J \subseteq J\Gamma S \subseteq J$. Let $x \in J$. Then $x \in (x\Gamma S)^2$. By Proposition 2.27, $x\Gamma S$ is the smallest right ideal containing x which implies that $x\Gamma S \subseteq J$. Thus

$$x \in (x\Gamma S)^2 \subseteq J^2.$$

Hence $J = J^2 = J\Gamma J$.

$(ii) \to (i)$ Suppose that $J^2 = J\Gamma J = J$ for all right ideals J of S. To show that S is right weakly regular, let $x \in S$. We know that $x\Gamma S$ is a right ideal of S. By assumption, $(x\Gamma S)^2 = (x\Gamma S)\Gamma(x\Gamma S) = (x\Gamma S)$. Then $x \in (x\Gamma S)^2$.

$(i) \to (iii)$ Suppose that S is right weakly regular. Let J be a right ideal and I an ideal of S. Then $J\Gamma I \subseteq J \cap I$. Let $x \in J \cap I$. By assumption, $x \in (x\Gamma S)^2$. Thus

$$x \in (x\Gamma S)\Gamma(x\Gamma S) \subseteq (J\Gamma S)\Gamma(I\Gamma S) \subseteq (J\Gamma S)\Gamma I \subseteq J\Gamma I.$$

Hence $J\Gamma I = J \cap I$.

$(iii) \to (i)$ Assume that $J \cap I = J\Gamma I$ for all right ideals J and ideals I of S. To show that S is right weakly regular, let $x \in S$. Then $x \in x\Gamma S \cap S\Gamma x\Gamma S$. Since

$x\Gamma S$ is a right ideal and $S\Gamma x\Gamma S$ an ideal of S, we obtain that

$$x \in x\Gamma S \cap S\Gamma x\Gamma S = (x\Gamma S)\Gamma(S\Gamma x\Gamma S) = (x\Gamma S\Gamma S)\Gamma(x\Gamma S) \subseteq (x\Gamma S)\Gamma(x\Gamma S).$$

Then $x \in (x\Gamma S)\Gamma(x\Gamma S) = (x\Gamma S)^2$. Therefore, S is right weakly regular. □

By virtue of Theorems 3.5 and 3.7, we obtain the following theorem.

Theorem 3.8. *Let S be a Γ-semiring with identity element. Then S is a right weakly regular Γ-semiring if and only if every ideal I of S is right pure.*

From this theorem, we can conclude that every ideal of S in Example 3.5 is right pure.

A sufficient condition on a Γ-semiring S that makes S a right pure ideal and a trivial right pure ideal are shown in the following proposition.

Proposition 3.9. *The following statements hold:*

(i) if S is a Γ-semiring with identity e, then S is a right pure ideal of S;

(ii) if S is a Γ-semiring with zero 0, then any finite intersection of right pure ideals of S is a right pure ideal of S.

Proof. (i) Let S be a Γ-semiring with identity e. Clearly, S is an ideal of S. For right ideal J of S, $J\Gamma S \subseteq J = J \cap S$. In contrast, since $e \in S$, for any $x \in J \cap S, x = x\alpha e \in J\Gamma S$. Thus $J \cap S \subseteq J\Gamma S$. Hence $J \cap S = J\Gamma S$. By Theorem 3.5, S is a right pure ideal of S.
(ii) Suppose that S is a Γ-semiring with zero 0. For $n \in \mathbb{N}$, let $A = \bigcap_{i=1}^{n}\{J_i \mid J_i$ is a right pure ideal of $S\}$. We obtain from $0 \in A$ that A is a nonempty set. For each i, J_i is an ideal of S implies that A is an ideal of S by Proposition 2.20. Let $x \in A$. Then there exist $y_i \in J_i$, $\alpha_i \in \Gamma$ such that $x = x\alpha_i y_i$ for all i. Thus

$$x = x\alpha_1 y_1 = (x\alpha_2 y_2)\alpha_1 y_1 = \ldots = (x\alpha_n y_n)\alpha_{n-1}y_{n-1}\ldots\alpha_2 y_2\alpha_1 y_1.$$

Since each i, J_i is an ideal of S, it follows that $y_n\alpha_{n-1}y_{n-1}\ldots\alpha_2y_2\alpha_1y_1 \in J_i$ for all i. Now we have $y_n\alpha_{n-1}y_{n-1}\ldots\alpha_2y_2\alpha_1y_1 \in A$, $\alpha_n \in \Gamma$ such that

$$x = x\alpha_n(y_n\alpha_{n-1}y_{n-1}\ldots\alpha_2y_2\alpha_1y_1).$$

Hence A is a right pure ideal of S. $\square$

In the same way, we can show that S is a left pure ideal of a Γ-semiring S with identity. Any finite intersection of left pure ideals of S is a left pure ideal of a Γ-semiring S with zero.

By virtue of above theorem, $\{0\}$ is a right pure ideal and a left pure ideal of a Γ-semiring with zero 0.

Next, we include some conditions on right pure ideals and investigate their properties.

Definition 3.10. A proper right pure ideal I of a Γ-semiring S is called **purely prime** if $I_1\Gamma I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ for any right pure ideals I_1 and I_2 of S.

Definition 3.11. A proper right pure ideal I of a Γ-semiring S is called **purely semiprime** if $I_1\Gamma I_1 \subseteq I$ implies $I_1 \subseteq I$ for any right pure ideals I_1 of S.

It is straightforward to verify that every purely prime ideal is purely semiprime.

Example 3.6. For $n \in \mathbb{N}$, let $S = M_{n\times 1}(\mathbb{Z}_{30})$ and $\Gamma = M_{1\times n}(\mathbb{Z}_{30})$. We have S forms a Γ-semiring with the usual multiplication of matrices $A\alpha B$ for all $A, B \in S$ and $\alpha \in \Gamma$. Then all right pure ideals of S are S,$\{[0]_{30}\}$, $M_{n\times 1}(2\mathbb{Z}_{30})$, $M_{n\times 1}(3\mathbb{Z}_{30})$, $M_{n\times 1}(5\mathbb{Z}_{30})$, $M_{n\times 1}(6\mathbb{Z}_{30})$, $M_{n\times 1}(10\mathbb{Z}_{30})$ and $M_{n\times 1}(15\mathbb{Z}_{30})$. Hence

- all purely prime ideals of S are $M_{n\times 1}(2\mathbb{Z}_{30})$, $M_{n\times 1}(3\mathbb{Z}_{30})$ and $M_{n\times 1}(5\mathbb{Z}_{30})$,
- all purely semiprime ideals of S are $M_{n\times 1}(2\mathbb{Z}_{30})$, $M_{n\times 1}(3\mathbb{Z}_{30})$, $M_{n\times 1}(5\mathbb{Z}_{30})$, $M_{n\times 1}(6\mathbb{Z}_{30})$, $M_{n\times 1}(10\mathbb{Z}_{30})$, $M_{n\times 1}(15\mathbb{Z}_{30})$ and $\{[0]_{30}\}$.

Example 3.7. Let $S = \mathbb{N}$ and $\Gamma = \{1, 2, 3\}$. By Example 3.2, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$ and all right pure ideals of S are $I_1 = \{1\}, I_2 = \{1, 2\}$ and $I_3 = \{1, 2, 3\}$. Consider sets $I_i\Gamma I_j$ where I_i and I_j are right pure ideals of S,

$$I_2\Gamma I_2 \subseteq I_2 \quad I_2\Gamma I_3 \subseteq I_2 \quad I_3\Gamma I_2 \subseteq I_2$$
$$I_3\Gamma I_3 \subseteq I_3 \quad I_i\Gamma I_1 \subseteq I_1 \quad I_1\Gamma I_i \subseteq I_1.$$

We obtain that

- all purely prime ideals of S are I_1, I_2 and I_3,
- all purely semiprime ideals of S are I_1, I_2 and I_3.

Using the include condition on right pure for purely prime, the finite intersection of purely prime ideals is just a purely semiprime. It's not a purely prime in general.

Proposition 3.12. *Let S be a Γ-semiring with zero 0. Then the finite intersection of purely prime (or purely semiprime) ideals of S is a purely semiprime ideal of S.*

Proof. For $n \in \mathbb{N}$, let $A = \bigcap_{i=1}^{n} \{P_i \mid P_i$ is a purely prime ideal of $S\}$. By Proposition 3.9, A is a right pure ideal of S. Assume that I is a right pure ideal of S such that $I\Gamma I \subseteq A$. Then $I\Gamma I \subseteq P_i$ for all i. Since P_i is purely prime for all i, P_i is a purely semiprime for all i. Hence $I \subseteq P_i$ for all i, it implies that $I \subseteq A$. Therefore, A is a purely semiprime ideal of S.

Likewise, we can show that if $B = \bigcap_{i=1}^{n} \{P_i \mid P_i$ is a purely semiprime ideal of $S\}$, then B is a purely semiprime ideal of S. □

Definition 3.13. A proper right pure ideal I of a Γ-semiring S is called **purely irreducible** if $I_1 \cap I_2 = I$ implies $I_1 = I$ or $I_2 = I$ for any right pure ideals I_1 and I_2 of S.

Definition 3.14. A proper right pure ideal I of a Γ-semiring S is called a **strongly irreducible pure ideal** if $I_1 \cap I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ for any right pure ideals I_1 and I_2 of S.

It is straightforward to verify that every strongly irreducible pure ideal is purely irreducible.

Example 3.8. For $n \in \mathbb{N}$, let $S = M_{n\times 1}(\mathbb{Z}_{30})$ and $\Gamma = M_{1\times n}(\mathbb{Z}_{30})$. We have S forms a Γ-semiring with the usual multiplication of matrices $A\alpha B$ for all $A, B \in S$ and $\alpha \in \Gamma$. Then all right pure ideals of S are S,$\{[0]_{30}\}$, $M_{n\times 1}(2\mathbb{Z}_{30})$, $M_{n\times 1}(3\mathbb{Z}_{30})$, $M_{n\times 1}(5\mathbb{Z}_{30})$, $M_{n\times 1}(6\mathbb{Z}_{30})$, $M_{n\times 1}(10\mathbb{Z}_{30})$ and $M_{n\times 1}(15\mathbb{Z}_{30})$. Hence

- all strongly irreducible pure ideals of S are $M_{n\times 1}(2\mathbb{Z}_{30})$, $M_{n\times 1}(3\mathbb{Z}_{30})$ and $M_{n\times 1}(5\mathbb{Z}_{30})$,
- all purely irreducible ideals of S are $M_{n\times 1}(2\mathbb{Z}_{30})$, $M_{n\times 1}(3\mathbb{Z}_{30})$ and $M_{n\times 1}(5\mathbb{Z}_{30})$.

Example 3.9. Let $S = \mathbb{N}$ and $\Gamma = \{1, 2, 3\}$. By Example 3.2, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$ and all right pure ideals of S are $I_1 = \{1\}, I_2 = \{1, 2\}$ and $I_3 = \{1, 2, 3\}$. Consider sets $I_i \cap I_j$ where I_i and I_j are right pure ideals of S,

$$I_2 \cap I_2 = I_2 \quad I_2 \cap I_3 = I_2 \quad I_3 \cap I_2 = I_2$$
$$I_3 \cap I_3 = I_3 \quad I_i \cap I_1 = I_1 \quad I_1 \cap I_i = I_1.$$

We obtain that

- all strongly irreducible pure ideals of S are I_1, I_2 and I_3,
- all purely irreducible ideals of S are I_1, I_2 and I_3.

On the space of right pure ideals, purely prime and strongly irreducible pure ideals are coincide.

Proposition 3.15. *Let I be an ideal of a Γ-semiring S with identity e. Then I is a strongly irreducible pure ideal of S if and only if I is a purely prime ideal of S.*

Proof. $(\rightarrow)$ Suppose that I is a strongly irreducible pure ideal of S. To show that I is a purely prime ideal of S, let I_1 and I_2 be right pure ideals of S such that $I_1\Gamma I_2 \subseteq I$. Since I_1 is a right ideal and I_2 is a right pure ideal of S, $I_1 \cap I_2 =$

$I_1\Gamma I_2 \subseteq I$ by Theorem 3.5. Since I is a strongly irreducible pure ideal of S, $I_1 \subseteq I$ or $I_2 \subseteq I$.

($\leftarrow$) Suppose that I is a purely prime ideal of S. To show that I is a strongly irreducible pure ideal of S, let I_1 and I_2 be right pure ideals of S such that $I_1 \cap I_2 \subseteq I$. By Theorem 3.5, $I_1 \cap I_2 = I_1\Gamma I_2$. So $I_1\Gamma I_2 \subseteq I$. Since I is a purely prime of S, $I_1 \subseteq I$ or $I_2 \subseteq I$. □

A **partially ordered set** is defined as a set P together with relation " $\leq$ " over set P if it is reflexive ($x \leq x$ for all $x \in P$), antisymmetric ($x \leq y$ and $y \leq x$ together imply $x = y$ for all $x, y \in P$) and transitive ($x \leq y$ and $y \leq z$ together imply $x \leq z$ for all $x, y, z \in P$). A **totally ordered set** is a subset T of a partially ordered set P such that any two elements of T are comparable with relation $\leq$ (for any $s, t \in T$ we have either $s \leq t$ or $t \leq s$) [10].

We recall that every strongly irreducible pure ideal is purely irreducible. The converse is not true generally. However, the converse is true on some assumption of the set of all right pure ideals of a Γ-semiring with zero.

Proposition 3.16. *Let S be a Γ-semiring with zero. Then the following assertions are equivalent.*

(i) The set of all right pure ideals of S is a totally ordered set under inclusion of sets.

(ii) Each right pure ideal of S is a strongly ireducible pure ideal of S.

(iii) Each right pure ideal of S is a purely ireducible ideal of S.

Proof. $(i) \rightarrow (ii)$ Suppose that the set of all right pure ideals of S is totally ordered under inclusion of sets and I is a right pure ideal of S. To show that I is a strongly irreducible pure ideal of S, let I_1 and I_2 be right pure ideals of S such that $I_1 \cap I_2 \subseteq I$. By assumption, $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. Then $I_1 \cap I_2 = I_1$ or $I_1 \cap I_2 = I_2$, it follows that $I_1 \subseteq I$ or $I_2 \subseteq I$. Hence I is a strongly irreducible pure ideal of S.

$(ii) \rightarrow (iii)$ Suppose that each right pure ideal of S is a strongly irreducible pure ideal of S. Let I be a right pure ideal of S and $I_1 \cap I_2 = I$ where I_1 and I_2 are

right pure ideals of S. By assumption, $I_1 \subseteq I$ or $I_2 \subseteq I$. Since $I = I_1 \cap I_2 \subseteq I_1$ and $I = I_1 \cap I_2 \subseteq I_2$, $I_1 = I$ or $I_2 = I$. Hence I is a purely irreducible ideal of S.
$(iii) \to (i)$ Suppose that each right pure ideal of S is a purely irreducible ideal. To show that the set of all right pure ideals of S is totally ordered set under inclusion of sets, let I_1 and I_2 be right pure ideals of S. Thus $I_1 \cap I_2$ is a right pure ideal of S. By assumption $I_1 \cap I_2$ is purely irreducible of S. Note that $I_1 \cap I_2 = I_1 \cap I_2$ we obtain that $I_1 = I_1 \cap I_2$ or $I_2 = I_1 \cap I_2$. Hence $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. Therefore, the set of all right pure ideals of S is totally ordered set under inclusion of sets. □

Proposition 3.17. *Let $\mathcal{X}$ be the set of right pure ideals of a Γ-semiring S, ordered by inclusion and $\mathcal{C}$ a totally ordered subset of $\mathcal{X}$. Then $\bigcup_{J\in\mathcal{C}} J$ is a right pure ideal of S.*

Proof. Let $M = \bigcup_{J\in\mathcal{C}} J$. To show that M is a right pure ideal of S, let $x_1, x_2 \in M$. Then $x_1 \in J_1$ and $x_2 \in J_2$ for some $J_1, J_2 \in \mathcal{C}$. Thus $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$; WLOG the former is assumed. We obtain $x_1 \in J_2$ so that $x_1 + x_2 \in J_2 \subseteq M$. Next, let $a \in S$ and $\alpha \in \Gamma$. Then $a\alpha x_2, x_2\alpha a \in J_2 \subseteq M$. It follows that M is an ideal of S. Since $x_1 \in J_1$ and J_1 is a right pure ideal of S, there exist $y_1 \in J_1 \subseteq M$ and $\alpha \in \Gamma$ such that $x_1 = x_1\alpha y_1$. Hence M is a right pure ideal of S. □

Zorn's lemma is a proposition of set theory which states that a partially ordered set containing upper bounds for every totally ordered subset necessarily contains at least one maximal element.

Proposition 3.18. *Let I be a right pure ideal of S and $a \in S$ such that $a \notin I$. Then there exists a purely irreducible ideal J of S such that $I \subseteq J$ and $a \notin J$.*

Proof. Let $X = \{J \mid J$ is a right pure ideal of S such that $I \subseteq J$ and $a \notin J\}$. We obtain that $X \neq \varnothing$ because $I \in X$. First, we will show that a partially ordered set X containing upper bounds for every totally ordered subset. Let T be a totally ordered subset of X and $M = \bigcup_{J\in\mathcal{C}} J$. By Proposition 3.17, M is a right pure ideal in S. We have $I \subseteq J_i$ and $a \notin J_i$ for all i implies that $I \subseteq M$ and $a \notin M$. Thus an upper bound M of T contain in X. We obtain that X has a maximal element, say J by Zorn's Lemma. Thus J is a right pure ideal such that $I \subseteq J$ and $a \notin J$. Next, show that J is a purely irreducible ideal of S. Let A and B are right pure ideals of S such that $A \cap B = J$. Then $J \subseteq A$ and $J \subseteq B$. Suppose that $J \subset A$ and $J \subset B$. By maximality of J, $A \notin X$ and $B \notin X$. Thus $I \nsubseteq A$ or $a \in A$ and $I \nsubseteq B$ or $a \in B$. Since $I \subset A$ and $I \subset B$, $a \in A \cap B = J$ which is a contradiction. Hence $J \nsubseteq A$ or $J \nsubseteq B$. Therefor, $J = A$ or $J = B$. □

We use Proposition 3.18 to find a relationship between right pure ideals and purely irreducible ideals in Γ- semirings.

Proposition 3.19. *Every proper right pure ideal I of S is the intersection of all purely irreducible ideals of S containing I.*

Proof. Suppose that I is a proper right pure ideal S. By Proposition 3.18, there exists a purely irreducible ideal containing I. Let $\{J_k\}_{k\in K}$ be the family of all purely irreducible ideals of S which contain I. Since $I \subseteq J_k$ for all $k \in K$, $I \subseteq \bigcap_{k\in K} J_k$. To show that $\bigcap_{k\in K} J_k \subseteq I$, let $a \notin I$. Then there exists a purely irreducible ideal J such that $I \subseteq J$ and $a \notin J$ by Proposition 3.18. It follows that $a \notin \bigcap_{k\in K} J_k$. We now conclude that $I = \bigcap_{k\in K} J_k$. □

Next, we introduce the concept of a purely maximal ideal of a Γ-semiring.

Definition 3.20. A proper right pure ideal I of a Γ-semiring S is said to be **purely maximal** if for any proper right pure ideals J of S, $I \subseteq J$ implies that $I = J$.

Example 3.10. For $n \in \mathbb{N}$, let $S = M_{n\times 1}(\mathbb{Z}_{30})$ and $\Gamma = M_{1\times n}(\mathbb{Z}_{30})$. We have S forms a Γ-semiring with the usual multiplication of matrices $A\alpha B$ for all $A, B \in S$ and $\alpha \in \Gamma$. Then all of right pure ideals of S are S,$\{[0]_{30}\}$, $M_{n\times 1}(2\mathbb{Z}_{30})$, $M_{n\times 1}(3\mathbb{Z}_{30})$, $M_{n\times 1}(5\mathbb{Z}_{30})$, $M_{n\times 1}(6\mathbb{Z}_{30})$, $M_{n\times 1}(10\mathbb{Z}_{30})$ and $M_{n\times 1}(15\mathbb{Z}_{30})$. We obtain that $M_{n\times 1}(2\mathbb{Z}_{30})$, $M_{n\times 1}(3\mathbb{Z}_{30})$ and $M_{n\times 1}(5\mathbb{Z}_{30})$ are purely maximal ideals of S.

Example 3.11. Let $S = \mathbb{N}$ and $\Gamma = \{1, 2, 3\}$. By Example 3.2, S is a Γ-semiring with $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S$ and $\alpha \in \Gamma$. All of proper right pure ideals of S are $\{1\}, \{1, 2\}$ and $\{1, 2, 3\}$. Thus $\{1, 2, 3\}$ is a purely maximal ideal of S but $\{1\}$ and $\{1, 2\}$ are not purely maximal ideals of S.

The following proposition guarantee existence of a purely maximal ideal in a Γ-semiring with identity element.

Proposition 3.21. *If I is a proper right pure ideal of a Γ-semiring S with identity element e, then S contains a purely maximal ideal M such that $I \subseteq M$.*

Proof. Let $\mathcal{X}$ be the set of proper right pure ideals of S, ordered by inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{X}$ and $M = \bigcup_{J \in \mathcal{C}} J$. By Proposition 3.17, M is a right pure ideal of S. If $e \in J$ for some $J \in \mathcal{C}$, then for all $x \in S$ and $\alpha \in \Gamma$, $x\alpha e, e\alpha x \in J$ which implies that $x \in J$. Thus $J = S$ which is a contradiction. We obtain from $e \notin J$ for all $J \in \mathcal{C}$ that $e \notin M$ and $M \neq S$. □

Proposition 3.22. *If I is a purely maximal ideal of a Γ-semiring S, then I is a purely irreducible ideal of S.*

Proof. Suppose that I is a purely maximal ideal of a Γ-semiring S. To show that I is a purely irreducible ideal of S, let J_1 and J_2 be right pure ideals of S such that $J_1 \cap J_2 = I$. Since I is a proper right pure ideal of S, J_1 or J_2 is a proper right pure ideal of S; WLOG the former is assumed. Then $I = J_1 \cap J_2 \subseteq J_1$. By assumption, $I = J_1$. Hence I is a purely irreducible ideal of S. □

The converse of Proposition 3.22 is not true. Counterexamples are $\{1, 2\}$ and $\{1\}$ in Example 3.11.

The last, from Theorem 3.5, I is a right pure ideal (left pure ideal) of a Γ-semiring S with identity provided $J\Gamma I = J \cap I$ for all right ideals (left ideals) J of S. Now, we construct right weakly pure ideals and left weakly pure ideals in a Γ-semiring S under the condition of all ideals of S.

Definition 3.23. An ideal A of a Γ-semiring S is called **right weakly pure** (**left weakly pure**) if $B \cap A = B\Gamma A$ ($A \cap B = A\Gamma B$) for all ideals B of S.

Every right pure ideal (left pure ideal) is a right weakly pure (left weakly pure).

Proposition 3.24. *If A and B are ideals of a Γ-semiring S with zero 0, then*

$$A_{-1}B = \{b \in S \mid b\alpha a \in B \textit{ for all } a \in A, \alpha \in \Gamma\}$$

and

$$BA^{-1} = \{b \in S \mid a\alpha b \in B \textit{ for all } a \in A, \alpha \in \Gamma\}$$

are ideals of S*.*

Proof. Suppose that A and B are ideals of a Γ-semiring S with zero. Since $0\alpha a = 0 \in B$ for all $a \in A$, $\alpha \in \Gamma$, $0 \in A_{-1}B$. So $A_{-1}B \neq \varnothing$. Let $b, b' \in A_{-1}B$, $a \in A$ and $\alpha \in \Gamma$. Then

$$(b + b')\alpha a = b\alpha a + b'\alpha a \in B + B.$$

Since B is a subsemigroup of $(S, +)$, $B + B \subseteq B$. Now we have $(b + b')\alpha a \in B + B \subseteq B$. So $b + b' \in A_{-1}B$. Thus $A_{-1}B$ is a subsemigroup of S. Next, let $x \in A_{-1}B, s \in S$ and $\beta \in \Gamma$. Then for any $y \in A$ and $\gamma \in \Gamma$, we obtain $s\gamma y \in A$ so that

$$(x\beta s)\gamma y = x\beta(s\gamma y) \in B$$

and

$$(s\beta x)\gamma y = s\beta(x\gamma y) \in S\Gamma B \subseteq B.$$

Hence $x\beta s, s\beta x \in A_{-1}B$. This shows that $A_{-1}B$ is an ideal of S. Similarly, BA^{-1} is an ideal of S. □

From above proposition, we can characterize right weakly pure ideals and left weakly pure ideals of Γ-semirings in the following theorem.

Theorem 3.25. *Let* S *be a* Γ*-semiring with zero* 0*. An ideal* A *of* S *is right weakly pure (left weakly pure) if and only if* $(A_{-1}B) \cap A = A \cap B$ $((BA^{-1}) \cap A = B \cap A)$ *for all ideals* B *of* S*.*

Proof. $(\rightarrow)$ Suppose that A is a right weakly pure ideal and B is an ideal of S. By Proposition 3.24, $A_{-1}B$ is an ideal of S. By assumption, $(A_{-1}B) \cap A = (A_{-1}B)\Gamma A$. For any $x \in (A_{-1}B)\Gamma A$, $x = \sum b_i\alpha_i a_i$ where $b_i \in A_{-1}B, \alpha_i \in \Gamma$ and $a_i \in A$ for all i. Then $b_i\alpha_i a_i \in B$ for all i. Since B is a subsemigroup of $(S, +)$, $x \in B$. Now we have $(A_{-1}B)\Gamma A \subseteq B$ and $(A_{-1}B)\Gamma A \subseteq A$. Thus

$$(A_{-1}B) \cap A = (A_{-1}B)\Gamma A \subseteq A \cap B.$$

Since B is an ideal of S, for any $b \in A \cap B$, $b\alpha a \in B$ for all $a \in A, \alpha \in \Gamma$. Thus $b \in A_{-1}B$. Hence $A \cap B \subseteq (A_{-1}B) \cap A$. Therefore, $(A_{-1}B) \cap A = A \cap B$.
$(\leftarrow)$ Assume that A is an ideal of S such that $(A_{-1}B) \cap A = A \cap B$ for all ideals B of S. To show that A is right weakly pure, let B be an ideal of S and $b \in B$. Thus for all $a \in A$ and $\alpha \in \Gamma$, $b\alpha a \in B\Gamma A$ and $B\Gamma A$ is an ideal of S because $(B\Gamma A)\Gamma S = B\Gamma(A\Gamma S) \subseteq B\Gamma A$ and $S\Gamma(B\Gamma A) = (S\Gamma B)\Gamma A \subseteq B\Gamma A$. We obtain $b \in A_{-1}(B\Gamma A)$ so that $B \subseteq A_{-1}(B\Gamma A)$. Thus

$$A \cap B \subseteq A \cap A_{-1}(B\Gamma A) \subseteq A \cap (B\Gamma A) \subseteq B\Gamma A.$$

Since $B\Gamma A \subseteq A \cap B$, $B\Gamma A = A \cap B$. Therefore, A is right weakly pure.

In the same way, we can shows an ideal A of S is left weakly pure if and only if $BA^{-1} \cap A = B \cap A$ for all ideals B of S. □

The following proposition shows the condition on the set of ideals in a Γ-semiring with zero that make left weakly pure ideals and right weakly pure ideals coincide.

Proposition 3.26. *Let S be a Γ-semiring with zero. Then the following assertions are equivalent:*

(i) each ideal of S is left weakly pure;

(ii) each ideal I of S, $I^2 = I\Gamma I = I$;

(iii) each ideal of S is right weakly pure.

Proof. $(i) \to (ii)$ Suppose that each ideal of S is left weakly pure. Let I be an ideal of S. Then $I \cap J = I\Gamma J$ for all ideals J of S. Hence

$$I = I \cap I = I\Gamma I = I^2.$$

$(ii) \rightarrow (i)$ Assume that each ideal I of S, $I^2 = I\Gamma I = I$. Let A and B be ideals of S. We have $A \cap B$ is an idael of S. Thus

$$A \cap B = (A \cap B)\Gamma(A \cap B) \subseteq A\Gamma B.$$

Since $A\Gamma B \subseteq A \cap B$, $A\Gamma B = A \cap B$. Therefore, A is left weakly pure.

$(ii) \rightarrow (iii)$ Similarly, as $(ii) \rightarrow (i)$.

$(iii) \rightarrow (ii)$ Suppose that each ideal of S is right weakly pure. Let I be an ideal of S. Then I is a right weakly pure ideal of S, it follows that $I \cap J = J\Gamma I$ for all ideals J of S. Thus

$$I = I \cap I = I\Gamma I = I^2.$$

□

CHAPTER IV

PURE IDEALS IN NEW Γ-SEMIRINGS

We devide this chapter into two parts. First, we find the conditions for preservation of right pure ideals and left pure ideals by Γ-semiring homomorphisms which were introduced by H. Hedayati and K. P. Shum in 2011. According to H. Hedayati and K. P. Shum [6], we investigate some properties of right pure ideals and left pure ideals in new Γ-semirings which are quotient Γ-semirings and the products of Γ-semirings in the last two parts.

The basic structure of a Γ-semiring homomorphism is a semigroup homomorphism.

Definition 4.1. [5] For semigroups S and T, a map $\phi : S \to T$ is a **homomorphism** if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in S$. **Monomorphisms**, **epimorphisms** and **isomorphisms** are defined as usual.

In 2011, H. Hedayati and K. P. Shum [6] introduced homomorphisms, epimorphisms, monomorphisms and isomorphisms in Γ-semirings.

Definition 4.2. [6] Let S_1 be a Γ_1-semiring and S_2 a Γ_2-semiring. A mapping $(\varphi, g) : S_1 \longrightarrow S_2$ is called a **Γ-semiring homomorphism** if $\varphi : S_1 \longrightarrow S_2$ and $g : \Gamma_1 \longrightarrow \Gamma_2$ are semigroup homomorphisms such that $\varphi(x\alpha y) = \varphi(x)g(\alpha)\varphi(y)$ for all $x, y \in S_1$ and $\alpha \in \Gamma_1$. The mapping (φ, g) is called a **Γ-semiring epimorphism** if (φ, g) is a Γ-semiring homomorphism and φ and g are epimorphisms of semigroups.

The definitions of **Γ-semiring monomorphisms** and **Γ-semiring isomorphisms** are defined usually.

In this research, we may write "homomorphism" instead of "Γ-semiring homomorphism". It is similar to write epimorphism, monomorphism and isomorphism instead of Γ-semiring epimorphism, Γ-semiring monomorphism and Γ-semiring isomorphism, respectively.

We need to introduce the following notations used throughout this thesis. For a nonempty subset X of a Γ_1-semiring S_1 and a nonempty subset Y of a Γ_2-semiring S_2,

$$(\varphi, g)(X) = \{(\varphi, g)(x) \mid x \in X\} = \varphi[X]$$
$$(\varphi, g)^{-1}(Y) = \{x \in S_1 \mid (\varphi, g)(x) \in Y\} = \varphi^{-1}[Y]$$

where φ is a semigroup homomorphism from S_1 to S_2 and g is a semigroup homomorphism from Γ_1 to Γ_2.

Example 4.1. Let $S_1 = \{1, 2, 3, 4, 5\}$, $\Gamma_1 = \{1, 2, 3\}$, $S_2 = \{7, 8, 9, 10\}$ and $\Gamma_2 = \{7, 8\}$. Then (S_1, max), (Γ_1, max), (S_2, max) and (Γ_2, max) are commutative semigroups. Define the mappings $S_1 \times \Gamma_1 \times S_1 \to S_1$ by $a\alpha b = min\{a, \alpha, b\}$ for all $a, b \in S_1$ and $\alpha \in \Gamma_1$ and $S_2 \times \Gamma_2 \times S_2 \to S_2$ by $x\beta y = min\{x, \beta, y\}$ for all $x, y \in S_2$ and $\beta \in \Gamma_2$. Thus S_1 is a Γ_1-semiring and S_2 is a Γ_2-semiring.
Consider the mapping $(\varphi, g) : S_1 \longrightarrow S_2$ defined by

$$1 \overset{\varphi}{\mapsto} 7,\ 2 \overset{\varphi}{\mapsto} 7,\ 3 \overset{\varphi}{\mapsto} 8,\ 4 \overset{\varphi}{\mapsto} 9,\ 5 \overset{\varphi}{\mapsto} 10$$
$$1 \overset{g}{\mapsto} 7,\ 2 \overset{g}{\mapsto} 7 \text{ and } 3 \overset{g}{\mapsto} 8.$$

We obtain that

$$\varphi\,(max\{1, 1\}) = \varphi(1) = 7 = max\{7, 7\} = max\{\varphi(1), \varphi(1)\},$$
$$\varphi\,(max\{1, 2\}) = \varphi(2) = 7 = max\{7, 7\} = max\{\varphi(1), \varphi(2)\},$$
$$\varphi\,(max\{1, 3\}) = \varphi(3) = 8 = max\{7, 8\} = max\{\varphi(1), \varphi(3)\},$$
$$\varphi\,(max\{1, 4\}) = \varphi(4) = 9 = max\{7, 9\} = max\{\varphi(1), \varphi(4)\},$$
$$\varphi\,(max\{1, 5\}) = \varphi(5) = 10 = max\{7, 10\} = max\{\varphi(1), \varphi(5)\},$$
$$\varphi\,(max\{2, 2\}) = \varphi(2) = 7 = max\{7, 7\} = max\{\varphi(2), \varphi(2)\},$$
$$\varphi\,(max\{2, 3\}) = \varphi(3) = 8 = max\{7, 8\} = max\{\varphi(2), \varphi(3)\},$$
$$\varphi\,(max\{2, 4\}) = \varphi(4) = 9 = max\{7, 9\} = max\{\varphi(2), \varphi(4)\},$$

$$\varphi\left(max\{2,5\}\right) = \varphi(5) = 10 = max\{7,10\} = max\{\varphi(2),\varphi(5)\},$$

$$\varphi\left(max\{3,3\}\right) = \varphi(3) = 8 = max\{8,8\} = max\{\varphi(3),\varphi(3)\},$$

$$\varphi\left(max\{3,4\}\right) = \varphi(4) = 9 = max\{8,9\} = max\{\varphi(3),\varphi(4)\},$$

$$\varphi\left(max\{3,5\}\right) = \varphi(5) = 10 = max\{8,10\} = max\{\varphi(3),\varphi(5)\},$$

$$\varphi\left(max\{4,4\}\right) = \varphi(4) = 9 = max\{9,9\} = max\{\varphi(4),\varphi(4)\},$$

$$\varphi\left(max\{4,5\}\right) = \varphi(5) = 10 = max\{9,10\} = max\{\varphi(4),\varphi(5)\},$$

$$\varphi\left(max\{5,5\}\right) = \varphi(5) = 10 = max\{10,10\} = max\{\varphi(5),\varphi(5)\}.$$

Hence $\varphi : S_1 \longrightarrow S_2$ is a semigroup homomorphism. In the same way, $g : \Gamma_1 \longrightarrow \Gamma_2$ is a semigroup homomorphism. We show that $\varphi(x\alpha y) = \varphi(x)g(\alpha)\varphi(y)$ for all $x, y \in S_1$ and $\alpha \in \Gamma_1$, let $x, y \in S_1$ and $\alpha \in \Gamma_1$.

Case 1 : if $x = y = \alpha$, then $\varphi(x) = g(\alpha) = \varphi(y)$. Hence $\varphi\left(min\{x,\alpha,y\}\right) = \varphi(x) = min\{\varphi(x), g(\alpha), \varphi(y)\}$.

Case 2 : if $x = y < \alpha$, then $\varphi(x) = \varphi(y) \leq g(\alpha)$. Hence $\varphi\left(min\{x,\alpha,y\}\right) = \varphi(x) = min\{\varphi(x), g(\alpha), \varphi(y)\}$.

Case 3 : if $x = \alpha < y$, then $\varphi(x) = g(\alpha) \leq \varphi(y)$. Hence $\varphi\left(min\{x,\alpha,y\}\right) = \varphi(x) = min\{\varphi(x), g(\alpha), \varphi(y)\}$.

Case 4 : if $\alpha < x = y$, then $g(\alpha) \leq \varphi(x) = \varphi(y)$. Hence $\varphi\left(min\{x,\alpha,y\}\right) = g(\alpha) = min\{\varphi(x), g(\alpha), \varphi(y)\}$.

Case 5 : if $x < \alpha < y$, then $\varphi(x) \leq g(\alpha) \leq \varphi(y)$. Hence $\varphi\left(min\{x,\alpha,y\}\right) = \varphi(x) = min\{\varphi(x), g(\alpha), \varphi(y)\}$.

Case 6 : if $\alpha < x < y$, then $g(\alpha) \leq \varphi(x) < \varphi(y)$. Hence $\varphi\left(min\{x,\alpha,y\}\right) = g(\alpha) = min\{\varphi(x), g(\alpha), \varphi(y)\}$.

Case 7 : if $x < y < \alpha$, then $\varphi(x) \leq \varphi(y) < g(\alpha)$. Hence $\varphi\left(min\{x,\alpha,y\}\right) = \varphi(x) = min\{\varphi(x), g(\alpha), \varphi(y)\}$.

Therefore, (φ, g) is a Γ-semiring homomorphism. Moreover, it is easy to see that φ and g are semigroup epimorphisms. Hence (φ, g) is a Γ-semiring epimorphism.

Example 4.2. Let $S_1 = M_{n\times 1}(\mathbb{Z}_{12})$, $\Gamma_1 = M_{1\times n}(\mathbb{Z}_{12})$, $S_2 = M_{n\times 1}(\mathbb{Z}_6)$ and $\Gamma_2 = M_{1\times n}(\mathbb{Z}_6)$. Then S_1 is a Γ_1-semiring and S_2 is a Γ_2-semiring. Define the mappings $\varphi : S_1 \longrightarrow S_2$ and $g : \Gamma_1 \longrightarrow \Gamma_2$ by $\big[[x_{i1}]_{12}\big]_{n\times 1} \overset{\varphi}{\longmapsto} \big[[x_{i1}]_6\big]_{n\times 1}$ and $\big[[x_{i1}]_{12}\big]_{1\times n} \overset{g}{\longmapsto} \big[[x_{1i}]_6\big]_{1\times n}$. Clearly, φ and g are well defined. We obtain that

$$\begin{aligned}
\varphi\Big(\big[[x_{i1}]_{12}\big]_{n\times 1} + \big[[y_{i1}]_{12}\big]_{n\times 1}\Big) &= \varphi\Big(\big[[x_{i1}]_{12} + [y_{i1}]_{12}\big]_{n\times 1}\Big) \\
&= \varphi\Big(\big[[x_{i1} + y_{i1}]_{12}\big]_{n\times 1}\Big) \\
&= \big[[x_{i1} + y_{i1}]_6\big]_{n\times 1} \\
&= \big[[x_{i1}]_6\big]_{n\times 1} + \big[[y_{i1}]_6\big]_{n\times 1} \\
&= \varphi\Big(\big[[x_{i1}]_{12}\big]_{n\times 1}\Big) + \varphi\Big(\big[[y_{i1}]_{12}\big]_{n\times 1}\Big),
\end{aligned}$$

where $\big[[x_{i1}]_{12}\big]_{n\times 1}, \big[[y_{i1}]_{12}\big]_{n\times 1} \in S_1$. This shows that φ is a semigroup homomorphism. Furthermore, we obtain that for every $\big[[x_{i1}]_6\big]_{n\times 1} \in S_2$ there exists $\big[[x_{i1}]_{12}\big]_{n\times 1} \in S_1$ such that $\varphi(\big[[x_{i1}]_{12}\big]_{n\times 1}) = \big[[x_{i1}]_6\big]_{n\times 1}$. Therefore, φ is a semigroup epimorphism. In the same way, g is a semigroup epimorphism. In particular,

$$\begin{aligned}
&\varphi\Big(\big[[x_{i1}]_{12}\big]_{n\times 1}\big[[\alpha_{1i}]_{12}\big]_{1\times n}\big[[y_{i1}]_{12}\big]_{n\times 1}\Big) \\
&\quad= \varphi\left(\left[\left[\sum_{k=1}^{n} x_{i1}\alpha_{1k}y_{k1}\right]_{12}\right]_{n\times 1}\right) \\
&\quad= \left[\left[\sum_{k=1}^{n} x_{i1}\alpha_{1k}y_{k1}\right]_{6}\right]_{n\times 1} \\
&\quad= \big[[x_{i1}]_6\big]_{n\times 1}\big[[\alpha_{1i}]_6\big]_{1\times n}\big[[y_{i1}]_6\big]_{n\times 1} \\
&\quad= \varphi\Big(\big[[x_{i1}]_{12}\big]_{n\times 1}\Big) g\Big(\big[[\alpha_{1i}]_{12}\big]_{1\times n}\Big) \varphi\Big(\big[[y_{i1}]_{12}\big]_{n\times 1}\Big).
\end{aligned}$$

where $\big[[x_{i1}]_{12}\big]_{n\times 1}, \big[[y_{i1}]_{12}\big]_{n\times 1} \in S_1$ and $\big[[\alpha_{1i}]_{12}\big]_{1\times n} \in \Gamma_1$. Now we conclude that (φ, g) is a Γ-semiring epimorphism.

Proposition 4.3. *Let S_1 be a Γ_1-semiring with zero 0_{S_1} and S_2 a Γ_2-semiring with zero 0_{S_2}. If (φ, g) is a homomorphism from S_1 into S_2 and $\varphi(x) = 0_{S_2}$ for some $x \in S_1$, then $\varphi(0_{S_1}) = 0_{S_2}$.*

Proof. If (φ, g) is a homomorphism from S_1 into S_2 and $\varphi(x) = 0_{S_2}$ for some

$x \in S_1$, then

$$0_{S_2} = 0_{S_2} g(\alpha)\varphi(0_{S_1}) = \varphi(x)g(\alpha)\varphi(0_{S_1}) = \varphi(x\alpha 0_{S_1}) = \varphi(0_{S1}).$$

□

Note that if (φ, g) is a homomorphism from Γ_1-semiring S_1 to Γ_2-semiring S_2, then $\varphi[S_1]$ is a $g[\Gamma_1]$-semiring.

Next, we investigate homomorphisms of right pure ideals and left pure ideals in Γ-semirings.

Theorem 4.4. *Let (φ, g) be a homomorphism from a Γ_1-semiring S_1 to a Γ_2-semiring S_2. If I is an ideal of S_1, then $(\varphi, g)(I)$ is an ideal of the $g[\Gamma_1]$-semiring $\varphi[S_1]$.*

Proof. Suppose that I is an ideal of S_1. Clearly, $(\varphi, g)(I)$ is a nonempty subset of S_2. Since φ is a semigroup homomorphism and I is a subsemigroup of S_1, $\varphi[I]$ is a subsemigroup of S_2. Hence $(\varphi, g)(I) = \varphi[I]$ is a subsemigroup of S_2. Next, let $s \in \varphi[S_1]$, $x \in \varphi[I]$ and $\alpha \in g[\Gamma_1]$. Then $s = \varphi(t)$, $x = \varphi(a)$ and $\alpha = g(\beta)$ for some $t \in S_1$, $a \in I$ and $\beta \in \Gamma_1$. Thus

$$s\alpha x = \varphi(t)g(\beta)\varphi(a) = \varphi(t\beta a) \text{ and } x\alpha s = \varphi(a)g(\beta)\varphi(t) = \varphi(a\beta t).$$

Since I is an ideal of S_1, $t\beta a, a\beta t \in I$. We obtain that $s\alpha x, x\alpha s \in \varphi[I] = (\varphi, g)(I)$. Therefore, $(\varphi, g)(I)$ is an ideal of the $g[\Gamma_1]$-semiring $\varphi[S_1]$. □

Corollary 4.5. *Let S_1 be a Γ_1-semiring and S_2 a Γ_2-semiring. If (φ, g) is an epimorphism, then $(\varphi, g)(I)$ is an ideal of S_2 where I is an ideal of S_1.*

Example 4.3. From Example 4.1, $(\varphi, g) : S_1 \longrightarrow S_2$ is a Γ-semiring epimorphism. Since $I = \{1, 2, 3\}$ is an ideal of S_1, $\{7, 8\} = (\varphi, g)(I)$ is an ideal of S_2.

Example 4.4. From Example 4.2, $(\varphi, g) : S_1 \longrightarrow S_2$ is a Γ-semiring epimorphism. We have $I = \left\{[x_{i1}]_{n\times 1} | x_{i1} \in \left\{[0]_{12}, [4]_{12}, [8]_{12}\right\}\right\}$ is an ideal of S_1. Thus $\left\{[x_{i1}]_{n\times 1} | x_{i1} \in \left\{[0]_6, [2]_6, [4]_6\right\}\right\} = (\varphi, g)(I)$ is an ideal of S_2.

On the other hand, if I is an ideal of a Γ_2-semiring S_2, we can prove that the set of all elements x in a Γ_1-semiring S_1 such that $\varphi(x) \in I$ is an ideal of S_1.

Theorem 4.6. *Let S_1 be a Γ_1-semiring with zero 0_{S_1} and S_2 be a Γ_2-semiring with zero 0_{S_2}. If $(\varphi, g) : S_1 \longrightarrow S_2$ is an epimorphism and I an ideal of S_2, then $(\varphi, g)^{-1}(I)$ is an ideal of S_1.*

Proof. Suppose that $(\varphi, g) : S_1 \longrightarrow S_2$ is an epimorphism and I is an ideal of S_2. Thus there exists $x \in S_1$ such that $\varphi(x) = 0_{S_2}$. By Proposition 4.3, $\varphi(0_{S_1}) = 0_{S_2}$. Since $0_{S_2} \in I$, $0_{S_1} \in \varphi^{-1}[I] = (\varphi, g)^{-1}(I) \neq \varnothing$. We show that $\varphi^{-1}[I]$ is an ideal of S_2. Since φ is a semigroup homomorphism and I is a subsemigroup of S_2, $\varphi^{-1}[I]$ is a subsemigroup of S_1. Next, Let $s \in S_1$, $x \in \varphi^{-1}[I]$ and $\alpha \in \Gamma_1$. Thus

$$\varphi(s\alpha x) = \varphi(s)g(\alpha)\varphi(x) \in I$$

and

$$\varphi(x\alpha s) = \varphi(x)g(\alpha)\varphi(s) \in I.$$

Hence $x\alpha s, s\alpha x \in \varphi^{-1}[I]$. Therefore, $(\varphi, g)^{-1}(I)$ is an ideal of S_1. □

Example 4.5. From Example 4.1, $(\varphi, g) : S_1 \longrightarrow S_2$ is an epimorphism. Since $I = \{7, 8\}$ is an ideal of S_2, $\{1, 2, 3\} = (\varphi, g)^{-1}(I)$ is an ideal of S_1.

Example 4.6. From Example 4.2, $(\varphi, g) : S_1 \longrightarrow S_2$ is an epimorphism. We have $J = \Big\{[x_{i1}]_{n\times 1} | x_{i1} \in \{[0]_6, [2]_6, [4]_6\}\Big\}$ is an ideal of S_2, Thus $\Big\{[x_{i1}]_{n\times 1} | x_{i1} \in \{[0]_{12}, [2]_{12}, [4]_{12}, [6]_{12}, [8]_{12}, [10]_{12}\}\Big\} = (\varphi, g)^{-1}(J)$ is an ideal of S_1.

The following theorem shows preservation of right pure ideals by Γ-semiring homomorphisms.

Theorem 4.7. *Let S_1 be a Γ_1-semiring and S_2 a Γ_2-semiring. If (φ, g) is an epimorphism, then $(\varphi, g)(I)$ is a right pure ideal of S_2 where I is a right pure ideal of S_1.*

Proof. Suppose that (φ, g) is an epimorphism and I is a right pure ideal of S_1. By Corollary 4.5, $(\varphi, g)(I)$ is an ideal of S_2. To show that $(\varphi, g)(I)$ is a right pure ideal of S_2, let $x \in (\varphi, g)(I)$. Then $x = \varphi(y)$ for some $y \in I$. Since I is a right pure ideal of S_1, there exist $a \in I$ and $\alpha \in \Gamma_1$ such that $y = y\alpha a$. We obtain that

$$x = \varphi(y) = \varphi(y\alpha a) = \varphi(y)g(\alpha)\varphi(a) = xg(\alpha)\varphi(a),$$

where $\varphi(a) \in \varphi[I]$ and $g(\alpha) \in \Gamma_2$. Therefore, $(\varphi, g)(I)$ is a right pure ideal of S_2. □

Similarly, we can show that $(\varphi, g)(I)$ is a left pure ideal of S_2 where I is a left pure ideal of S_1.

Example 4.7. From Example 4.1, $(\varphi, g) : S_1 \longrightarrow S_2$ is an epimorphism. Since $I = \{1, 2, 3\}$ is a right pure ideal and a left pure ideal of S_1, $\{7, 8, 9\} = (\varphi, g)(I)$ is a right pure ideal and a left pure ideal of S_2.

Example 4.8. From Example 4.2, $(\varphi, g) : S_1 \longrightarrow S_2$ is an epimorphism. Both of $J = \Big\{[x_{i1}]_{n\times 1} | x_{i1} \in \big\{[0]_{12}, [4]_{12}, [8]_{12}\big\}\Big\}$ and $I = \Big\{[x_{i1}]_{n\times 1} | x_{i1} \in \big\{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\big\}\Big\}$ are right pure ideals and left pure ideals of S_1. Thus $\Big\{[x_{i1}]_{n\times 1} | x_{i1} \in \big\{[0]_6, [2]_6, [4]_6\big\}\Big\} = (\varphi, g)(J)$ and $\Big\{[x_{i1}]_{n\times 1} | x_{i1} \in \big\{[0]_6, [3]_6\big\}\Big\} = (\varphi, g)(I)$ are right pure ideals and left pure ideals of S_2. Now we have $K = \Big\{[x_{i1}]_{n\times 1} | x_{i1} \in \big\{[0]_6, [2]_6, [4]_6\big\}\Big\}$ is a right pure ideal of S_2 but $(\varphi, g)^{-1}(K) = \Big\{[x_{i1}]_{n\times 1} | x_{i1} \in \big\{[0]_{12}, [2]_{12}, [4]_{12}, [6]_{12}, [8]_{12}, [10]_{12}\big\}\Big\}$ is not a right pure ideal of S_1.

For $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$, the sufficient conditions to prove that $(\varphi, g)^{-1}(I)$ is a right pure ideal in $\mathbb{Z}_n$ are given.

Proposition 4.8. *For $k \in \mathbb{N}$, $n = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ and $m = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ where $p_1, p_2, \ldots, p_k$ are pairwise distinct primes and $m_i \leq n_i$. Let $\varphi([x]_n) = [x]_m$ and $g([\alpha]_n) = [\alpha]_m$ for all $[x]_n, [\alpha]_n \in \mathbb{Z}_n$. Then $(\varphi, g)^{-1}(\{c[p_i^t]_m \mid c \in \mathbb{Z}\}) = \{l[p_i^t]_n \mid l \in \mathbb{Z}\}$ where $0 \leq t \leq m_i$.*

Proof. It is easy to show that φ and g are semigroup homomorphisms and $\varphi(a\alpha b) = \varphi(a)g(\alpha)\varphi(b)$ for any $a, b, \alpha \in \mathbb{Z}_n$. Thus (φ, g) is a Γ-semiring homomorphism. Set $0 \leq t \leq m_i$, $I_n = \{l[p_i^t]_n \mid l \in \mathbb{Z}\}$ and $I_m = \{c[p_i^t]_m \mid c \in \mathbb{Z}\}$. We will show that $(\varphi, g)^{-1}(I_m) = I_n$. Clearly, $\varphi[I_n] \subseteq I_m$ implies that $I_n \subseteq \varphi^{-1}[I_m]$. Next, let $[x]_n \in (\varphi, g)^{-1}(I_m)$. Then $[x]_m = \varphi([x]_n) \in I_m$. It follows that $[x]_m = c[p_i^t]_m = [cp_i^t]_m$ for some $c \in \mathbb{Z}$ implies that $x - cp_i^t = ma$ for some $a \in \mathbb{Z}$. We obtain from $x = ma + cp_i^t$ that $[x]_n = [ma + cp_i^t]_n = [p_i^t(\frac{m}{p_i^t}a + cp_i^t)]_n$. Since $\frac{m}{p_i^t}a + cp_i^t \in \mathbb{Z}$, $[x]_n \in I_n$. □

By Proposition 3.3, $\{c[p_i^t]_m \mid c \in \mathbb{Z}\}$ is a right pure ideal in the $\mathbb{Z}_m$-semiring $\mathbb{Z}_m$ if $t = m_i$. The above proposition makes we conclude that if $t = m_i = n_i$, then $(\varphi, g)^{-1}(I_m)$ is a right pure ideal in the $\mathbb{Z}_n$-semiring $\mathbb{Z}_n$. For instance, we consider $n = (2^2)(3)$ and $m = (2)(3)$. A right pure ideal $I_6 = \{[0]_6, [3]_6\}$ in the $\mathbb{Z}_6$-semiring $\mathbb{Z}_6$ is $\{c[3]_6 \mid c \in \mathbb{Z}\}$. By this conclusion, $\{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\} = (\varphi, g)^{-1}(I_6)$ is a right pure ideal in the $\mathbb{Z}_{12}$-semiring $\mathbb{Z}_{12}$. Thus $M_{n\times 1}((\varphi, g)^{-1}(I_6)) = \{[x_{i1}]_{n\times 1} \mid x_{i1} \in \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}\}$ is a right pure ideal in $M_{n\times 1}(\mathbb{Z}_{12})$ from Proposition 3.4. In Example 4.8, if $I = \{[x_{i1}]_{n\times 1} \mid x_{i1} \in \{[0]_6, [3]_6\}\}$, we obtain that $(\varphi, g)^{-1}(I) = \{[x_{i1}]_{n\times 1} \mid x_{i1} \in \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}\}$ is a right pure ideal in $M_{n\times 1}(\mathbb{Z}_{12})$.

Next, we show that the sufficient condition for $(\varphi, g)^{-1}(J)$ is a right pure ideal of S_1 where J is a right pure ideal of S_2 in the following theorem.

Theorem 4.9. *Let S_1 be a Γ_1-semiring with zero and S_2 a Γ_2-semiring with zero. If $(\varphi, g) : S_1 \longrightarrow S_2$ is an isomorphism and I is a right pure ideal of S_2, then $(\varphi, g)^{-1}(I)$ is a right pure ideal of S_1.*

Proof. Suppose that $(\varphi, g) : S_1 \longrightarrow S_2$ is an isomorphism and I is a right pure ideal of S_2. By Theorem 4.6, $(\varphi, g)^{-1}(I)$ is an ideal of S_1. To show that $(\varphi, g)^{-1}[I]$ is a right pure ideal of S_2, let $x \in \varphi^{-1}[I]$. Then $\varphi(x) \in I$. So there exist $y \in I$ and $\alpha \in \Gamma_2$ such that $\varphi(x) = \varphi(x)\alpha y$. Since φ and g are onto, $\varphi(x) = \varphi(x)g(\beta)\varphi(b)$ for some $b \in \varphi^{-1}[I]$ and $\beta \in \Gamma_1$. Now we obtain that $\varphi(x) = \varphi(x\beta b)$. Since φ is one to one, $x = x\beta b$. Therefore, $\varphi^{-1}[I]$ is a right pure ideal of S_1. □

Similarly, we can show that $(\varphi, g)^{-1}(I)$ is a left pure ideal of S_1 where I is a left pure ideal of S_2.

The second part, we study some properties of a quotient Γ-semiring and construct right pure ideals and left pure ideals in quotient Γ-semirings.

In 2011, H. Hedayati and K. P. Shum [6] introduced the quotient Γ-semiring on a Γ-semiring with zero and correspondence theorem.

Suppose that S is a Γ-semiring with zero and I is an ideal of S. H. Hedayati and K. P. Shum defined

$$\frac{S}{I} = \{x + I \mid x \in S\}$$

where

$$x + I = \{x + a \mid a \in I\}.$$

Theorem 4.10. [6] *Let S be a Γ-semiring with zero and I an ideal of S. Then the operators $\oplus$ and $*$, given by*

$$(x + I) \oplus (y + I) = x + y + I \text{ and } (x + I) * \gamma * (y + I) = x\gamma y + I,$$

*for all $x, y \in S$ and $\gamma \in \Gamma$, make $\frac{S}{I}$ into a Γ-semiring, called a **quotient Γ-semiring**.*

For convenience, we write $(x + I)\gamma(y + I)$ instead of $(x + I) * \gamma * (y + I)$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Note that I is a zero of the quotient Γ-semiring $\frac{S}{I}$. Moreover, if S is a Γ-semiring with zero and identity e, we obtain that $e + I$ is an identity of $\frac{S}{I}$.

Example 4.9. Let $S = \mathbb{Z}_{12}$ and $\Gamma = S$. By Example 2.6, S is a Γ-semiring. All of ideals of S are $\{[0]_{12}\}$, $\{[0]_{12}, [6]_{12}\}$, $\{[0]_{12}, [4]_{12}, [8]_{12}\}$, $\{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}$, $\{[0]_{12}, [2]_{12}, [4]_{12}, [6]_{12}, [8]_{12}, [10]_{12}\}$ and S. If $I = \{\bar{0}, \bar{6}\}$, then

$$\begin{aligned}
[0]_{12}+I = && I && = [6]_{12}+I,\\
[1]_{12}+I = && \{[1]_{12},[7]_{12}\} && = [7]_{12}+I,\\
[2]_{12}+I = && \{[2]_{12},[8]_{12}\} && = [8]_{12}+I,\\
[3]_{12}+I = && \{[3]_{12},[9]_{12}\} && = [9]_{12}+I,\\
[4]_{12}+I = && \{[4]_{12},[10]_{12}\} && = [10]_{12}+I,\\
[5]_{12}+I = && \{[5]_{12},[11]_{12}\} && = [11]_{12}+I.
\end{aligned}$$

We have $\dfrac{S}{I} = \{x+I \mid x \in S\}$ is a quotient Γ-semiring.

Example 4.10. Let $S = \mathbb{N}$ and $\Gamma = \{1,2,3\}$. By Example 2.16, $I = \{1,2\}$ is an ideal of S.

$$\begin{aligned}
1+I &= I,\\
2+I &= \{2\},\\
3+I &= \{3\},\\
\vdots &= \vdots\\
n+I &= \{n\}.
\end{aligned}$$

We have $\dfrac{S}{I} = \{x+I \mid x \in S\}$ is a quotient Γ-semiring.

Example 4.11. Let $S = \mathbb{Z}$ and $\Gamma = 2\mathbb{Z}$. Then S is a Γ-semiring with $x\alpha y$, which is the usual multiplication on $\mathbb{Z}$ for all $x, y \in \mathbb{Z}$ and $\alpha \in \Gamma$. Let $I = 10\mathbb{Z}$. Then I is an ideal of S. We have $\dfrac{S}{I} = \{I, 1+I, 2+I, ..., 9+I\}$ is a quotient Γ-semiring.

Example 4.12. Let $S = M_{2\times 1}(\mathbb{Z}_{12})$ and $\Gamma = M_{1\times 2}(\mathbb{Z}_{12})$. By Example 2.17, we see that $I = \left\{[x_{i1}]_{2\times 1} \mid x_{i1} \in \{[0]_{12},[6]_{12}\}\right\}$ is an ideal of S. We have $\dfrac{S}{I} = \left\{[x_{i1}]_{2\times 1} + I \mid\ [x_{i1}]_{2\times 1} \in S\right\}$ is a quotient Γ-semiring.

Some basic properties to prove our work are as follows.

Proposition 4.11. *Let I be ideals of a Γ-semiring S with zero and J a right ideal (left ideal) of S such that $I \subseteq J$. If $x + I \in \dfrac{J}{I}$, then $x \in J$.*

Proof. Suppose that $x + I \in \dfrac{J}{I}$. Thus $x + I = y + I$ for some $y \in J$. We obtain $0 \in I$ so that

$$x = x + 0 \in x + I = y + I \subseteq J.$$

□

Proposition 4.12. *Let I be an ideal of a Γ-semiring S with zero. If H and K are right ideals (left ideals) of S such that $I \subseteq H$ and $I \subseteq K$, then the following statements hold:*

(i) $H \subseteq K$ *if and only if* $\dfrac{H}{I} \subseteq \dfrac{K}{I}$;

(ii) $\dfrac{H \cap K}{I} = \dfrac{H}{I} \cap \dfrac{K}{I}$;

(iii) $\dfrac{H\Gamma K}{I} = \dfrac{H}{I}\Gamma\dfrac{K}{I}$.

Proof. (i) $(\rightarrow)$ Suppose that $H \subseteq K$ and $x + I \in \dfrac{H}{I}$. Then $x \in H \subseteq K$. It follows that $x + I \in \dfrac{K}{I}$.
$(\leftarrow)$ For $x \in H$, $x + I \in \dfrac{H}{I} \subseteq \dfrac{K}{I}$. So $x \in K$.
(ii) $(\subseteq)$ Note that $H \cap K \subseteq H$ and $H \cap K \subseteq K$ we obtain that $\dfrac{H \cap K}{I} \subseteq \dfrac{H}{I}$ and $\dfrac{H \cap K}{I} \subseteq \dfrac{K}{I}$ by (i). Thus $\dfrac{H \cap K}{I} \subseteq \dfrac{H}{I} \cap \dfrac{K}{I}$.
$(\supseteq)$ Let $x + I \in \dfrac{H}{I} \cap \dfrac{K}{I}$. It follows that $x \in H \cap K$. So $x + I \in \dfrac{H \cap K}{I}$.
Therefore, $\dfrac{H \cap K}{I} = \dfrac{H}{I} \cap \dfrac{K}{I}$.
(iii) $(\subseteq)$ Let $x + I \in \dfrac{H\Gamma K}{I}$. We obtain that $x \in H\Gamma K$. So $x = \sum h_i\alpha_i k_i$ where $h_i \in H, \alpha_i \in \Gamma$ and $k_i \in K$ for all i. Thus

$$x + I = \left(\sum h_i\alpha_i k_i\right) + I = \sum(h_i\alpha_i k_i + I) = \sum(h_i + I)\alpha_i(k_i + I) \in \frac{H}{I}\Gamma\frac{K}{I}.$$

$(\supseteq)$ Let $x \in \dfrac{H}{I}\Gamma\dfrac{K}{I}$. Then $x = \sum a_i\alpha_i b_i$ where $a_i \in \dfrac{H}{I}, \alpha_i \in \Gamma$ and $b_i \in \dfrac{K}{I}$ for all i. For each i, we have $a_i = x_i + I$ and $b_i = y_i + I$ for some $x_i \in H$ and $y_i \in K$. Then

$$x + I = \sum a_i\alpha_i b_i = \sum(x_i\alpha_i y_i + I) = \left(\sum x_i\alpha_i y_i\right) + I \in \frac{H\Gamma K}{I}.$$

We now conclude that $\dfrac{H\Gamma K}{I} = \dfrac{H}{I}\Gamma\dfrac{K}{I}$. $\square$

Theorem 4.13. [6]*(Correspondence Theorem) Let I and J be ideals of a Γ-semiring S with zero such that $I \subseteq J$. Then $\dfrac{J}{I}$ is an ideal of $\dfrac{S}{I}$. Conversely, if K is an ideal of $\dfrac{S}{I}$, then there exists an ideal J of S such that $I \subseteq J$ and $K = \dfrac{J}{I}$.*

This conclusion is true for right ideals and left ideals.

Proposition 4.14. *Let I be an ideal of a Γ-semiring S with zero and J a right ideal (left ideal) of S such that $I \subseteq J$. Then $\dfrac{J}{I}$ is a right ideal (left ideal) of $\dfrac{S}{I}$. Conversely, if K is a right ideal (left ideal) of $\dfrac{S}{I}$, then there exists a right ideal (left ideal) J of S such that $I \subseteq J$ and $K = \dfrac{J}{I}$.*

Proof. Let $x+I, y+I \in \dfrac{J}{I}$. Then $x, y \in J$. It follows $x+y \in J$ because J is a subsemigroup of S. Thus $(x+I)+(y+I) = (x+y)+I \in \dfrac{J}{I}$. So $\dfrac{J}{I}$ is a subsemigroup of $\dfrac{S}{I}$. Next, let $a+I \in \dfrac{S}{I}$ and $\alpha \in \Gamma$. Since J is a right ideal of S, $(x+I)\alpha(a+I) = (x\alpha a)+I \in \dfrac{J}{I}$. Hence $\dfrac{J}{I}$ is a right ideal of $\dfrac{S}{I}$.

Conversely, let $\xi = \{J \mid J \text{ is a right ideal of } S \text{ such that } I \subseteq J\}$ and $\zeta = \left\{H \mid H \text{ is a right ideal of } \dfrac{S}{I}\right\}$. We define the mapping $f : \xi \to \zeta$ by $f(J) = \dfrac{J}{I}$ for all $J \in \xi$. Let $J_1, J_2 \in \xi$ be such that $\dfrac{J_1}{I} = \dfrac{J_2}{I}$. Then $J_1 = J_2$. Hence f is one to one. Finally, we show that f is onto. Let $H \in \zeta$. Choose $J = \cup H$. Since I is a zero of $\dfrac{S}{I}$, $I \in H$. Thus $I \subseteq \cup H = J$. To show that J is a right ideal of S, let $x, y \in J$. So $x+I, y+I \in \dfrac{J}{I} = \dfrac{\cup H}{I} = H$. Thus $(x+y)+I = (x+I)+(y+I) \in H$ implies that $x+y \in \cup H = J$. Hence J is a subsemigroup of S. For $a \in S$ and $\alpha \in \Gamma$, $x\alpha a + I = (x+I)\alpha(a+I) \in H$. Thus $x\alpha a \in \cup H = J$. Therefore, J is a right ideal of S such that $f(J) = \dfrac{J}{I} = H$. This prove is true for left ideal. $\square$

Example 4.13. Let $S = \mathbb{Z}_{12}$ and $\Gamma = S$. By Example 4.9, $\dfrac{S}{I} = \{x+I \mid x \in S\}$ is a quotient Γ-semiring where $I = \{[0]_{12}, [6]_{12}\}$. Since $J = \{[0]_{12}, [2]_{12}, [4]_{12}, [6]_{12}, [8]_{12},$

$[10]_{12}\}$ and $K = \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}$ are ideals of S such that $I \subseteq J$ and $I \subseteq K$, $\dfrac{J}{I} = \{I, \{[2]_{12}, [8]_{12}\}, \{[4]_{12}, [10]_{12}\}\}$ and $\dfrac{K}{I} = \{I, \{[3]_{12}, [9]_{12}\}\}$ are ideals of $\dfrac{S}{I}$.

Example 4.14. Let $S = \mathbb{N}$ and $\Gamma = \{1, 2, 3\}$. By Example 4.10, we see that $\dfrac{S}{I} = \{x + I \mid x \in S\}$ is a quotient Γ-semiring where $I = \{1, 2\}$. Since $J = \{1, 2, 3\}$ is an ideal of S and $I \subseteq J$, $\dfrac{J}{I} = \{I, \{2\}, \{3\}\}$ is an ideal of $\dfrac{S}{I}$.

Example 4.15. Let $S = \mathbb{Z}$ and $\Gamma = 2\mathbb{Z}$. By Example 4.11, $\dfrac{S}{I} = \{x + I \mid x \in S\}$ is a quotient Γ-semiring where $I = 10\mathbb{Z}$. Since $J = 2\mathbb{Z}$ is an ideal of S, $\dfrac{J}{I} = \{I, 2 + I, 4 + I, 6 + I, 8 + I\}$ is an ideal of $\dfrac{S}{I}$.

Example 4.16. Let $S = M_{2\times 1}(\mathbb{Z}_{12})$ and $\Gamma = M_{1\times 2}(\mathbb{Z}_{12})$. By Example 4.12, $\dfrac{S}{I} = \{x + I \mid x \in S\}$ is a quotient Γ-semiring where $I = \left\{ [x_{i1}]_{2\times 1} \mid x_{i1} \in \{[0]_{12}, [6]_{12}\}\right\}$. Since $J = \left\{ [x_{i1}]_{2\times 1} \mid x_{i1} \in \{[0]_{12}, [2]_{12}, [4]_{12}, [6]_{12}, [8]_{12}, [10]_{12}\}\right\}$ and $K = \left\{ [x_{i1}]_{2\times 1} \mid x_{i1} \in \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}\right\}$ are ideals of S, $\dfrac{J}{I} = \{x + I \mid x \in J\}$ and $\dfrac{K}{I} = \{x + I \mid x \in K\}$ are ideals of $\dfrac{S}{I}$.

Next, we can characterize right pure ideals in a quotient Γ-semiring as in the following theorem.

Theorem 4.15. *Let K and I be ideals of a Γ-semiring S with zero and identity such that $I \subseteq K$. An ideal $\dfrac{K}{I}$ of $\dfrac{S}{I}$ is right pure if and only if $\dfrac{H \cap K}{I} = \dfrac{H\Gamma K}{I}$ for all right ideals H of S such that $I \subseteq H$.*

Proof. $(\rightarrow)$ Suppose that $\dfrac{K}{I}$ is a right pure ideal of $\dfrac{S}{I}$ and H is a right ideal of S such that $I \subseteq H$. Since H is an ideal such that $I \subseteq H$, $\dfrac{H}{I}$ is an ideal of $\dfrac{S}{I}$. We obtain that $\dfrac{S}{I}$ is a Γ-semiring with identity. By Proposition 3.5 and Proposition 4.12,

$$\frac{K\Gamma H}{I} = \frac{K}{I}\Gamma\frac{H}{I} = \frac{K}{I} \cap \frac{H}{I} = \frac{K \cap H}{I}.$$

$(\leftarrow)$ Suppose that $\dfrac{K}{I}$ is an ideal of $\dfrac{S}{I}$ and $\dfrac{H \cap K}{I} = \dfrac{H\Gamma K}{I}$ for all right ideals H of S such that $I \subseteq H$. To show that $\dfrac{K}{I}$ is a right pure ideal of $\dfrac{S}{I}$, let A be a right ideal of $\frac{S}{I}$. Then there exists a right ideal J of S such that $I \subseteq J$ and $A = \frac{J}{I}$. It follows that

$$A \cap \frac{K}{I} = \frac{J}{I} \cap \frac{K}{I} = \frac{J \cap K}{I} = \frac{J\Gamma K}{I} = \frac{J}{I}\Gamma\frac{K}{I} = A\Gamma\frac{K}{I}.$$

Therefore, $\frac{K}{I}$ is a right pure ideal of $\frac{S}{I}$. $\square$

Similarly, an ideal $\frac{K}{I}$ of $\frac{S}{I}$ is left pure if and only if $\frac{H \cap K}{I} = \frac{K\Gamma H}{I}$ for all left ideals H of S such that $I \subseteq H$.

Next, we construct a right pure ideal and a left pure ideal in a quotient Γ-semiring $\frac{S}{I}$ from a right pure ideal and a left pure ideal in a Γ-semiring S.

Theorem 4.16. *Let I and J be ideals of a Γ-semiring S with zero. If J is a right pure ideal of S such that $I \subseteq J$, then $\frac{J}{I}$ is a right pure ideal of $\frac{S}{I}$.*

Proof. Suppose that J is a right pure ideal of S such that $I \subseteq J$. By Theorem 4.13, $\frac{J}{I}$ is an ideal of $\frac{S}{I}$. To show that $\frac{J}{I}$ is a right pure ideal of $\frac{S}{I}$, let $x + I \in \frac{J}{I}$. Since $x \in J$, there exist $a \in J, \alpha \in \Gamma$ such that $x = x\alpha a$. Thus

$$x + I = x\alpha a + I = (x + I) * \alpha * (a + I).$$

where $a + I \in \frac{J}{I}$ and $\alpha \in \Gamma$. Therefore, $\frac{J}{I}$ is a right pure ideal of $\frac{S}{I}$. $\square$

Likewise, if J is a left pure ideal of S such that $I \subseteq J$, then $\frac{J}{I}$ is a left pure ideal of $\frac{S}{I}$.

Example 4.17. Let $S = \mathbb{Z}_{12}$ and $\Gamma = S$. By Example 4.9, $\frac{S}{I} = \{x + I \mid x \in S\}$ is a quotient Γ-semiring where $I = \{[0]_{12}, [6]_{12}\}$. Since $K = \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}$ is a right pure ideal and a left pure ideal of S such that $I \subseteq K$, $\frac{K}{I} = \{I, \{[3]_{12}, [9]_{12}\}\}$ is a right pure ideal and a left pure ideal of $\frac{S}{I}$. Moreover, $H = \{I, \{[2]_{12}, [8]_{12}\}, \{[4]_{12}, [10]_{12}\}\}$ is a right pure ideal of $\frac{S}{I}$ because

$$
\begin{aligned}
[0]_{12}+I &= ([0]_{12}+I)[1]_{12}([2]_{12}+I),\\
[8]_{12}+I &= ([8]_{12}+I)[1]_{12}([4]_{12}+I),\\
[4]_{12}+I &= ([4]_{12}+I)[1]_{12}([4]_{12}+I).
\end{aligned}
$$

But there are no right pure ideals A of S such that $I \subseteq A$, $K = \frac{A}{I}$.

Example 4.18. Let $S = \mathbb{N}$ and $\Gamma = \{1,2,3\}$. By Example 4.10, we see that $\frac{S}{I} = \{x+I \mid x \in S\}$ is a quotient Γ-semiring where $I = \{1,2\}$. Since $J = \{1,2,3\}$ is a right pure ideal of S and $I \subseteq J$, $\frac{J}{I} = \{I, \{2\}, \{3\}\}$ is a right pure ideal of $\frac{S}{I}$.

Example 4.19. Let $S = M_{2\times 1}(\mathbb{Z}_{12})$ and $\Gamma = M_{1\times 2}(\mathbb{Z}_{12})$. By Example 4.12, $\frac{S}{I} = \{x+I \mid x \in S\}$ is a quotient Γ-semiring where $I = \left\{ [x_{i1}]_{2\times 1} \mid x_{i1} \in \{[0]_{12}, [6]_{12}\}\right\}$. Since $K = \left\{ [x_{i1}]_{2\times 1} \mid x_{i1} \in \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}\right\}$ is a right pure ideal and a left pure ideal of S, $\frac{K}{I} = \{x+I \mid x \in K\}$ is a right pure ideal and a left pure ideal of $\frac{S}{I}$. Moreover, $\left\{[x_{i1}]_{2\times 1} + I \mid x_{i1} \in \{[0]_{12}, [4]_{12}, [8]_{12}\}\right\}$ is a right pure ideal of $\frac{S}{I}$ because

$$
\begin{aligned}
\begin{pmatrix}[0]_{12}\\ [0]_{12}\end{pmatrix}+I &= \left(\begin{pmatrix}[0]_{12}\\ [0]_{12}\end{pmatrix}+I\right)\begin{pmatrix}[1]_{12} & [0]_{12}\end{pmatrix}\left(\begin{pmatrix}[0]_{12}\\ [0]_{12}\end{pmatrix}+I\right),\\
\begin{pmatrix}[4]_{12}\\ [0]_{12}\end{pmatrix}+I &= \left(\begin{pmatrix}[4]_{12}\\ [0]_{12}\end{pmatrix}+I\right)\begin{pmatrix}[1]_{12} & [0]_{12}\end{pmatrix}\left(\begin{pmatrix}[4]_{12}\\ [0]_{12}\end{pmatrix}+I\right),\\
\begin{pmatrix}[0]_{12}\\ [4]_{12}\end{pmatrix}+I &= \left(\begin{pmatrix}[0]_{12}\\ [4]_{12}\end{pmatrix}+I\right)\begin{pmatrix}[1]_{12} & [0]_{12}\end{pmatrix}\left(\begin{pmatrix}[0]_{12}\\ [4]_{12}\end{pmatrix}+I\right),\\
\begin{pmatrix}[4]_{12}\\ [4]_{12}\end{pmatrix}+I &= \left(\begin{pmatrix}[4]_{12}\\ [4]_{12}\end{pmatrix}+I\right)\begin{pmatrix}[1]_{12} & [0]_{12}\end{pmatrix}\left(\begin{pmatrix}[4]_{12}\\ [0]_{12}\end{pmatrix}+I\right),\\
\begin{pmatrix}[8]_{12}\\ [0]_{12}\end{pmatrix}+I &= \left(\begin{pmatrix}[8]_{12}\\ [0]_{12}\end{pmatrix}+I\right)\begin{pmatrix}[1]_{12} & [0]_{12}\end{pmatrix}\left(\begin{pmatrix}[4]_{12}\\ [0]_{12}\end{pmatrix}+I\right),\\
\begin{pmatrix}[0]_{12}\\ [8]_{12}\end{pmatrix}+I &= \left(\begin{pmatrix}[0]_{12}\\ [8]_{12}\end{pmatrix}+I\right)\begin{pmatrix}[1]_{12} & [0]_{12}\end{pmatrix}\left(\begin{pmatrix}[4]_{12}\\ [0]_{12}\end{pmatrix}+I\right),\\
\begin{pmatrix}[8]_{12}\\ [8]_{12}\end{pmatrix}+I &= \left(\begin{pmatrix}[8]_{12}\\ [8]_{12}\end{pmatrix}+I\right)\begin{pmatrix}[1]_{12} & [0]_{12}\end{pmatrix}\left(\begin{pmatrix}[4]_{12}\\ [0]_{12}\end{pmatrix}+I\right).
\end{aligned}
$$

But there are no right pure ideals A of S such that $I \subseteq A$, $K = \frac{A}{I}$.

The sufficient conditions for the converse of Theorem 4.16 is furnished in the following theorem.

Theorem 4.17. *Let J and I be ideals of a Γ-semiring S with zero such that $I \subseteq J$ and $\frac{J}{I}$ a right pure ideal of $\frac{S}{I}$. If for every $A \in \frac{J}{I}$ there exists a unique $x \in J$ such that $A = x + I$, then J is a right pure ideal of S.*

Proof. Suppose that I is a right ideal of S and for each $A \in \frac{J}{I}$ there exists a unique $x \in J$ such that $A = x + I$. To show that J is a right pure ideal of S, let $x \in J$. Then $x + I \in \frac{J}{I}$. It follows that there exist $a + I \in \frac{J}{I}$ and $\alpha \in \Gamma$ such that

$$x + I = (x + I)\alpha(a + I) = x\alpha a + I.$$

By assumption, $x = x\alpha a$. Hence J is a right pure ideal of S. □

In this same way, if $\frac{J}{I}$ is a left pure ideal of $\frac{S}{I}$ and for each $A \in \frac{J}{I}$ there exists a unique $x \in J$ such that $A = x + I$, then J is a left pure ideal of S.

The last part, we examine some properties in the products of Γ-semirings which the one new Γ-semiring was constructed by H. Hedayati and K. P. Shum. Later on, we show the construction of right pure ideals and left pure ideals in the products of Γ-semirings.

Suppose that $X_1, ..., X_n$ be nonempty sets. Let $X_1 \times \cdots \times X_n$ be the cartesian product of $X_1, ..., X_n$, i.e,

$$X_1 \times \cdots \times X_n = \{(x_1, ..., x_n) \mid x_i \in X_i \text{ for all } i\}.$$

Lemma 4.18. [6] *Let R_i be a Γ_i-semiring $(1 \leq i \leq n)$. Then the operations on $R_1 \times \cdots \times R_n$ defined by*

$$(x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n)$$

and

$$\circ : (R_1 \times \cdots \times R_n) \times (\Gamma_1 \times \cdots \times \Gamma_n) \times (R_1 \times \cdots \times R_n) \longrightarrow (R_1 \times \cdots \times R_n)$$

by

$$(x_1, ..., x_n) \circ (\gamma_1, ..., \gamma_n) \circ (y_1, ..., y_n) = (x_1\gamma_1 y_1, ..., x_n\gamma_n y_n),$$

for all $(x_1, ..., x_n), (y_1, ..., y_n) \in R_1 \times \cdots \times R_n$ *and* $(\gamma_1, ..., \gamma_n) \in \Gamma_1 \times \cdots \times \Gamma_n$ *make* $R_1 \times \cdots \times R_n$ *into a* $\Gamma_1 \times \cdots \times \Gamma_n$*-semiring, called the* ***products of*** Γ***-semirings****.*

Example 4.20. Let S_i be a semiring $(1 \leq i \leq n)$. Then S_i is a Γ_i-semiring where Γ_i is a subsemiring of S_i. Therefore, $S_1 \times \cdots \times S_n = \{(x_1, ..., x_n) \mid x_i \in S_i\}$ is a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring.

Example 4.21. Let $R_1 = \mathbb{Z}_{12}$ and $R_2 = \mathbb{N}$. We obtain that R_i is a Γ_i-semiring where $\Gamma_1 = R_1$ and $\Gamma_2 = \{1, 2, 3\}$ so that $R_1 \times R_2 = \{(x_1, x_2) \mid x_1 \in R_1$ and $x_2 \in R_2\}$ is a $\Gamma_1 \times \Gamma_2$-semiring.

Some basic properties in the products of Γ-semirings are given.

Proposition 4.19. *Let* X_i *and* Y_i *be nonempty subsets of a* Γ_i*-semiring* R_i $(1 \leq i \leq n)$*. Then the following statements hold:*

(i) for each i*,* $X_i \subseteq Y_i$ *if and only if* $X_1 \times \cdots \times X_n \subseteq Y_1 \times \cdots \times Y_n$*;*

(ii) $(X_1\Gamma_1 Y_1) \times \cdots \times (X_n\Gamma_n Y_n) = (X_1 \times \cdots \times X_n)(\Gamma_1 \times \cdots \times \Gamma_n)(Y_1 \times \cdots \times Y_n)$*;*

(iii) $(X_1 \cap Y_1) \times \cdots \times (X_n \cap Y_n) = (X_1 \times \cdots \times X_n) \cap (Y_1 \times \cdots \times Y_n)$*.*

Proof. *(i)* $(\rightarrow)$ For each i, $X_i \subseteq Y_i$, we obtain that for $(x_1, ..., x_n) \in X_1 \times \cdots \times X_n$, $x_i \in X_i \subseteq Y_i$ for all i. So that $(x_1, ..., x_n) \in Y_1 \times \cdots \times Y_n$.
$(\leftarrow)$ Suppose that $X_1 \times \cdots \times X_n \subseteq Y_1 \times \cdots \times Y_n$. Let $x_i \in X_i$ for all i. Then $(x_1, ..., x_n) \in X_1 \times \cdots \times X_n \subseteq Y_1 \times \cdots \times Y_n$. So that $x_i \in Y_i$ for all i.
(ii) $(\subseteq)$ Let $(x_1, ..., x_n) \in (X_1\Gamma_1 Y_1) \times \cdots \times (X_n\Gamma_n Y_n)$. Thus $x_i = \sum a_{ij}\alpha_{ij}b_{ij}$ where

$a_{ij} \in X_i$, $b_{ij} \in Y_i$ and $\alpha_{ij} \in \Gamma_i$ for all i, j. So that

$$\begin{aligned}(x_1, ..., x_n) &= (\sum a_{1j}\alpha_{1j}b_{1j}, ..., \sum a_{nj}\alpha_{nj}b_{nj}) \\ &= \sum (a_{1j}\alpha_{1j}b_{1j}, ..., a_{nj}\alpha_{nj}b_{nj}) \\ &= \sum (a_{1j}, ..., a_{nj})(\alpha_{1j}, ..., \alpha_{nj})(b_{1j}, ..., b_{nj}).\end{aligned}$$

Hence $(x_1, ..., x_n) \in (X_1 \times \cdots \times X_n)(\Gamma_1 \times \cdots \times \Gamma_n)(Y_1 \times \cdots \times Y_n)$.
($\supseteq$) By symmetry, $(X_1 \times \cdots \times X_n)(\Gamma_1 \times \cdots \times \Gamma_n)(Y_1 \times \cdots \times Y_n) \subseteq (X_1\Gamma_1Y_1) \times \cdots \times (X_n\Gamma_nY_n)$.
(iii) ($\subseteq$) Let $(x_1, ..., x_n) \in (X_1 \cap Y_1) \times \cdots \times (X_n \cap Y_n)$. Then $x_i \in X_i$ and $x_i \in Y_i$ for all i. We obtain from $(x_1, ..., x_n) \in (X_1 \times \cdots \times X_n) \cap (Y_1 \times \cdots \times Y_n)$ that $(X_1 \cap Y_1) \times \cdots \times (X_n \cap Y_n) \subseteq (X_1 \times \cdots \times X_n) \cap (Y_1 \times \cdots \times Y_n)$.
($\supseteq$) By symmetry, $(X_1 \times \cdots \times X_n) \cap (Y_1 \times \cdots \times Y_n) \subseteq (X_1 \cap Y_1) \times \cdots \times (X_n \cap Y_n)$. □

Proposition 4.20. *Let H_i be a nonempty subset of a Γ_i-semiring R_i $(1 \leq i \leq n)$. Then $H_1 \times \cdots \times H_n$ is an ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$ if and only if each i, H_i is an ideal of R_i.*

Proof. ($\rightarrow$) Suppose that $H_1 \times \cdots \times H_n$ is an ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$. We show that for each i, H_i is an ideal of R_i. First, let $x_i, y_i \in H_i$ for all i. Then $(x_1, ..., x_n), (y_1, ..., y_n) \in H_1 \times \cdots \times H_n$. By assumption, $(x_1, ..., x_n) + (y_1, ..., y_n) \in H_1 \times \cdots \times H_n$. Thus $x_i + y_i \in H_i$ for all i. Next, let $a_i \in R_i$ and α_i for all i. We obtain that

$$(a_1\alpha_1x_1, ..., a_n\alpha_nx_n) = (a_1, ..., a_n)(\alpha_1, ..., \alpha_n)(x_1, ..., x_n) \in H_1 \times \cdots \times H_n$$

and

$$(x_1\alpha_1a_1, ..., x_n\alpha_na_n) = (x_1, ..., x_n)(\alpha_1, ..., \alpha_n)(a_1, ..., a_n) \in H_1 \times \cdots \times H_n.$$

Hence $x_i\alpha_ia_i, a_i\alpha_ix_i \in H_i$ for all i. Therefore, for each i, H_i is an ideal of R_i.
($\leftarrow$) Similarly, if each i, H_i is an ideal of R_i, then $H_1 \times \cdots \times H_n$ is an ideal of a

$\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$. □

Example 4.22. Let $R_1 = \mathbb{Z}_{12}$, $R_2 = \mathbb{N}$, $\Gamma_1 = R_1$ and $\Gamma_2 = \{1, 2, 3\}$. By Example 4.21, $R_1 \times R_2 = \{(x_1, x_2) \mid x_1 \in R_1$ and $x_2 \in R_2\}$ is a $\Gamma_1 \times \Gamma_2$-semiring. Let $I_1 = \{[0]_{12}, [6]_{12}\}$, $I_2 = \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}$ and $K = \{1, 2, 3\}$. We have I_1 and I_2 are ideals of R_1 and K is an ideal of R_2 so that $I_1 \times K$ and $I_2 \times K$ are ideals of $R_1 \times R_2$.

The construction of right pure ideals in $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$ from right pure ideals in a Γ_i-semiring R_i for all i are proved.

Theorem 4.21. *Let H_i be a nonempty subset of a Γ_i-semiring R_i $(1 \leq i \leq n)$. Then $H_1 \times \cdots \times H_n$ is a right pure ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$ if and only if for each i, H_i is a right pure ideal of R_i.*

Proof. $(\rightarrow)$ Suppose that $H_1 \times \cdots \times H_n$ is a right pure ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$. By Proposition 4.20, H_i is an ideal of R_i for all i. We show that for each i, H_i is a right pure ideal of R_i. First, let $x_i \in H_i$ for all i. Thus $(x_1, ..., x_n) \in H_1 \times \cdots \times H_n$. We obtain that there exist $(a_1, ..., a_n) \in H_1 \times \cdots \times H_n$ and $(\alpha_1, ..., \alpha_n) \in \alpha_1 \times \cdots \times \alpha_n$ such that

$$(x_1, ..., x_n) = (x_1, ..., x_n)(\alpha_1, ..., \alpha_n)(a_1, ..., a_n) = (x_1\alpha_1a_1, ..., x_n\alpha_na_n).$$

Hence $x_i = x_i\alpha_ia_i$. Therefore, each i, H_i is a right pure ideal of R_i.
$(\leftarrow)$ Suppose that each i, H_i is a right pure ideal of R_i. By Proposition 4.20, $H_1 \times \cdots \times H_n$ is an ideal of $R_1 \times \cdots \times R_n$. Let $(x_1, ..., x_n) \in H_1 \times \cdots \times H_n$. We obtain $x_i = x_i\alpha_ia_i$ for some $a_i \in H_i$ and $\alpha_i \in \Gamma_i$ so that

$$(x_1, ..., x_n) = (x_1\alpha_1a_1, ..., x_n\alpha_na_n) = (x_1, ..., x_n)(\alpha_1, ..., \alpha_n)(a_1, ..., a_n).$$

We conclude that $H_1 \times \cdots \times H_n$ is a right pure ideal of $R_1 \times \cdots \times R_n$. □

Similarly, we can show that $H_1 \times \cdots \times H_n$ is a left pure ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$ if and only if each i, H_i is a left pure ideal of R_i.

Example 4.23. From Example 4.22, $I_1 \times K$ and $J_1 \times K$ are ideals of $R_1 \times R_2$. Since J_1 and K are right pure ideals of R_1 and R_2, respectively, $J_1 \times K$ is a right pure ideal of $R_1 \times R_2$. But $I_1 \times K$ is not a right pure ideal of $R_1 \times R_2$ because I_1 is not a right pure ideal of R_1.

Moreover, we show relationship between purely prime in $\Gamma_1 \times \cdots \times \Gamma_n$ -semiring $R_1 \times \cdots \times R_n$ and Γ_i-semiring R_i.

Theorem 4.22. *Let $H_1 \times \cdots \times H_n$ be a purely prime ideal (purely semiprime ideal) of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n (1 \leq i \leq n)$. Then for each i, H_i is a purely prime ideal (purely semiprime ideal) of a Γ_i-semiring R_i.*

Proof. By Theorem 4.21, each i, H_i is a right pure ideal of R_i. To show that for each i, H_i is a purely prime ideal of R_i, let K_i and P_i be right pure ideals of R_i such that $K_i \Gamma_i P_i \subseteq H_i$ for all i. We obtain that

$$(K_1 \times \cdots \times K_n)(\Gamma_1 \times \cdots \times \Gamma_n)(P_1 \times \cdots \times P_n) = (K_1 \Gamma_1 P_1 \times \cdots \times K_n \Gamma_n P_n) \subseteq H_1 \times \cdots \times H_n.$$

Since $K_1 \times \cdots \times K_n$ and $P_1 \times \cdots \times P_n$ are right pure ideals of $R_1 \times \cdots \times R_n$, $K_1 \times \cdots \times K_n \subseteq H_1 \times \cdots \times H_n$ or $P_1 \times \cdots \times P_n \subseteq H_1 \times \cdots \times H_n$ for all i. It follows that $K_i \subseteq H_i$ or $P_i \subseteq H_i$ for all i. Therefore, for each i, H_i is a right pure ideal of R_i.

Similarly, we can show that if $H_1 \times \cdots \times H_n$ is a purely semiprime ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n (1 \leq i \leq n)$, then for each i, H_i is a purely semiprime ideal of a Γ_i-semiring R_i. □

Example 4.24. Consider a right pure ideal $(\{1, 2\}, 2\mathbb{Z}_{30})$ in a $\{1, 2, 3\} \times \mathbb{Z}_{30}$-semiring $\mathbb{N} \times \mathbb{Z}_{30}$. We know that $\{1, 2\}$ and $2\mathbb{Z}_{30}$ are purely prime ideals of $\mathbb{N}$ and $\mathbb{Z}_{30}$, respectively. But $(\{1, 2\}, 2\mathbb{Z}_{30})$ is not a purely prime ideal of $\mathbb{N} \times \mathbb{Z}_{30}$ because there exist right pure ideals $(\{1\}, 3\mathbb{Z}_{30})$ and $(\{1, 2, 3\}, 2\mathbb{Z}_{30})$ of $\mathbb{N} \times \mathbb{Z}_{30}$ such that $(\{1\}, 3\mathbb{Z}_{30})(\{1, 2, 3\}, \mathbb{Z}_{30})(\{1, 2, 3\}, 2\mathbb{Z}_{30}) \subseteq (\{1, 2\}, 2\mathbb{Z}_{30})$ but $(\{1\}, 3\mathbb{Z}_{30}) \nsubseteq (\{1, 2\}, 2\mathbb{Z}_{30})$ and $(\{1, 2, 3\}, 2\mathbb{Z}_{30}) \nsubseteq (\{1, 2\}, 2\mathbb{Z}_{30})$.

In purely semiprimes, the properties of them make the converse true.

Theorem 4.23. *Let H_i be a nonempty subset of a Γ_i-semiring R_i $(1 \leq i \leq n)$. Then $H_1 \times \cdots \times H_n$ is a purely semiprime ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$ if and only if each i, H_i is a purely semiprime ideal of R_i.*

Proof. $(\rightarrow)$ It is clear by Theorem 4.22.
$(\leftarrow)$ Assume that for each i, H_i is a purely semiprime ideal of R_i. We will show that $H_1 \times \cdots \times H_n$ is a purely semiprime ideal of $R_1 \times \cdots \times R_n$. Let $A_1 \times \cdots \times A_n$ be a right pure ideal of $R_1 \times \cdots \times R_n$ such that $(A_1 \times \cdots \times A_n)(\Gamma_1 \times \cdots \times \Gamma_n)(A_1 \times \cdots \times A_n) \subseteq H_1 \times \cdots \times H_n$. We obtain $A_1\Gamma_1A_1 \times \cdots \times A_n\Gamma_nA_n \subseteq H_1 \times \cdots \times H_n$ so that $A_i\Gamma_iA_i \subseteq H_i$ for all i. By assumption, $A_i \subseteq H_i$ for all i. It follows that $A_1 \times \cdots \times A_n \subseteq H_1 \times \cdots \times H_n$. Therefore, $H_1 \times \cdots \times H_n$ is a purely semiprime ideal of $R_1 \times \cdots \times R_n$. □

Likewise, a relationship between purely irreducible ideals in $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$ and Γ_i-semiring R_i is shown.

Theorem 4.24. *Let $H_1 \times \cdots \times H_n$ be a purely irreducible ideal (strongly irreducible pure ideal) of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n (1 \leq i \leq n)$. Then each i, H_i is a purely irreducible ideal (strongly irreducible pure ideal) of a Γ_i-semiring R_i.*

Proof. Clearly, each i, H_i is a right pure ideal of R_i by Theorem 4.21. Let A_i and B_i be right pure ideals of R_i such that $A_i \cap B_i = H_i$ for all i. Then

$$(A_1 \times \cdots \times A_n) \cap (B_1 \times \cdots \times B_n) = (A_1 \cap B_1) \times \cdots \times (A_n \cap B_n) = H_1 \times \cdots \times H_n.$$

Since $H_1 \times \cdots \times H_n$ is a purely irreducible ideal of $R_1 \times \cdots \times R_n$, $A_1 \times \cdots \times A_n = H_1 \times \cdots \times H_n$ or $B_1 \times \cdots \times B_n = H_1 \times \cdots \times H_n$, Hence $A_i = H_i$ or $B_i = H_i$ for all i. Therefore, each i, H_i is a purely irreducible ideal of R_i.

Similarly, we can show that if $H_1 \times \cdots \times H_n$ is a strongly irreducible pure ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n (1 \leq i \leq n)$, then each i, H_i is a strongly irreducible pure ideal of a Γ_i-semiring R_i. □

The converse is not true. Here is an example.

Example 4.25. Consider a right pure ideal $(\{1,2\}, 2\mathbb{Z}_{30})$ in a $\{1,2,3\} \times \mathbb{Z}_{30}$-semiring $\mathbb{N} \times \mathbb{Z}_{30}$. We know that $\{1,2\}$ and $2\mathbb{Z}_{30}$ are purely irreducible ideals of $\mathbb{N}$ and $\mathbb{Z}_{30}$, respectively. But $(\{1,2\}, 2\mathbb{Z}_{30})$ is not a purely irreducible ideal of $\mathbb{N} \times \mathbb{Z}_{30}$ because there exist right pure ideals $(\{1\}, 3\mathbb{Z}_{30})$ and $(\{1,2,3\}, 2\mathbb{Z}_{30})$ of $\mathbb{N} \times \mathbb{Z}_{30}$ such that $(\{1\}, 3\mathbb{Z}_{30}) \cap (\{1,2,3\}, 2\mathbb{Z}_{30}) \subseteq (\{1,2\}, 2\mathbb{Z}_{30})$ but $(\{1\}, 3\mathbb{Z}_{30}) \nsubseteq (\{1,2\}, 2\mathbb{Z}_{30})$ and $(\{1,2,3\}, 2\mathbb{Z}_{30}) \nsubseteq (\{1,2\}, 2\mathbb{Z}_{30})$.

Finally, the characterization of purely maximal ideals are proved.

Theorem 4.25. *Let H_i be a nonempty subset of a Γ_i-semiring R_i $(1 \leq i \leq n)$. Then $H_1 \times \cdots \times H_n$ is a purely maximal ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$ if and only if each i, H_i is a purely maximal ideal of R_i.*

Proof. $(\rightarrow)$ Assume that $H_1 \times \cdots \times H_n$ is a purely maximal ideal of $R_1 \times \cdots \times R_n$. We show that each i, H_i is a purely maximal ideal of R_i. For each i, let K_i be a proper right pure ideal of R_i such that $H_i \subseteq K_i$. Then $K_1 \times \cdots \times K_n$ is a proper right pure ideal of $R_1 \times \cdots \times R_n$ such that $H_1 \times \cdots \times H_n \subseteq K_1 \times \cdots \times K_n$. By assumption, $H_1 \times \cdots \times H_n = K_1 \times \cdots \times K_n$. Hence $H_i = K_i$ for all i. Therefore, each i, H_i is a purely maximal ideal of R_i.
$(\leftarrow)$ Suppose that each i, H_i is a purely maximal ideal of R_i. To Show that $H_1 \times \cdots \times H_n$ is a purely maximal ideal of $R_1 \times \cdots \times R_n$, let $K_1 \times \cdots \times K_n$ be a proper right pure ideal of $R_1 \times \cdots \times R_n$ such that $H_1 \times \cdots \times H_n \subseteq K_1 \times \cdots \times K_n$. We obtain $H_i \subseteq K_i$ so that $K_i = H_i$ for all i. It follows that $H_1 \times \cdots \times H_n = K_1 \times \cdots \times K_n$. Therefore, $H_1 \times \cdots \times H_n$ is a purely maximal ideal of $R_1 \times \cdots \times R_n$. □

In conclusion, in quotient Γ-semirings, our goal is a $1-1$ correspondence between the set of right pure ideals of a quotient Γ-semiring $\frac{R}{I}$ and the set of right pure ideals of a Γ-semiring R containing I but it is not complete. We obtain only an $1-1$ function from the set of right pure ideals of a Γ-semiring R containing I, say A, to the set of right pure ideals of a quotient Γ-semiring $\frac{R}{I}$, say B that is $\theta : A \rightarrow B$ by $\theta(J) = \frac{J}{I}$ for all $J \in A$. Example 4.17 shows that it is not onto. However, if for every $A \in \frac{J}{I}$ there exists a unique $x \in J$ such that $A = x + I$ for all right pure ideals $\dfrac{J}{I}$ of $\dfrac{S}{I}$, then J is a right pure ideal of S. This conditions make θ onto.

In the products of Γ-semirings, we want to construct a right pure ideal in the products of Γ-semirings from the products of right pure ideals in Γ-semirings. Furthermore, we can conclude that $H_1 \times \cdots \times H_n$ is a right pure (purely semiprime, purely maximal) ideal of a $\Gamma_1 \times \cdots \times \Gamma_n$-semiring $R_1 \times \cdots \times R_n$ if and only if for each i, H_i is a right pure (purely semiprime, purely maximal) ideal of R_i.

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