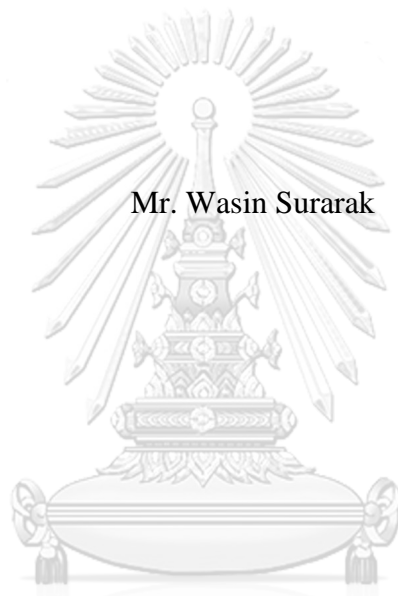


A STUDY OF THE EFFECT OF DARK POOL USING MARKET SIMULATION



Mr. Wasin Surarak

จุฬาลงกรณ์มหาวิทยาลัย

บทคัดย่อและแฟ้มข้อมูลฉบับเต็มของวิทยานิพนธ์ตั้งแต่ปีการศึกษา 2554 ที่ให้บริการในคลังปัญญาจุฬาฯ (CUIR)
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การศึกษาผลกระทบของแอมมีด โดยใช้การจำลองสถานการณ์ตลาด



วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาวิศวกรรมการเงิน ภาควิชาการธนาคารและการเงิน

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ลิขสิทธิ์ของจุฬาลงกรณ์มหาวิทยาลัย

วคิน สุรารักษ์ : การศึกษาผลกระทบของแอ่งมืด โดยใช้การจำลองสถานการณ์ตลาด (A STUDY OF THE EFFECT OF DARK POOL USING MARKET SIMULATION) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ. ดร. สิริระ สุจินตะบัณฑิต, 49 หน้า.

งานวิจัยฉบับนี้ได้พัฒนาแบบจำลองการซื้อขายหลักทรัพย์ของ Goettler et. al (2005) โดยการเพิ่มตลาดแอ่งมืดที่ไม่มีการแสดงคำสั่งซื้อขายไปยังผู้ค้าหลักทรัพย์คนอื่นๆ ในตลาด แบบจำลองนี้เปรียบเสมือนเกมของผู้ค้าหลักทรัพย์ที่เป็นกลางต่อความเสี่ยง เราใช้ลำดับขั้นตอนแบบเฟ้นสุ่มของ Pakes and McGuire (2001) เพื่อหาจุดดุลยภาพมาร์คอฟแบบสมบรูณ์ เราพบว่าเมื่อมีตลาดแอ่งมืด คุณภาพของตลาดแย่ลง (ความลึกของตลาดลดลง และส่วนต่างระหว่างอัตราเสนอซื้อและขายกว้างขึ้น) และอัตราการจับคู่ของคำสั่งซื้อขายลดลง การพังทลายของคุณภาพตลาดเป็นผลมาจากคำสั่งซื้อขายที่ถูกย้ายไปยังตลาดแอ่งมืดเพื่อมองหาผลตอบแทนคาดหวังที่สูงขึ้น เมื่อผู้ค้าหลักทรัพย์สามารถลดค่าส่วนต่างการซื้อขายได้ครั้งหนึ่งในตลาดแอ่งมืด แต่ความพึงพอใจของผู้ค้าหลักทรัพย์โดยรวมกลับเพิ่มสูงขึ้น



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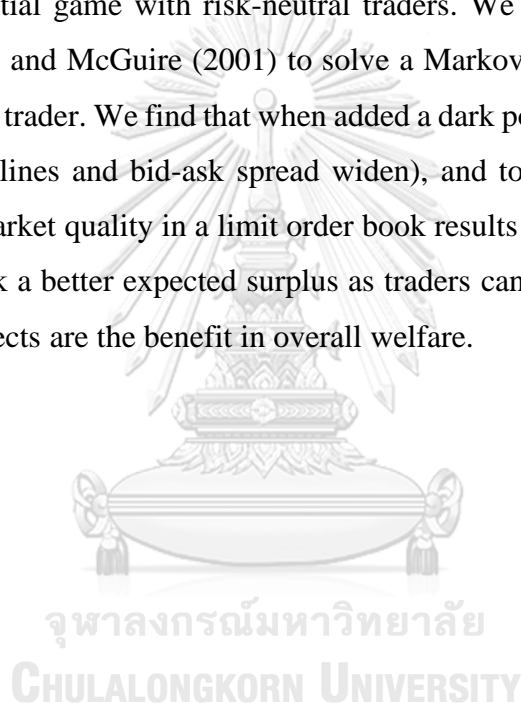
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This paper extend a limit order book model of security trading from Goettler, et al. (2005) by adding a dark pool which does not publicly display trading orders. This model is a sequential game with risk-neutral traders. We use a stochastic algorithm followed by Pakes and McGuire (2001) to solve a Markov-perfect equilibrium as the optimal action of a trader. We find that when added a dark pool, market quality worsens (market depth declines and bid-ask spread widen), and total fill rate decreases. The deterioration of market quality in a limit order book results from an order migration to a dark pool to seek a better expected surplus as traders can save half-spread in a dark pool, but these effects are the benefit in overall welfare.



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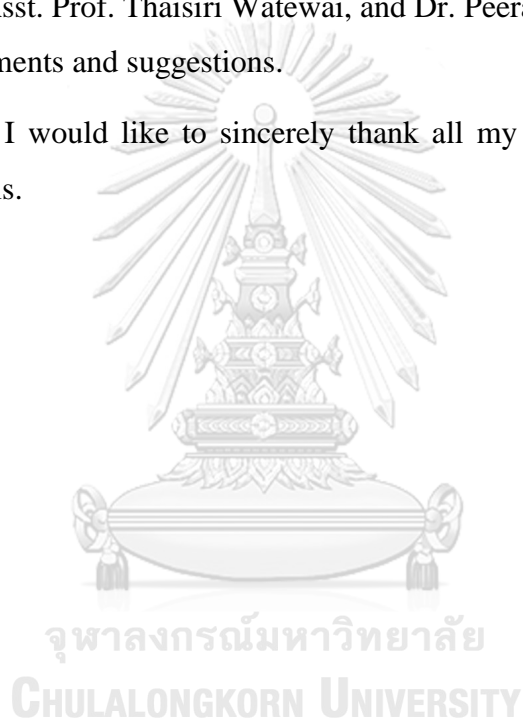
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1. INTRODUCTION

Nowadays, trading venues are not only limited to the exchange where traders submit an order to a limit order book (LOB) which publicly display orders to all market participants. There are alternative trading systems (ATSs) which are off-exchange trading platforms. Dark pool (DP) is one of the ATSs that does not publicly display their orders. At first intention, DP is mostly used by institutional investors who trade large orders without showing their inventory to others and avoiding the price impact on the exchange along with front-running by brokers. Since other market participants cannot see at the trade transactions before the orders are executed, as this causes the market to be no longer transparent. Currently, there are more than 40 DPs registered with the Commission in the US from Securities and Exchange Commission (2015). In October 2017, DPs executed nearly 15% of US equities trading volume compared to roughly 4% in 2005 from Rosenblatt Securities Inc. (2017). Regarding European Commission (2010), DP diverts trading volume from LOB rather than attracting new order flow to the market, resulting in higher trading costs. So, using more DPs have raised concerns of regulators that they may harm market quality and welfare.

This paper uses a theoretical model based on a LOB model of Goettler, et al. (2005) which is an infinite time horizon of Parlour (1998). This model is a dynamic sequential game and we use it as a base model. Traders in our model are fully rational and arrive sequentially to the market. Each trader has a different private valuation additive to the fundamental value of the asset. Traders with high individual values are willing to buy the asset, whereas traders with low private values are willing to sell the asset. Each trader is restricted to trade only one share in a price grid. In each period, a

trader chooses whether to submit a limit order, a market order, or to refrain from trading to maximize his expected profit based on the current state of the LOB and the fundamental value of the asset which are both publicly observed. A limit order is an order to buy or sell an asset at a specific price but not guaranteed to execute. Therefore, the expected profit is calculated using belief about the execution probability and the change in the fundamental value conditional on execution. For a market order, an order will immediately match at the best price available so that the profit can be calculated. We use a stochastic algorithm followed by Pakes and McGuire (2001) to solve their beliefs in the form of Markov-perfect equilibrium. This algorithm technique can deal with the curse of dimensionality since it computes equilibrium values from the average when the order is executed or canceled at the recurrent states of the time in the simulation.

We extend the LOB model by adding a DP operates alongside a LOB and use it as an extension model (LOB&DP). In the DP, traders can only submit buy or sell orders without a specific price, and the submitted orders will execute continuously at the midpoint of the best bid and ask on the LOB. This method is used by many DPs such as NASDAQ, LiquidNet, and Direct Edge in the US. Since traders cannot observe the state of the DP, and they do not know the actual execution price will be for an order sent to the DP as it relies on the future book of the LOB. Therefore, the expected profit of dark orders is calculated using belief about the execution probability, the change in the fundamental value and the difference in the midpoint of the LOB conditional on execution.

In the equilibrium, we analyze the optimal submission strategies for six states of the LOB that the most occur in the simulation by comparing the results from the base

model (LOB) to the results from the extension model (LOB&DP). Traders are more likely to use limit orders when faced with the empty book in the LOB. When the book is filled up by the limit orders, the execution probability of limit orders in the LOB tends to decline, and market orders are more attractive. The presence of the DP stimulates these effects and traps traders with low willingness to trade away from the LOB. These traders have switched from the limit orders to dark orders since the saving half-spread in the DP outweighs the lower execution probability of limit orders in the LOB. For traders with high willingness, they still prefer to use the market orders in the LOB because the execution certainty is more important than the higher trading costs.

In this research, we investigate the effects of a DP on market quality, fill rate and welfare by comparing the results from the base model (LOB) when there is only a LOB to the results from the extension model (LOB&DP) when added a DP. The market quality is considered in two measures. First, market depth is defined as the average number of buy or sell orders at each price level in the LOB. The depth of market provides the liquidity for absorbing large market orders or when traders continuously trade in one-side of the market without price impact. Second, bid-ask spread is defined as the average difference between the ask and bid prices on the LOB book of the time in the simulation. Spread indicates the transaction costs in the LOB when traders submitted market orders. Next, the fill rate is defined as the percentage number of executed orders over the number of submitted orders. It indicates the probability that the orders will be matched in the LOB or the DP. We show that the introduction of the DP results in lower total fill rate, but higher LOB fill rate. The intuition of this result is that the orders migration from the LOB to the DP, causing the depth of LOB declines. Including, the execution probability of limit orders in the LOB is lower when added the

DP given the same state of the LOB. Traders tend to use more market orders, causing LOB fill rate increases with a broader LOB spread.

For welfare, we compute using the average surplus of each trade in the market since all traders maximize utility. We find that the overall welfare increase, and also the LOB welfare. Traders with low willingness to trade tend to use the dark orders to save half-spread, leave traders with high willingness to trade in the LOB. Therefore, the matching orders between traders with high willingness to buy and sell in the LOB make the LOB welfare increases.

Finally, we study the effect of increasing in volatility of the asset. We show that the increase in volatility of the asset results in a lower total fill rate. However, DP fill rate in a high volatile is higher than DP fill rate in a low volatile as traders significantly migrate their orders to the dark venue and lessen the market depth. Using more dark orders will increase the chance of execution in the DP even if traders cannot observe the state of the DP. However, the overall welfare decreases from the lower LOB welfare. Because traders with high willingness to buy or sell the asset who use market orders in the LOB to guarantee their execution, they face the higher trading costs from a wider spread.

This paper is organized as follows: we review the literature on a DP in Section 2. In Section 3 we present our research objectives. In Section 4 we review the LOB model from Goettler, et al. (2005) and we extend for the DP. In Section 5 we examine the optimal strategies to compute market microstructure when DP is existed to answer our research questions in fill rate and welfare. We also increase the volatility of the asset to analyze the effect in Section 6. Finally, the effect of the DP has been concluded in Section 7.

2. LITERATURE REVIEW ON DARK POOL

Our model is the theoretical model, the paper which related to ours is Degryse, et al. (2009), who analyze a dynamics model on a dealer market and a crossing network (dark pool) for three information settings regarding the degree of transparency. The dealer market is only allowed traders to submit market orders where our model is a LOB where traders can also submit limit orders or market orders to the market. Also, the crossing network is similar to our DP where the execution price derives from the midpoint of the prevailing market but Degryse, et al. (2009) assumes that the dark orders cross at the end of the trading day. The results show systematic patterns in order flow depend on the level of transparency. Introducing a DP, it attracts a new order from patient traders now submit dark orders instead of refraining from trade. Including, traders with low willingness to trade switch from submitted a dealer market to the DP instead. Our results have contributed to these effects as the DP attracts traders with low private values instead of submitted limit orders.

Another related paper is Buti, et al. (2011) extend the model from Degryse, et al. (2009) by adding a limit order market instead of a dealer market. However, our model has a richer model such as they restricted only conservative limit orders, limit buy (sell) orders lower (higher) than the fundamental value, the asset value can be changed, and the remaining orders can be canceled to reflect the delay costs. They show that there is no order creation, only patient trades can submit a limit order instead of no submit any order. For market quality, since order migration occurs when a DP is presented, the market depth in the LOB declines, but total volume increases. Bid-ask spread increase for illiquid stocks but decrease for liquid stocks.

Regarding the regulators' concern about the DP might harm price discovery. There are two recent literatures with contrast results between Ye (2011) and Zhu (2014). Ye (2011) extended the Kyle (1985) framework to model the competition between one informed trader and market makers. In Ye (2011) model, only informed traders can choose the DP to trade while uninformed traders provide liquidity in a dealer market. Ye (2011) concluded that the uninformed traders would drive the dealer market in a wrong direction if informed traders migrate to the DP.

Zhu (2014) extend Glosten and Milgrom (1985) model to find the competition between informed traders and uninformed traders where both trader's types can trade in both a dealer market and a DP. Zhu (2014) disagrees with Ye (2011), he concludes that a DP improves price discovery together with declined in market quality (bid-ask spread widen and market depth decreases). Because informed traders tend to trade on the heavy side of the market and they do not like the execution risk in a DP. The driven of informed traders in a dealer market makes a DP improve price discovery as a DP attracts uninformed traders to trade away from a dealer market. The reason that Zhu (2014) contrast from Ye (2011) is Zhu assumes both informed and liquidity traders can freely select trading venues. Conversely, Ye (2011) assumes only informed traders can choose freely trading venues.

From two recent models about the effect of DP on price discovery if we extend our model to include informed and uninformed traders. We conjecture that the DP would not necessarily cause a wider spread as informed traders can use both the LOB and the DP, unlike the dealer market where restricted only market orders. The optimal strategies of traders depend on how to deviate from the actual value to the current asset price as the private value and the fundamental value in our model.

3. RESEARCH OBJECTIVES

3.1. Does the dark pool increase or decrease fill rate?

Fill rate is defined as the percentage number of executed orders over the number of submitted orders. Everyone knows the presence of the DP migrates the orders from the LOB to the DP and decreases the book depth of the LOB, including the reduction in execution probability of limit orders given the same state of the LOB. These effects drive traders to use more market orders. We believe that the LOB fill rate should increase. However, we focus on the total fill rate of the market, and the DP fill rate is ambiguous as traders cannot observe the state of the DP.

3.2. Does the dark pool increase or decrease welfare?

Welfare is defined as the average surplus of each trade in the market since all traders maximize utility. Obviously, DP makes the spread wider from the orders migration. We believe that the LOB welfare should decrease from the higher transaction costs in the DP. However, the saving half-spread in the DP can compensate the loss in LOB welfare and increases the overall welfare.

3.3. Will the results in two questions above remain the same under the increase in volatility of the asset?

Due to the exogenous of the volatility of the asset, our model can compare between the low and high volatility of the asset. In the high volatility, traders should use less the limit orders to avoid adverse selection when the fundamental value changed. As a result, the market depth should be thinner and spread should be wider than the low volatility. Thus, traders should migrate more to the DP to save the transaction costs. We believe that DP fill rate should increase because more traders are using the DP, on the other hand, the total fill rate should be lower since fewer traders

use market orders to guarantee their orders executed in the LOB. Also, we believe that the overall welfare should increase with the benefit of trading in the DP dominates the higher trading costs in the LOB because most of the transactions occur at the DP.



4. METHODOLOGY

In this section, we review the model of a LOB from Goettler, et al. (2005) and use it as a base model (LOB). In this model, rational traders arrive sequentially to the market. Each trader is restricted to trade only one share in one asset. If any submitted orders are not executed, traders can cancel their orders by cancellation factor. We extend the base model by adding a DP which the orders cross at the midpoint of the best bid and ask price on the LOB. In the extension model (LOB&DP), traders not only submit the orders to the LOB but also send the dark orders to the DP. We use a stochastic algorithm based on Pakes and McGuire (2001) to solve an equilibrium of the optimal order submission strategies.

4.1. Review of the Existing Limit Order Book Model (LOB)

This LOB model is a discrete time model with an infinite horizon ($t = t_1, t_2, t_3, \dots$) adopted from Goettler, et al. (2005). In each time, t , the asset has a fundamental value, denoted by v_t , which is public information distributed to all traders in the market. This fundamental value can be changed along trading periods as new information in the future will reflect the asset. This value, v_t , increases or decreases each time by one tick size with probability $\sigma/2$, where $\sigma \leq 1$.

The market structure of the LOB is characterized by a vector set of four discrete prices, denoted by $P = \{p^i\}$, and $i = \{1,2,3,4\}$ is the tick level on the price grid:

$$p^4 = \frac{3}{2}d$$

$$p^3 = \frac{1}{2}d$$

$$p^2 = -\frac{1}{2}d$$

$$p^1 = -\frac{3}{2}d$$

where the minimum tick size, denoted by d , is a constant between any two consecutive prices that traders can place their orders on the LOB. The prices are defined relative to the fundamental value of the asset, assuming v_t places between price p^2 and p^3 . At time t , if traders want to buy one share at price p^i , traders will have to pay $v_t + p^i$. The book depth of orders at each price is characterized by a vector set of outstanding orders, denoted by $L_t = \{l_t^i\}$, and tick level i associated to each price p^i . For price p^1 and p^4 , assuming unlimited buy orders and unlimited sell orders, respectively, to absorb any amount at the lowest bid and the highest ask prices on the LOB as Seppi (1997). To simplify, we ignore the unlimited orders at price p^1 and p^4 in L_t , then, the state of the LOB at each time t is $L_t = \{l_t^2, l_t^3\}$. We use positive quantities for buy orders and negative quantities for sell orders. Given the book depth, L_t , the bid price, denoted by B_t , is the highest price of a limit buy order on the book, and the ask price, denoted by A_t , is the lowest price of a limit sell order on the book:

$$B_t = v_t + \max\{p^i \mid l_t^i > 0\}$$

and

$$A_t = v_t + \min\{p^i \mid l_t^i < 0\}$$

The execution of the LOB is prioritized by the principle of price and time priority in which a buy order greater than or equal to the ask price will execute immediately at the ask price and is called a market order. Similarly, for a sell order less than or equal to the bid price. If traders submit a buy order less than the ask price or a sell order greater than the bid price, this order is prioritized by price and has been waited for a market order for matching in the future and is called a limit order. If two limit orders are at time same price, time priority is used.

Assuming at time t_1 , an empty book depth in the LOB $L_t = \{0,0\}$ and the fundamental value $v_1 = 0$. For other parameters that do not depend on time, we assume that the minimum tick size $d = 0.10$, and the changing probability in fundamental value $\sigma = 0.05$.

In each time t , a new risk-neutral trader arrives with a private value in the asset, denoted by β_t , where β_t drawn from a normal distribution with a normal distribution with a mean of 0 and a standard deviation of 0.20, $\beta \sim Normal(0, 0.20)$. Each trader has a different private valuation additive to the fundamental value. Traders with a positive β are willing to buy the asset and tend to use a market order if extremely high β , whereas traders with a negative β are eager to sell the asset and tend to use a market order if extremely low β . Each trader is restricted to trade at most one share of the asset or not submit an order. If he chooses to trade that share, he will select to either buy or sell, and either a market or limit order which maximizes his expected profit. An action of a trader at time t , denoted by $X_t = \{x_t^i\}$, in which tick level i associated to a price p^i on the book:

$$x_t^i = \begin{cases} 1 & \text{a buy order is submitted to the LOB at time } t \\ 0 & \text{no order is submitted to the LOB at time } t \\ -1 & \text{a sell order is submitted to the LOB at time } t \end{cases}$$

After the trader submitted an order to the LOB, the state of the LOB updates at price p^i accords to his action. For example, at time t_1 trader submits a limit buy order at price p^2 , the state of the LOB will update from $L_1 = \{0,0\}$ to $L_2 = \{1,0\}$.

This model has an exogenous cancellation factor denoted as δ which is the probability that the limit orders are canceled if they are not executed. The cancellation factor can act as a discount factor or a delay cost. This factor reduces the expected profit

of traders at a constant rate. Assuming the cancellation factor $\delta = 0.05$, so the expected time before it is canceled is 20 order-arrival periods.

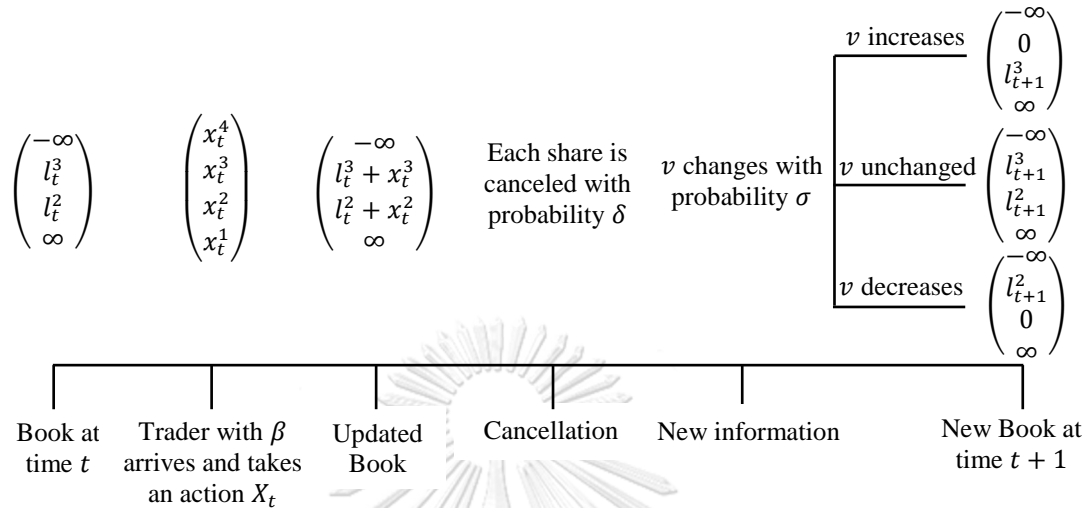


Figure 1. Timeline of the base model (LOB).

In Figure 1, we show the sequence of events at each time t of the base model (LOB). Starting with the book depth at time t , l_t^i , where i associated to price p^i . Next, a new risk-neutral trader with private value, β_t , arrives at the market and observes the state of the LOB, L_t , then takes an optimal action X_t given the state of the LOB which maximizes his expected payoff. The book depth at price p^i is now updating to $(l_t^i + x_t^i)$. After trader submits an order, each remaining order in the LOB is canceled with probability δ . If the fundamental value does not change, it will be the end of time t and the new book at time $t+1$ will be the second case in Figure 1. Otherwise, suppose the fundamental value of the asset increases by one tick. All orders at price p^i are now changed to price p^{i-1} since this model assume all prices relative to the fundamental value. Buy orders at price p^2 will be placed at price p^1 and are immediately canceled from unlimited buy orders at price p^1 ; vice versa, sell orders at price p^2 will be listed

at p^1 , and are immediately executed against unlimited buy orders and the new book at time $t + 1$ will be the first case in Figure 1. Similarly, when the fundamental value of the asset decreases by one tick, all orders at price p^i are now changed to price p^{i+1} instead and the new book at time $t + 1$ will be the third case in Figure 1.

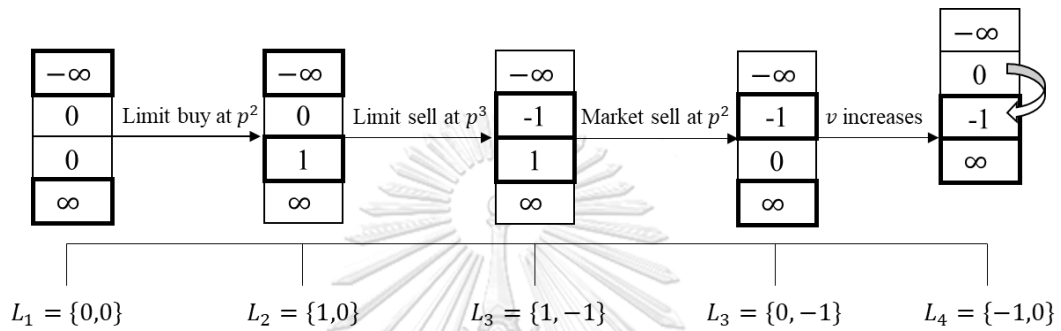


Figure 2. Example of the base model (LOB).

In Figure 2, we show the example of the LOB model in three trading periods when no any orders are canceled between t_1 and t_4 and the fundamental value increases at t_3 after trader submitted a market sell order at price p^2 . All orders at price p^i are now changed to price p^{i-1} since this model assume all prices relative to the fundamental value, the limit sell order at price p^3 has shifted to price p^2 . Then, the new book at time t_4 before trader acts will be $L_4 = \{-1, 0\}$.

The utility or surplus of the trader in the LOB that trader will earn only if an order executes in the LOB. Let t' be the time that trader with private values, $\beta_{t'}$, submitted an order to the LOB and t is the time when an order executed where $t \geq t'$. The utility of the LOB, u_t^{LOB} , is calculated from the difference between the transaction price ($v_{t'} + p^i$) and the current value of each trader ($v_t + \beta_{t'}$) when his order is

executed. The utility or payoff of a trader who submits an order to the LOB at time t' and price p^i is:

$$u_t^{LOB} = \begin{cases} (v_t + \beta_{t'}) - (v_{t'} + p^i) & \text{his buy order at price } p^i \text{ and executed at time } t \\ (v_{t'} + p^i) - (v_t + \beta_{t'}) & \text{his sell order at price } p^i \text{ and executed at time } t \\ 0 & \text{his order is canceled before it is executed} \end{cases}$$

4.2. Extension for Dark Pool (LOB&DP)

We extend the base model of the LOB by adding the DP which operates alongside with the LOB. The DP is an opaque exchange that traders cannot observe the book of the DP. In our model, this DP executes every time t , and the execution price derives from the midpoint of the current bid and ask price on the LOB. We defined p_t^{mid} as the midpoint of the tick level on the price grid:

$$p_t^{mid} = \frac{\max\{p^i \mid l_t^i > 0\} + \min\{p^i \mid l_t^i < 0\}}{2}$$

Since the execution price is based on the midpoint of the best bid and ask on the LOB, the DP is prioritized by the principle of time only. At time t , if traders want to buy one share in the DP and there is a counterparty available, traders will have to pay $v_t + p_t^{mid}$. We form the state of the DP in each time t from the number of outstanding buys or sells in the DP, denoted by q_t . When $q_t > 0$, there are buy orders available in the DP, and $q_t < 0$ there are sell orders available. We assume an empty book depth in the DP at time t_1 as $q_1 = \{0\}$.

In this extension model (LOB&DP), all traders can access the DP to trade the asset. Traders are not restricted to trade in the LOB, but they can choose to buy or sell in the DP to maximize their expected profit. However, traders can only observe the state of the LOB, L_t , and have no any information regarding the orders previously

submitted by the other traders to the DP. If they choose to trade in the DP, an action of a trader at time t denotes by $Y_t = \{y_t\}$:

$$y_t = \begin{cases} 1 & \text{a buy order is submitted to the DP at time } t \\ 0 & \text{no order is submitted to the DP at time } t \\ -1 & \text{a sell order is submitted to the DP at time } t \end{cases}$$

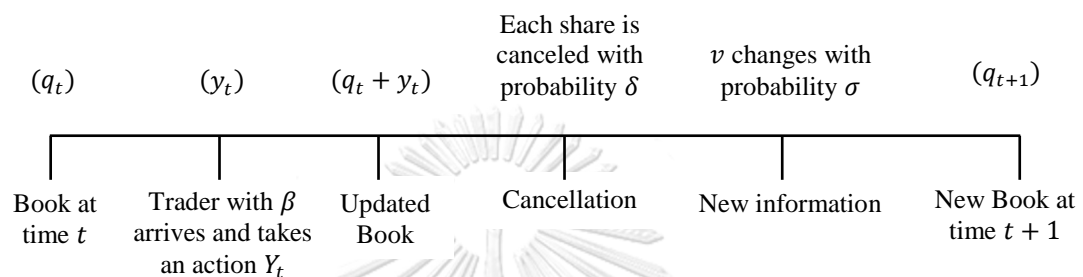


Figure 3. Timeline of the DP in the extension model (LOB&DP).

In Figure 3, we show the sequence of events in the DP at each time t of the extension model (LOB&DP). Starting with the book of the DP at time t , q_t . Next, a new risk-neutral trader with private value, β_t , arrives at the market and observes the state of the LOB, then takes an action Y_t if he chooses to trade in the DP. The DP book is now updating to $(q_t + y_t)$. After trader submits an order, each remaining order in the DP is canceled with probability δ . Because the changing of the fundamental value does not affect the DP book the new DP book at time $t + 1$, q_{t+1} will be $(q_t + y_t)$.

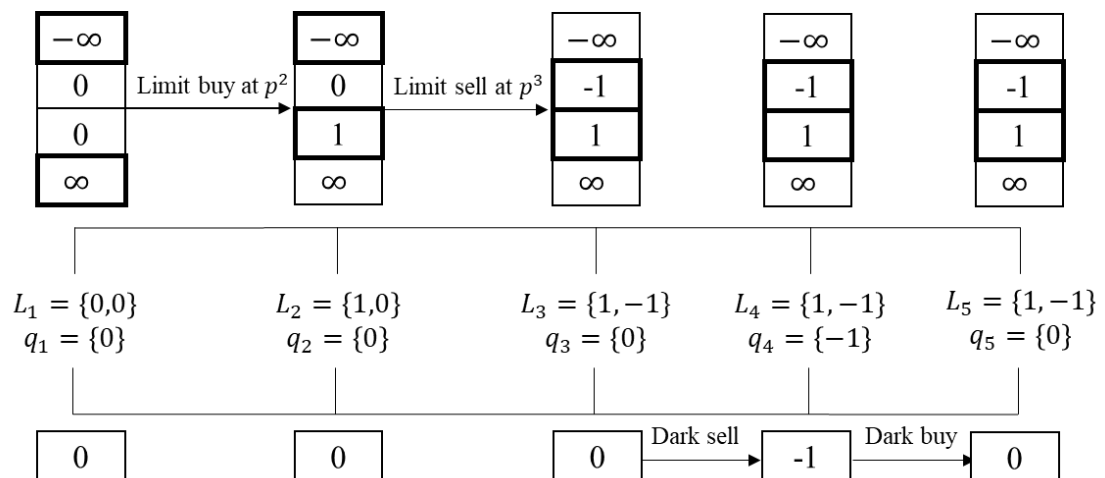


Figure 4. Example of the extension model (LOB&DP).

Figure 4 shows the example of the extension model (LOB&DP) in four trading periods when no orders are canceled, together with no change in the fundamental value. Traders are not restricted to the LOB, but they can submit buy or sell dark orders to the DP. Starting with an empty book of the DP at time t_1 , $q_1 = \{0\}$. In the example, Trader begins trading in the DP at time t_3 with a dark sell order. The DP book now updates from $q_3 = \{0\}$ to $q_4 = \{-1\}$. At time t_4 trader also uses a dark buy order so that it will be matched against the dark sell order in the DP book. Then, the new DP book at time t_5 before the new trader acts will be $q_5 = \{0\}$.

The utility or surplus of the trader in the DP that trader will earn only if an order executes in the DP. Let t' be the time that trader with private values, $\beta_{t'}$ submitted an order to the DP and t is the time when an order executed where $t \geq t'$. The utility of the DP, u_t^{DP} , is calculated from the difference between the transaction price from the DP ($v_{t'} + p_t^{mid}$) and the current value of each trader ($v_t + \beta_{t'}$) when his order is executed. The utility or payoff of a trader who submits an order to the DP at time t' is:

$$u_t^{DP} = \begin{cases} (v_t + \beta_{t'}) - (v_{t'} + p_t^{mid}) & \text{his dark buy order executed at time } t \\ (v_{t'} + p_t^{mid}) - (v_t + \beta_{t'}) & \text{his dark sell order executed at time } t \\ 0 & \text{his order is canceled before it is executed} \end{cases}$$

4.3. Equilibrium

Each time t , a new risk-neutral trader with private valuation, β_t , arrives at the market and chooses an optimal action that maximizes his expected utility or surplus, given the fundamental value, v_t , and the current state of the LOB, L_t , that he can observe at time t . The trader does not know the future actions of other traders, the change in the fundamental value, and the change in the midpoint of the LOB. These factors determine whether his limit orders or dark orders will execute or not and also the benefit from trading.

For LOB, trader will form beliefs of the execution probability of his action, X_t , given the current state of the LOB, L_t , that can be denoted as $\mu_t^{LOB}(L_t, X_t)$. In addition, he also forms the expectation of the net change in the fundamental value until his order is executed, that can be denoted as $\Delta_t^{LOB}(L_t, X_t)$. If trader submits an order at time t and his order is executed at time $\tau \geq t$, the equations can be expressed as:

$$\mu_t^{LOB}(L_t, X_t) = P(\tau < \infty | L_t, X_t)$$

$$\Delta_t^{LOB}(L_t, X_t) = \mathbb{E}[v_\tau - v_t | \tau < \infty, L_t, X_t]$$

For DP, trader forms beliefs about the execution probability, denoted by $\mu_t^{DP}(L_t, Y_t)$, given the current state of the LOB, L_t , and action taken, Y_t . Likewise, the expectation of the net change in v_t and p_t^{mid} , denoted by $\Delta_t^{DP}(L_t, Y_t)$, will habit the same until the order is executed. These can be expressed in equations as:

$$\mu_t^{DP}(L_t, Y_t) = P(\tau < \infty | L_t, Y_t)$$

$$\Delta_t^{DP}(L_t, Y_t) = \mathbb{E}[(v_\tau - v_t) + (p_\tau^{mid} - p_t^{mid}) | \tau < \infty, L_t, Y_t]$$

Since all traders are risk-neutral, trader observes the state of the LOB and takes an action $X_t = \{x_t^i\}$ to the LOB, his expected payoff when submitted an order to the LOB at price p^i is

$$\pi_t^{LOB}(L_t, X_t) = \begin{cases} \mu_t^{LOB}(L_t, X_t) \{ \Delta_t^{LOB}(L_t, X_t) + (\beta_t - p^i) \} & \text{for a buy order at } p^i \\ \mu_t^{LOB}(L_t, X_t) \{ (p^i - \beta_t) - \Delta_t^{LOB}(L_t, X_t) \} & \text{for a sell order at } p^i \end{cases}$$

Since market orders execute immediately at time t , then the execution probability is $\mu_t^{LOB}(\cdot) = 1$ and the net change in value is $\Delta_t^{LOB}(\cdot) = 0$. A limit order submits at time t may cancel before this share is executed, so the execution probability is $\mu_t^{LOB}(\cdot) < 1$.

In the DP, traders can observe only the state of the LOB and takes an action $Y_t = \{y_t\}$ to the DP. His expected payoff, when submitted an order to the DP is

$$\pi_t^{DP}(L_t, Y_t) = \begin{cases} \mu_t^{DP}(L_t, Y_t) \{ \Delta_t^{DP}(L_t, Y_t) + (\beta_t - p_t^{mid}) \} & \text{for a buy order at } p_t^{mid} \\ \mu_t^{DP}(L_t, Y_t) \{ (p_t^{mid} - \beta_t) - \Delta_t^{DP}(L_t, Y_t) \} & \text{for a sell order at } p_t^{mid} \end{cases}$$

Since traders cannot observe the state of the DP at time t and the orders may cancel before this share is executed, so the execution probability is $\mu_t^{DP}(\cdot) < 1$.

Given these beliefs, in each time t the risk-neutral trader optimally chooses an action either in the LOB or the DP, or not submit any order conditional on the current state of the LOB that gives his maximum expected payoff:

$$\max_{X_t, Y_t} \{ \pi_t^{LOB}(L_t, X_t), \pi_t^{DP}(L_t, Y_t), 0 \}$$

4.4. Solving for Equilibrium

Solving for equilibrium is to find trader's beliefs, μ^{LOB} , μ^{DP} , Δ^{LOB} , and Δ^{DP} , such that each trader played his best response when he faced each state. We do not use a general solution as closed-form or backward induction since these techniques will suffer a curse of dimensionality. We follow Pakes and McGuire (2001) which use a

stochastic algorithm to find Markov-Perfect equilibrium. Therefore, time is no longer a state variable, and the equilibrium must be stationary so that $\mu_t^{LOB} = \mu^{LOB}$, $\mu_t^{DP} = \mu^{DP}$, $\Delta_t^{LOB} = \Delta^{LOB}$, and $\Delta_t^{DP} = \Delta^{DP}$, respectively.

We update the values of μ^{LOB} , μ^{DP} , Δ^{LOB} , and Δ^{DP} when the order is executed or canceled by averaging the result with the previous results for shares submitted at this state like perform Monte-Carlo evaluation in a visited state. For example, if the order is executed the result will be 1, and the order is canceled the result will be 0. Then the execution probability, μ , will be updated the result given the submitted state of this share.

Considering the orders in the LOB, if a limit order submitted at time t' is matched at time $t > t'$. We update μ_t^{LOB} and Δ_t^{LOB} given the state of the LOB $L_{t'}$ and acts $X_{t'}$ of the limit order.

$$\mu_{t+1}^{LOB}(L_{t'}, X_{t'}) = \frac{n}{n+1} \mu_t^{LOB}(L_{t'}, X_{t'}) + \frac{1}{n+1} \quad (1)$$

$$\Delta_{t+1}^{LOB}(L_{t'}, X_{t'}) = \frac{n}{n+1} \Delta_t^{LOB}(L_{t'}, X_{t'}) + \frac{v_t - v_{t'}}{n+1} \quad (2)$$

where n is the number of shares submitted at state $(L_{t'}, X_{t'})$ that have either executed or been canceled between time 0 and t .

If a limit order at time t' is canceled at time $t > t'$, we update μ_t^{LOB} given the state of the LOB $L_{t'}$ and acts $X_{t'}$ of the limit order. This update uses only the first term in equation (1) since the numerator in the second term is zero for canceled orders. Note that we do not update Δ_t^{LOB} because it requires the change in v conditional on execution.

$$\mu_{t+1}^{LOB}(L_{t'}, X_{t'}) = \frac{n}{n+1} \mu_t^{LOB}(L_{t'}, X_{t'}) \quad (3)$$

Considering the orders in the DP, if a dark order submitted at time t' is matched at time $t > t'$. We update μ_t^{DP} and Δ_t^{DP} given the state of the LOB $L_{t'}$ and acts $Y_{t'}$ of the dark order.

$$\mu_{t+1}^{DP}(L_{t'}, Y_{t'}) = \frac{n}{n+1} \mu_t^{DP}(L_{t'}, Y_{t'}) + \frac{1}{n+1} \quad (4)$$

$$\Delta_{t+1}^{DP}(L_{t'}, Y_{t'}) = \frac{n}{n+1} \Delta_t^{DP}(L_{t'}, Y_{t'}) + \frac{v_t - v_{t'}}{n+1} + \frac{p_t^{mid} - p_{t'}^{mid}}{n+1} \quad (5)$$

where n is the number of shares submitted at state $(L_{t'}, Y_{t'})$ that have either executed or been canceled between time 0 and t .

If a dark order at time t' is canceled at time $t > t'$, we update μ_t^{DP} given the state of the LOB $L_{t'}$ and acts $Y_{t'}$ of the dark order. This update uses only the first term in equation (4) since the numerator in the second term is zero for canceled orders. Note that we do not update Δ_t^{DP} because it requires the change in v conditional on execution.

$$\mu_{t+1}^{DP}(L_{t'}, Y_{t'}) = \frac{n}{n+1} \mu_t^{DP}(L_{t'}, Y_{t'}) \quad (6)$$

We set an initial traders' beliefs to be optimistic to avoid the false beliefs at states outside the recurrent state and ensure our results does not converge to that equilibrium. We set $\Delta_1^{LOB}(\cdot)$ to $(p^4 - p^i)$ for a buy order and to $(p^1 - p^i)$ for a sell order in the LOB. These $\Delta_1^{LOB}(\cdot)$ represent the most possible of changes in fundamental value before the orders executed. We set $\Delta_1^{DP}(\cdot)$ to 0 for a buy and sell order in the DP. We also set $\mu_1^{LOB}(\cdot)$ for a limit buy and sell order at price p^i in the LOB, and $\mu_1^{DP}(\cdot)$ for a buy and sell dark order at the DP equal to 0.95.

We choose the initial book depth of the LOB $L_1 = \{0,0\}$, the initial book depth of the DP $q_1 = \{0\}$, and the initial fundamental value $v_1 = 0$. Recalled that the other parameters that do not depend on time are the minimum tick size $d = 0.10$, the

changing probability in fundamental value $\sigma = 0.05$, the cancellation factor $\delta = 0.05$, and the private values of trader distributed $\beta \sim Normal(0, 0.20)$. The flowchart of the stochastic algorithm has been shown in Figure 5.

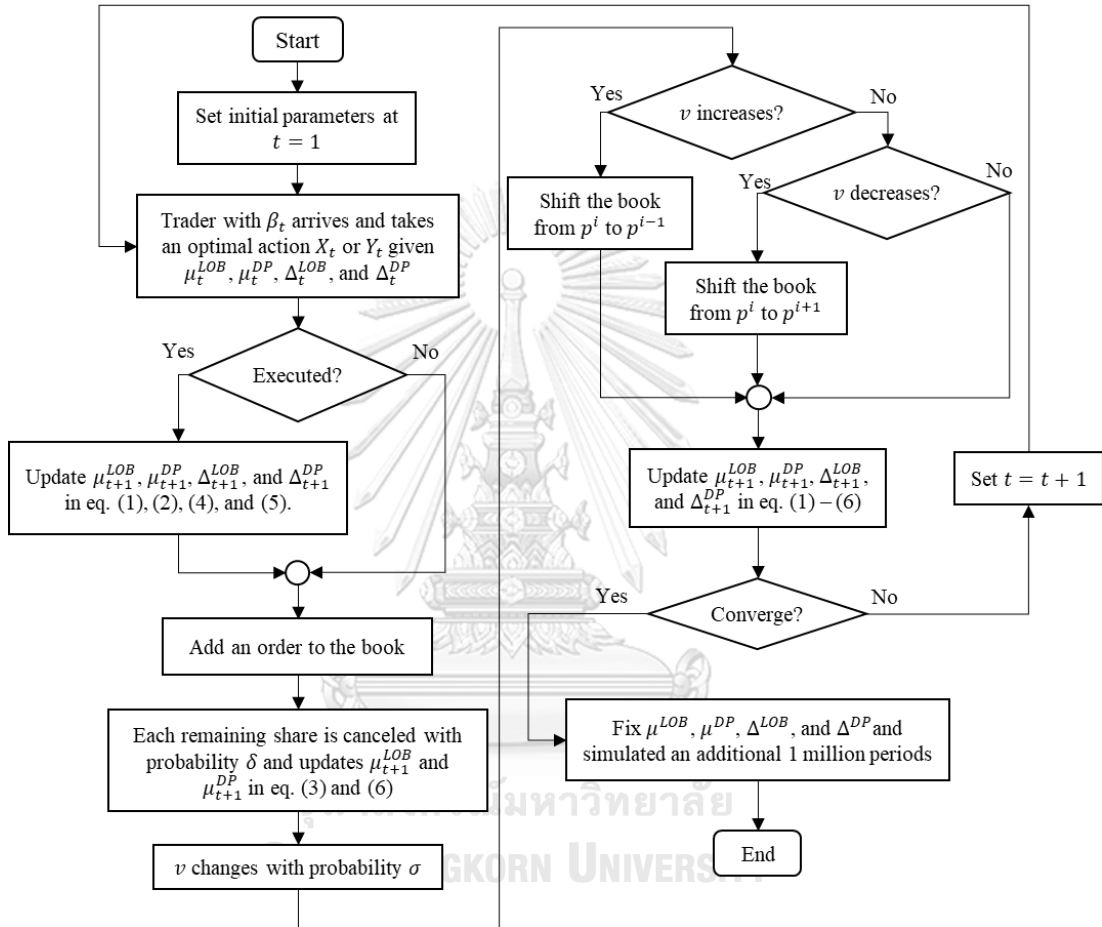


Figure 5. Flowchart of the stochastic algorithm for solving the equilibrium.

We iterate over the following steps starting with $t = 1$:

Step 1: At time t , a new rational trader with private values, β_t , arrives at the market and choose the optimal action, given μ_t^{LOB} , μ_t^{DP} , Δ_t^{LOB} , and Δ_t^{DP} .

Step 2: For each market order in the LOB, we update μ_t^{LOB} and Δ_t^{LOB} in equation (1) and (2) for the state $(L_{t'}, X_{t'})$ of the limit order at time t' is executed at time $t > t'$ and price p^i . For each matched order in the DP, we update μ_t^{DP} and Δ_t^{DP} in the equation (4) and (5) for the state $(L_{t'}, Y_{t'})$ of the dark order at time t' is executed at time $t > t'$ and price p_t^{mid} .

Step 3: Add the limit order, x_t^i , or the dark order, y_t , to the end of the queue in l_t^i and q_t , respectively. The book now is $(l_t^i + x_t^i)$ for the LOB and $(q_t + y_t)$ for the DP.

Step 4: Each remaining share in the LOB and the DP is canceled with probability δ . If the order at time t' is canceled, we shall update μ_t^{LOB} and μ_t^{DP} from the equation (3) and (6) for the state $(L_{t'}, X_{t'})$ or $(L_{t'}, Y_{t'})$ of the canceled order.

Step 5: The fundamental value, v_t , changes for the next period as follows:

$$v_t = \begin{cases} v_t + d & \text{with probability } \sigma/2 \\ v_t & \text{with probability } 1 - \sigma \\ v_t - d & \text{with probability } \sigma/2 \end{cases}$$

If v increases, the sell orders at price p^2 will be shifted to price p^1 and immediately executed against unlimited buy orders, but the buy orders at price p^2 are canceled. The states at which these orders were submitted have been updated in the equation (1) and (2) for the executed orders and update in the equation (3) for the canceled orders. When v decreases, the orders at price p^3 are processed similarly.

Step 6: Set $\mu_{t+1}^{LOB} = \mu_t^{LOB}$, $\mu_{t+1}^{DP} = \mu_t^{DP}$, $\Delta_{t+1}^{LOB} = \Delta_t^{LOB}$, and $\Delta_{t+1}^{DP} = \Delta_t^{DP}$ for the states that not updated in Step 2, 4 or 5. We then set $t = t + 1$, and return to Step 1.

Note that we reset t and n to one for every 10 million periods. Resetting n makes the algorithm to converge quickly from the optimism of initial beliefs. Resetting t uses for checking the convergence criteria. We evaluate the change in the execution probability every 10 million periods, $|\mu_{10^7}^{LOB}(L, X) - \mu_1^{LOB}(L, X)|$ and $|\mu_{10^7}^{DP}(L, Y) - \mu_1^{DP}(L, Y)|$, for each pair of (L, X) and (L, Y) . If the change in the execution probability is below 0.05 for both criteria, we will stop the iterations. Then, we deem the algorithm to have converged.



5. NUMERICAL RESULTS

We solve both the base model (LOB) when there is only a LOB and the extension model (LOB&DP) when added a continuous DP operated alongside a LOB. After the algorithm converged, we hold beliefs fixed and simulate an additional 1 million trading periods. First, we analyze the optimal submission strategies for each state of the LOB that the most occur in the simulation by comparing the results from the base model (LOB) to the results from the extension model (LOB&DP). Then, we find the market quality in two measures: market depth and bid-ask spread. The optimal strategies and the market quality provide useful information to answer our research questions in fill rate and welfare. In this Section, we present only the set of parameters from Section 4, and we will change the volatility by increasing the changing probability of the fundamental value in Section 6.

5.1. Optimal Strategies

There are 229 different books arise in the base model (LOB), and there are 144 different books in the extension model (LOB&DP) during the simulation. We demonstrate how a trader's action depends on the state of the LOB and his private value by picking up the six cases of the state of the LOB that the most occur during the simulation. The first three cases are order-balance between buy and sell limit orders on the book. Later, the last three cases are order-imbalance between buy and sell limit orders on the book as follows:

Case 1: State = $\{0, 0\}$, the empty book depth.¹

¹ The empty book arises at 5.3% in the base model (LOB) and 8.7% in the extension model (LOB&DP) of the time in the simulation.

Case 2: State = {1, -1}, the book depth of one limit buy and sell order at price p^2 and p^3 , respectively.¹

Case 3: State = {2, -2}, the book depth of two limit buy and sell orders at price p^2 and p^3 , respectively.

Case 4: State = {1, 0}, the book depth of one limit buy order at price p^2 .²

Case 5: State = {2, 0}, the book depth of two limit buy orders at price p^2 .

Case 6: State = {2, -1}, the book depth of two limit buy orders and one limit sell order at price p^2 and p^3 , respectively.

Given beliefs μ^{LOB} , μ^{DP} , Δ^{LOB} , and Δ^{DP} , we can demonstrate the optimal submission strategies for all values of β_t . The simulated values of a β lie in the interval (-0.6, 0.6). Recalling that β is drawn from a normal distribution with mean 0 and standard deviation of 0.2 so that the range will cover the 99.7% of trader's type. From Figure 6 to Figure 11, we show the optimal action of a trader from Case 1 to Case 6. We defined market orders as upper-case letters: "B" stands for market buy orders and "S" stands for market sell orders, and defined limit orders and dark orders as lower-case letters: "b" stands for limit buy or dark buy orders and "s" stands for limit sell or dark sell orders. We set the dark pool price level between the price p^2 and p^3 as the midpoint in our model is in the range $p^2 \leq p^{mid} \leq p^3$.

Starting with the Case 1, when traders faced the empty book depth in the LOB, traders can submit whether market buys at price p^1 , or market sells at price p^4 , or limit orders at price p^2 , p^3 . Otherwise, they may choose to trade in the DP. In the

¹ The book depth of one limit buy and sell order at price p^2 and p^3 , respectively, is the most common book in the base model (LOB) arises 7.5% of the time in the simulation and arises 6.5% in the extension model (LOB&DP).

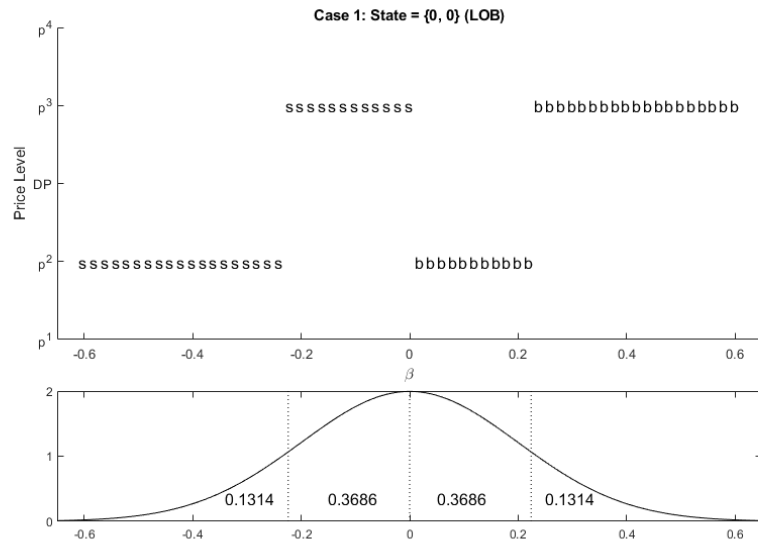
² The book depth of one limit buy order at price p^2 is the most common book in the extension model (LOB&DP) arises 10.6% of the time in the simulation and arises 6.3% in the base model (LOB).

equilibrium, traders with low willingness to trade or $\beta \in (-0.22, 0.22)$, they submit conservative limit orders: limit buys one tick below the fundamental value at price p^2 for a positive β , and limit sells one tick above the fundamental value at price p^3 for a negative β . For the traders with high willingness to trade, they submit aggressive limit orders: limit buys one tick above the fundamental value at price p^3 or limit sells one tick below the fundamental value at price p^2 . When introduced the DP, the execution probability of limit orders slightly decreases. Because they expect that some orders will migrate to the DP and the LOB activity will decline. Some traders have switched from conservative limit orders to aggressive limit orders to trade off this effect. So, the proportion of traders who submit conservative limit orders decrease from 73.7% to 70.2% and an increase in aggressive limit orders instead. In this case, the orders will be sent only limit orders at price p^2 or p^3 . The results from Case 1 are shown in Figure 6.

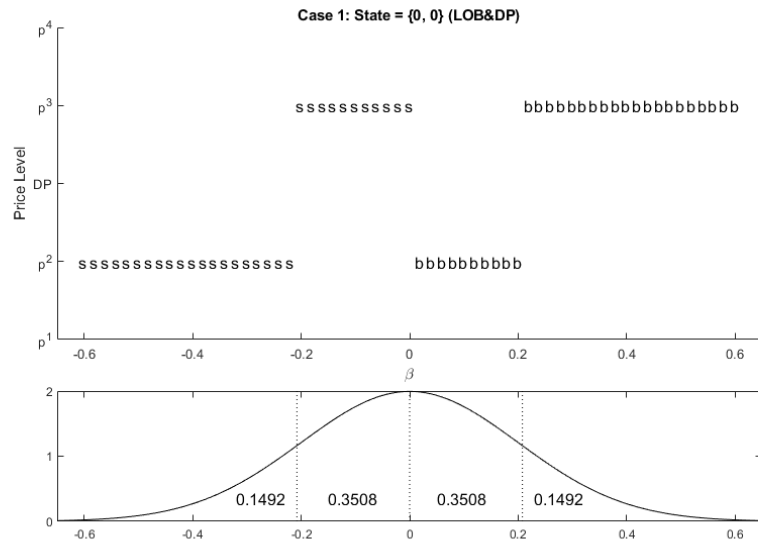
In Case 2 and Case 3, the book depth is filled by limit buys and sells at price p^2 and p^3 , respectively, each one share for Case 2 and two shares for Case 3. Traders can submit whether market sells at price p^3 or a market buys at price p^2 , or aggregate limit buys at price p^2 or aggregate limit sells price p^3 . Otherwise, they may trade away to the DP instead. Comparing the base model (LOB) between Case 2 and Case 3 in Figure 7a and Figure 8a, the results can be predictable as more liquidity in book depth, causing more traders from the limit orders to diverge to the market orders. However, when comparing the extension model (LOB&DP) between Case 2 and Case 3 in Figure 7b and Figure 8b, the results are impressive since some traders appear to use the DP in Case 3 but not in Case 2. When the book depth of two limit buys and sells, the DP attracts traders with private values $\beta \in (-0.14, 0.14)$ from the limit orders and some part of market orders in the base model (LOB). The migration of limit orders is a result of

the benefit of trading in the DP overcomes the execution probability of limit orders in the LOB. The DP also attracts some part of market orders from the LOB to the DP as the percentage of market orders in Case 3 decreases from 56.2% in the base model (LOB) to 49.7% in the extension model (LOB&DP) since the DP can save half of the trading costs from market orders. However, some traders with high willingness to trade still submit a market order to the LOB since their order can guarantee to execute even if they have to pay the bid-ask spread.



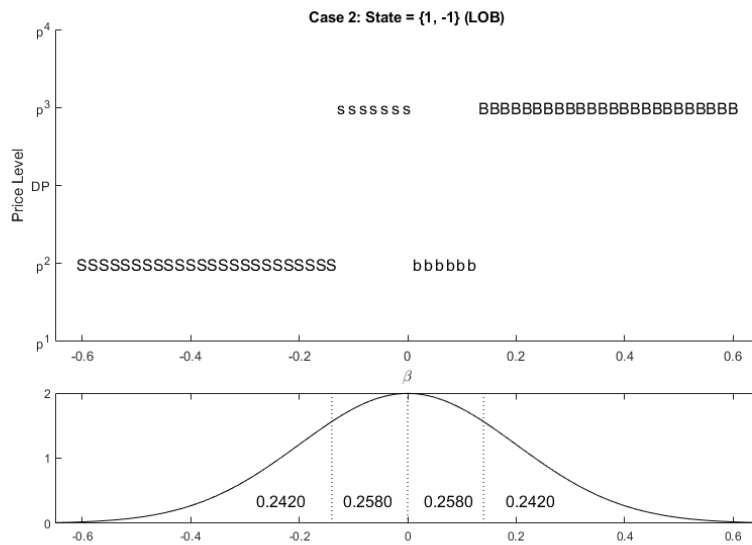


(a) The base model (LOB)

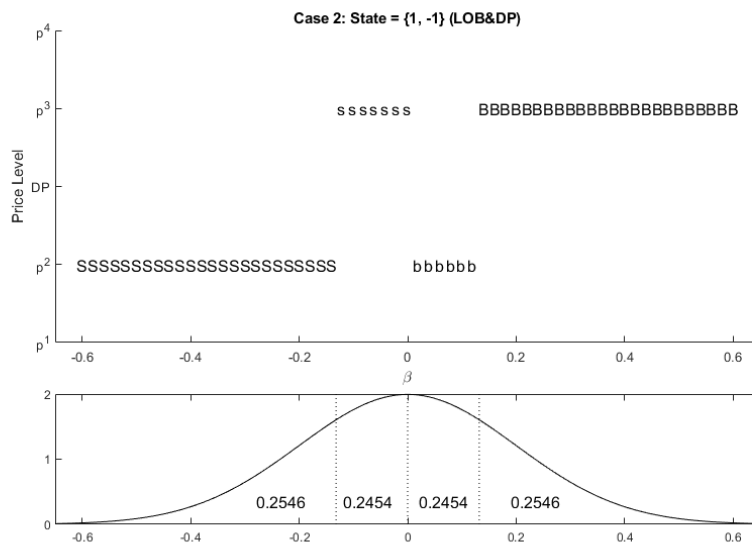


(b) The extension model (LOB&DP)

Figure 6: The optimal action of a trader in Case 1: State = {0, 0}

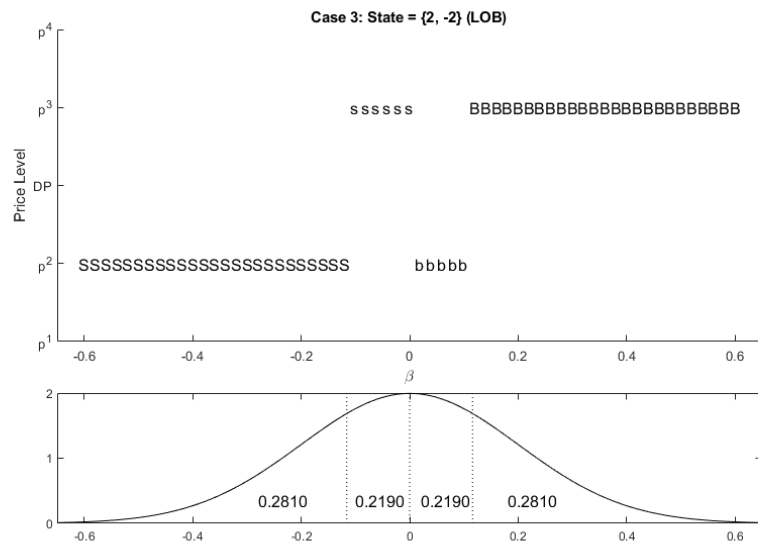


(a) The base model (LOB)

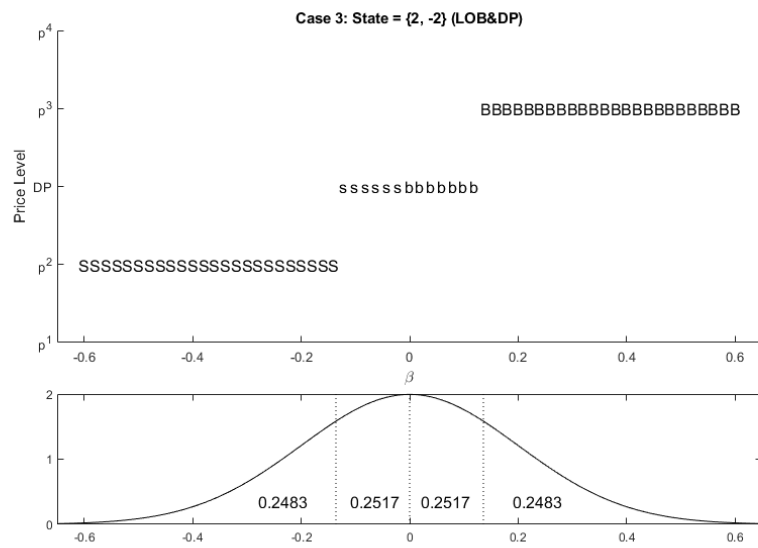


(b) The extension model (LOB&DP)

Figure 7: The optimal action of a trader in Case 2: State = {1, -1}.



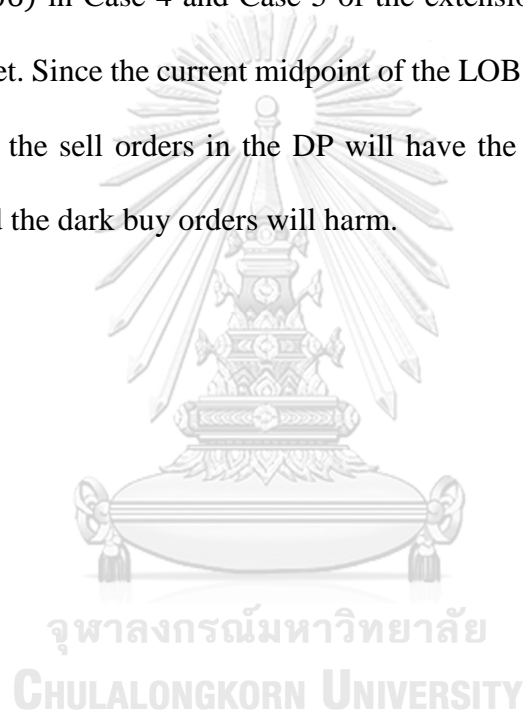
(a) The base model (LOB)

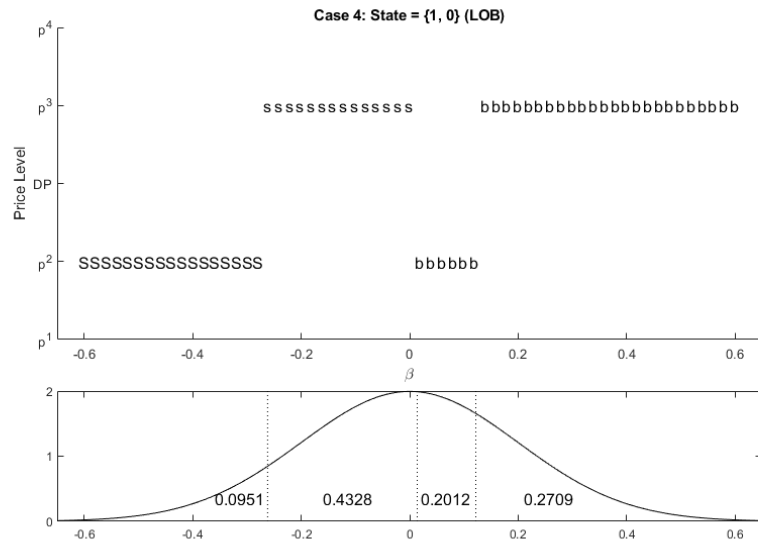


(b) The extension model (LOB&DP)

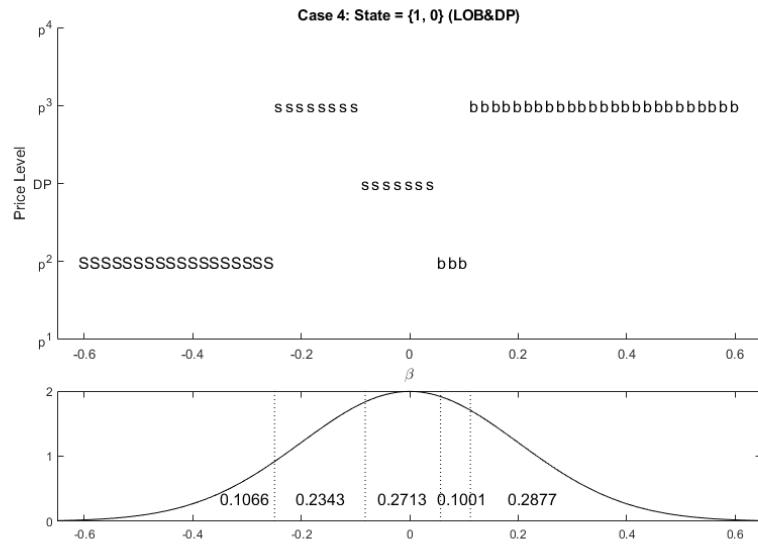
Figure 8: The optimal action of a trader in Case 3: State = {2, -2}.

Next, we consider an order-imbalance of the book depth in Case 4 to Case 6. The results in the base model (LOB) are correspondent to the order-balance in Case 1 to Case 3. In Figure 9a to 11a, the graph shows that traders divert from the limit orders on the heavy-side of the book depth to aggressive limit orders or market orders as we expected from the order-balance cases. When introduced the DP, the DP attracts trader with low willingness to trade like the first three cases. Surprisingly, traders with positive values of $\beta \in (0, 0.06)$ in Case 4 and Case 5 of the extension model submits dark sell orders to the market. Since the current midpoint of the LOB in Case 4 and Case 5 equal to price p^3 . Thus, the sell orders in the DP will have the benefit from the higher in crossing price, and the dark buy orders will harm.



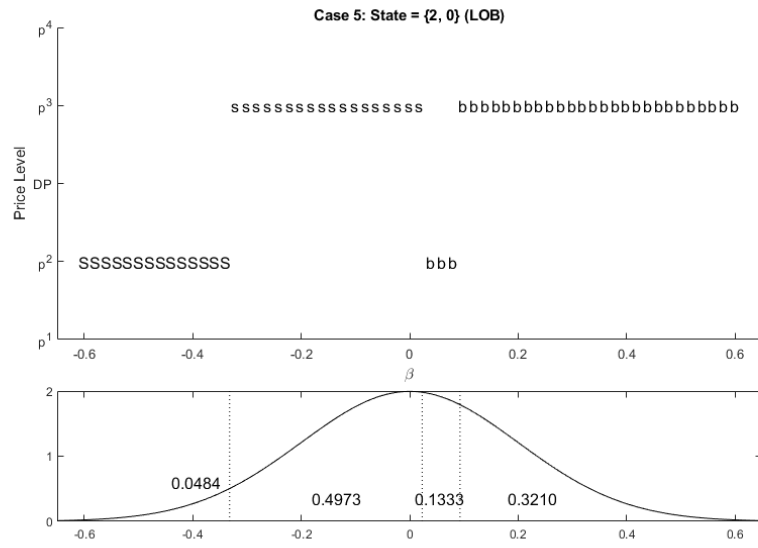


(a) The base model (LOB)

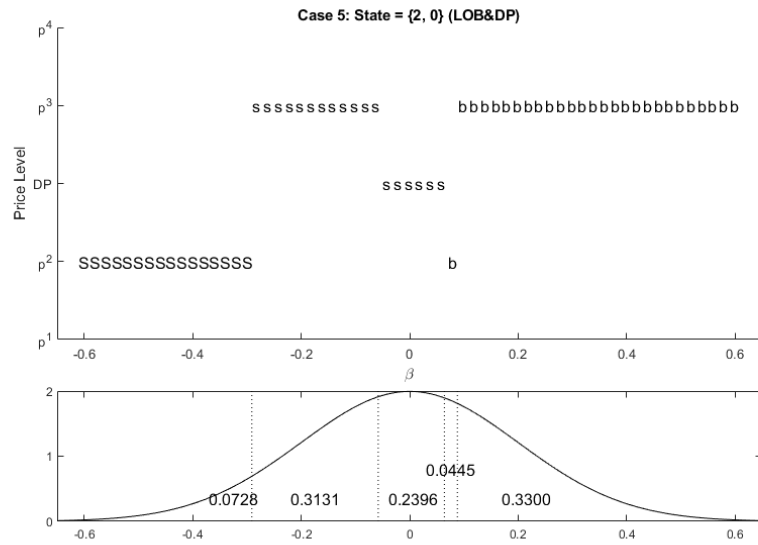


(b) The extension model (LOB&DP)

Figure 9: The optimal action of a trader in Case 4: State = {1, 0}.



(a) The base model (LOB)



(b) The extension model (LOB&DP)

Figure 10: The optimal action of a trader in Case 5: State = {2, 0}.

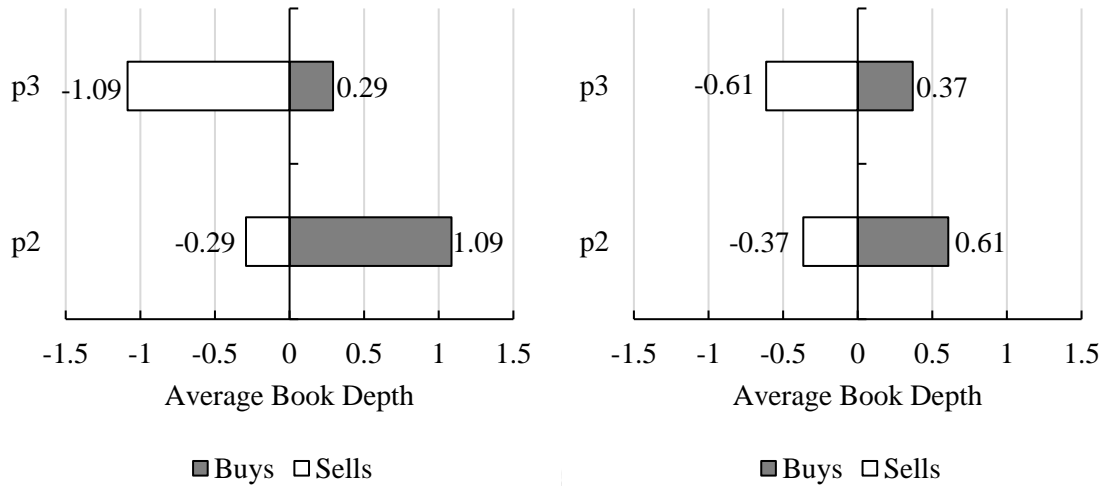
5.2. Market Quality

As the DP has been increased the recent years, regulators are concerned about the effects on market quality when widespread use of the DPs. We consider the impact of the DP on market quality of the LOB with two measures. First, market depth (D) is defined as the average number of buy or sell orders at each price in the LOB. Market depth indicates the liquidity of the asset. The higher the number of buy and sell orders at each price, the higher the thickness of the market:

$$D_{\text{buy at } p^i}^{LOB, LOB\&DP} = \frac{1}{T} \sum_{t=1}^T (l_t^i | l_t^i > 0)$$

$$D_{\text{sell at } p^i}^{LOB, LOB\&DP} = \frac{1}{T} \sum_{t=1}^T (l_t^i | l_t^i < 0)$$

The average buy and sell sides of the book is depicted in Figure 12. The average book has a total of 1.38 on the buy side and 1.38 on the sell side in the base model (LOB) and a total of 0.98 on the buy side and 0.98 on the sell side for the extension model (LOB&DP). As expected, given our symmetric model, the book is symmetric. The decreasing of average book depth from the DP is strong at conservative limit orders from 1.09 in the base model (LOB) to 0.61 in the extension model (LOB&DP). But, the aggressive limit orders incline from 0.29 in the base model (LOB) to 0.37 in the extension model (LOB&DP). The optimal strategies support these results. Since the execution probability of limit orders tends to decline when added the DP given the same state. Traders tend to use more aggressive limit orders or market orders to compensate for the lower in execution probability.



(a) The base model (LOB)

(b) The extension model (LOB&DP)

Figure 12: The average book depth at each price level, the positive value is averaged over limit buys and the negative values is averaged over limit sells at price p^3 (one-tick above the fundamental value) and price p^2 (one-tick below the fundamental value) in the LOB market only between (a) the base model and (b) the extension model.

Second, bid-ask spread (S) represents the trading cost when traders submit market orders to the LOB. We define as the average difference between the ask and bid prices of the LOB:

$$S^{LOB,LOB\&DP} = \frac{1}{T} \sum_{t=1}^T (A_t - B_t)$$

In Figure 13, we show that the bid-ask spread increases from 0.137 in the base model (LOB) to 0.147 in the extension model (LOB&DP). Because the DP attracts traders with low willingness who provide the liquidity of the LOB, and the order migrations make a wider spread.

As a result, in the market depth shows that traders switch from conservative limit orders to aggressive limit orders, traders also use more market orders. In the long-run when aggressive limit orders are executed, the decreasing in conservative limit orders will lead to a broader spread.

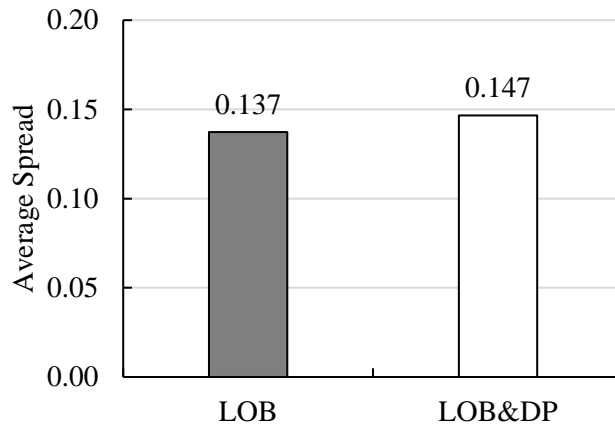


Figure 13: The average bid-ask spread in the LOB market between the base model (LOB) on the left bar and the extension model (LOB&DP) on the right bar.

5.3 Fill Rate

Fill rate indicates the probability that the orders will be matched when submitted to the market. We demonstrate how the optimal strategies drive the total fill rate and break down fill rate of each trading venue in the extension model (LOB&DP).

The fill rate is defined as the percentage of the number of executed orders over the number of submitted orders:

$$FR^{LOB} = \frac{\text{the number of executed orders}}{\sum_{t=1}^T (1|X_t \neq 0)}$$

$$FR^{LOB\&DP} = \frac{\text{the number of executed orders}}{\sum_{t=1}^T (1|X_t \neq 0 \vee Y_t \neq 0)}$$

In the extension model (LOB&DP), we also defined the fill rate in each market between LOB fill rate and DP fill rate as:

$$FR_{LOB}^{LOB\&DP} = \frac{\text{the number of executed orders in the LOB}}{\sum_{t=1}^T (1|X_t \neq 0)}$$

$$FR_{DP}^{LOB\&DP} = \frac{\text{the number of executed orders in the DP}}{\sum_{t=1}^T (1|Y_t \neq 0)}$$

In Figure 14, we show the total fill rate between the base model (LOB) and the extension model (LOB&DP) of the time in the simulation and focus on each trading venue in the extension model (LOB&DP). The total fill rate decreases from 79% to 75% when introduced the DP, but LOB fill rate increases. The higher in LOB fill rate results from traders are using less the conservative limit orders from the lower in execution probability. Then, traders have switched to aggressive limit orders or market orders, causing LOB fill rate increases. The lower in total fill rate come from the low level of DP fill rate as the DP usage in our model is around 25% of trading periods, and the dark orders are executed only 52% of total dark orders.

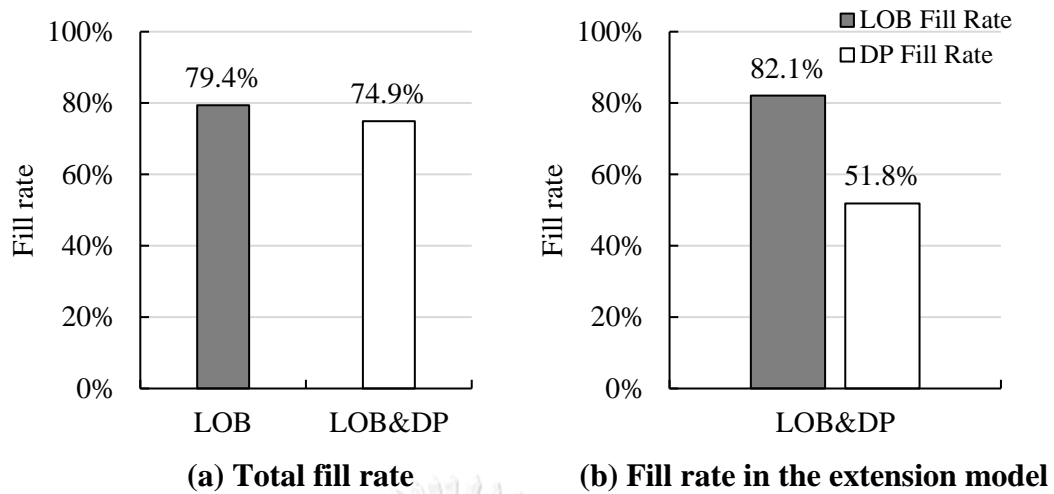


Figure 14: The total fill rate between two models and fill rate in each market in the extension model (LOB&DP), (a) The total fill rate; the left bar is the percentage of the executed orders over the LOB market, and over both markets (LOB&DP) are on the right bar. (b) The fill rate in the extension model (LOB&DP); the left bar is the percentage of the executed orders over the LOB market, and over the DP market is on the right bar.

5.4. Welfare

Recall the surplus or utility of a trader in the LOB and the DP that trader will earn only if his order executes as follows:

$$u_t^{LOB} = \begin{cases} (v_t + \beta_{t'}) - (v_{t'} + p^i) & \text{his buy order at price } p^i \text{ and executed at time } t \\ (v_{t'} + p^i) - (v_t + \beta_{t'}) & \text{his sell order at price } p^i \text{ and executed at time } t \\ 0 & \text{his order is canceled before it is executed} \end{cases}$$

$$u_t^{DP} = \begin{cases} (v_t + \beta_{t'}) - (v_{t'} + p_t^{mid}) & \text{his dark buy order executed at time } t \\ (v_{t'} + p_t^{mid}) - (v_t + \beta_{t'}) & \text{his dark buy order executed at time } t \\ 0 & \text{his order is canceled before it is executed} \end{cases}$$

where t' be the time that trader with private values, $\beta_{t'}$, submitted an order to the LOB and t is the time when an order executed where $t \geq t'$. The utility of trader is

calculated from the difference between the transaction price ($v_{t'} + p^i$) for the LOB and ($v_{t'} + p_t^{mid}$) for the DP, and the current value of each trader ($v_t + \beta_{t'}$) when his order is executed.

We define welfare (W) as an average surplus or utility of all traders in the market as:

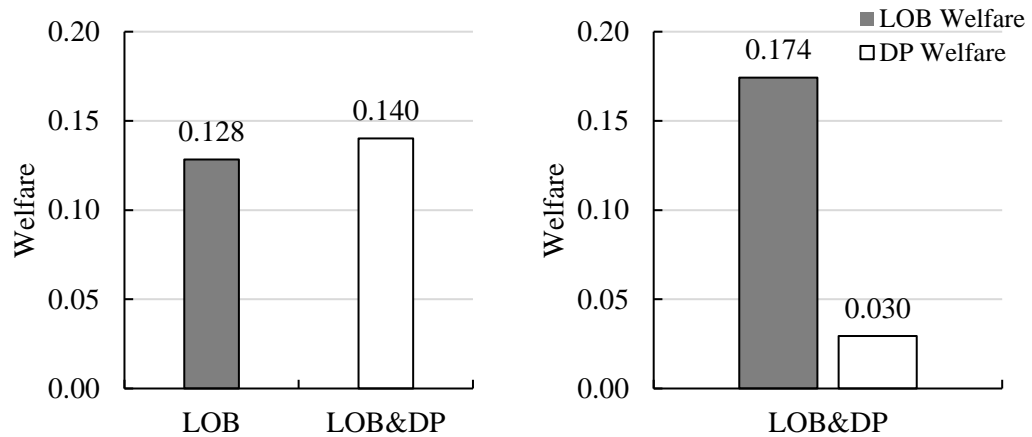
$$W^{LOB,LOB\&DP} = \frac{1}{T} \sum_{t=1}^T u_t^{LOB,DP}$$

In the extension model (LOB&DP), we also defined the welfare in each trading venue between LOB welfare and DP welfare as:

$$W_{LOB}^{LOB\&DP} = \frac{\sum_{t=1}^T u_t^{LOB}}{\sum_{t=1}^T (1|X_t \neq 0)}$$

$$W_{DP}^{LOB\&DP} = \frac{\sum_{t=1}^T u_t^{DP}}{\sum_{t=1}^T (1|Y_t \neq 0)}$$

In Figure 15, the graph shows that the overall welfare increases from 0.128 in the base model (LOB) to 0.140 in the extension model (LOB&DP), also LOB welfare rises. According to the optimal strategies, the DP lures traders with low willingness and leaves the LOB for traders with high willingness. When more traders with a high willingness in the LOB have two effects. First, these traders are likely to use the market orders for execution certainty or aggressive limit orders for price improvement. Second, the LOB fill rate increases as traders have a shorter waiting line in the LOB. For the low level in DP welfare, the DP welfare is only 0.030 because the traders who migrate to the DP have low intention to trade. They send orders to the DP for the benefit of half spread-saving and more chances to execute than conservative orders in the LOB.



(a) Overall welfare

(b) Welfare in the extension model

Figure 15: Overall welfare between two models and welfare in each market in the extension model (LOB&DP), (a) Overall welfare; the left bar is averaged utility over the LOB market, and the right bar is averaged utility over both markets (LOB&DP). (b) Welfare in the extension model (LOB&DP); the left bar is averaged utility over the LOB market, and the right bar is averaged utility over the DP market.

6. ROBUSTNESS: EFFECT OF VOLATILITY

For the flexibility in our model, we can increase the volatility of the asset by increasing the changing probability in the fundamental value σ . In this section, we found the effect of high volatility of the asset by increasing σ from 0.05 to 0.10, and we called it low and high volatility asset, respectively. Other parameters remain the same.

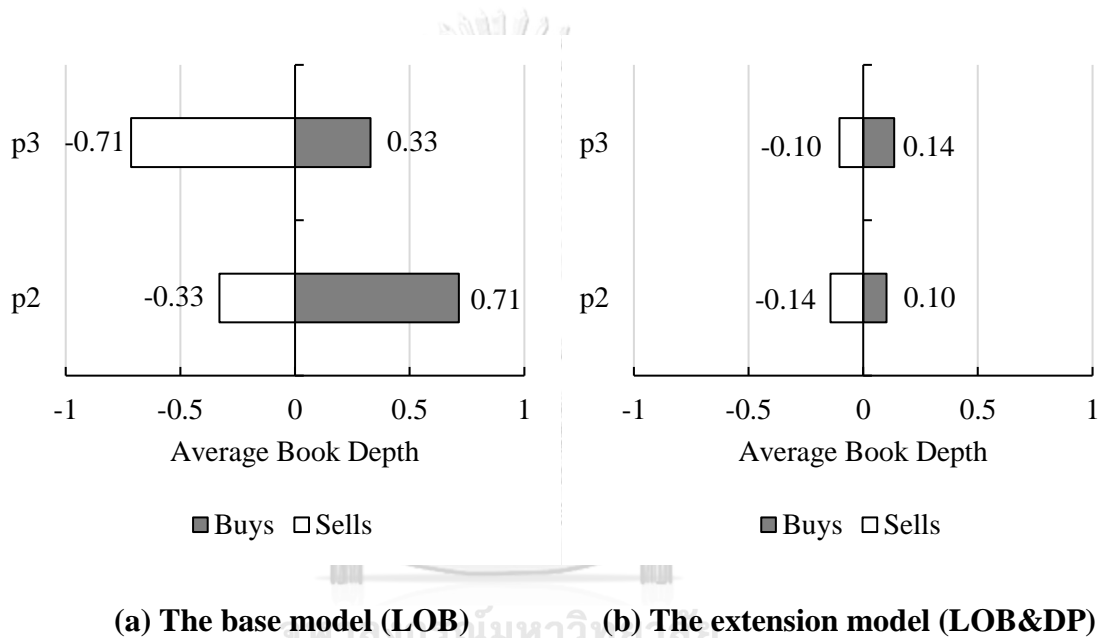


Figure 16: The average book depth at each price level of high volatility asset, the positive value is averaged over limit buys and the negative values is averaged over limit sells at price p^3 (one-tick above the fundamental value) and price p^2 (one-tick below the fundamental value) in the LOB market between the two models.

Regarding high volatility, traders should avoid conservative limit orders as there are more chances to cancel when the fundamental value changed in the same direction as their orders. The suffering of the movement in the fundamental value will drive more

traders using the DP. In high volatility asset, there are around 80% of traders using dark orders in our model. Not surprisingly that the market depth declines and spread widens when introduced the DP as we shown in Figure 16 and Figure 17.

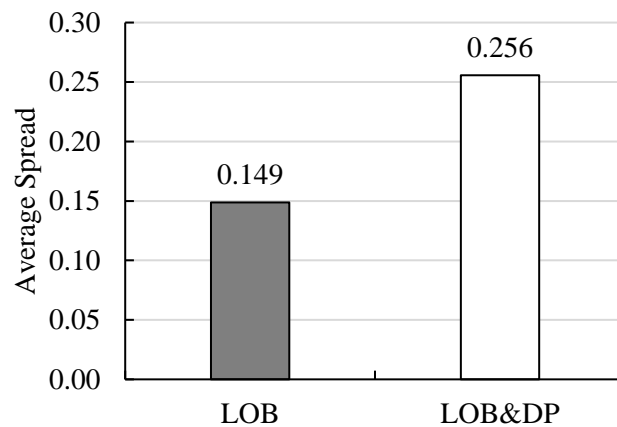


Figure 17: The average bid-ask spread in the LOB market of high volatility asset between the base model (LOB) on the left bar and the extension model (LOB&DP) on the right bar.

The total fill rate is around 81% in both models, including the fill rate in each market as we shown in Figure 18. The higher in DP usage supports this result. Although traders cannot observe the state of the DP, they can imply from the low liquidity of the LOB that previous traders might submit the orders to the DP. Due to the wider spread in the extension model (LOB&DP) increases trading costs of market orders in the LOB, the traders who trade in the LOB must have the very high willingness for crossing market orders against the trading crowd or aggressive limit orders. Since these traders' type still have the high level of execution, LOB fill rate does not diminish.

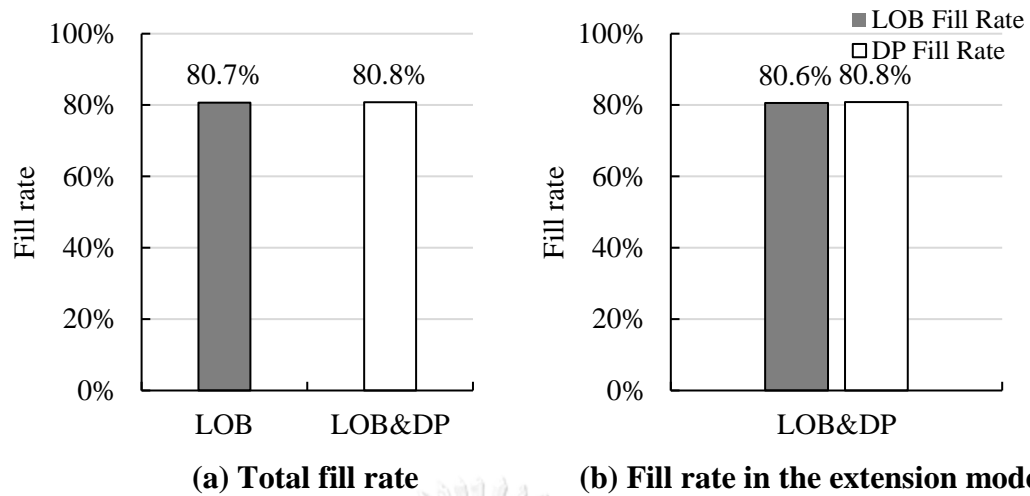
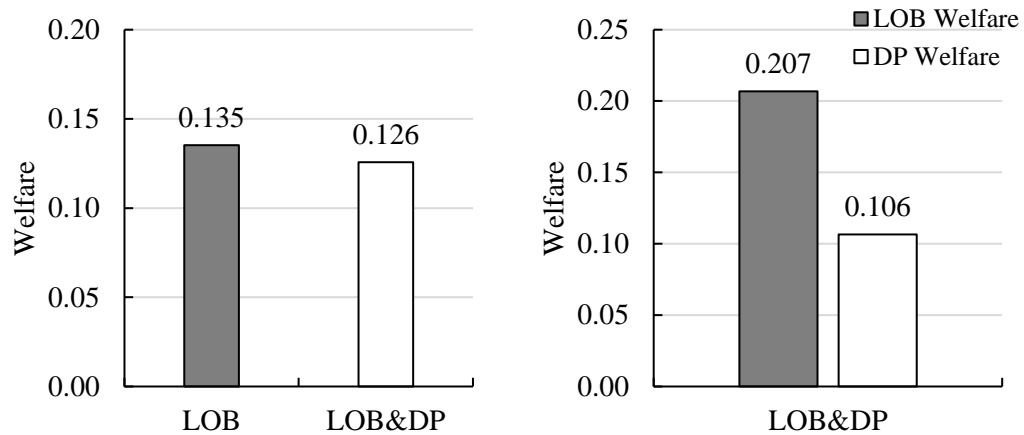


Figure 18: The total fill rate between two models and fill rate in each market in the extension model (LOB&DP) of high volatility asset. (a) The total fill rate; the left bar is the percentage of the executed orders over the LOB market, and over both markets (LOB&DP) are on the right bar. (b) The fill rate in the extension model (LOB&DP); the left bar is the percentage of the executed orders over the LOB market, and over the DP market is on the right bar.

Finally, the overall welfare has been deteriorated by increasing the volatility of the asset as we shown as depicted in Figure 19. The overall welfare decreases from 0.135 in the base model (LOB) to 0.126 in the extension model (LOB&DP), surprisingly higher LOB welfare. The intuition of this result is that the DP appeals to most traders and leave the LOB for traders with very high willingness since these traders' type can gain the utility even if they have to pay the full-spread. Since the crossing price in DP can be varied from the limit orders in the LOB and harm traders in the DP. The higher bid-ask spread supports the sensitivity of midpoint. As there is the higher proportion of traders in the DP, the DP welfare overcomes the overall welfare.



(a) Overall welfare

(b) Welfare in the extension model

Figure 19: Overall welfare between two models and welfare in each market in the extension model (LOB&DP) of high volatility asset, (a) Overall welfare; the left bar is averaged utility over the LOB market, and the right bar is averaged utility over both markets (LOB&DP). (b) Welfare in the extension model (LOB&DP); the left bar is averaged utility over the LOB market, and the right bar is averaged utility over the DP market.

7. CONCLUSIONS

Regarding the concern of regulators on the existence of a DP, the base model is created when only the LOB and the extension model when adding the DP operates parallel. We use a stochastic algorithm to form the optimal strategies in this dynamic trading game. After deriving the optimal submission strategies, we analyze the state of the market in six cases between order-balance and order-imbalance, then demonstrating how the DP impacts trader with the different types of book depth. We also found the market quality in two measures: market depth and bid-ask spread. The results in the optimal strategies and market quality are correspondent to the answer of our research questions in the effects of the DP on fill rate and welfare.

In our model, there is one-fourth of all traders in the simulation diverge from the LOB to the DP, causing the lower market depth and broader spread in the LOB. We show that the DP decreases total fill rate, however, there is an increment in the LOB fill rate since traders have switched from the limit orders to market orders for offsetting the lower in execution probability of limit orders when added the DP. Using more market orders also supports the result that the DP harm market quality in both market depth and bid-ask spread. Besides, the reduction in the total fill rate is caused by the DP fill rate. Since the DP traps traders with low willingness to trade who frequently submit conservative limit orders from the LOB, and the attractiveness of the DP is stronger when the liquidity of the book depth is built up. Due to the percentage of these traders' types is only one-fourth of all traders of the time in the simulation, the dark orders are usually canceled before it executes. Resulting in the DP fill rate is low, and the total fill rate declines.

We also find that the overall welfare increases, mainly from the improvement in LOB welfare. Since traders with high willingness drive the orders in the LOB and low willingness drive the orders in the DP. Traders with high willingness prefer to submit market orders and aggressive limit orders. Also, these traders will face the shorter waiting line in the LOB as traders with low willingness move to the DP. The higher in the LOB fill rate supports this result. While traders with a low willingness in the DP can gain more welfare than submitted to the LOB.

As a robustness check, we modify the parameter in our model by increasing the volatility of the asset in Section 6. We found that DP usage is around 80% in the high volatility and making the book depth less. Also, the bid-ask spread is wider. By using more DP, total fill rate is not demolished by the DP fill rate as the low volatility asset. Moreover, overall welfare has been harmed by broader spread as it is sensitive to the execution price in the DP. As the high usage of the DP, the overall welfare is pulled by DP welfare.

As a policy implication of this work, our result suggests that DP increases welfare when the market volatility is not too high, but policy-makers will have to balance between the increase in welfare and the decrease in LOB's market quality (widen spread and lower market depth).

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