การออกแบบการควบคุมเชิงทำนายแบบจำลองไม่เชิงเส้นสำหรับแขนเพนดูลัมผกผันโดยใช้การป้อนกลับสัญญาณขาออก

นายเพชรกฤษณ์ ภิญโญภาวศุทธิ

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DESIGN OF NONLINEAR MODEL PREDICTIVE CONTROL FOR INVERTED PENDULUM USING OUTPUT FEEDBACK

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A Thesis Submitted in Partial Fulfillment of the Requirements
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เพชรกฤษณ์ ภิญโญภาวศุทธิ : การออกแบบการควบคุมเชิงทำนายแบบจำลองไม่เชิงเส้นสำหรับแขนเพนดูลัมผกผันโดยใช้การป้อนกลับสัญญาณขาออก (DESIGN OF NONLINEAR MODEL PREDICTIVE CONTROL FOR INVERTED PENDULUM USING OUTPUT FEEDBACK) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ศ. ดร.เดวิด บรรเจิดพงศ์ชัย, หน้า.

วิทยานิพนธ์นี้เสนอการออกแบบของการควบคุมเชิงทำนายแบบจำลองไม่เชิงเส้นสำหรับแขนเพนดูลัมผกผันโดยใช้การป้อนกลับสัญญาณขาออก การควบคุมเชิงทำนายแบบจำลองไม่เชิงเส้นได้รับความสนใจอย่างยิ่ง เนื่องจากการควบคุมกระบวนการสามารถประยุกต์กับระบบไม่เชิงเส้น การควบคุมเชิงทำนายแบบจำลองไม่เชิงเส้นมีรูปแบบเป็นปัญหาการหาค่าเหมาะที่สุดที่เวลาสุ่ม การควบคุมเชิงทำนายแบบจำลองไม่เชิงเส้นมีวัตถุประสงค์เพื่อคุมค่าสัญญาณขาออกและตามรอยสัญญาณอ้างอิง ในทางปฏิบัติตัวแปรสถานะบางตัวเท่านั้นที่วัดได้เมื่อมีเพียงการวัดสัญญาณขาออกเราจึงออกแบบตัวสังเกตแบบไม่เชิงเส้นเพื่อประมาณสถานะที่วัดค่าไม่ได้ จากนั้นจะนำตัวแปรสถานะที่ประมาณได้ เป็นสัญญาณป้อนกลับเพื่อคำนวณหาสัญญาณควบคุมเหมาะที่สุด เราประยุกต์ใช้การควบคุมเชิงทำนายแบบจำลองไม่เชิงเส้นอิงตัวสังเกตกับแขนเพนดูลัมผกผันบนรถผลลัพธ์เชิงตัวเลขแสดงการควบคุมเชิงทำนายแบบจำลองไม่เชิงเส้นโดยเปรียบเทียบระหว่างการป้อนกลับสัญญาณขาออกกับการป้อนกลับตัวแปรสถานะพบว่า ตัวสังเกตไม่เชิงเส้นให้สถานะที่ประมาณใกล้เคียงกับสถานะจริง ซึ่งยืนยันเสถียรภาพของตัวสังเกตไม่เชิงเส้นนอกจากนั้น การควบคุมเชิงทำนายแบบจำลองที่ใช้การป้อนกลับสัญญาณขาออกให้สมรรถนะใกล้เคียงกับผลตอบสนองที่ใช้การป้อนกลับสถานะ

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PETCHAKRIT PINYOPAWASUTTHI: DESIGN OF NONLINEAR MODEL PREDICTIVE CONTROL FOR INVERTED PENDULUM USING OUTPUT FEEDBACK. ADVISOR: PROF.DAVID BANJERDPONGCHAI, Ph.D., pp.

This thesis proposes design of nonlinear model predictive control (NMPC) for inverted pendulum using output feedback. NMPC has received a great attention because process control can be applied to nonlinear systems. NMPC is formulated as the optimal control problem for nonlinear systems at every sampling time. The main objectives of NMPC design are regulation of response and reference tracking. In practice, only some states of the system are measurable. Based on the output measurement, we employ nonlinear observer to estimate the other states. Then estimated state will be feedback to calculate the optimal control input. We apply observer-based NMPC to inverted pendulum on cart. Numerical results show a comparison of NMPC using output feedback and state feedback. The nonlinear observer gives estimated state which is close to the real state which ensures the stability of nonlinear observer. Output feedback NMPC gives performance similar to that of state feedback NMPC.

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CONTENTS

CONTENS OF TABLES

CONTENTS OF FIGURES

Chapter 1

Introduction

1.1 Background

Optimal control problem of nonlinear systems is one of the most attractive problem in control theory because of the complexity with the classical control theory occurs when the plant we used is nonlinear. As we know, If the plant of the dynamic systems is linear, it is possible to find the optimal control by solving the Riccati equation. In nonlinear systems, The Hamilton-Jacobi-Bellman (HJB) from [1], [2] partial differential equations (PDE) will become more challenge. It is difficult to find the closed form of solution or rarely an analytical solution. Many researches proposed the method to approximate the HJB solution. Power series approximation [3], State-Dependent Riccati equation (SDRE) [4], Successive Garlerkin approximation [5], approximating sequence of Riccati equation (ASRE) [3]. All of above is based on some hypothesis in the dynamic system. From [6] proposed the numerical methods for solving the optimal control problem into 3 types. First, dynamical programming approach compute the Hessian matrix and satisfy the necessary condition. However, the computation increases exponentially with the number of states and control variables. Second, indirect method approach solves the necessary condition in the form of two-point boundary value problems (TPBV). Finally, the essence of direct method is the discretization of the optimal control problem. In this study, we interest to solve optimal control problem in nonlinear system with TPBV. SDRE becomes useful when we solved TPBV because SDRE can satisfy TPBV by some condition. Almost application can be expressed in state-dependent linear like which can be solved by SDRE so it is suitable to compare it with indirect method for solving TPBV. You can see the example of application by [7], [8], [9].

Model predictive control (MPC) approach [10], [11], [9] has received a great attention over the past. The main idea of model based predictive is to use open-loop optimal control instead of closed-loop optimal control with finite time in the future. MPC is a kind of optimal control problem for finding a state feedback law which does not only depend on time. Time interval in performance measure of MPC is finite. The state feedback law depends on the state. The optimal feedback law the best performance as it is possible in the sense that performance measure is minimized. When nonlinear MPC (NMPC) is considered, the open-loop optimal control is calculated from the current to the finite future time. And the current control input is based on the MPC strategy [12], that is, (TPBV) [1], [2] or SDRE [4] need to be solved in on line to finding the control input at every sampling time. There are many people try to apply it to nonlinear systems and prove that closed-loop is asymptotically stable if terminal state is zero [11]. It is well known that the controller from finite optimal control do not guaranteed the closed-loop stability. The MPC can be guaranteed by adding some condition into finite horizon optimal control problem which can be prove in [9]. However, the most of NMPC are based on the knowledge of full state. This mean that the NMPC can be applied when you can measure all the state of the systems. In real environment, we cannot observe all the states of the system.

To overcome the above problem, we propose to use an observer to observe the state of the system. Many researches proposed the observer to use with linear systems such as [13] which is solving Sylvester observer equation. In this study, the reduced-observer can be solved by using Kalman-type observe. The reduced-form need to be found and use Kalman gain to be feedback to the reduced-system. The stability of the observer need to be concern to ensure the convergence of the reduced-system.

Observer based NMPC use NMPC with output feedback which is found by using observer. Inverted pendulum, inverted pendulum on linear motion cart and inverted pendulum on circular motion cart are the example for optimal control problem, NMPC, observer and output feedback NMPC.

This paper proposes you to use observer in feedback control [14], [15], [16] to overcome the above problem. The nonlinear systems can be written in form as reduced-form.

1.2 Objective

1. To solve the optimal control problem with TPBV.
2. To estimate all states of the systems by using observer.
3. To apply output feedback to nonlinear model predictive control with guaranteed stability of the closed loop system of NMPC.
4. To achieve the regulation and reference tracking criteria.

1.3 Scope of research

1. Simulation and programming development are carried out on MATLAB.
2. Nonlinear systems can be expressed in affine control input.
3. Final states are free and the terminal time is fixed.
4. Control input has constraints from physical limitation.
5. Nonlinear systems can be observed and controlled.
6. Measured output is only some of the states.

1.4 Methodology

1. Review the optimal control theory, nonlinear model predictive control, observer for nonlinear systems, output feedback for nonlinear systems with observer.
2. Develop the optimal control and test with examples and compare the results.
3. Develop the nonlinear model predictive control and apply to the same examples and compare the results between optimal control problem and NMPC.
4. Develop observer of nonlinear systems and apply to the same examples and compare the results.
5. Apply the output feedback nonlinear model predictive control using observer.
6. Compare all the results and discussion.
7. Complete thesis writing.

1.5 Expected outcome

Most of MPC approximated the process dynamics as linear system and applied linear MPC. In real environment, many applications are nonlinear systems. To conserve nonlinearity, it is useful to use NMPC. The observer becomes useful when we cannot measure all the output because NMPC can be achieved only when we know all the states. We expect to design observer to estimate the states and feedback to NMPC, Thus, we will be able to control the nonlinear systems. We focus on the regulation and reference tracking as the main control objectives.

Chapter 2

Optimal Control for Nonlinear Systems

In the past, classical control is process that deals with the characteristic of system by solving differential equation in a frequency domain. In fact, systems in real environment have more complexity than we learned in a classical control. The knowledge about classical control is not enough to explain the characteristic of those systems. Optimal control become useful to have an effect on those complex systems. The objective of optimal control is to find the control input that gives the minimum cost function or performance measure and satisfy physical constraints of the systems.

2.1 Problem formulation

To find the control input which minimize the cost function

$$\boldsymbol{J} = h\left(\boldsymbol{x}\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)dt \quad (1)$$

where h is terminal cost scalar function and g is scalar function. t_0 and t_f are initial and terminal time, respectively.

Constraints are

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \quad (2)$$

$$\boldsymbol{x}(t) = x_0 \quad (3)$$

Initial state and control input for interval time $t \in \left[t_0, t_f\right]$ drive the system by two parameters. The minimum performance is achieved by some control input. We call it as the optimal control.

2.2 The Hamilton-Jacobi-Bellman equation

Let the optimal control be $(\boldsymbol{u}(t)^*)$ and it depends on the state.

$$\boldsymbol{u}^*(t) = \boldsymbol{a}(\boldsymbol{x}(t), t) \quad (4)$$

We can rewrite the minimum cost function as

$$J^*(\boldsymbol{x}(t), t) = min_{\boldsymbol{u}(t)_{t_0 \le t \le t_f}} \left\{ h\left(\boldsymbol{x}\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)dt \right\} \quad (5)$$

where
$$\dot{x}^*(t) = f(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), t)$$
$$\boldsymbol{x}(t) = x_0$$

Using the principle of optimality and Taylor series to rewrite the new formulation

$$0 = \boldsymbol{J^*}_{\boldsymbol{t}}(\boldsymbol{x}(t), t) + min_{\boldsymbol{u}(t)}\left\{ g(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + {\boldsymbol{J^*}_{\boldsymbol{x}}}^T(\boldsymbol{x}(t), t)[f(\boldsymbol{x}(t), \boldsymbol{u}(t), t)] \right\} \quad (6)$$

where $J^*{}_t \triangleq \frac{\partial J^*}{\partial t}$, $J^*{}_x \triangleq \frac{\partial J^*}{\partial x}$. Define the Hamiltonian $\mathcal{H}$ as

$$\mathcal{H}\left(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{J^*}_{\boldsymbol{x}}, t\right) \triangleq g(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + {\boldsymbol{J^*}_{\boldsymbol{x}}}^T(\boldsymbol{x}(t), t)[f(\boldsymbol{x}(t), \boldsymbol{u}(t), t)] \quad (7)$$

It can be shown that

$$\mathcal{H}\big(\boldsymbol{x}(t), \boldsymbol{u}^*\big(\boldsymbol{x}(t), \boldsymbol{J}^*{}_{\boldsymbol{x}}, t\big), \boldsymbol{J}^*{}_{\boldsymbol{x}}, t\big) = min_{\boldsymbol{u}(t)}\mathcal{H}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{J}^*{}_{\boldsymbol{x}}, t\big) \tag{8}$$

The Hamilton-Jacobi-Bellman equation is

$$0 = \boldsymbol{J}^*{}_{\boldsymbol{t}}(\boldsymbol{x}(t), t) + \mathcal{H}\big(\boldsymbol{x}(t), u^*\big(\boldsymbol{x}(t), \boldsymbol{J}^*{}_{\boldsymbol{x}}, t\big), \boldsymbol{J}^*{}_{\boldsymbol{x}}, t\big) \tag{9}$$

Unfortunately, above equation is based on knowledge of the optimal control input. In fact, the solution of the Hamilton-Jacobi-Bellman equation (HJB) is not easy to solve but we have many ways to approximate the solution of HJB equation. In the next section, we review some of the methods to approximate the HJB solution.

2.3 The variational to optimal control

The HJB equation is difficult sufficient condition to solve. Fortunately, the optimal control can be derived using a necessary condition for optimal control also called “Pontryagin’s minimum principle”.

Necessary condition for optimal control

The objective of the optimal control problem is to find the control input which makes the system (2) drives the trajectories. The minimum cost function occurs when the optimal control is found with the optimal trajectories.

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$$

We can rewrite (1) as

$$\boldsymbol{J} = h(\boldsymbol{x}(t_0), t_0) + \int_{t_0}^{t_f} [\frac{dh(\boldsymbol{x}(t), t)}{dt} + g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)]dt \tag{10}$$

In this work, we are interested in a fix terminal condition. Eq. (10) will not change by the $h(\boldsymbol{x}(t_0), t_0)$ term so we can relax this term and rewrite (10) by using chain rule as follows

$$\boldsymbol{J} = \int_{t_0}^{t_f} [\frac{\partial h(\boldsymbol{x}(t), t)}{\partial t} + \left[\frac{\partial h(\boldsymbol{x}(t), t)}{\partial x}\right]^T \dot{\boldsymbol{x}}(t) + g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)]dt \tag{11}$$

Applying the Lagrange multiplier to (11) and we define it as J_a

$$\begin{aligned} J_a = \int_{t_0}^{t_f} \Big\{ & g(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + \left[\frac{\partial h}{\partial x}(x(t), t)\right]^T \dot{x}(t) + \frac{\partial h}{\partial t}(x(t),\ t) \\ & + p^T(t)[f(\boldsymbol{x}(t), \boldsymbol{u}(t), t) - \dot{\boldsymbol{x}}(t)] \Big\} dt \end{aligned} \tag{12}$$

where $p(t)$ is the Lagrange multiplier. Sometimes we call it as “co-state of systems”. For easy understanding, define

$$g_a(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t) = g(\boldsymbol{x}(t), \boldsymbol{u}(t), t) +$$
$$\left[\frac{\partial h}{\partial x}(x(t), t)\right]^T \dot{x}(t) + \frac{\partial h}{\partial t}(x(t),\ t) + p^T(t)[f(\boldsymbol{x}(t), \boldsymbol{u}(t), t) - \dot{\boldsymbol{x}}(t)]$$

(12) can be written as

$$J_a = \int_{t_0}^{t_f} \{g_a(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)\} dt \tag{13}$$

From the calculus of variations, we can apply it with the chain rule and write it as the variation of J_a. While $\delta x_f, \delta t_f, \delta x(t), \delta u(t), \delta p(t)$ are the variation of terminal state, terminal time, state, input, Lagrange multiplier, respectively.

The variation of J_a is

$$\begin{aligned}
\delta J_a(u^*) = 0 = &\left[\frac{\partial g_a}{\partial \dot{x}}\left(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), \boldsymbol{u}^*(t_f), \boldsymbol{p}^*(t_f), t_f\right)\right]^T \delta x_f \\
&+ \Bigg[g_a\left(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), \boldsymbol{u}^*(t_f), \boldsymbol{p}^*(t_f), t_f\right) \\
&- \left[\frac{\partial g_a}{\partial \dot{x}}\left(\boldsymbol{x}^*(t_f), \dot{\boldsymbol{x}}^*(t_f), \boldsymbol{u}^*(t_f), \boldsymbol{p}^*(t_f), t_f\right)\right]^T \dot{\boldsymbol{x}}^*(t_f)\Bigg]^T \delta t_f \\
&+ \int_{t_0}^{t_f} \Bigg\{\Bigg[\left[\frac{\partial g_a(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)}{\partial x}\right]^T \\
&- \frac{\partial}{\partial t}\left[\frac{\partial g_a(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)}{\partial \dot{x}}\right]^T\Bigg]^T \delta x(t) \\
&+ \left[\frac{\partial g_a(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)}{\partial u}\right]^T \delta u(t) \\
&+ \left[\frac{\partial g_a(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)}{\partial p}\right]^T \delta p(t)\Bigg\} dt
\end{aligned} \tag{14}$$

The inside integral term must be zero if it is an extremal. An outside integral term will be considered as boundary conditions. In this study, we specify that t_f is fixed. Finally, we get these necessary conditions. Compare the HJB equation with these conditions you will see that you can consider $\boldsymbol{J}^*{}_x$ in (7) as the Lagrange multiplier $p(t)$ in (12)

$$\dot{\boldsymbol{x}}^*(t) = \frac{\boldsymbol{\partial\mathcal{H}}}{\boldsymbol{\partial p}} = \boldsymbol{f}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), t) \tag{15}$$

$$\dot{\boldsymbol{p}}^*(t) = -\frac{\boldsymbol{\partial\mathcal{H}}}{\boldsymbol{\partial x}} = -\left[\frac{\boldsymbol{\partial f}}{\boldsymbol{\partial x}}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), t)\right]^T \boldsymbol{p}^*(t) - \frac{\boldsymbol{\partial g}}{\boldsymbol{\partial x}}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), t) \tag{16}$$

$$0 = \frac{\boldsymbol{\partial\mathcal{H}}}{\boldsymbol{\partial u}} = \left[\frac{\boldsymbol{\partial f}}{\boldsymbol{\partial u}}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), t)\right]^T \boldsymbol{p}^*(t) + \frac{\boldsymbol{\partial g}}{\boldsymbol{\partial u}}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), t) \tag{17}$$

$$\boldsymbol{x}^*(t_0) = \boldsymbol{x}_0 \tag{18}$$

$$\boldsymbol{p}^*(t_f) = \frac{\boldsymbol{\partial h}}{\boldsymbol{\partial x}}\left(\boldsymbol{x}^*(t_f)\right) \tag{19}$$

where $\boldsymbol{u}^*(t)$ is optimal control input which makes optimal trajectories $\boldsymbol{x}^*(t)$ and optimal Lagrange multipliers $\boldsymbol{p}^*(t)$.

The next section, we will present techniques to approximate the solution which satisfy all of these necessary conditions (15)-(19). In this work, we introduce two iterative techniques for finding the optimal control input and one approach which is not involve to iterative method. These two iterative techniques use the concept of an open loop optimal control.

2.4 Two-point boundary-value problems

If optimal control problem is linear systems with quadratic cost function, it is not difficult to find the optimal control by solving the Riccati equation. When systems are nonlinear system, we have to solve the HJB equation which is generally impossible to find the closed form of the HJB solution. The variational approach gives us a means to solve nonlinear two-point-boundary-value (TPBV) problem.
Assume that control input and final state are free and terminal time is fixed.

$$\dot{\boldsymbol{x}}^{(i)}(t) = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \boldsymbol{f}(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t) \tag{20}$$

$$\begin{aligned}\dot{\boldsymbol{p}}^{(i)}(t) &= -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} \\ &= -\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t\right)\right]^T \boldsymbol{p}^{(i)}(t) \\ &\quad - \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t\right)\end{aligned} \tag{21}$$

$$0 = \frac{\partial \mathcal{H}}{\partial \boldsymbol{u}} = \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\left(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t\right)\right]^T \boldsymbol{p}^{(i)}(t) + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\left(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t\right) \tag{22}$$

$$\boldsymbol{x}^{(i)}(t_0) = \boldsymbol{x}_0 \tag{23}$$

$$\boldsymbol{p}^{(i)}(t_f) = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^{(i)}(t_f)\right) \tag{24}$$

The main principle uses the following idea.

“The idea is starting from an initial guess is used to obtain the solution which is not satisfied all of the necessary conditions. Then adjust the next control input to make the solution come closer to all of the necessary conditions. We will stop updating the control input until all of the necessary conditions are satisfied.”

2.4.1 The steepest descent method

This method is developed by using knowledge of optimization of function by using steepest descent. Let f be a function of y_1, y_2 which are independent. A necessary condition is

$$\begin{aligned}\partial f({y_1}^*, {y_2}^*) &= \left[\frac{\partial f}{\partial y_1}({y_1}^*, {y_2}^*)\right]\Delta y_1 + \left[\frac{\partial f}{\partial y_2}({y_1}^*, {y_2}^*)\right]\Delta y_2 = 0 \\ &\triangleq \frac{\partial f}{\partial y}(y^*)^T \Delta y = 0\end{aligned} \tag{25}$$

This implies that $\frac{\partial f}{\partial y}(y^*) = 0$ for any $\Delta y \neq 0.$ Let $z(y^{(i)})$ be the unit vector in the gradient direction at the point $y^{(i)}$.

$$\Delta y = y^{(i+1)} - y^{(i)}$$

However, we can proceed to the next iteration in the opposite direction $z(y^{(i)})$ with some value, give the variable τ is that meaning. Thus,

$$\Delta y = y^{(i+1)} - y^{(i)} = -\tau z(y^{(i)}) \tag{26}$$

From (25), (26), it can be shown that

$$\partial f\left(y^{(i)}\right) = -\tau \frac{\partial f}{\partial y}(y^{*})^{T} z(y^{(i)})$$

Because $z(y^{(i)})$ is the unit vector in the gradient direction. As a result, we have

$$\partial f\left(y^{(i)}\right) \leq 0$$

Figure 2-1 shows the procedure of the steepest descent method. It starts from the starting point and go through the deepest path in the opposite gradient direction repetitively until the criterion is satisfied.

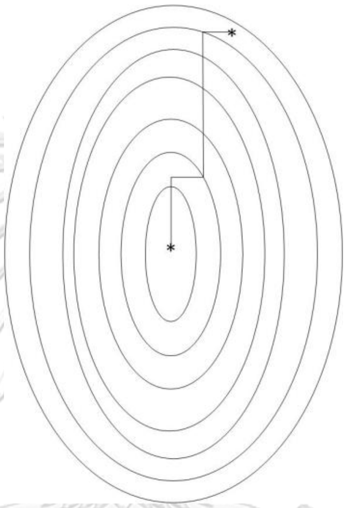

Figure 2-1 Steepest descent behavior

Next, we will apply this knowledge to the optimization of functional by using steepest descent. From the necessary conditions (20)-(24)

$$\dot{\boldsymbol{x}}^{(i)}(t) = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \boldsymbol{f}(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t) \tag{20}$$

$$\begin{aligned} \dot{\boldsymbol{p}}^{(i)}(t) = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} \\ &= -\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t\right)\right]^{T} \boldsymbol{p}^{(i)}(t) \\ &\quad - \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t\right) \end{aligned} \tag{21}$$

$$0 = \frac{\partial \mathcal{H}}{\partial \boldsymbol{u}} = \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\left(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t\right)\right]^{T} \boldsymbol{p}^{(i)}(t) + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\left(\boldsymbol{x}^{(i)}(t), \boldsymbol{u}^{(i)}(t), t\right) \tag{22}$$

$$\boldsymbol{x}^{(i)}(t_0) = \boldsymbol{x}_0 \tag{23}$$

$$\boldsymbol{p}^{(i)}(t_f) = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^{(i)}(t_f)\right) \tag{24}$$

First, let discuss about the variation of J_a in (14). Consider the control input.

To find the control input which satisfies all of the necessary conditions is impossible. However, we can adjust some condition to make the solution comes closer to satisfy the necessary condition. Suppose (22) is not satisfied so the variation of J_a is

$$\delta J_a = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial \mathcal{H}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)}{\partial u} \right]^T \delta u(t) \right\} dt \tag{27}$$

Let

$$\delta u(t) = \boldsymbol{u}^{(\boldsymbol{i}+\mathbf{1})}(t) - \boldsymbol{u}^{(\boldsymbol{i})}(t) = -\tau \frac{\partial \mathcal{H}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)}{\partial u} \tag{28}$$

Thus,

$$\delta J_a = -\tau \int_{t_0}^{t_f} \left\{ \left[\frac{\partial \mathcal{H}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)}{\partial u} \right]^T \frac{\partial \mathcal{H}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t)}{\partial u} \right\} dt \leq 0$$

The equality will be hold if and only if (22) is satisfied.

The steepest descent algorithm

1. Select the discrete approximation of the control input.

$$\boldsymbol{u}^{(\boldsymbol{i})}(t) = \boldsymbol{u}^{(\boldsymbol{i})}(t_k), \qquad t \in [t_0, t_f] \quad k = 0,1, \dots, N-1 \tag{29}$$

2. Using input (29) to integrate the state equation (20) from t_0 to t_f with initial condition (23).
3. Calculate $\boldsymbol{p}^{(\boldsymbol{i})}(t_f)$ in (24) and let it be terminal condition for integration backward from t_f to t_0 of co-state equation.
4. Calculate equation (22) and let it be stopping criterion.

   If stopping criterion is satisfied, then that control will be the optimal control.
   If stopping criterion is not satisfied, then update new control by

$$\boldsymbol{u}^{(\boldsymbol{i}+\mathbf{1})}(t_k) = \boldsymbol{u}^{(\boldsymbol{i})}(t_k) - \tau \frac{\partial \mathcal{H}^{(\boldsymbol{i})}}{\partial \boldsymbol{u}}(t_k), \qquad k = 0,1, \dots, N-1 \tag{30}$$

2.5 State dependent Riccati equation

Consider nonlinear affine control systems

$$\dot{x}(t) = f(x) + g(x)u(t) \tag{40}$$

The nonlinear systems can be expressed in state dependent linear like form as

$$\dot{x}(t) = A(x)x(t) + B(x)u(t) \tag{41}$$

where $f(x) = A(x)x(t)$, $g(x) = B(x)$. The objective is to minimize the performance measure (1) subject to (40) or (41) with initial condition (3).

SDRE which satisfies the necessary condition (15)-(19) can be proved by assume that co-state of the system is state dependent linear like in $x(t)$

$$p(t) = P(x)x(t) \tag{42}$$

Consider with the nonlinear discrete-time system

$$x_{k+1} = f_d(x_k, u_k, t_k) \tag{43}$$

$$x_{k+1} = A(x_k)x_k + B(x_k)u_k \tag{44}$$

the control feedback law as

$$u_k = -k(x_k)x_k = -R^{-1}B(x_k)P(x_k) \tag{45}$$

where $P(x)$ is the positive definite matrix and is the solution of state dependent differential Riccati equation (SDDRE) in discrete-time.

$$\begin{gathered} A(x_k)^T\big(P(x_k) - P(x_k)B(x_k)(R + B(x_k)P(x_k)B(x_k)^{-1})^{-1}B(x_k)^T P(x_k)\big) \\ A(x_k) - P(x_k) + Q = 0 \end{gathered} \tag{46}$$

As you can see above, control feedback law of SDRE is similar to optimal control when we consider linear system with quadratic cost function. Only matrix in SDRE depends on state at each sampling time.

2.6 Numerical examples

Inverted pendulum

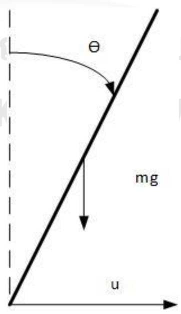

Figure 2-2 Inverted pendulum

The state equations of inverted pendulum are described by

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = \frac{mgl\sin x_1(t) - ml^2x_2^2(t)\sin x_1(t)\cos x_1(t) - (l\cos x_1(t))u(t)}{J + ml^2(\sin x_1(t))^2}$$

where $x_1(t)$ is angular position and $x_2(t)$ is angular velocity.

Figure 2-3 and 2-4 compare the results between the steepest descent method and SDRE where initial control signal is zero and sin wave for the steepest descent method.

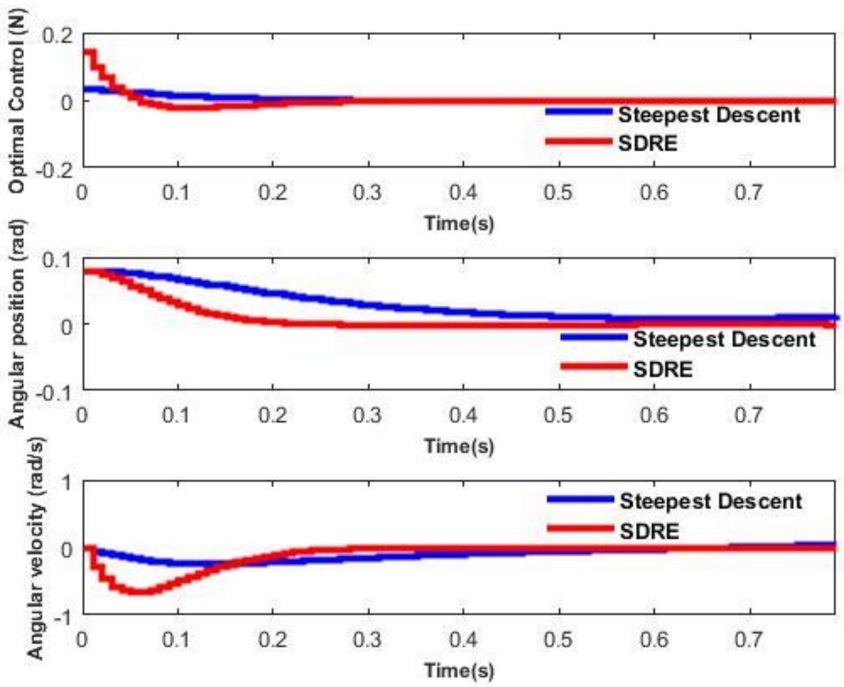

Figure 2-3 Optimal control, angular position and velocity with zero input guess for steepest descent and SDRE

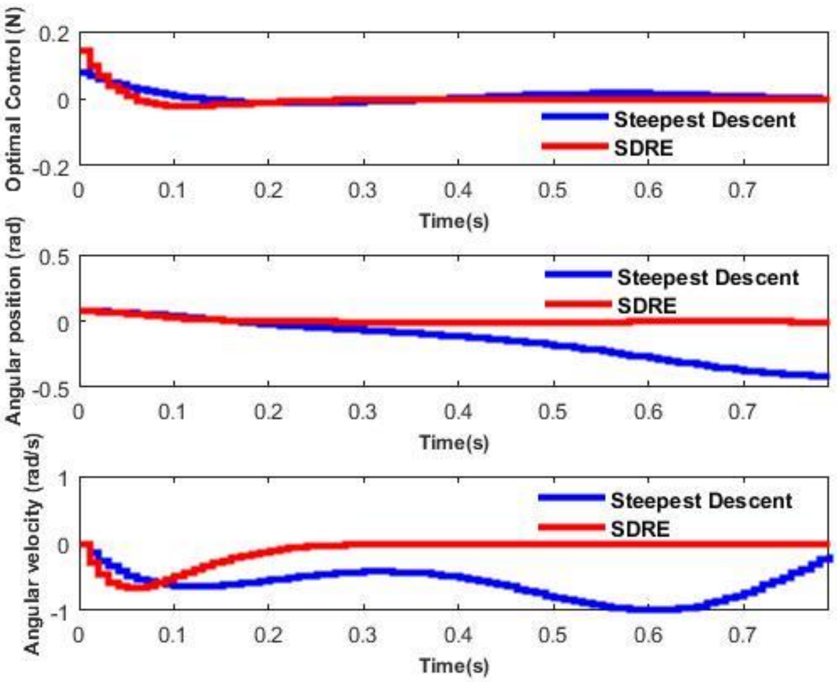

Figure 2-4 Optimal control, angular position and velocity with sin input guess for steepest descent and SDRE

The cost function with steepest descent is 1.5239×10^{-4} when guess input is zero and is 0.0021 when guess input is sin wave while cost function with SDRE is 6.1298×10^{-4}.

Table 2-1 Comparison of three methods for solving nonlinear optimal control problem.

	Steepest descent	The variation of extremals	SDRE
Cost function	1.5239×10^{-4}	1.2492×10^{-4}	6.1298×10^{-4}
Initial guess	$u(t)\ t \in [t_0, t_f]$	$p(0)$	-
Advantage	It gives minimum cost function when initial guess is close to optimal control	Once converge, it is fast.	It has closed form solution.
Disadvantage	Convergence is slow when it converges	May diverge for poor guess	The systems must be expressed in state dependent linear like form.

Conclusion

If the initial co-state is far from the optimal condition, then it takes more time to converge to the optimal solution. In the case we guess initial co-state close to the optimal solution, it takes a few iterations to find the optimal control the process. This method is based on free final state like the steepest descent method. Notice that steepest descent approach can minimize the cost function lower than that of SDRE. In this case, we have a good initial control input. If the time horizon is longer or we lack a good control input, SDRE is the better method to solve the optimal control problem. SDRE still has the limitation on the form of the dynamic system must be expressed as state dependent linear like form.

Chapter 3

Nonlinear Model Predictive Control

3.1 Problem formulation

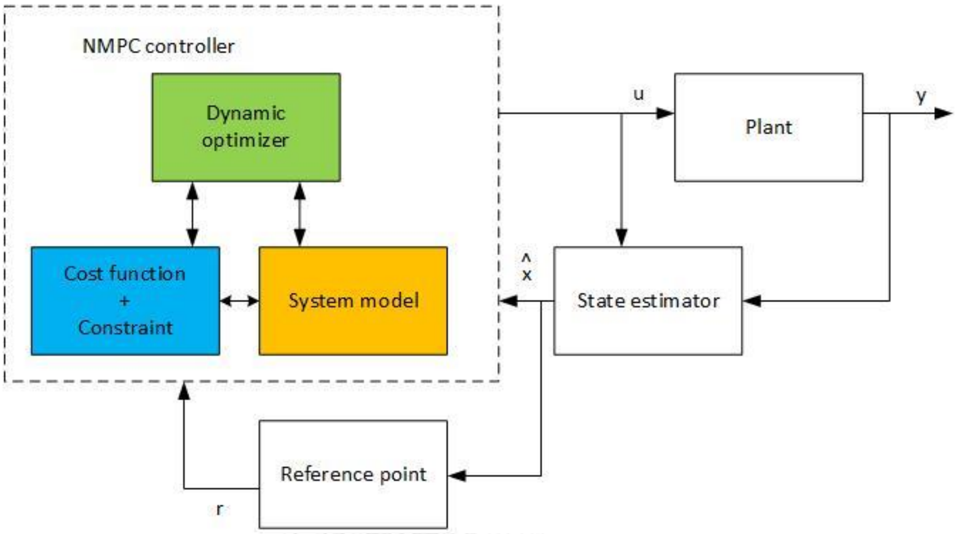

Figure 3-1 Structure of MPC [17]

Since NMPC is designed to discrete-time system, we shall discretize the systems by the discretization method [10].

$$x(t_{k+1}) = f(x(t_k), u(t_k), t_k) = f_d(x(t_k)) + B_d u(t_k)$$
$$f_d(x(t_k)) = x(t_k) + T_s f_c(x(t_k))$$
$$B_d = T_s B_c$$

where T_s is the sampling time. $x_{i,k}$ is state x in element ith at time index k.
Let the cost function be defined as

$$J_D = T_s\left(\sum_{k=0}^{N-1}\left(x(t_k)^T Q x(t_k) + u(t_k)^T R u(t_k)\right) + x(t_N)^T Q x(t_N)\right)$$

where Q and R are weighting matrices of state and control input. We discretize nonlinear system (1) using a fourth order Runge Kutta method and subdividing the interval time $[t_0, t_f]$into N subinterval . Let T_s be a sampling time and t_k be the kth sample, $t_k = t_0 + kT_s$.

$$x(t_{k+1}) = f_d(x(t_k), u(t_k)) = x(t_k) + \frac{T_s}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{47}$$

where $k_1 = f(x(t_k), u(t_k), t_k)$

$$k_2 = f\left(x(t_k) + \frac{T_s}{2}k_1, u(t_k), t_k + \frac{T_s}{2}\right)$$
$$k_3 = f\left(x(t_k) + \frac{T_s}{2}k_2, u(t_k), t_k + \frac{T_s}{2}\right)$$

$k_4 = f(x(t_k) + T_s k_3, u(t_k), t_k + T_s)$

so the objective of NMPC is to minimize the performance measure as follows

$$\begin{aligned} &T_s \sum_{i=0}^{N-1} \left(g\left(x(t_{k+i|k}), u(t_{k+i|k}) \right) + h\left(x(t_{k+N|k}), u(t_{k+N|k}) \right) \right) \\ &= T_s \left(\sum_{i=0}^{N-1} \left(x(t_{k+i|k})^T Q x(t_{k+i|k}) + u(t_{k+i|k})^T R u(t_{k+i|k}) \right) \right. \\ &\qquad \left. + x(t_{k+N|k})^T Q x(t_{k+N|k}) \right) \end{aligned} \tag{48}$$

with dynamical constraint

$$x(t_{k+i+1|k}) = f_d\left(x(t_{k+i+1|k}), u(t_{k+i+1|k}) \right)$$
$$x(t_{k|k}) = x(t_k)$$

The above problem is called regulation problem. The performance measure is quadratic with state and control input. We can track state of the system to reference signal by using the performance measure which is quadratic function of error between state and reference signal and control input. The objective is to minimize the performance measure

$$\begin{aligned} J = &T_s \sum_{i=0}^{N-1} \left(g\left(x_e(t_{k+i|k}), u(t_{k+i|k}) \right) + h\left(x_e(t_{k+N|k}), u(t_{k+N|k}) \right) \right) \\ &= T_s \left(\sum_{i=0}^{N-1} \left(x_e(t_{k+i|k})^T Q x_e(t_{k+i|k}) + u(t_{k+i|k})^T R u(t_{k+i|k}) \right) \right. \\ &\qquad \left. + x_e(t_{k+N|k})^T Q x_e(t_{k+N|k}) \right) \end{aligned} \tag{49}$$

$$\begin{aligned} \text{subject to } \; &x\left(t_{k+i+1|k}\right) = f_d\left(x\left(t_{k+i|k}\right), u\left(t_{k+i|k}\right) \right) \\ &x(t_{k|k}) = x(t_k) \end{aligned} \tag{50}$$

where $x_e(t_{k+i|k}) = x(t_{k+i|k}) - r(t_{k+i})$, $r(t)$ is reference signal.
In chapter 2, the algorithm to solving the optimal control for regulation problem is described. In this chapter, solving the NMPC problem is similar to the regulation. The different are Hamiltonian in (7) and necessary conditions (15)-(19). The control feedback law in SDRE algorithm is $u(t_k) = -R^{-1}B(x_k)(P(x_k) - r(t_k))$

3.2 NMPC algorithm

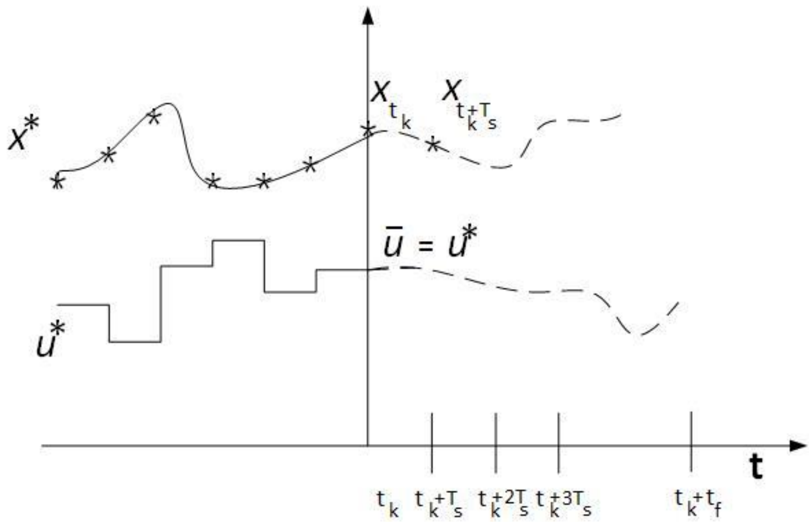

Figure 3-2 Process of MPC [17]

Figure 3-2 shows the concept of MPC that is
1. Obtain the present state of plant $x(t_i)$
2. Determine the open-loop optimal control problem by solving optimization with the cost function (48) for regulation problem (49) for the tracking problem

$$\hat{u}(t_k) = u^*(t_0 + kT_s)$$

where T_s is sampling time.
3. Apply only the first control input $\bar{u}(t) = \hat{u}(t_i)$ on interval $[t_i, t_i + T_s]$ to the plant.
4. Repeat from Step 1 to 3 for the next sampling t_{i+1}.

As the MPC concept, we see that MPC is a kind of optimal control problem. In nonlinear model predictive control, an open-loop optimal control law from the current time to the future time is calculated. The optimal control law can be achieved by solving the nonlinear optimal control problem at every sampling time.

3.3 Stability analysis

It is well known that the controller from NMPC with finite horizon is not guaranteed to be stable [18]. The stability can be achieved by adding terminal state term to cost function. The following part shows the stability analysis when we add terminal state term to cost function

$$J = h\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt$$

where $h(0) = 0$ and $h(\boldsymbol{x}(t), t) \geq 0$ for all $\boldsymbol{x}(t), t \neq 0$

Stability theorem [9] suggests that the NMPC is asymptotically stable if the terminal state exists such that the following condition satisfied

$$\dot{h}\big(x(t_f), t_f\big) + g(x(t), u(t), t) \leq 0 \tag{51}$$

Define $V^*(x^*(t), t)$ as the cost function of open-loop optimal control problem.

Proof: Define $V(x(t),t)$ as the cost function of NMPC. $V(x(t),t)$ is nonincreasing function. For time period $\tau \in [t_0, T_s)$

$$\begin{aligned} V(x(t_0),\tau) &= V^*(x^*(t_0),\tau) \\ &= V^*(x^*(t_0),\tau) - \int_{t_0}^{\tau} g(x^*(s),u^*(s),s)ds \\ &= V(x(t_0),\tau) - \int_{t_0}^{\tau} g(x(s),u(s),s)ds \leq V(x(t_0),\tau) \end{aligned} \tag{52}$$

For time period $t_0 + T_s$

$$V^*(x(t_0+T_s),t_0+T_s) \geq V^*(x^*(t_0+T_s),t_0+T_s) \tag{53}$$

we know that

$$V(x(t_0+T_s),t_0+T_s) = V^*(x^*(t_0+T_s),t_0+T_s) \tag{54}$$

$$\begin{aligned} V^*(x(t_0+T_s),t_0+T_s) - V(x(t_0),t_0) \\ = h\big(x\big(t_0+T_s+t_f\big),t_0+T_s+t_f\big) - h\big(x^*\big(t_0+t_f\big),t_0+t_f\big) \\ + \int_{t_0+T_s}^{t_0+T_s+t_f} g(x(s),u(s),s)ds - \int_{t_0}^{t_0+T_s} g(x^*(s),u^*(s),s)ds \end{aligned} \tag{55}$$

Integrating (51), we have

$$\begin{aligned} h\big(x\big(t_0+T_s+t_f\big),t_0+T_s+t_f\big) - \ h\big(x\big(t_0+t_f\big),t_0+t_f\big) \\ + \int_{t_0+T_s}^{t_0+T_s+t_f} g(x(s),u(s),s)ds \leq \end{aligned} \tag{56}$$

From (53), (54), (55), (56), we have

$$V(x(t_0+T_s),t_0+T_s) - V(x(t_0),t_0) \leq -\int_{t_0}^{t_0+T_s} g(x(s),u(s),s)ds \tag{57}$$

Let $t_0 = 0, T_s = t$

$$V(x(t\ ),t\ ) - V(x(0),0) \leq -\int_{0}^{t} g(x(s),u(s),s)ds \tag{58}$$

Now $V(x(t\ ),t\ )$ is nonincreasing function so stability theorem is satisfied.

3.4 Numerical examples

Design of state feedback NMPC for inverted pendulum

The objective is to minimize the performance measure (48) and to control angular position and velocity to converge to zero. Let parameters be chosen as

$$Q = 100 \times I_2\ , R = 10,\ t_f = 0.8, T_s = 0.01, x_0 = [0.08 \quad 0]^T.$$

Figure 3-3 shows responses of the system converge to zero.

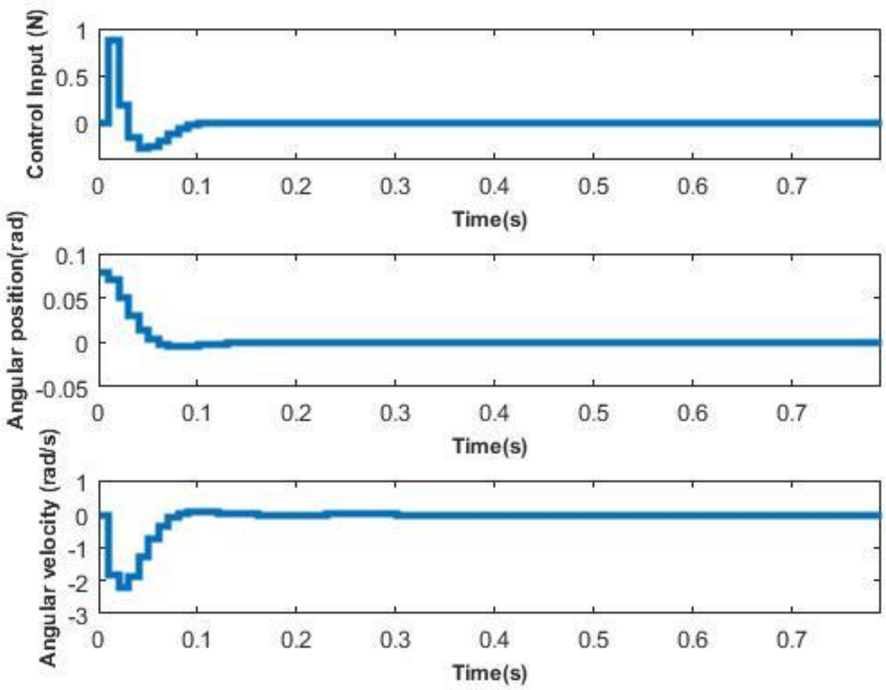

Figure 3-3 Control input, angular position and velocity of inverted pendulum for regulation problem

NMPC can stabilize pendulum at the vertically upright position. Notice that angular position starts at 0.08 rad. To stabilizing pendulum, control input forces the system with positive value at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position becomes negative, control input forces in the opposite direction to keep pendulum into vertically upright position.

Design of state feedback NMPC to inverted pendulum for tracking problem

We assume that all states of system are measurable. The objective is to minimize the performance measure (49) and track sinusoid signal. As shown in the figure 3-4, responses of the system can track sinusoid signal. NMPC can track sinusoid signal. Notice that angular position starts at 0.08 rad. To control pendulum, control input forces the system with positive value at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. Then control input becomes negative to drive pendulum to go through 0.2 rad. As soon as angular position can track sinusoid signal, amplitude of control input decreases. Trajectory of control input is still sinusoid like angular position. Control input overlaps with angular position.

Next, we change to reference input to square wave signal. As shown in figure 3-5, responses of the system can track square wave signal. Notice that angular position starts at 0.08 rad. To control pendulum, control input forces the system with negative value at the bottom of pendulum.

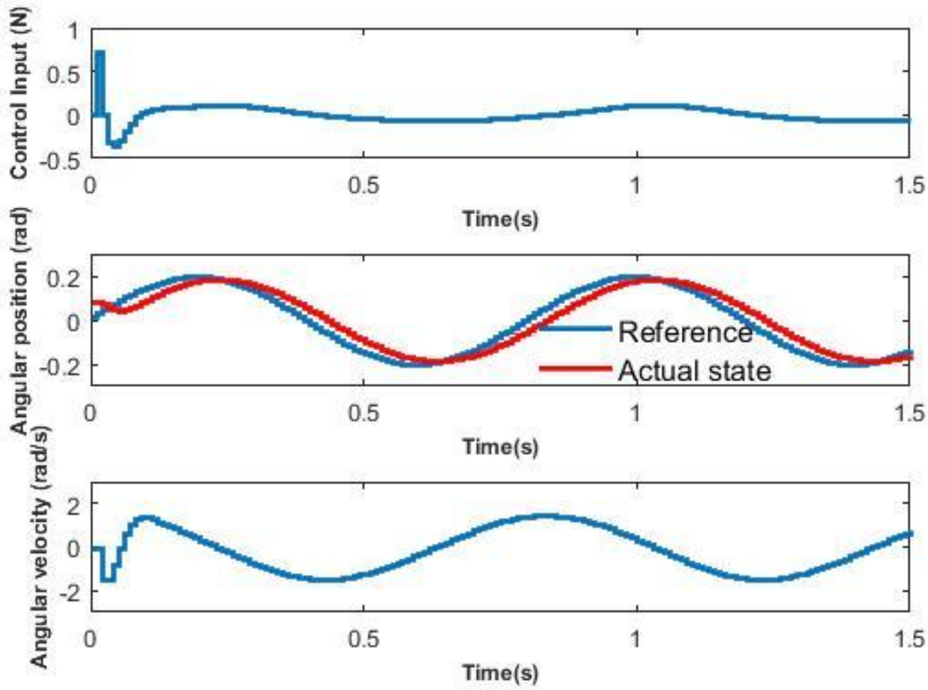

Figure 3-4 Control, angular position and velocity of inverted pendulum for tracking sinusoid signal

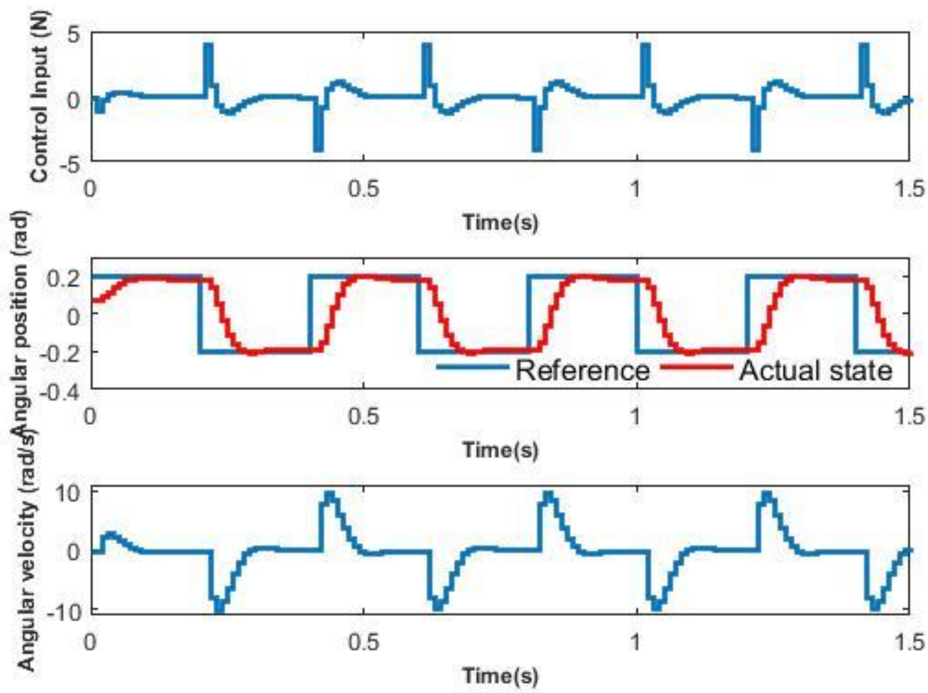

Figure 3-5 Control, angular position and velocity of inverted pendulum for tracking square wave signal

Angular velocity should be positive value because of control input force. Positive value of angular velocity drives pendulum to go through 0.2 rad. As soon as angular position can track sinusoid signal, amplitude of control input decreases. Control input forces close to zero for short time and change value to positive value suddenly to drive pendulum into -0.2 rad.

On regulation problem, state feedback NMPC is applied to inverted pendulum. It takes a long-time horizon than the optimal control to make the response converge to zero. On tracking problem, NMPC can control the angular position of inverted pendulum to track the reference signal. Notice that the angular position moves all the time so the angular velocity must move all the time too for tracking sinusoid signal. On the other hand, tracking square wave has a short time for constant value at 0.2 and -0.2 so the angular position tracks the value for short time. The angular velocity will be close to zero at that time before changing the direction.

Inverted pendulum on linear motion cart

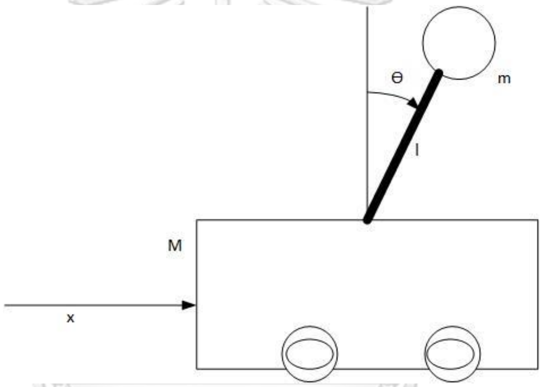

Figure 3-6 Inverted pendulum on linear motion cart

The state observation of the single inverted pendulum on cart is described as

$$\dot{x}_1(t) = x_3(t)$$

$$\dot{x}_2(t) = \frac{g \sin x_1(t)}{l}$$

$$\frac{-ml \sin x_1(t) \cos x_1(t) x_3^2(t) + mg(t) \sin x_1(t) \cos x_1(t) - ml^2 \cos x_1(t)\, u(t)}{l(M + m\,(\sin x_1(t))^2)}$$

$$\dot{x}_3(t) = x_4(t)$$

$$\dot{x}_4(t) = \frac{ml \sin x_1(t)\, x_3^2(t) - mg \sin x_1(t) \cos x_1(t) + u(t)}{(M + m\,(\sin x_1(t))^2)}$$

where $x_1(t), x_2(t), x_3(t), x_4(t)$ are angular position, angular velocity, cart position and cart velocity, respectively. The cart moves only on the horizon plane.

Table 3-1 Paramete values of inverted pendulum on linear motion cart

Parameters	Symbol	Value	Unit
Mass of pendulum	m	0.23	kg
Mass of cart	M	0.6096	kg
Length of pendulum	l	0.94	m
Gravity force	g	9.81	m/s^2

Design of state feedback NMPC for inverted pendulum on linear motion cart

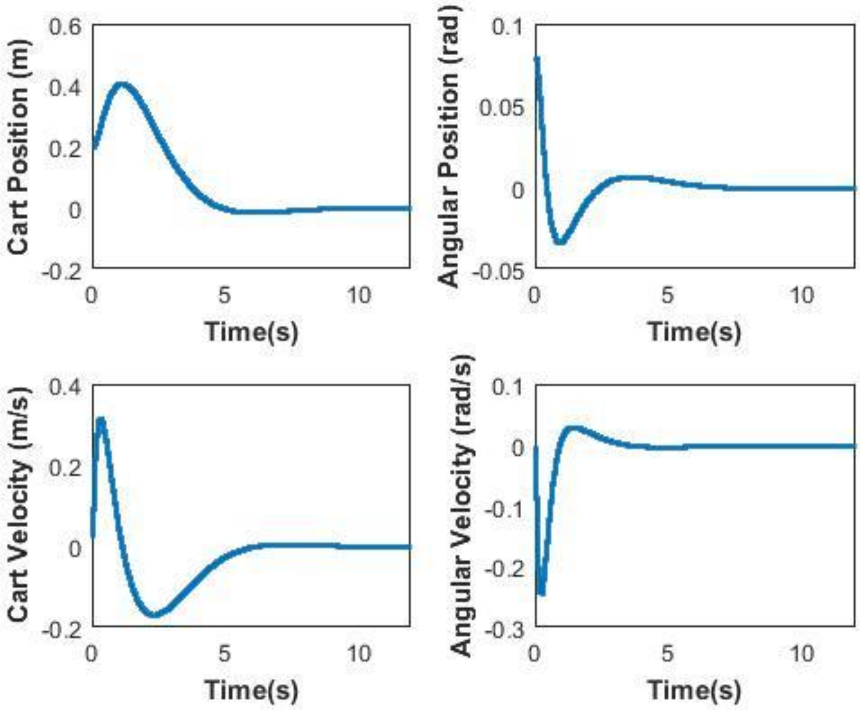

Figure 3-7 Cart, angular position and cart, angular velocity of inverted pendulum on linear motion cart for regulation problem

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$, and $t_f = 7.2, T_s = 0.01, x_0 = [0.08 \quad 0 \quad 0.2 \quad 0.02]^T$. The objective is to minimize the performance measure (48) and to control all responses converge to zero. As the figure 3-7 shown, responses of the system converge to zero.

NMPC can stabilize pendulum into vertically upright position while cart moves in horizon axes. Notice that angular position starts at 0.08 rad. To stabilizing pendulum, control input forces cart to moves in positive position. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position close to zero, control input forces cart to move in the opposite direction to keep pendulum into vertically upright position. Figure 3-8 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 3-9 shows phase plane trajectories between angular position and cart position.

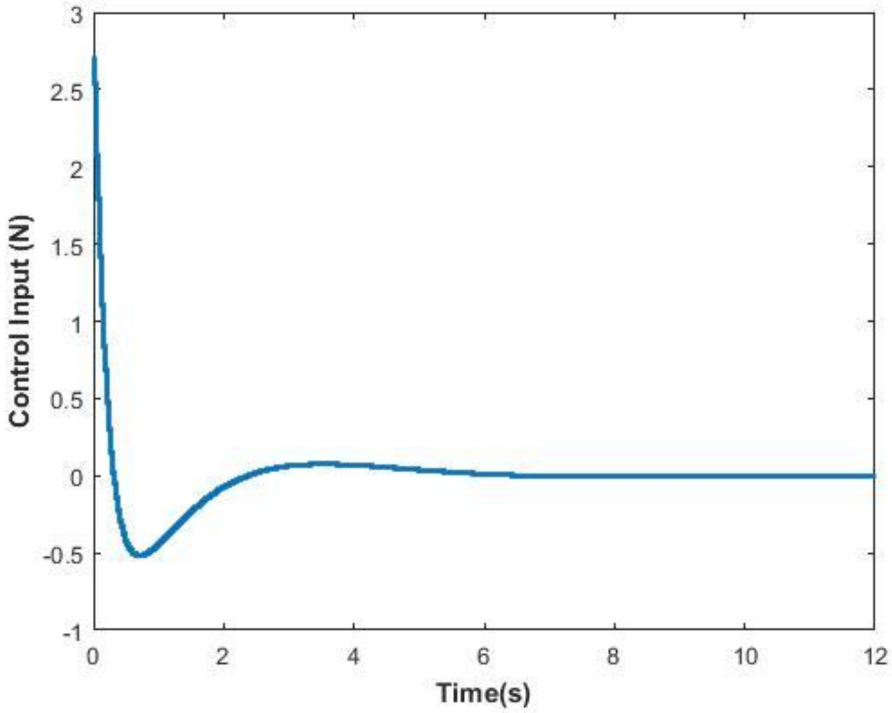

Figure 3-8 Control input of inverted pendulum on linear motion cart for regulation problem.

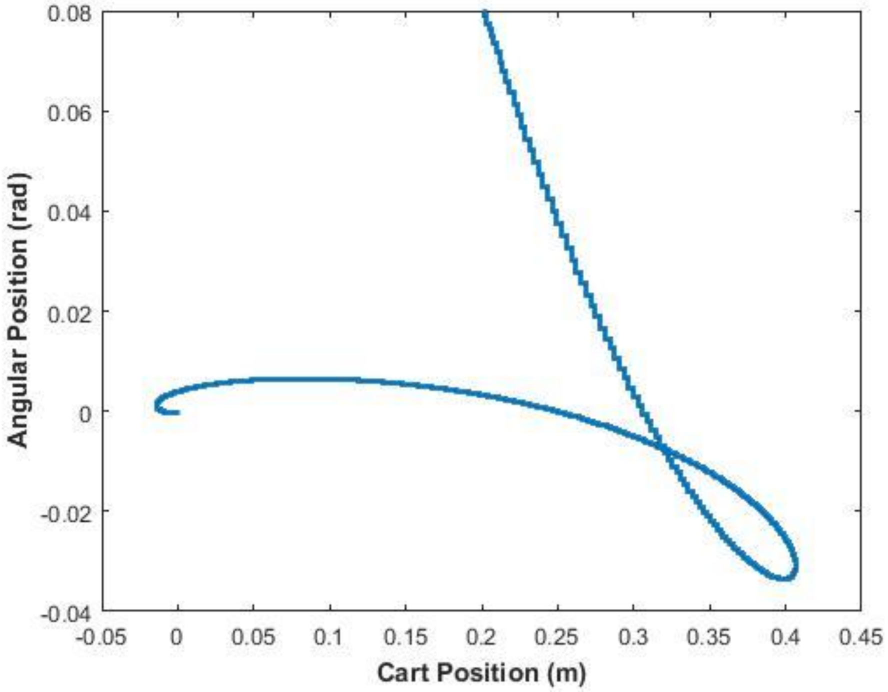

Figure 3-9 Phase plane trajectories of inverted pendulum on linear motion cart for regulation problem

Control input and phase plane of angular position and cart position are shown by figures 3-8, 3-9. Initial point is $[0.08 \quad 0.2]^T$, then control input at each time moves cart while stabilizing pendulum in vertically upright position is continue. When response close to zero, control input close to zero.

Design of state feedback NMPC for inverted pendulum on linear motion cart for tracking problem

We assume that we can measure all states of system. The objective is to drive cart position track reference signal.

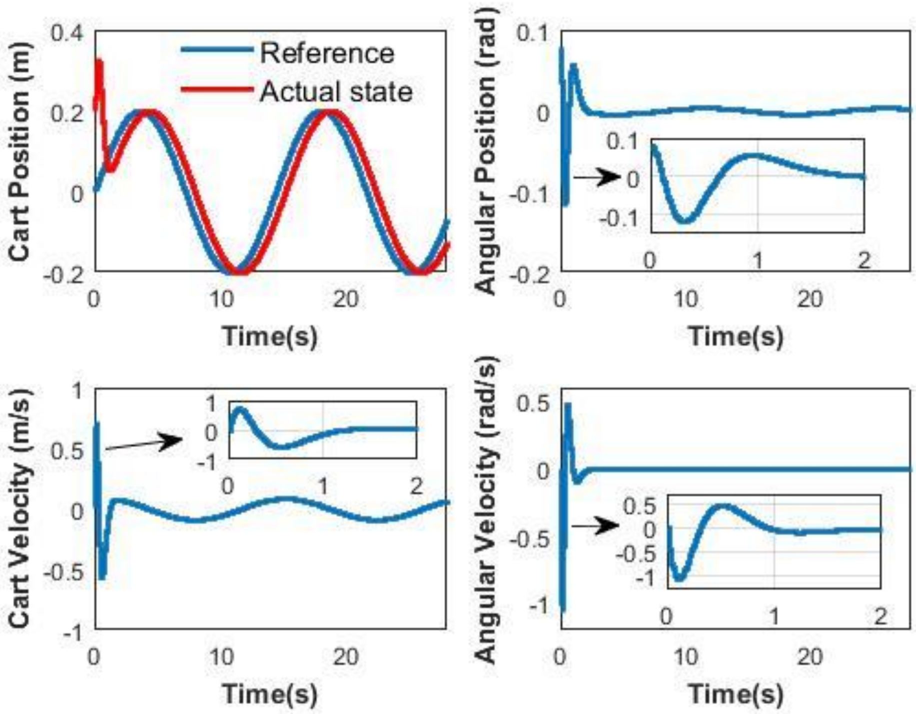

Figure 3-10 Control, angular position and velocity of inverted pendulum on linear motion cart for tracking sinusoid signal

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$ and $t_f = 7.2, T_s = 0.01, x_0 = [0.08 \quad 0 \quad 0.2 \quad 0.02]^T$. The objective is to minimize the performance measure (49) and track sinusoid. As the figure 3-10 shown, responses of the system can track sin wave signal.

NMPC can stabilize pendulum into vertically upright position while cart moves in horizon axes. Notice that angular position starts at 0.08 rad. To stabilizing pendulum, control input forces cart to moves in positive position. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position close to zero, control input forces cart to track sinusoid signal while moving cart still keep pendulum into vertically upright position. Figure 3-11 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 3-12 shows phase plane trajectories between angular position and cart position.

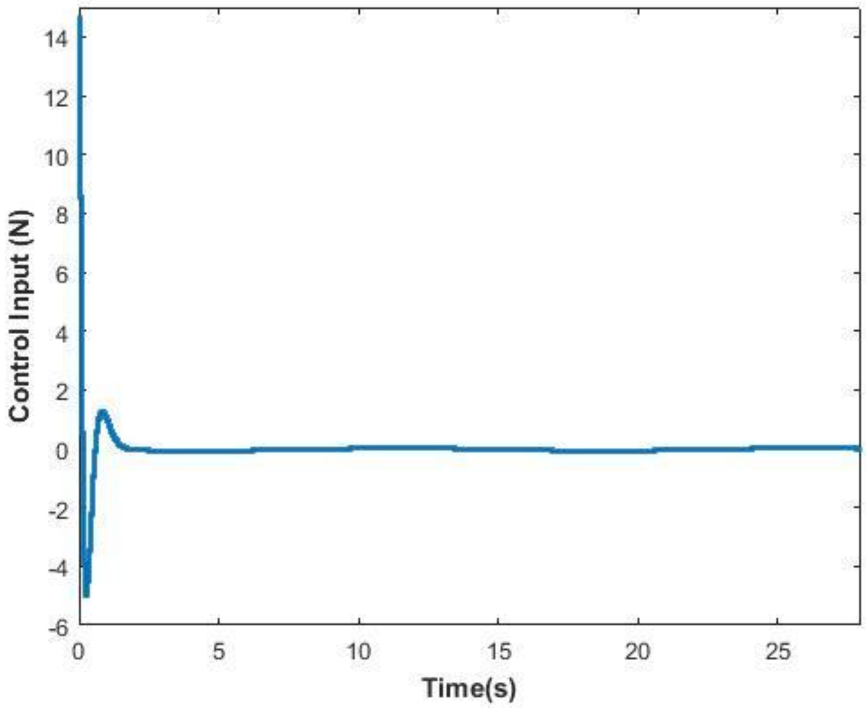

Figure 3-11 Control input of inverted pendulum on linear motion cart for tracking sinusoid signal

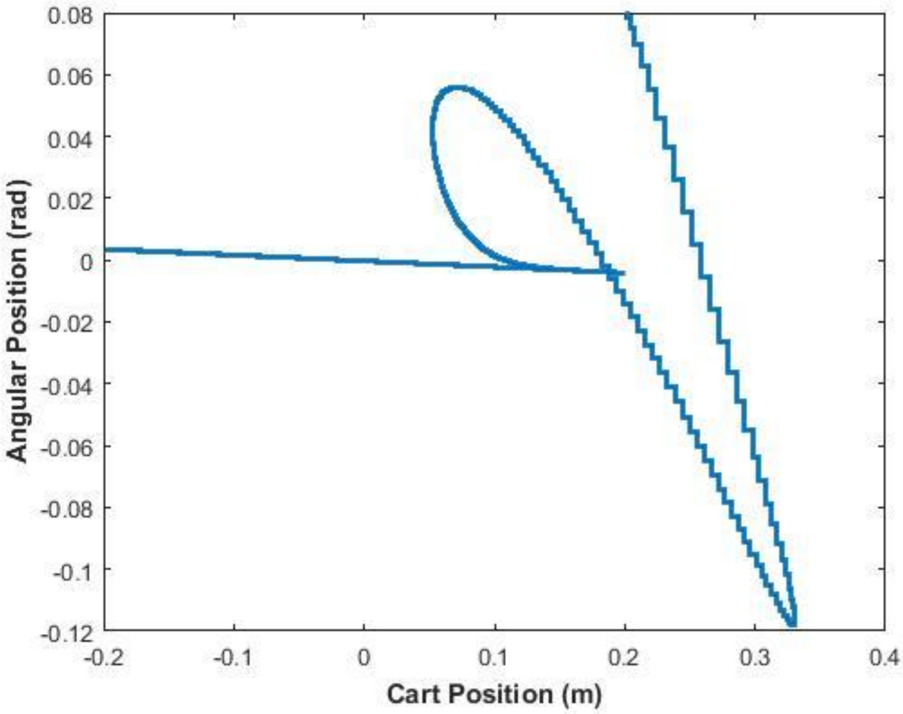

Figure 3-12 Phase plane trajectories of inverted pendulum on linear motion cart for tracking sinusoid signal

Control input and phase plane of angular position and cart position are shown in figure 3-13. Initial point is $[0.08 \quad 0.2]^T$, then control input at each time moves cart in sinusoid trajectories while stabilizing pendulum in vertically upright position is continue. Only earliest stage is different because initial point of cart position is far from reference signal. Control input in earliest stage drives cart moves to track reference signal. As soon as cart can track to reference signal, control input value decreases in order to drive cart slowly while stabilize pendulum in vertically upright position.

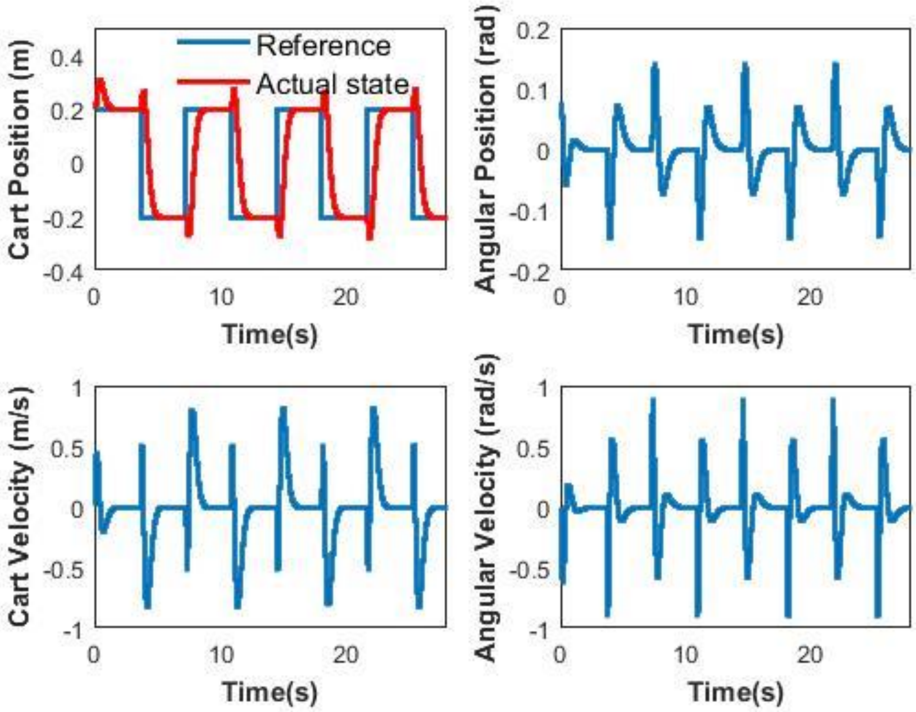

Figure 3-13 Control, angular position and velocity of inverted pendulum on linear motion cart for tracking square wave signal

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$ and $t_f = 7.2, T_s = 0.01, x_0 = [0.08 \quad 0 \quad 0.2 \quad 0.02]^T$. The objective is to minimize the performance measure (49) and track square wave signal. As shown in figure 3-16, responses of the system can track square wave signal.

NMPC can stabilize pendulum into vertically upright position while cart moves in horizon axes. Notice that angular position starts at 0.08 rad. To stabilizing pendulum, control input forces cart to moves in positive position. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position close to zero, control input forces cart to track square wave signal while moving cart still keep pendulum into vertically upright position. Figure 3-14 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 3-15 shows phase plane trajectories between angular position and cart position.

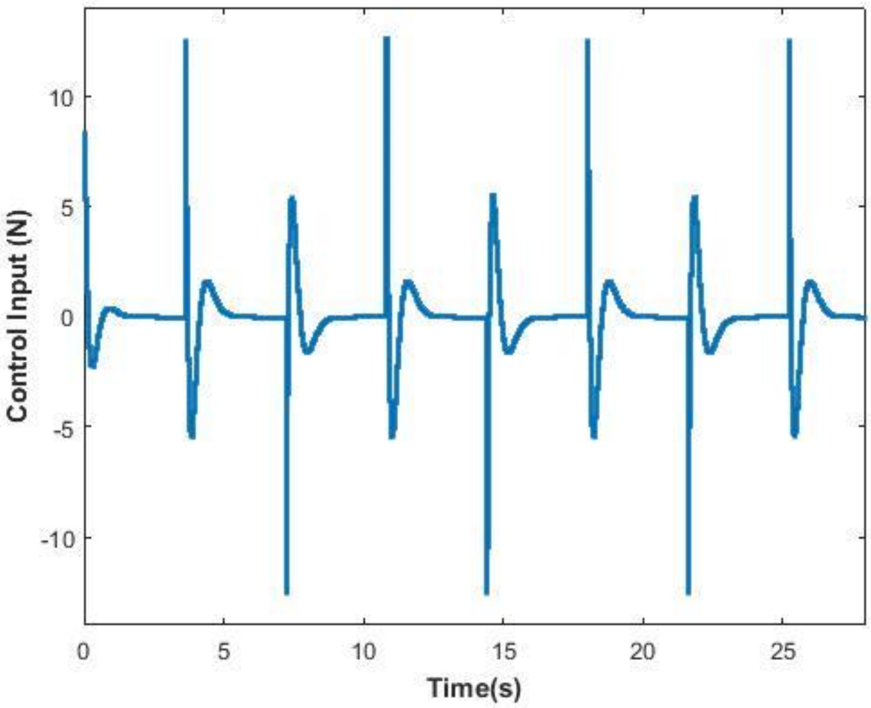

Figure 3-14 Control input of inverted pendulum on linear motion cart for tracking square wave signal

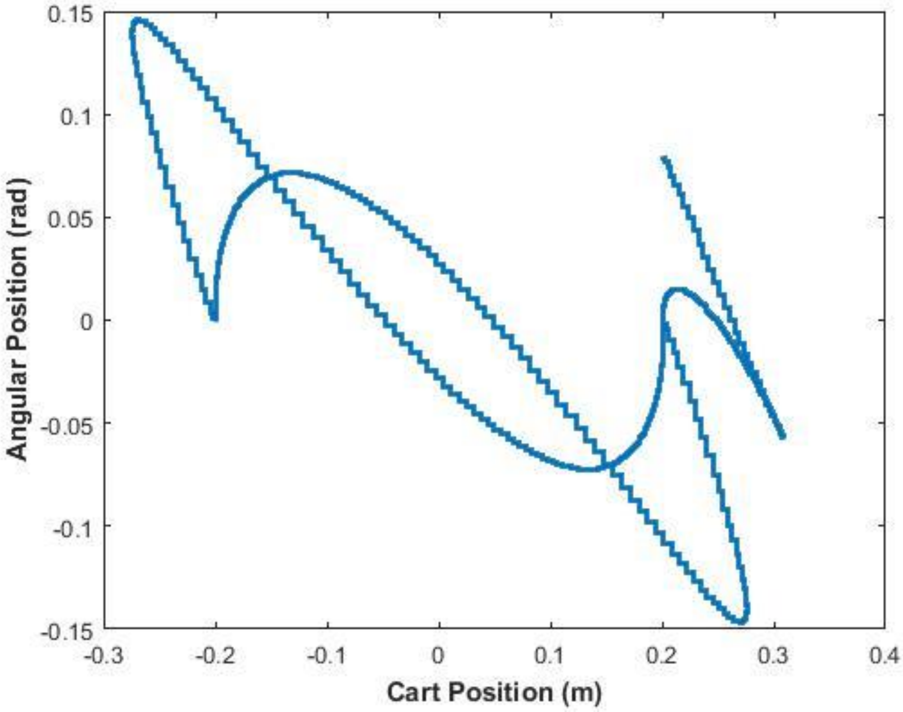

Figure 3-15 Phase plane trajectories of inverted pendulum on linear motion cart for tracking square wave signal

Control input and phase plane of angular position and cart position are shown by figure 3-15. Initial point is $[0.08 \quad 0.2]^T$, then control input at each time moves cart in square trajectories while stabilizing pendulum in vertically upright position is continue. Control input drives cart moves to track square wave signal. Notice that cart always moves over amplitude value of square wave because cart has to stabilize pendulum in vertically upright position. When cart moves in opposition direction, force of gravity has effect on pendulum. To stabilize pendulum, cart moves over amplitude value to compensate gravity force.

On regulation problem, NMPC can be applied with the single inverted pendulum on linear motion cart. As shown in figure 3-8, initial position starts at initial point and move to the zero point by applying control input.

On tracking problem, NMPC can control the cart position of inverted pendulum on linear motion cart to track the reference signal. Notice that the cart position moves all the time so the cart velocity must move all the time too for tracking sin wave signal. The angular position and velocity still close to zero because the objective of problem is to minimize the cost function (49) which means that we want x_e converge to zero. On the other hand, tracking square wave has a short time for constant value at 0.2 and -0.2 so the cart position tracks the value for short time. The cart velocity will be close to zero at that time before changing the direction. When the direction is changed, the force will make the angular position and velocity move opposite direction as they before. You can see that the angular position has trajectory similar to opposite direction to the car velocity.

Design of state feedback NMPC for inverted pendulum on circular motion cart

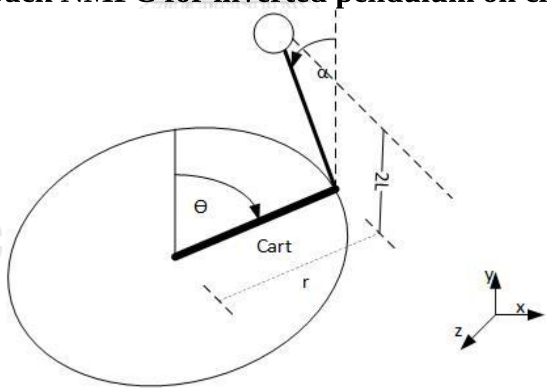

Figure 3-16 Inverted pendulum on circular motion cart

The state observation of the inverted pendulum on circular cart

$$\dot{x}_1(t) = x_3(t)$$

$$\dot{x}_2(t) = \frac{T_{op} - Bex_2(t) - mLr\sin x_3(t)x_4(t) + \frac{3}{4}mgr\sin x_3(t)\cos x_3(t)}{\left(J_e + mr^2 - \frac{3}{4}mr^2(\cos x_3(t))^2\right)}$$

$$\dot{x}_3(t) = x_4(t)$$

$$\dot{x}_4(t) = \frac{3r\cos x_3(t)\dfrac{\tau(t) - Bex_2(t) - mLr\sin x_3(t)x_4(t) + \frac{3}{4}mgr\sin x_3(t)\cos x_3(t)}{\left(J_e + mr^2 - \frac{3}{4}mr^2(\cos x_3(t))^2\right)}}{4L} + \frac{+3g\sin x_3(t)}{4L}$$

$$\tau(t) = \frac{\eta_m\eta_g K_t K_g\left(u(t) - K_g K_m x_1(t)\right)}{R_m}$$

where $x_1(t), x_2(t), x_3(t), x_4(t)$ are cart position, cart velocity, angular position and angular velocity, respectively. Cart moves only on the circle plane.

Table 3-2 Parameter values of inverted pendulum on circular motion cart

Parameters	Symbol	Value	Unit
Equivalent inertia	J_e	0.0023	Kgm^2
Mass of pendulum	m	0.15	Kg
Length of pendulum	L	0.3	M
Length of cart	r	0.2	M
Equivalent viscous friction	β_e	0.004	$\frac{\mathrm{rad}}{\mathrm{s}}$
Motor efficiency due to rotational loss	η_m	0.87	-
Gearbox efficiency	η_g	0.85	-
Motor torque constant	K_t	0.00767	$\frac{\mathrm{Nm}}{\mathrm{A}}$
Gearbox ratio	K_g	3.7	-
Motor voltage constant	K_m	0.00767	$\frac{\mathrm{Nm}}{\mathrm{A}}$
Motor armature resistance	R_m	2.6	Ohm
Gravity force	g	9.8	$\frac{\mathrm{m}}{\mathrm{s}^2}$

Design of state feedback NMPC for the single inverted pendulum on circular motion cart

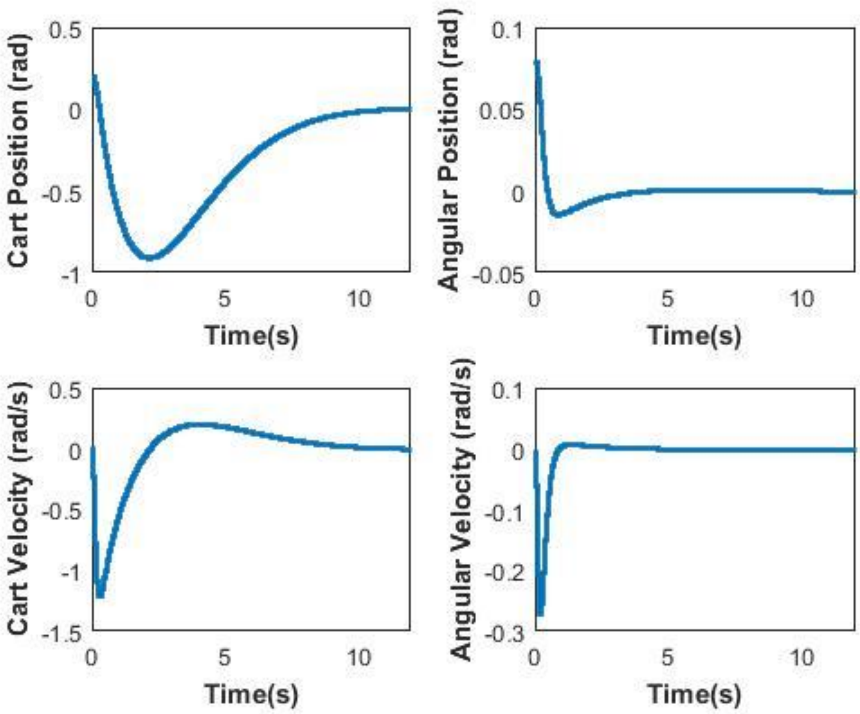

Figure 3-17 Cart, angular position and cart, angular velocity of inverted pendulum on circular motion cart for regulation problem

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$ and $t_f = 7.2, T_s = 0.01, x_0 = [0.2 \quad 0.02 \quad 0.08 \quad 0]^T$. The objective is to minimize the performance measure (48) and to control all the response to converge to zero. As the figure 3-17 shown, responses of the system converge to zero.

NMPC can stabilize pendulum into vertically upright position while cart moves in circle track. Notice that angular position starts at 0.08 rad. To stabilizing pendulum, control input forces cart to moves in negative position. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position close to zero, control input forces cart to move in the opposite direction to keep pendulum into vertically upright position. Figure 3-18 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 3-19 shows phase plane trajectories between angular position and cart position.

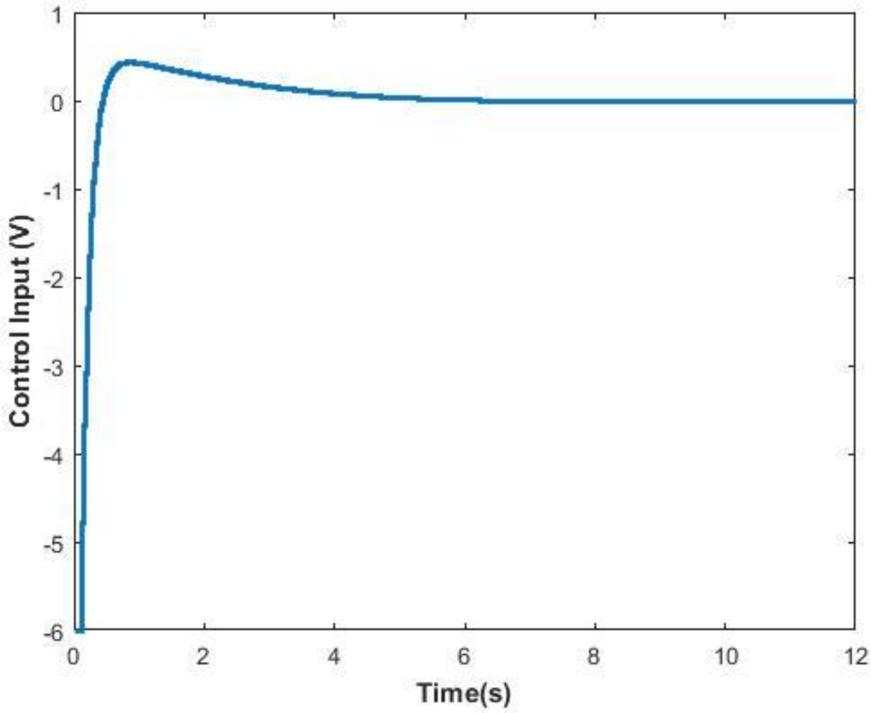

Figure 3-18 Control input of inverted pendulum on circular motion cart for regulation problem

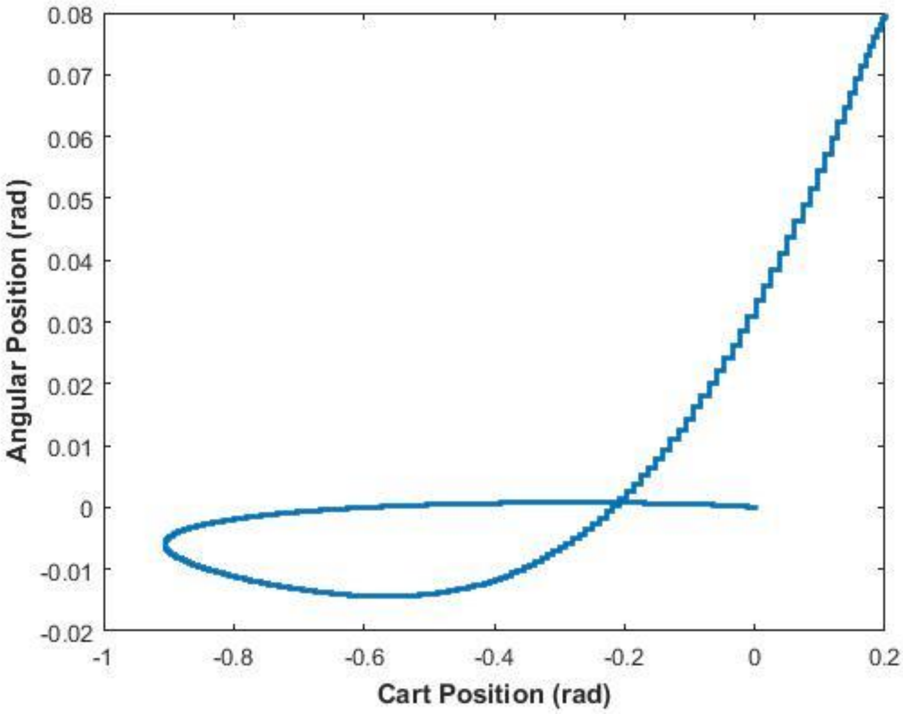

Figure 3-19 Phase plane trajectories of inverted pendulum on circular motion cart for regulation problem

Control input and phase plane of angular position and cart position are shown by figure 3-18, 3-19. Initial point is $[0.08 \quad 0.2]^T$, then control input at each time moves cart while stabilizing pendulum in vertically upright position is continue. When response close to zero, control input close to zero.

Design of state feedback NMPC for inverted pendulum on circular motion cart for tracking problem

We assume that we can measure all states of system. The objective is to drive cart position track the reference signal.

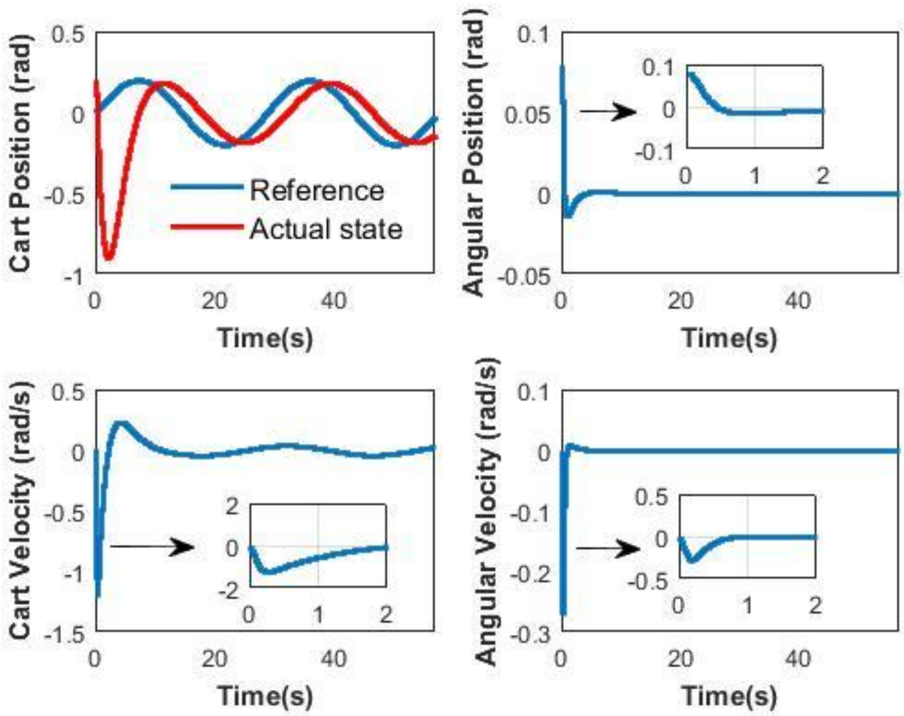

Figure 3-20 Control, angular position and velocity of inverted pendulum on circular motion cart for tracking sinusoid signal

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$ and $t_f = 7.2, T_s = 0.01, x_0 = [0.2 \quad 0.02 \quad 0.08 \quad 0]^T$. The objective is to minimize the performance measure (48) and track sinusoid signal. As the figure 3-20 shown, responses of the system can track sinusoid signal.

NMPC can stabilize pendulum into vertically upright position while cart moves in circle track. Notice that angular position starts at 0.08 rad. To stabilizing pendulum, control input forces cart to moves in negative position. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position close to zero, control input forces cart to track sinusoid signal while moving cart still keep pendulum into vertically upright position. Figure 3-21 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 3-22 shows phase plane trajectories between angular position and cart position.

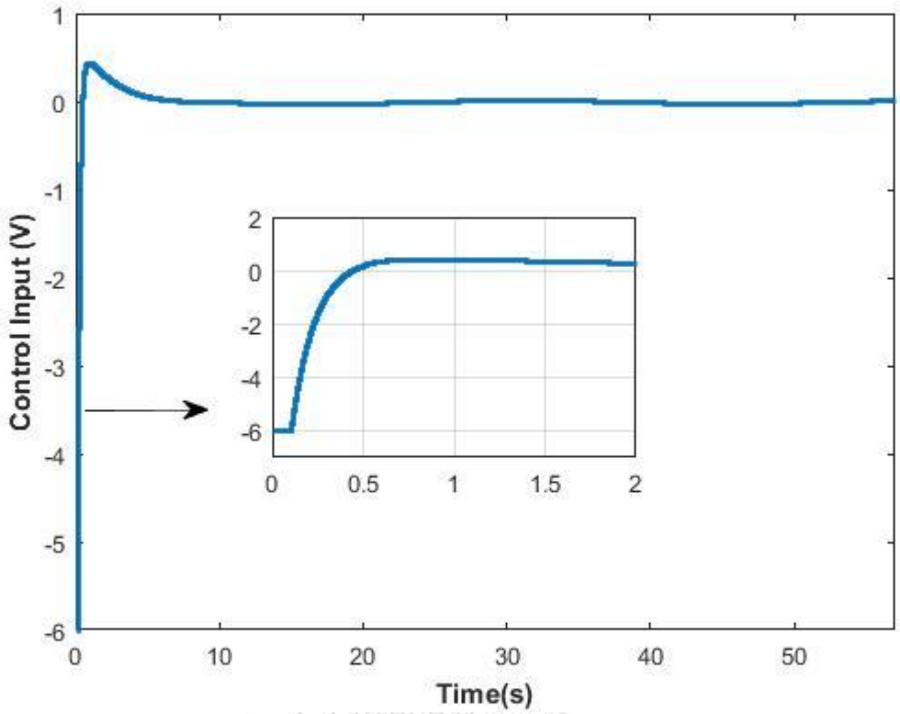

Figure 3-21 Control input of inverted pendulum on circular motion cart for tracking sinusoid signal

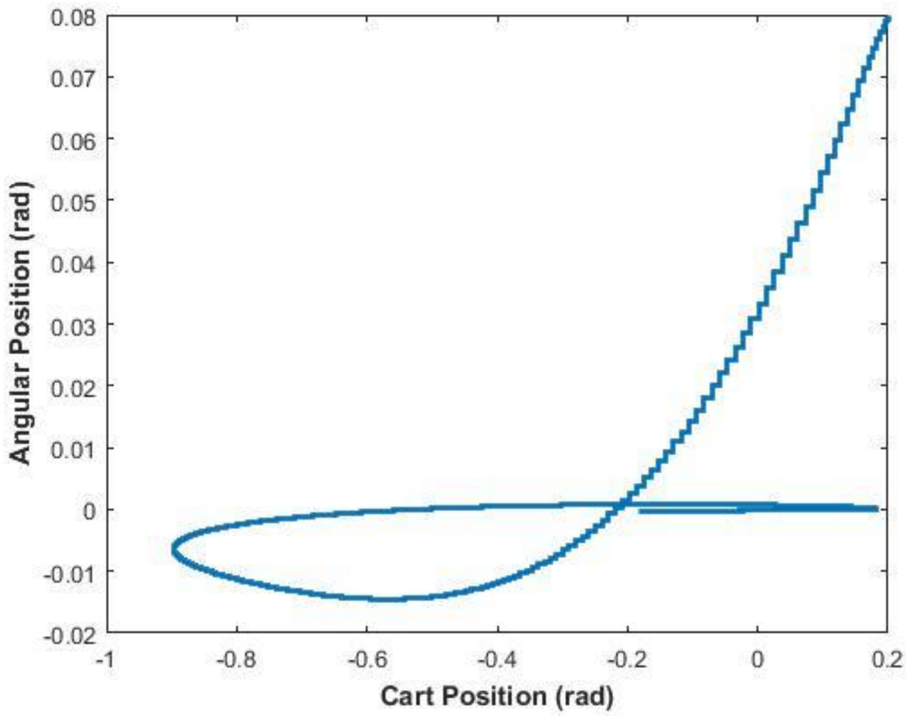

Figure 3-22 Phase plane trajectories of inverted pendulum on circular motion cart for tracking sinusoid signal

Control input and phase plane of angular position and cart position are shown by figure 3-22, 3-23. Initial point is $[0.08 \quad 0.2]^T$, then control input at each time moves cart in sinusoid trajectories while stabilizing pendulum in vertically upright position is continue. As soon as cart can track to sinusoid signal, control input value decreases in order to drive cart slowly while stabilize pendulum in vertically upright position.

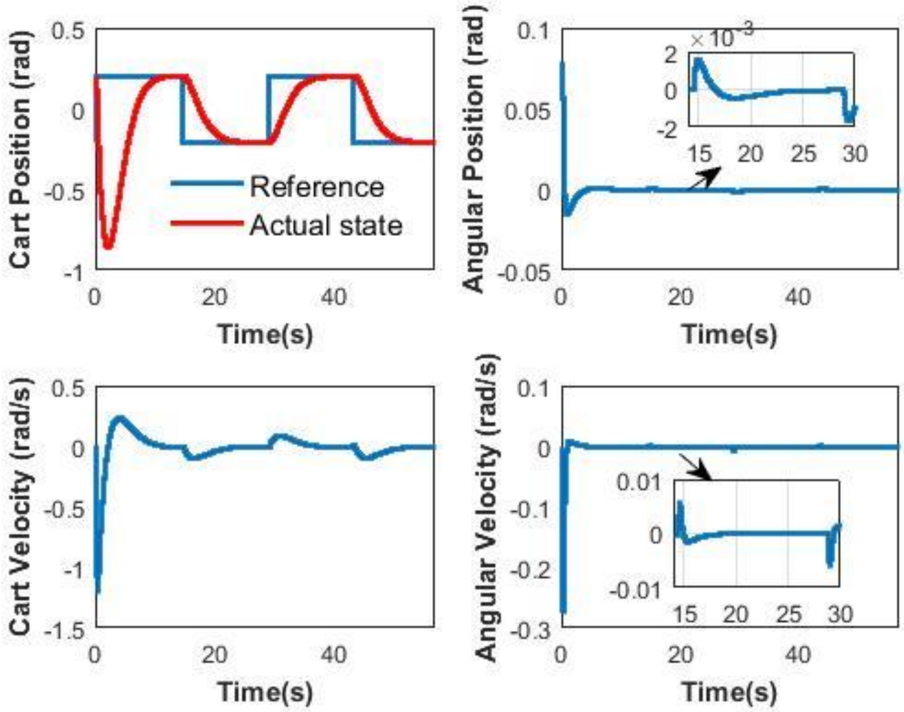

Figure 3-23 Control, angular position and velocity of inverted pendulum on circular motion cart for tracking square wave signal

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$ and $t_f = 7.2, T_s = 0.01, x_0 = [0.2 \quad 0.02 \quad 0.08 \quad 0]^T$. The objective is to minimize the performance measure (48) and track square wave signal. As the figure 3-23 shown, responses of the system can track square wave signal.

NMPC can stabilize pendulum into vertically upright position while cart moves in circle track. Notice that angular position starts at 0.08 rad. To stabilizing pendulum, control input forces cart to moves in negative position. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position close to zero, control input forces cart to track square wave signal while moving cart still keep pendulum into vertically upright position. Figure 3-24 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 3-25 shows phase plane trajectories between angular position and cart position.

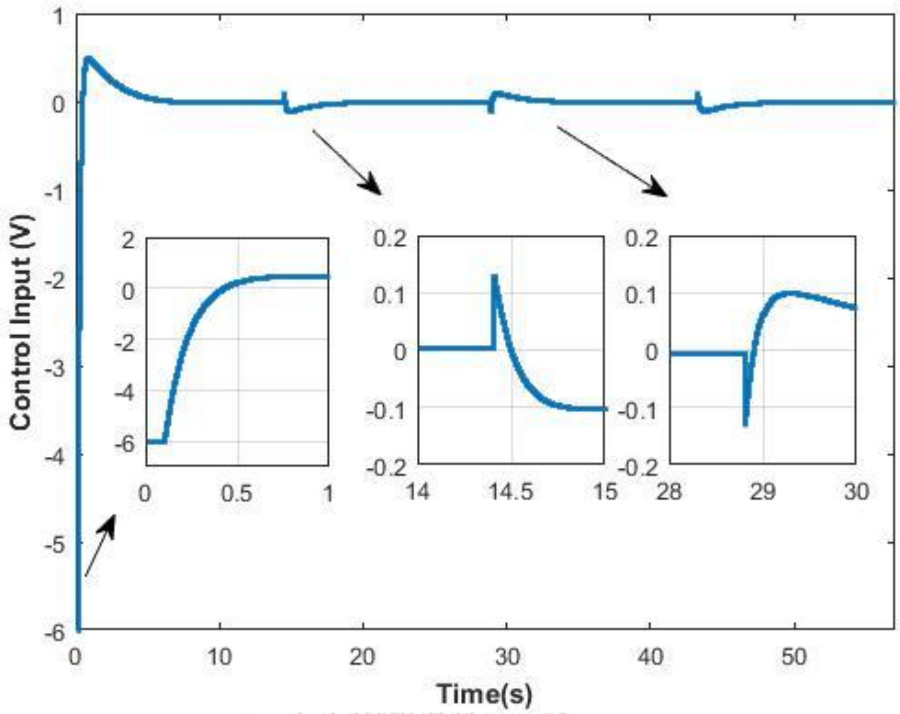

Figure 3-24 Control input of inverted pendulum on circular motion cart for tracking square wave signal

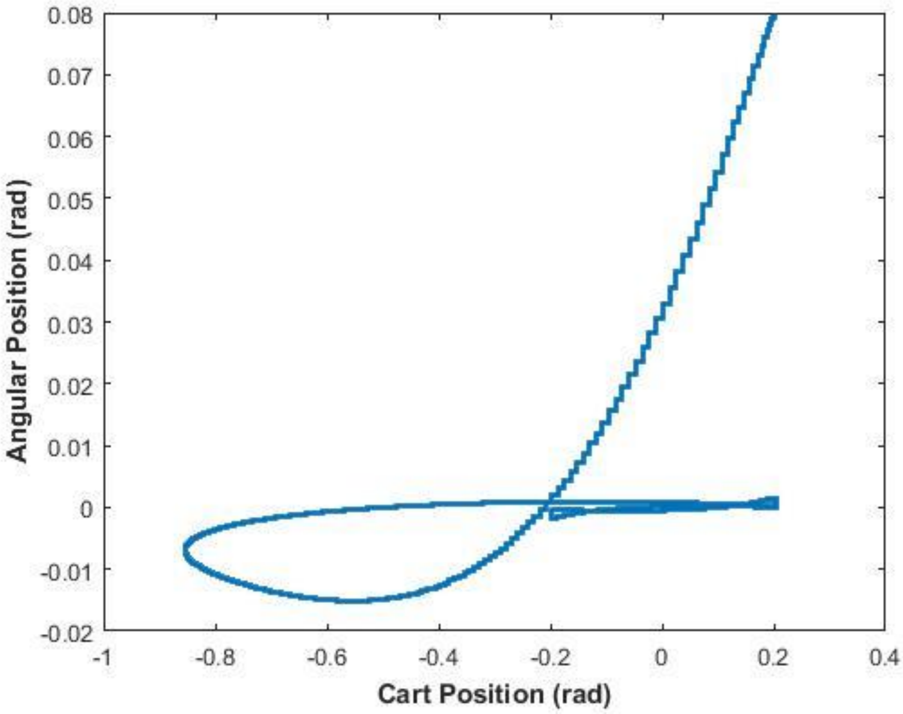

Figure 3-25 Phase plane trajectories of inverted pendulum on circular motion cart for tracking square wave signal

Control input and phase plane of angular position and cart position are shown by figure 3-25, 3-26. Initial point is $[0.08 \quad 0.2]^T$, then control input at each time moves cart in sinusoid trajectory while stabilizing pendulum in vertically upright position is continue. When cart moves in opposition direction, force of gravity has effect on pendulum. To stabilize pendulum, cart moves over amplitude value to compensate gravity force. As soon as cart can track to square wave signal, control input value decreases in order to drive cart slowly while stabilize pendulum in vertically upright position.

On regulation problem, NMPC can be applied with inverted pendulum on circular motion cart. As figure 3-9, 3-19 shown, the initial position starts at initial point and move to the zero point by applying control input.

On tracking problem, NMPC can control cart position of inverted pendulum on linear motion, circular motion cart to track reference signal. Notice that cart position moves all the time so cart velocity must move all the time for tracking sinusoid signal. Angular position and velocity still close to zero because the objective of problem is to minimize the cost function (49) which means that we want x_e converge to zero. On the other hand, tracking square wave has a short time for constant value at 0.2 and -0.2 so cart position tracks the value for short time. Cart velocity will be close to zero at that time before changing the direction. When the direction is changed, the force will make angular position and velocity move opposite direction as they before. You can see that angular position has trajectory similar to direction to car velocity.

Chapter 4
Output Feedback NMPC

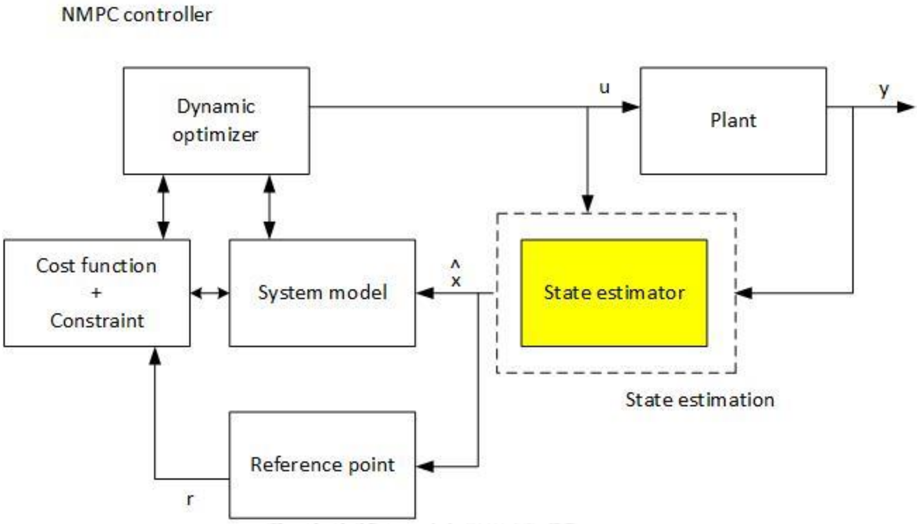

Figure 4-1 State estimation with NMPC

In Chapter 3 shows the process of state feedback NMPC to control the systems. State feedback for NMPC problem is based on the measurement of all states. Unfortunately, we cannot measure all the states of the systems in real environment. Thus, observer come to play an important role to overcome this problem.
Consider nonlinear systems (2) with the output

$$\begin{aligned} x_{k+1} &= f_d(x_k, u_k) \\ y_k &= Cx_k \end{aligned} \tag{59}$$

To simplify the notation, we use x_k to represent $x(t_k)$.

Many observers were proposed for nonlinear systems such as high gain observer [32]. High gain parameters have to be carefully designed to protect peak phenomenon. Peak phenomenon can cause the observed state outside the region of attraction. Extended Kalman filter (EKF) was proposed as an observer which first order approximation from nonlinear systems at each sampling time. It is suitable to use EKF with NMPC because NMPC is real-time optimization.

4.1 State observer of nonlinear systems

In this thesis, we consider observer.

$$z_k = Lx_k \tag{60}$$

where $z_k \in \mathbb{R}^{n-p}$ is the reduced state vector and $L \in \mathbb{R}^{(n-p)\times n}$ is a matrix which makes $\begin{pmatrix} L \\ C \end{pmatrix}$ invertible. Reduced-form of observer can be expressed as

$$\begin{pmatrix} z_k \\ y_k \end{pmatrix} = \begin{pmatrix} L \\ C \end{pmatrix} x_k \tag{61}$$

Thus,

$$\begin{pmatrix} z_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} L \\ C \end{pmatrix} x_{k+1} = \begin{pmatrix} L \\ C \end{pmatrix} (f_d(x_k, u_k)) \tag{62}$$

$$\begin{pmatrix} \hat{z}_{k+1} \\ \hat{y}_{k+1} \end{pmatrix} = \begin{pmatrix} g(\hat{z}_k, u_k, y_k) \\ h(\hat{z}_k, u_k, y_k) \end{pmatrix} \tag{63}$$

Kalman observer can be divided into 2 parts.

1.Measurement update

$$\hat{z}_{k+1} = \hat{z}_{k+1|k} + K_{k+1} e_{k+1} \tag{64}$$

$$K_{k+1} = F_k P_{k+1|k} {H_{k+1}}^T \left(H_{k+1} P_{k+1|k} {H_{k+1}}^T + v_{k+1} \right)^{-1} \tag{65}$$

$$P_{k+1} = (F_k - K_{k+1} H_{k+1}) P_{k+1|k} \tag{66}$$

2.Time update

$$\hat{z}_{k+1|k} = g(\hat{z}_k, u_k, y_k) \tag{67}$$

$$P_{k+1|k} = F_k P_k {F_k}^T + w_k \tag{68}$$

where $e_{k+1} = y_{k+1} - h(\hat{z}_k, u_k, y_k)$, $F_k = \left. \frac{\partial g(z_k, u_k, y_k)}{\partial z_k} \right|_{z_k = \hat{z}_k}$,

$H_{k+1} = \left. \frac{\partial h(z_{k+1}, u_{k+1}, y_{k+1})}{\partial z_k} \right|_{z_{k+1} = \hat{z}_{k+1|k}}$, w_k, v_k is positive definite matrix.

In Kalman filter of linear systems, w_k, v_k are covariance matrices of systems and measurement noise. However, the EKF which is not optimal estimator for nonlinear systems. w_k, v_k are important for stability of observer.

4.2 Stability analysis of observer

The Lyapunov approach is employed to achieve stability analysis of EKF. The values of w_k, v_k play important role to the convergence of EKF of nonlinear systems.
Define $\tilde{e}_{k+1}$, $\tilde{e}_{k+1|k}$ as state estimation error and state prediction error, respectively.

$$\tilde{e}_{k+1} = z_{k+1} - \hat{z}_{k+1} \tag{69}$$

$$\tilde{e}_{k+1|k} = z_{k+1} - \hat{z}_{k+1|k} \tag{70}$$

The candidate Lyapunov function $V_{k+1}$is

$$V_{k+1} = {\tilde{e}_{k+1}}^T {P_{k+1}}^{-1} \tilde{e}_{k+1} \tag{71}$$

Proof: V_{k+1} is nonincreasing function.

$$V_{k+1} - V_k \leq -\delta V_k \tag{72}$$

The EKF is used by first order approximation. We assume that

$$\tilde{e}_{k+1|k} = \alpha_k F_k \tilde{e}_k \tag{73}$$

$$e_{k+1} = \beta_k H_{k+1} \tilde{e}_k \tag{74}$$

then, we substitute (64) by (65) and subtract it from z_{k+1}. We have

$$\tilde{e}_{k+1} = \tilde{e}_{k+1|k} - K_{k+1} e_{k+1} \tag{75}$$

$$\tilde{e}_{k+1} = \alpha_k F_k \tilde{e}_k - K_{k+1} \beta_k H_{k+1} \tilde{e}_k \tag{76}$$

Substituting (75) into (71)

$$V_{k+1} = (\alpha_k F_k \tilde{e}_k - K_{k+1} \beta_k H_{k+1} \tilde{e}_k)^T {P_{k+1}}^{-1} (\alpha_k F_k \tilde{e}_k - K_{k+1} \beta_k H_{k+1} \tilde{e}_k) \tag{77}$$

From (71) and (77), we have linear matrix inequality (LMI)

$$(\alpha_k F_k - K_{k+1} \beta_k H_{k+1})^T {P_{k+1}}^{-1} \alpha_k F_k - K_{k+1} \beta_k H_{k+1} \leq (1 - \delta) {P_k}^{-1} \tag{78}$$

If (77) is satisfied, then we can obtain nonincreasing Lyapunov function.

Lemma [16]. Define

$$\mu = \max(|\alpha_k|, |\beta_k|) \leq \frac{(1 - \delta)\underline{\sigma}(F_k - K_{k+1} H_{k+1}) P_k {F_k}^T + w_k}{\overline{\sigma}(P_k)(\overline{\sigma}(((F_k - K_{k+1} H_{k+1}) P_k)))^2}$$

where $\overline{\sigma}$, $\underline{\sigma}$ are the maximum eigenvalues and minimum eigenvalues, respectively. then the LMI (78) is satisfied.

Theorem [16]. If the following statements hold

1. There exists a finite number $M \geq 0$ for all $k \geq M$, positive real numbers n_1, n_2

$$n_1 I_n \leq \mathrm{O}_e(k - M, k)^T \mathrm{P}(k - M, k) \mathrm{O}_e(k - M, k) \leq n_2 I_n$$

where

$$\mathrm{O}_e(k - M, k) = \begin{bmatrix} H_{k-M} \\ H_{k-M+1} F_{k-M} \\ \vdots \\ H_k F_{k-1} F_{k-2} \dots F_{k-M} \end{bmatrix}, \mathrm{P}(k - M, k) = diag({v_{k-M}}^{-1}, \dots, {v_k}^{-1})$$

2. F_k, H_k are bounded matrix.
3. (78) is satisfied by choosing the proper w_k

then the local asymptotic convergence is achieved for EkF.

Note that the first statement is observability of linearized nonlinear systems [19].

4.3 Observer based NMPC

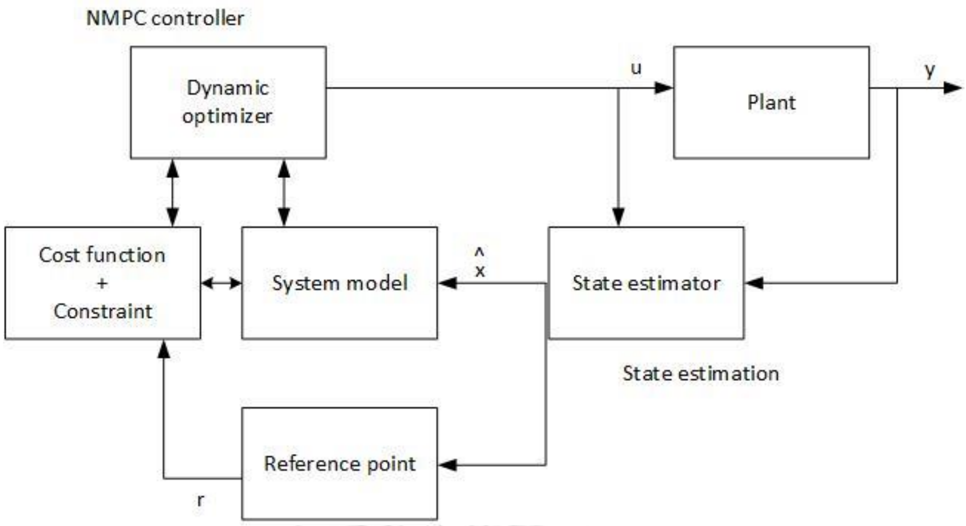

Figure 4-2 Observer based NMPC [17]

In chapter 3, we present the state feedback NMPC approach to nonlinear control system. This approach is based on the knowledge of states. In previous section, we present EKF to estimate the unknow state by using the information of the output. This section presents applying EKF to feedback for NMPC.

Problem formulation

The objective is to minimize the cost function (48) or (49)

$$J_D = T_s\left(\sum_{i=0}^{N-1}\left(\hat{x}_{k+i|k}{}^T Q\hat{x}_{k+i|k} + u_{k+i|k}{}^T R u_{k+i|k}\right) + \hat{x}_{k+N|k}{}^T Q\hat{x}_{k+N|k}\right) \tag{48}$$

or

$$J_D = T_s\left(\sum_{i=0}^{N-1}\left(\hat{x}_{e_{k+i|k}}{}^T Q\hat{x}_{e_{k+i|k}} + u_{k+i|k}{}^T R u_{k+i|k}\right) + \hat{x}_{e_{k+N|k}}{}^T Q\hat{x}_{e_{k+N|k}}\right) \tag{49}$$

$$\begin{aligned} \text{subject to } x_{k+i+1|k} &= f_d\left(x_{k+i|k}, u_{k+i|k}\right) \\ \hat{x}_0 &= x_0 \\ y_{k+i|k} &= Cx_{k+i|k} \\ \hat{z}_{k+i|k} &= Lx_{k+i|k} \end{aligned} \tag{79}$$

where $\hat{x}_{e_{k+i|k}} = \hat{x}_{k+i|k} - r_{k+i}$, r_k is reference signal at time instant k.

$$\begin{pmatrix} \hat{z}_{k+1+i|k} \\ \hat{y}_{k+1+i|k} \end{pmatrix} = \begin{pmatrix} g\left(\hat{z}_{k+i|k}, u_{k+i|k}, y_{k+i|k}\right) \\ h\left(\hat{z}_{k+i|k}, u_{k+i|k}, y_{k+i|k}\right) \end{pmatrix} \tag{80}$$

$$\hat{z}_0 = z_0 \tag{81}$$

Then use EKF to estimate the state.

1. Measurement update

$$\hat{z}_{k+1} = \hat{z}_{k+1|k} + K_{k+1}e_{k+1} \tag{64}$$

$$K_{k+1} = F_k P_{k+1|k} H_{k+1}{}^T \left(H_{k+1} P_{k+1|k} H_{k+1}{}^T + v_{k+1}\right)^{-1} \tag{65}$$

$$P_{k+1} = (F_k - K_{k+1}H_{k+1})P_{k+1|k} \tag{66}$$

2. Time update

$$\hat{z}_{k+1|k} = g(\hat{z}_k, u_k, y_k) \tag{67}$$

$$P_{k+1|k} = F_k P_k F_k{}^T + w_k \tag{68}$$

where $e_{k+1} = y_{k+1} - h(\hat{z}_k, u_k, y_k)$, $F_k = \left.\frac{\partial g(z_k,u_k,y_k)}{\partial z_k}\right|_{z_k=\hat{z}_k}$,

$H_{k+1} = \left.\frac{\partial h(z_{k+1},u_{k+1},y_{k+1})}{\partial z_k}\right|_{z_{k+1}=\hat{z}_{k+1|k}}$, w_k, v_k is positive definite matrix.

$$\hat{x}_{k+1} = \begin{pmatrix} L \\ C \end{pmatrix}^{-1} \begin{pmatrix} \hat{z}_{k+1} \\ y_{k+1} \end{pmatrix} \tag{82}$$

4.4 Numerical examples

First, we design EKF for nonlinear systems with zero input and step input to compare behavior of the observer.

Inverted pendulum

We assume that we can measure angular position. Figure 4-3 shows comparison between real output and estimated output from zero input while figure 4.3 is response from step input. Figure 4-3, 4-4 have the same parameters which are $t_f = 0.8, T_s = 0.01, x_0 = [0.08 \quad 0]^T$.

We introduce to choose parameter w_k, v_k in terms of error of output $\tilde{e}_k$ as

$w_k = (\gamma(\tilde{e}_k)^T\tilde{e}_k + \varepsilon)I_{n-p}$, $v_k = \zeta H_k P_k H_k{}^T + \varepsilon I_{n-p}$, $\gamma = 10^{15}, \varepsilon = 10^{-3}, \zeta = 10^{-2}$

Blue line is real output, red line is estimated output when initial state $\hat{z}_0 = 1$ and green line is estimated output when the initial state $\hat{z}_0 = 10$.

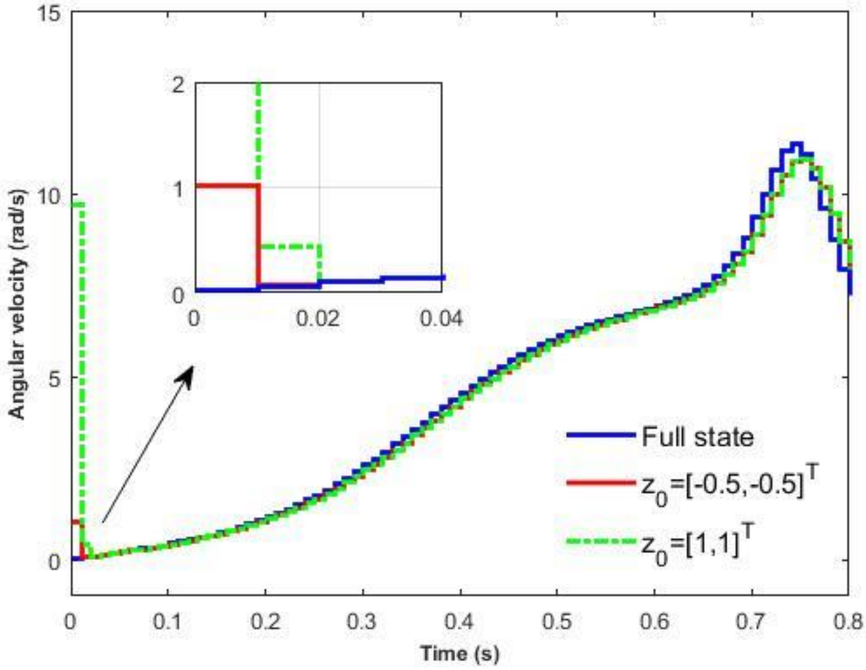

Figure 4-3 Comparison between real output and estimated output of inverted pendulum with zero input

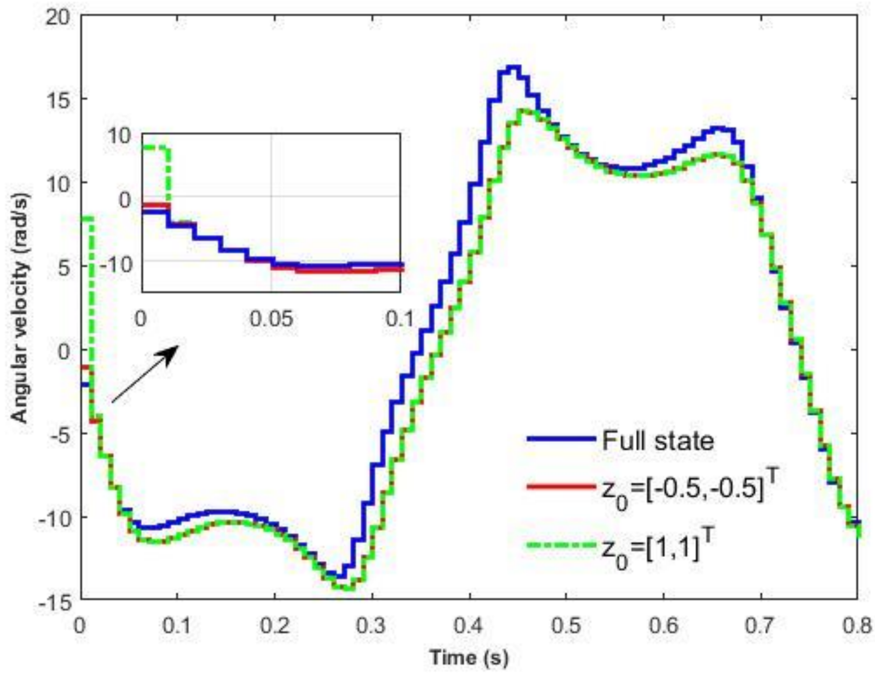

Figure 4-4 Comparison between real output and estimated output of inverted pendulum with step input

Inverted pendulum on linear motion cart
We assume that we can measure cart and angular position.

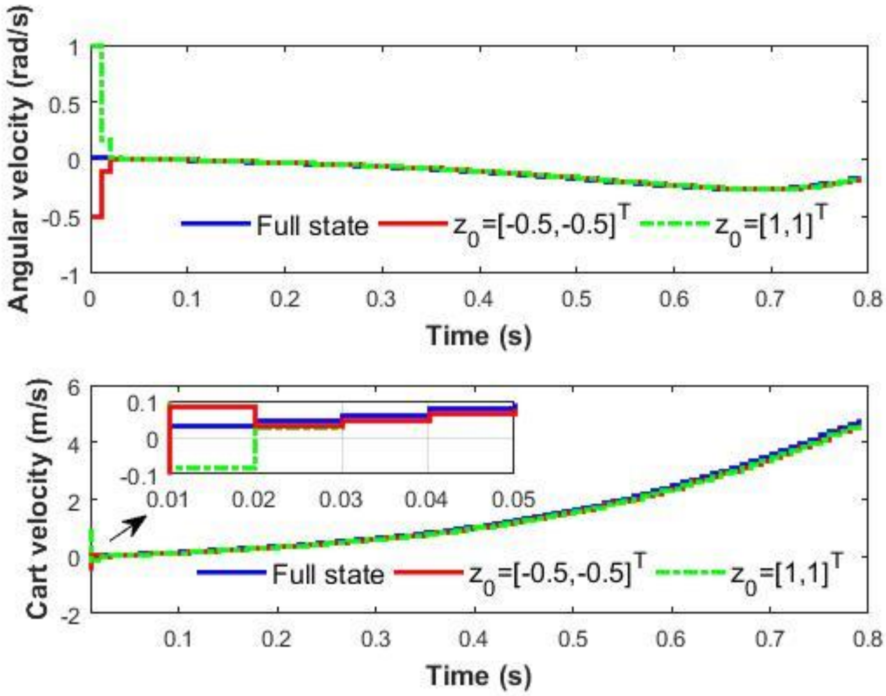

Figure 4-5 Comparison between real output and estimated output of inverted pendulum on linear motion cart with zero input

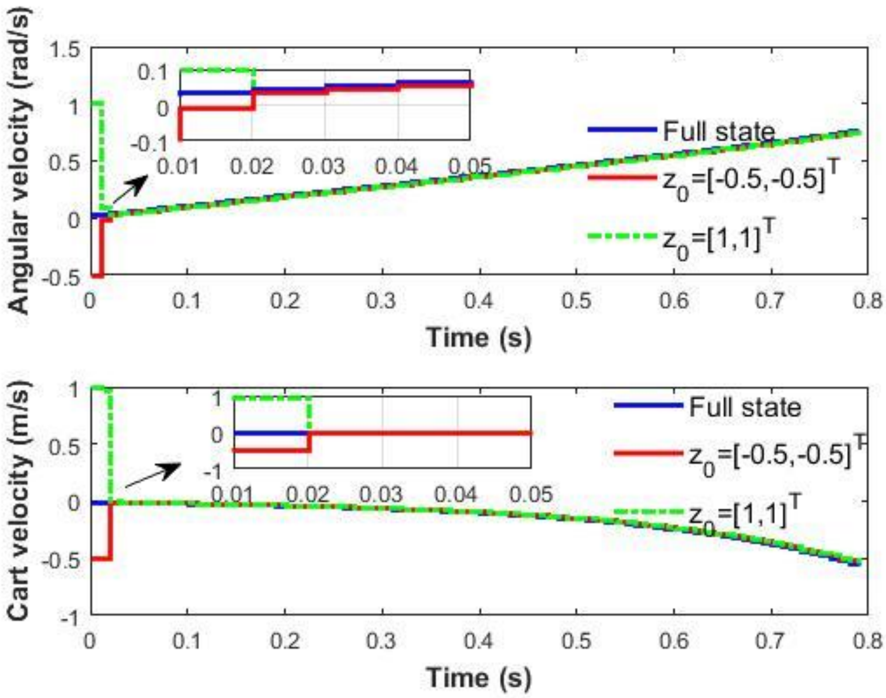

Figure 4-6 Comparison between real output and estimated output of inverted pendulum on linear motion cart with step input

Figure 4-5 shows comparison between real output and estimated output from zero input while figure 4.6 is response from step input. Figure 4-5 and 4-6 have the same parameters which are $t_f = 7.2, T_s = 0.01, x_0 = [0.08 \quad 0 \quad 0.2 \quad 0.2]^T$.

We choose parameter w_k, v_k in terms of error of output $\tilde{e}_k$ as

$w_k = (\gamma(\tilde{e}_k)^T \tilde{e}_k + \varepsilon)I_{n-p}, v_k = \zeta H_k P_k {H_k}^T + \varepsilon I_{n-p}, \gamma = 10^{15}, \varepsilon = 10^{-3}, \zeta = 10^{-2}$

Blue line is real output, red line is estimated output when $\hat{z}_0 = [-0.5 \quad -0.5]^T$ and green line is estimated output when $\hat{z}_0 = [1 \quad 1]^T$.

Inverted pendulum on circular motion cart

We assume that we can measure angular position and cart position.

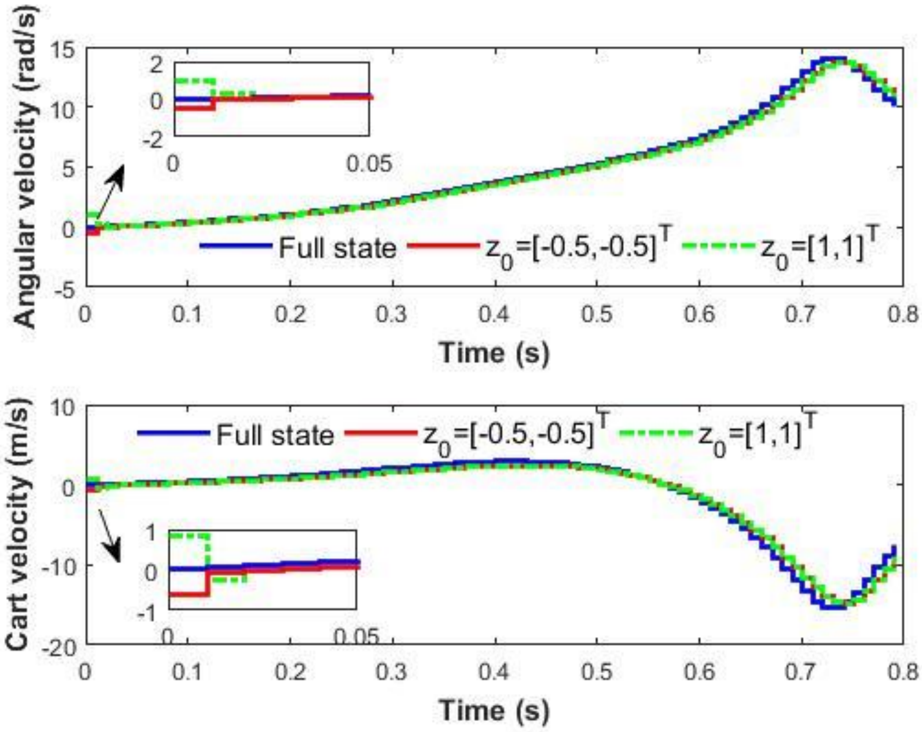

Figure 4-7 Comparison between real output and estimated output of inverted pendulum on circular motion cart with zero input

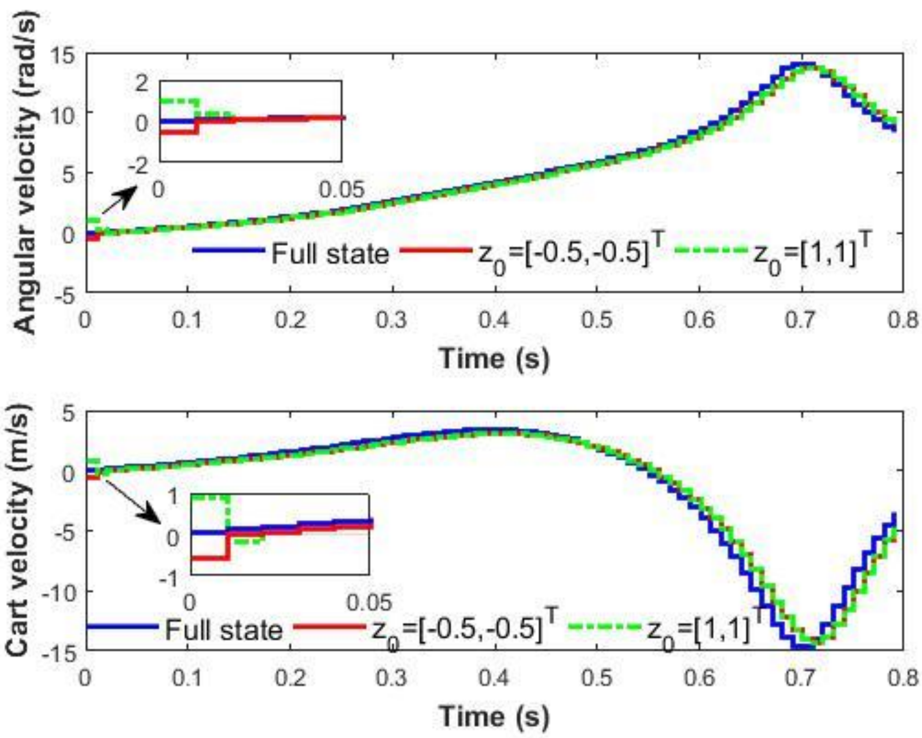

Figure 4-8 Comparison between real output and estimated output of inverted pendulum on circular motion cart with step input

Figure 4-7 shows comparison between real output and estimated output from zero input while figure 4-8 is response from step input. Figure 4-7, 4-8 have the same parameters which are $t_f = 7.2, T_s = 0.01, x_0 = [-0.08 \quad 0 \quad 0.2 \quad 0.2]^T$.
We choose the parameter w_k, v_k in terms of error of output $\tilde{e}_k$ as
$w_k = (\gamma(\tilde{e}_k)^T\tilde{e}_k + \varepsilon)I_{n-p}, v_k = \zeta H_k P_k {H_k}^T + \varepsilon I_{n-p}, \gamma = 10^{15}, \varepsilon = 10^{-3}, \zeta = 10^{-2}$
Blue line is real output, red line is estimated output when $\hat{z}_0 = [-0.5 \quad -0.5]^T$ and green line is estimated output when $\hat{z}_0 = [1 \quad 1]^T$.

Observer can be applied to inverted pendulum. It can track to real outputs if it has measured outputs by using extended Kalman observer. Although initial reduced-state is far from real points, it still can track real outputs like closed initial reduced-states. Because measured outputs are the same value, Kalman gain multiply by error between measured outputs and estimated outputs. If initial reduced-state is far from real points, it makes estimated outputs far from the real points. When Kalman gain multiply by errors, it makes reduced-systems closed to real systems.

Comparison between state and output feedback NMPC with inverted pendulum

Let design parameters be chosen as $Q = 100 \times I_2, R = 10$ and $t_f = 0.8, T_s = 0.01, x_0 = [0.08 \quad 0]^T$. Measured output is angular position so we want to estimate angular velocity to be output feedback in the next step. On the estimation process, let initial angular velocity be 10 rad/s and let matrix w_k and v_k be expressed in terms of error of output as

$w_k = (\gamma(\tilde{e}_k)^T \tilde{e}_k + \varepsilon) I_{n-p}$, $v_k = \zeta H_k P_k {H_k}^T + \varepsilon I_{n-p}$, $\gamma = 10^{10}, \varepsilon = 1, \zeta = 10^{-2}$. The objective is to minimize the performance measure (48) and to control angular position and angular velocity to converge to zero. Control input for output feedback NMPC is not similar to state feedback NMPC as shown in figure 4-9 because angular velocity of the observer is different from the real value.

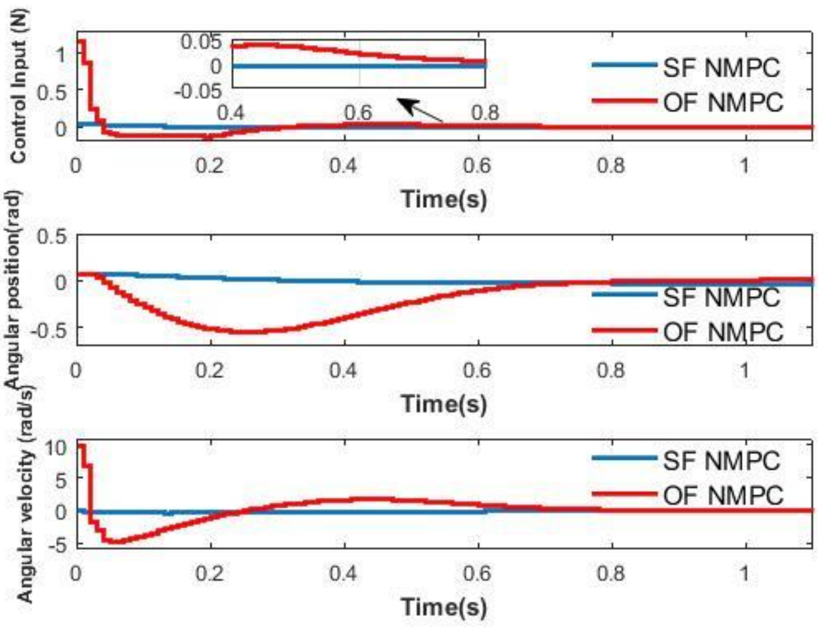

Figure 4-9 Control input, angular position and velocity of inverted pendulum for regulation problem

Output feedback NMPC can stabilize pendulum into vertically upright position. Angular position starts at 0.08 rad. Control input forces the system with positive value at the bottom of pendulum to stabilize pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position becomes negative, control input forces in the opposite direction to keep pendulum into vertically upright position. Control input for output feedback NMPC has amplitude larger than that of state feedback NMPC because angular velocity of observer starts at 10 rad/s. Control input of output feedback NMPC forces with larger magnitude than that of state feedback NMPC which the angular velocity starts at 0 rad/s.

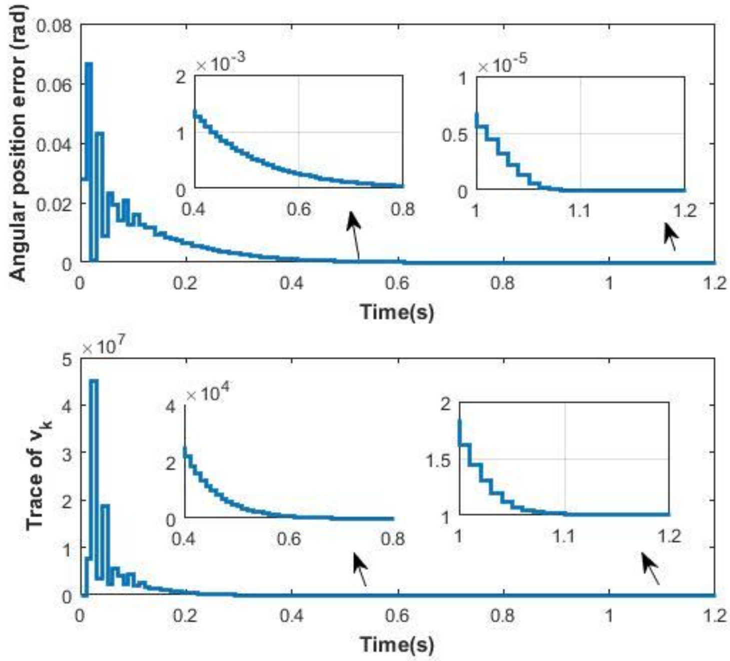

Figure 4-10 Angular position error, trace of matrix v_k of inverted pendulum for regulation problem

Figure 4-10 shows error of angular position and trace value of matrix v_k. Error of angular position is very small value when we consider w_k in terms of error of angular position and covariance error of system. It means w_k, v_k converge to 1.

Comparison between state and output feedback NMPC for inverted pendulum for tracking problem

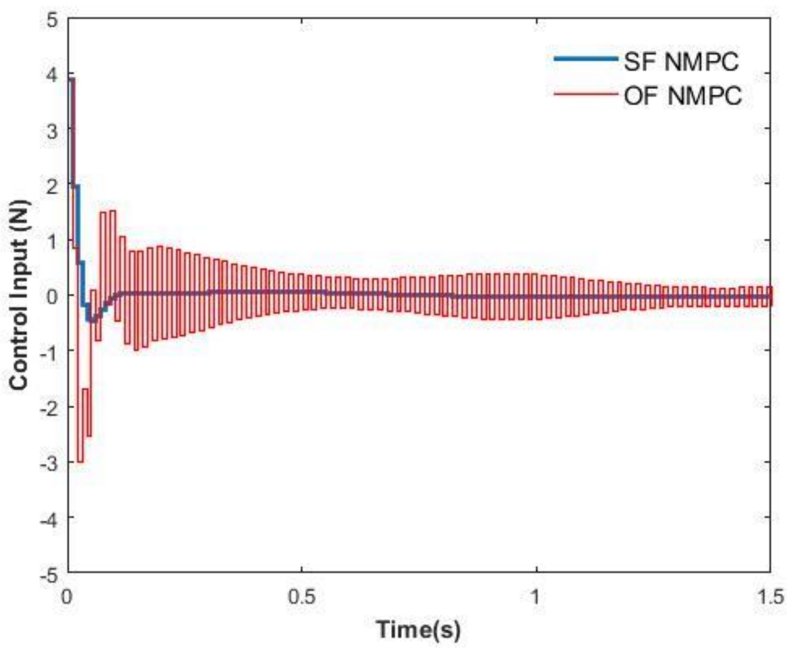

Figure 4-11 Control input of inverted pendulum for tracking sinusoid signal

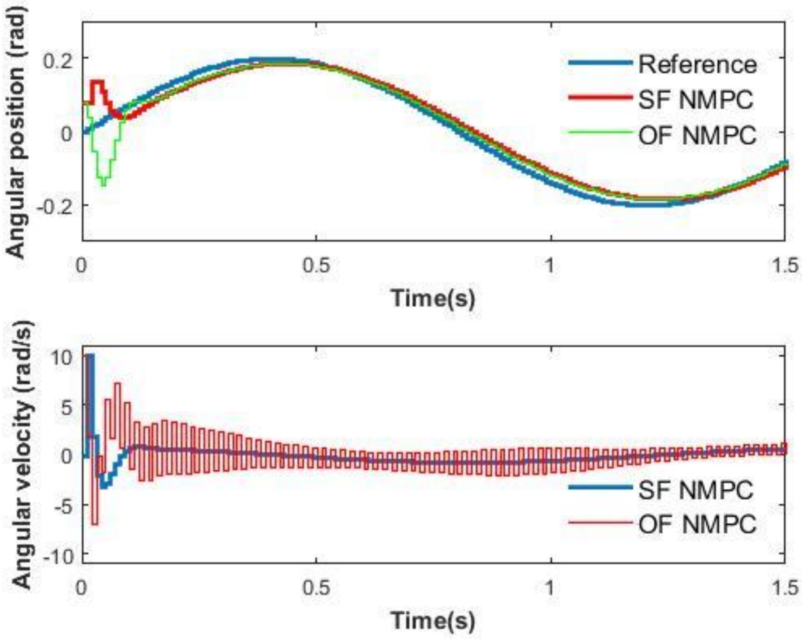

Figure 4-12 Angular position and velocity of inverted pendulum for tracking sinusoid signal

Let design parameters be chosen as $Q = 100 \times I_2$, $R = 10$, and $t_f = 0.8$, $T_s = 0.01$, $x_0 = [0.08 \quad 0]^T$. Measured output is angular position so we want to estimate angular velocity to be output feedback in the next step. On estimation process, let initial angular velocity be 10 rad/s and let w_k and v_k be expressed in terms of error of output as $w_k = (\gamma(\tilde{e}_k)^T \tilde{e}_k + \varepsilon)I_{n-p}$, $v_k = \zeta H_k P_k {H_k}^T + \varepsilon I_{n-p}$, $\gamma = 10^{10}$, $\varepsilon = 1$, $\zeta = 10^{-2}$.

The objective is to minimize the performance measure (49) and track sinusoid signal. As shown in figure 4-12, responses of the system can track sinusoid signal. Control input for output feedback NMPC is similar to state feedback NMPC except the magnitude in a few initial steps because initial estimated state is different from real value.

Output feedback NMPC can track sinusoid signal. Angular position starts at 0.08 rad. Control input forces the system with positive value at the bottom of pendulum to stabilize pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. Then control input becomes negative to drive pendulum to go through 0.2 rad. As soon as angular position can track sinusoid signal, amplitude of control input decreases. Trajectory of control input is similar to angular position. Control input overlaps with angular position. Control input for output feedback NMPC has amplitude larger than that of state feedback NMPC because angular velocity of observer starts at 10 rad/s. Control input forces larger magnitude than that of state feedback which the angular velocity starts at 0 rad/s.

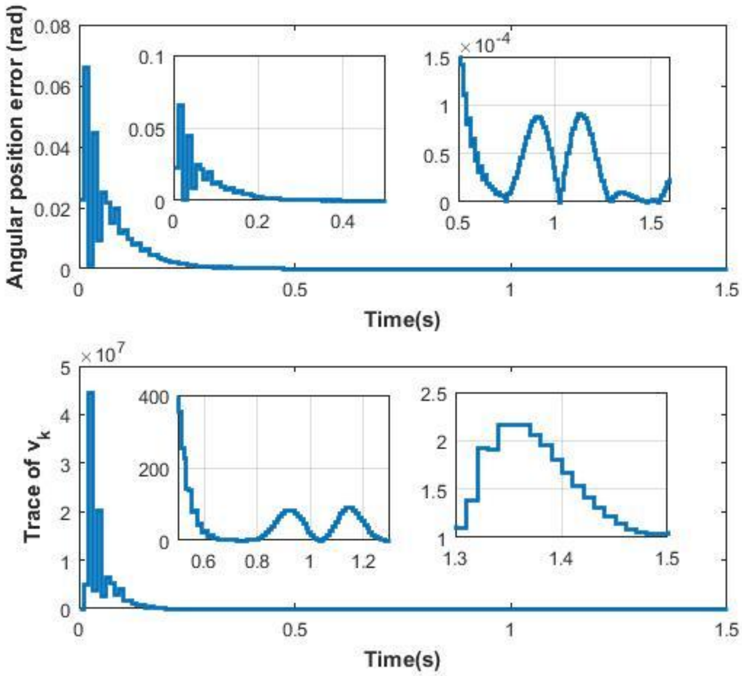

Figure 4-13 Angular position error, trace of matrix v_k of inverted pendulum for tracking sinusoid signal

Figure 4-13 shows error of angular position and trace value of matrix v_k. Error of angular position is very small value when we consider $w_k$in terms of error of angular position and covariance error of system. It means w_k, v_k converge to 1. Although trace of v_k converges to 1, it has deviation because of sinusoid signal.

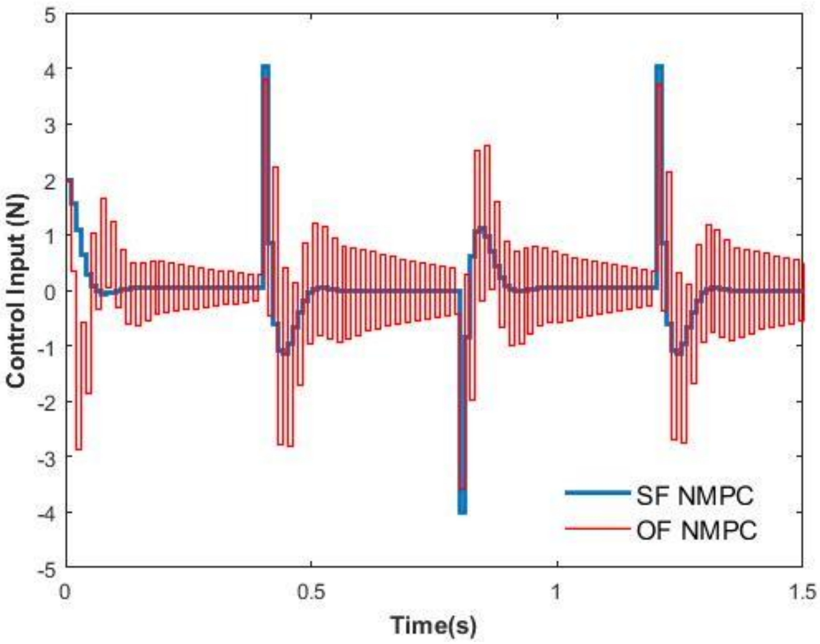

Figure 4-14 Control input of inverted pendulum for tracking square wave signal

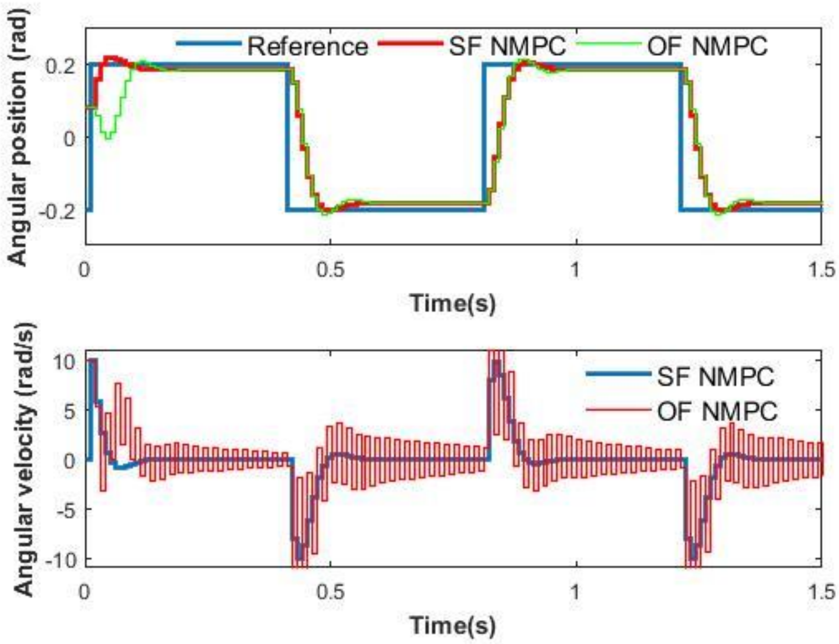

Figure 4-15 Angular position and velocity of inverted pendulum for tracking square wave signal

The objective is to minimize the performance measure (49) and track square wave signal. As the figure 4-13 shown, responses of the system can track square wave signal. Control input for output feedback NMPC is similar to state feedback NMPC except the magnitude because a few initial estimated state is different from the actual state.

Output feedback NMPC can track square wave signal. Angular position starts at 0.08 rad. To control pendulum, control input forces the system with negative value at the bottom of pendulum. Angular velocity should be positive value because of control input force. Positive value of angular velocity drives pendulum to go through 0.2 rad. As soon as angular position can track sinusoid signal, amplitude of control input decreases. Control input forces with negative value suddenly to drive pendulum into -0.2 rad. Notice that control input for output feedback NMPC has amplitude larger than state feedback NMPC because angular velocity of observer starts at 10 rad/s. To stabilize pendulum for output feedback, control input forces with higher magnitude than state feedback NMPC which angular velocity starts at 0 rad/s.

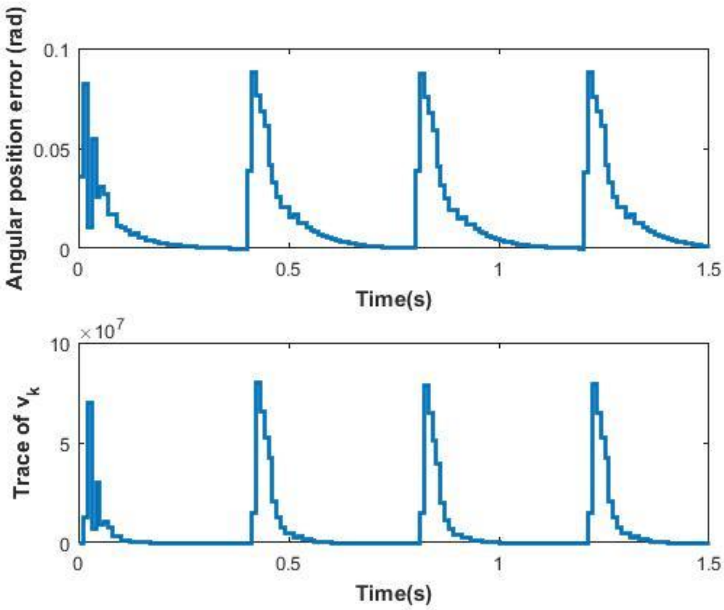

Figure 4-16 Angular position error, trace of matrix v_k of inverted pendulum for tracking square wave signal

Figure 4-16 shows error of angular position and trace value of matrix v_k. Error of angular position is very small when we consider $w_k$in terms of error of angular position and covariance error of system. Although trace of v_k converges, it has deviation because of square wave reference. As it is suddenly changed from 0.2 rad to -0.2 rad or -0.2 rad to 0.2 rad, trace of v_k and error of angular position is higher than when cart stops.

Comparison state and output feedback NMPC with inverted pendulum on linear motion cart

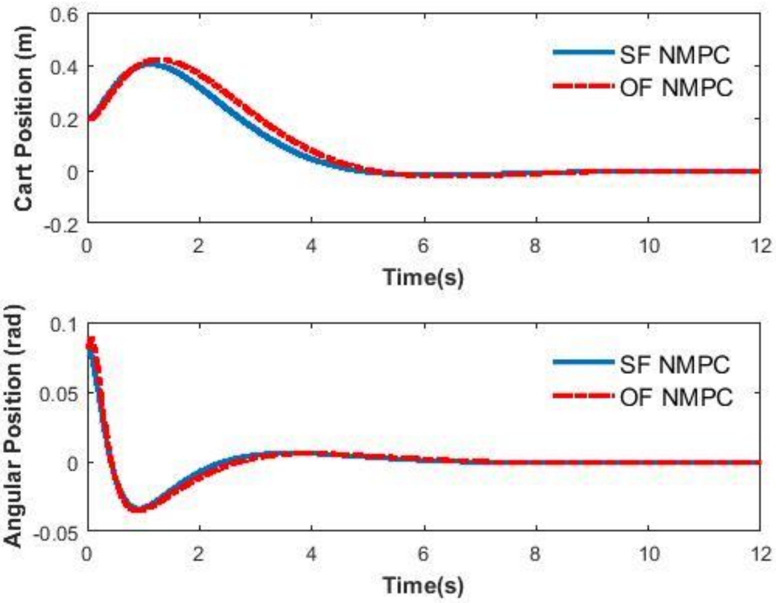

Figure 4-17 Cart and angular position of inverted pendulum on linear motion for regulation problem

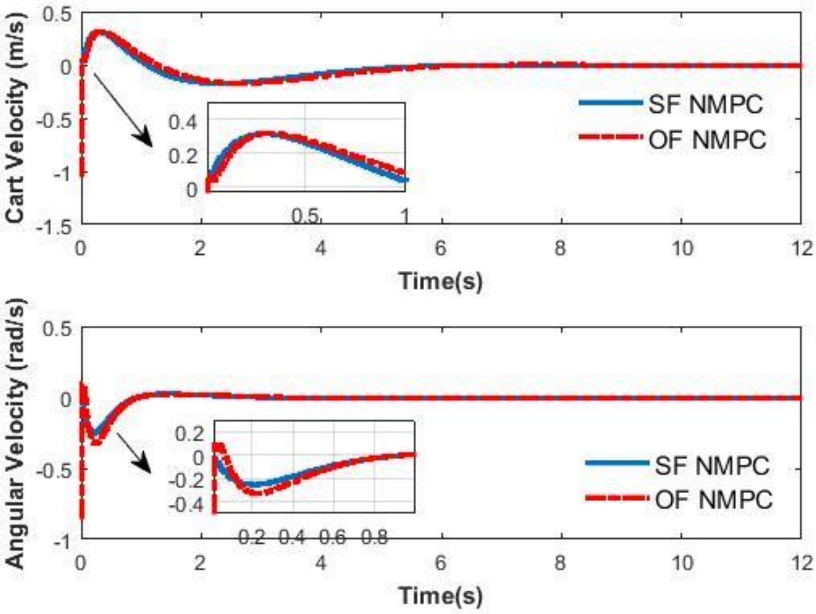

Figure 4-18 Cart and angular position of inverted pendulum on linear motion for regulation problem

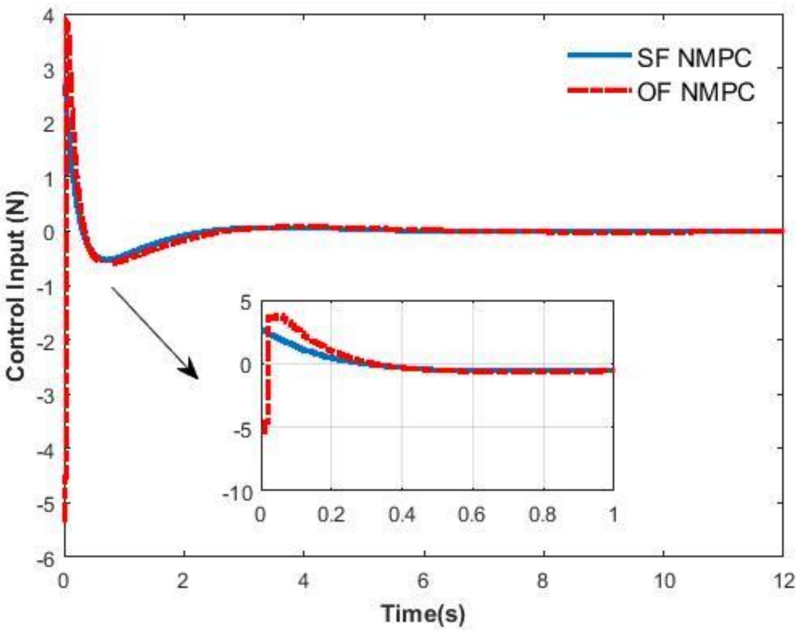

Figure 4-19 Control input of inverted pendulum on linear motion for regulation problem

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$, and $t_f = 7.2, T_s = 0.01, x_0 = [0.08 \quad 0 \quad 0.2 \quad 0.02]^T$. Measured outputs are cart and angular position so we want to estimate cart and angular velocity to be output feedback in the next step. On the estimation process, let initial cart and angular velocity to be -1 rad/s and let matrix w_k and v_k are in terms of error of output as

$w_k = (\gamma(\tilde{e}_k)^T \tilde{e}_k + \varepsilon) I_{n-p}$,$v_k = \zeta H_k P_k {H_k}^T + \varepsilon I_{n-p}$, $\gamma = 10^{10}, \varepsilon = 1, \zeta = 10^{-2}$

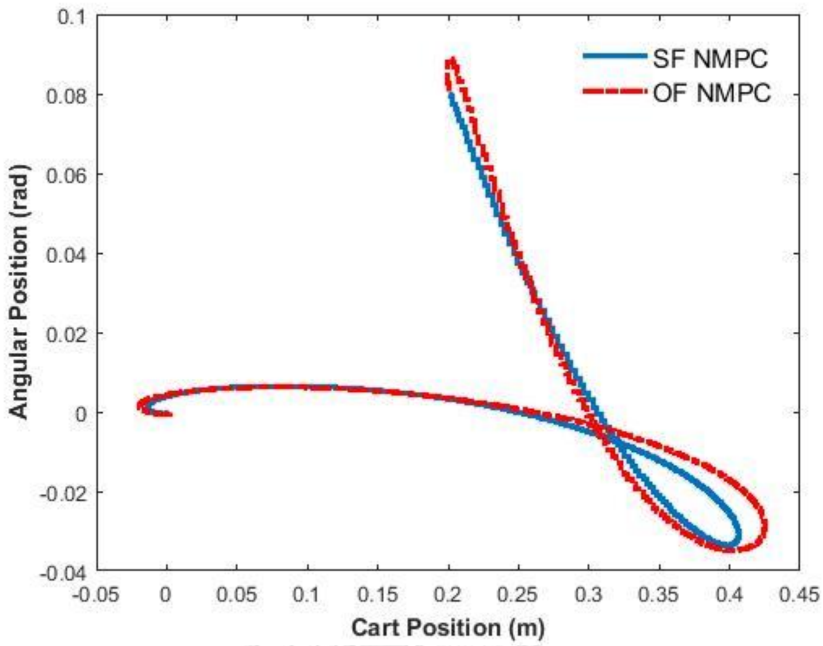

Figure 4-20 Phase plane trajectories of inverted pendulum on linear motion for regulation problem

The objective is to minimize the performance measure (48) and to control angular position and velocity to converge to zero. Control input for output feedback NMPC is not similar to NMPC full state feedback because initial estimated states are different from real values.

Output feedback NMPC can stabilize pendulum into vertically upright position while cart moves in horizon axes. Angular position starts at 0.08 rad. Control input forces cart to moves in positive position to stabilize pendulum. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position is close to zero, control input forces cart to move in the opposite direction to keep pendulum into vertically upright position. Figure 4-19 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 4-20 shows phase plane trajectories between angular position and cart position. Only earliest stage, control input for output feedback NMPC has amplitude more than state feedback NMPC because angular velocity starts at -1 rad/s and cart velocity at -1 rad/s. Control input forces with higher magnitude than that of state feedback at earliest stage. As soon as observer uses output information to estimate cart and angular velocity, control input of output feedback NMPC converges to control input of state feedback NMPC.

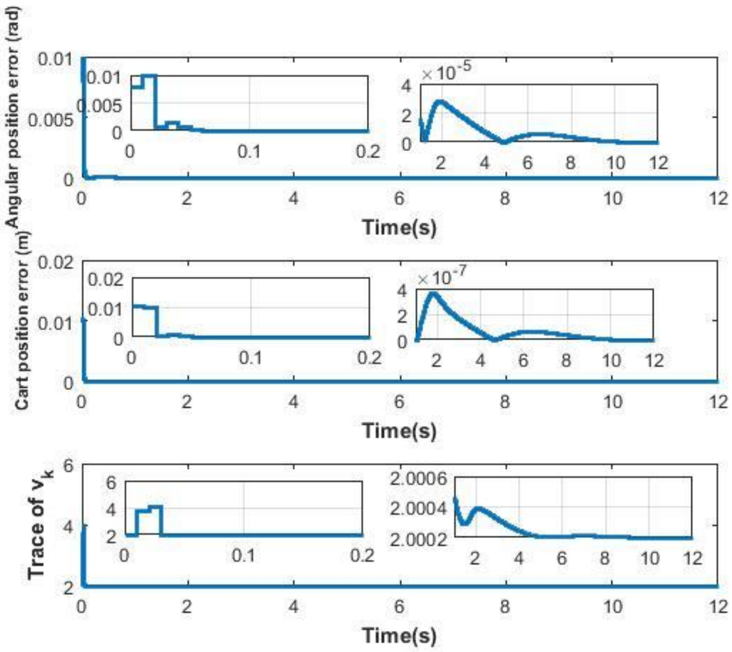

Figure 4-21 Cart and angular position error, trace of matrix $v_k$of inverted pendulum on linear motion cart for regulation problem

Figure 4-21 shows error of cart and angular position and trace value of matrix $\mathrm{v_k}$. Error of cart and angular position are very small when we consider $w_k$in terms of error of cart and angular position and covariance error of system. It means $w_k, \mathrm{v_k}$ converge to 2.

Comparison state and output feedback NMPC with inverted pendulum on linear motion cart for tracking problem

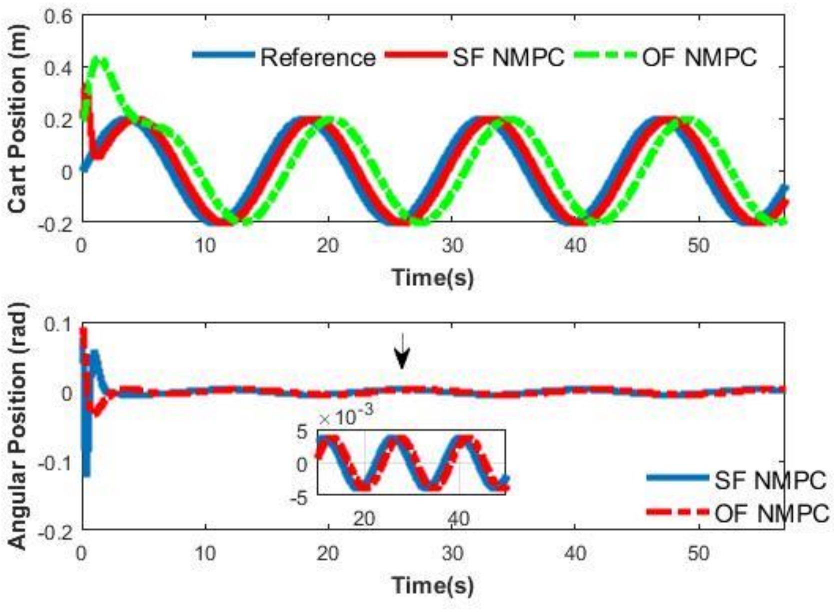

Figure 4-22 Cart and angular position of inverted pendulum on linear motion for tracking sinusoid signal

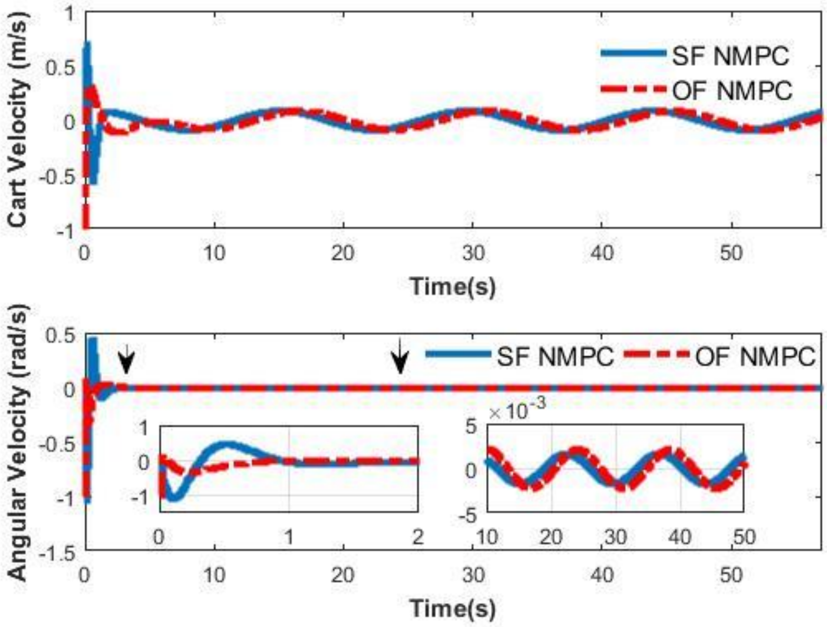

Figure 4-23 Cart and angular velocity of inverted pendulum on linear motion for tracking sinusoid signal

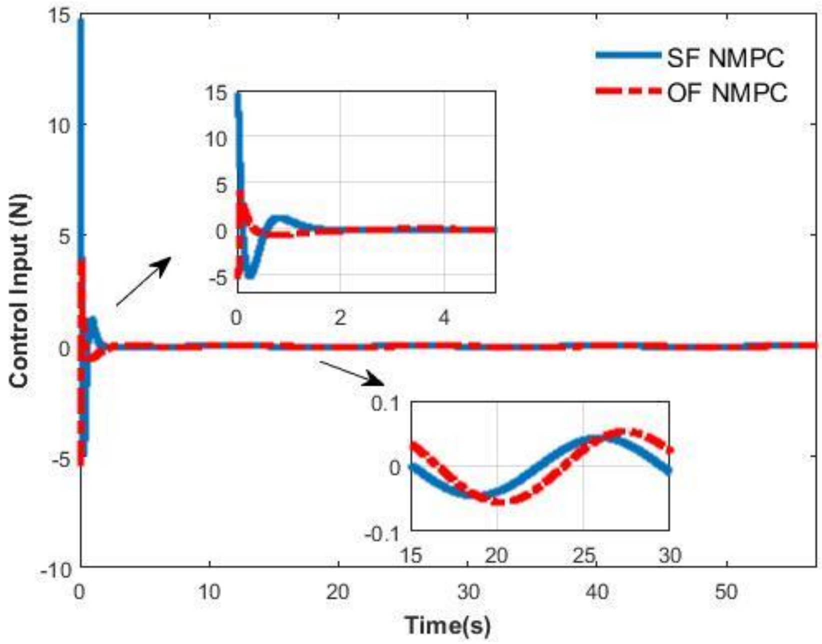

Figure 4-24 Control input of inverted pendulum on linear motion for tracking sinusoid signal

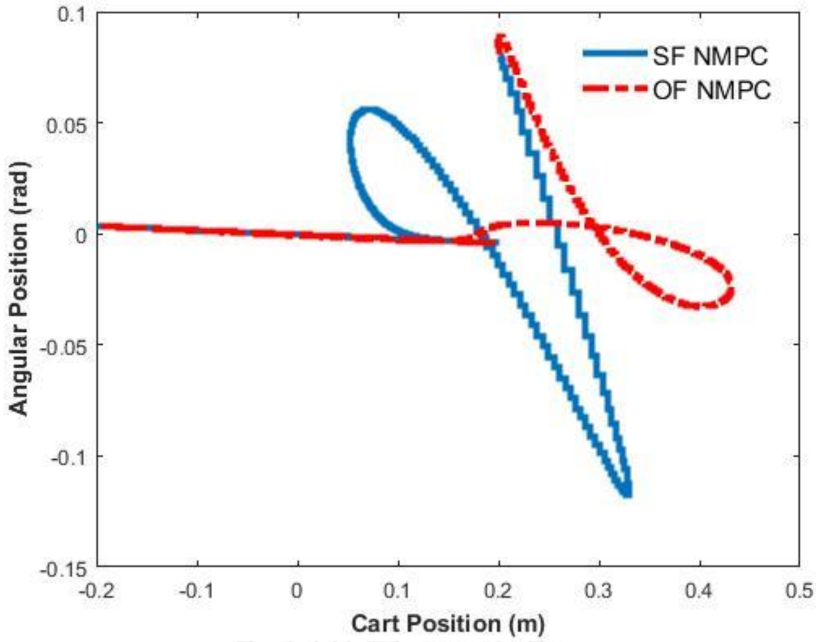

Figure 4-25 Phase plane trajectories of inverted pendulum on linear motion for tracking sinusoid signal

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$, and $t_f = 7.2, T_s = 0.01, x_0 = [0.08 \quad 0 \quad 0.2 \quad 0.02]^T$. Measured outputs are cart and angular position so we want to estimate cart and angular velocity to be output feedback in the next step. On the estimation process, let initial cart and angular velocity be -1 rad/s and let w_k and v_k be expressed in terms of error of output as

$w_k = (\gamma(\tilde{e}_k)^T \tilde{e}_k + \varepsilon) I_{n-p}$, $v_k = \zeta H_k P_k H_k{}^T + \varepsilon I_{n-p}$, $\gamma = 10^{15}, \varepsilon = 10^{-3}, \zeta = 10^{-2}$

The objective is to minimize the performance measure (49) and track sinusoid signal. As the figure 4-22, 4-23 shown, responses of the system can track sinusoid signal. Control input for output feedback NMPC is not similar to NMPC full state feedback because initial estimated states are different from real points.

Output feedback NMPC can stabilize pendulum into vertically upright position while cart moves in horizon axes. Angular position starts at 0.08 rad. Control input forces cart to moves in positive position to stabilize pendulum. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position close to zero, control input forces cart to track sinusoid signal while moving cart keeps pendulum into vertically upright position. Figure 4-24 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 4-25 shows phase plane trajectories between angular position and cart position. Output feedback NMPC has different trajectory from state feedback NMPC because we assume that angular velocity starts at -1 rad/s and cart velocity at -1 rad/s which have the different sign when comparing to the actual state.

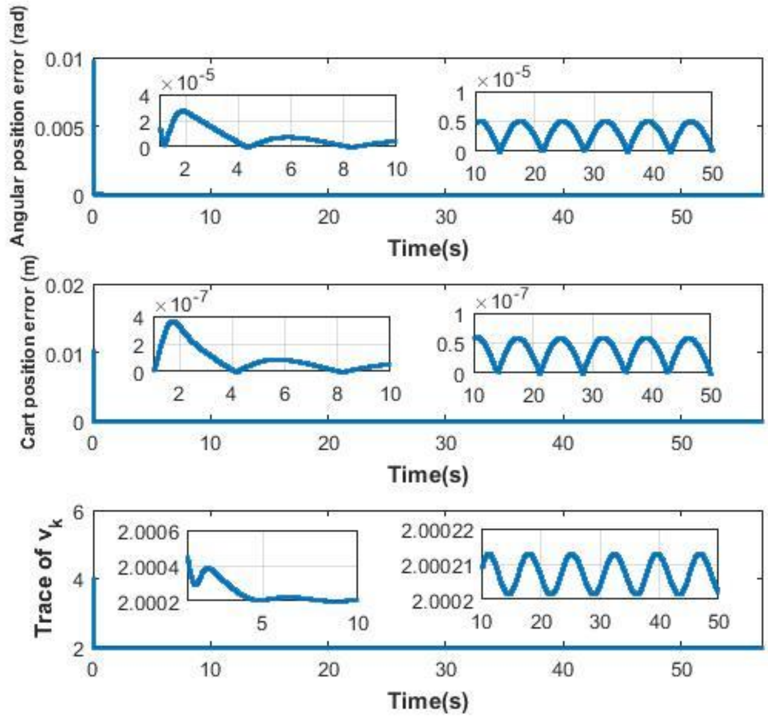

Figure 4-26 Cart and angular position error, trace of matrix $v_k$of inverted pendulum on linear motion cart for tracking sinusoid signal

Figure 4-26 shows error of cart and angular position and trace value of matrix $\mathrm{v_k}$. Error of cart and angular position are very small value when we consider $w_\mathrm{k}$in terms of error of cart and angular position and covariance error of system. It means $w_\mathrm{k}, \mathrm{v_k}$ converge to 2. Although trace of $\mathrm{v_k}$ is close to 2, it has deviation because of sinusoid signal. As soon as cart moves trace of $\mathrm{v_k}$ and error of cart and angular position change with sinusoid function.

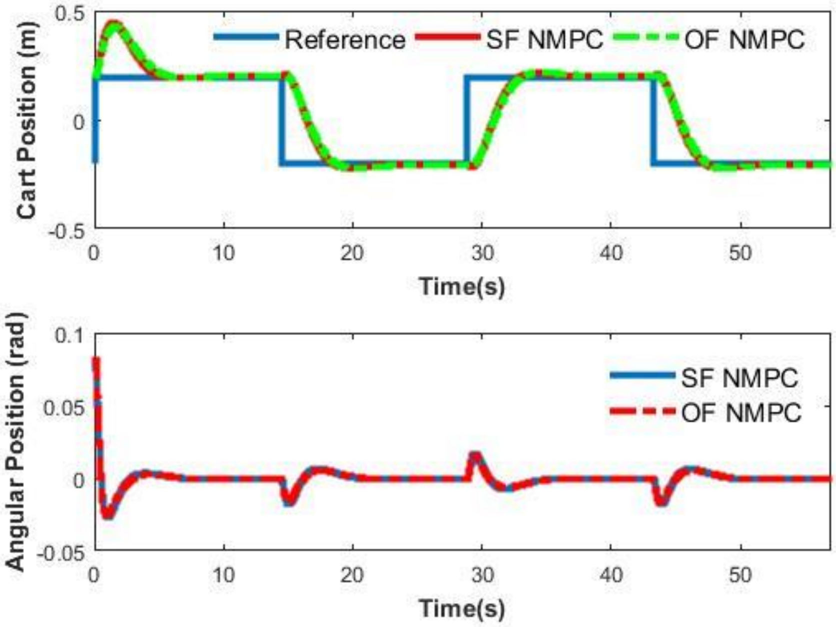

Figure 4-27 Cart and angular position of inverted pendulum on linear motion for tracking square wave signal

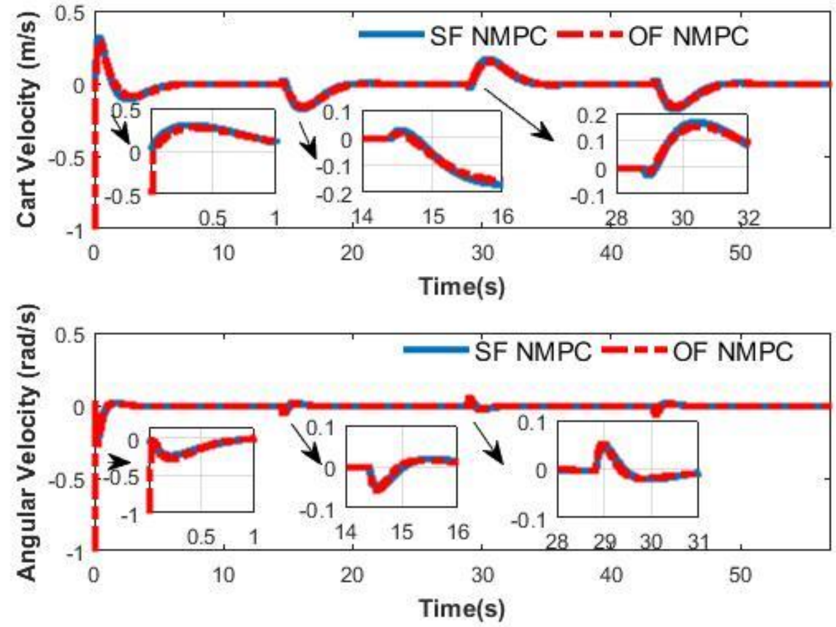

Figure 4-28 Cart and angular velocity of inverted pendulum on linear motion for tracking square wave signal

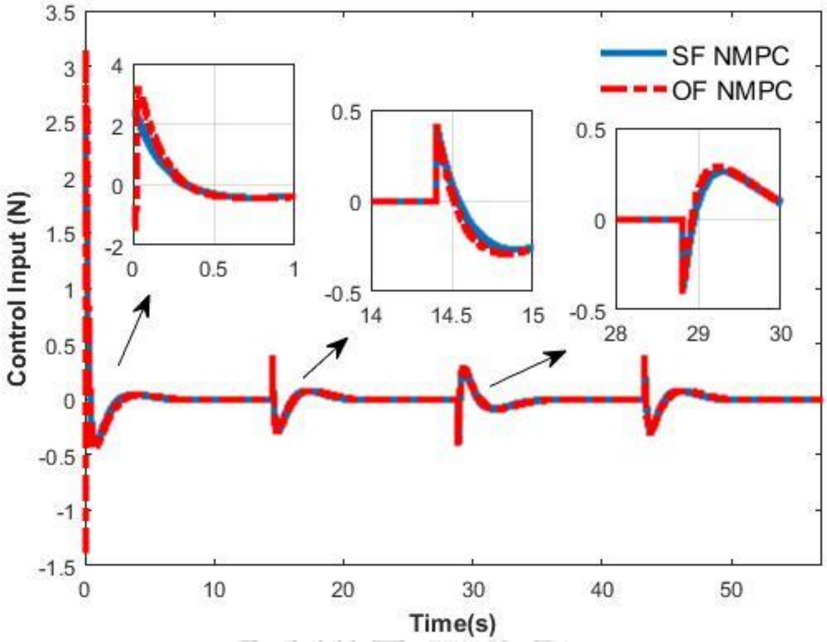

Figure 4-29 Control input of inverted pendulum on linear motion for tracking square wave signal

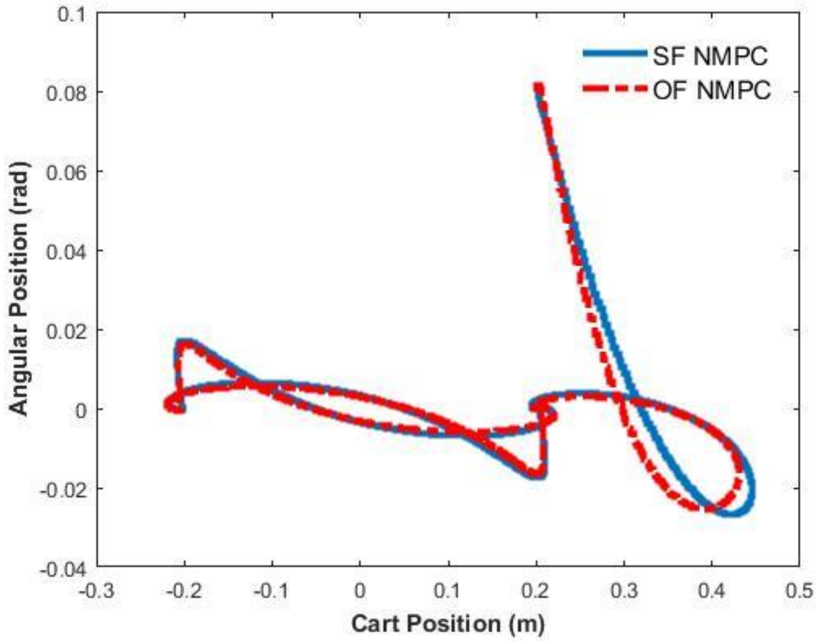

Figure 4-30 Phase plane trajectories of inverted pendulum on linear motion for tracking square wave signal

The objective is to minimize the performance measure (49) and track the reference signal which is square wave. As shown in figure 4-27, 4-28, responses of the system can track square wave signal. Notice that control input for output feedback NMPC is not similar to that of state feedback NMPC because the initial estimated states are different from the real states.

Output feedback NMPC can stabilize pendulum into vertically upright position while cart moves in horizon axes. Angular position starts at 0.08 rad. To stabilizing pendulum, control input forces cart to moves in positive position. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position is close to zero, control input forces cart to track square wave signal while moving cart still keep pendulum into vertically upright position. Figure 4-29 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 4-30 shows phase plane trajectories between angular position and cart position. Output feedback NMPC has the same trajectory from state feedback NMPC because we assume that angular velocity starts at -1 rad/s and cart velocity at -1 rad/s which are have the same sign with the actual state. As soon as observer uses output information to estimate cart and angular velocity, control input of output feedback converges to control input of state feedback.

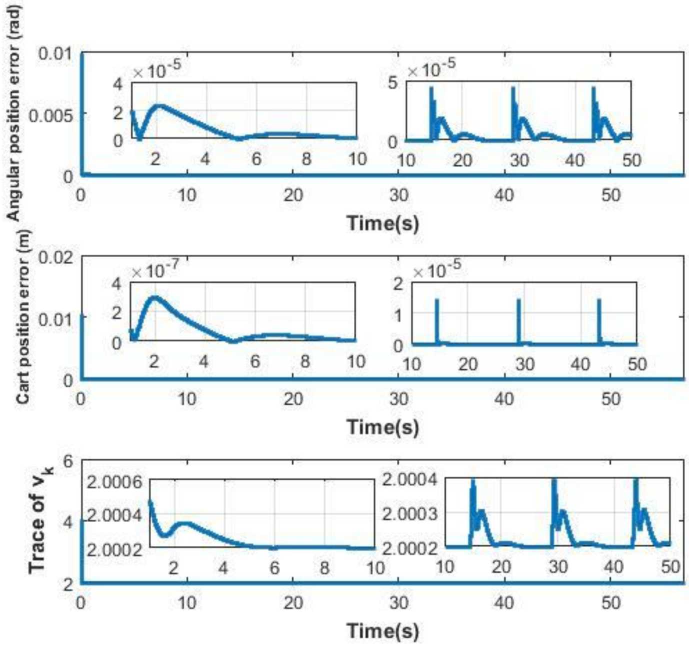

Figure 4-31 Cart and angular position error, trace of matrix v_k of inverted pendulum on linear motion cart for tracking square wave signal

Figure 4-31 shows error of cart and angular position and trace value of matrix v_k. Error of cart and angular position are very small value when we consider w_k in terms of error of cart and angular position and covariance error of system. It means w_k, v_k converge to 2. Although trace of v_k is close to 2, it has deviation because of square wave signal. As soon as cart moves suddenly from 0.2 rad to -0.2 rad or -0.2 rad to 0.2 rad, trace of v_k and error of cart and angular position change.

Comparison state and output feedback NMPC with inverted pendulum on circular motion cart

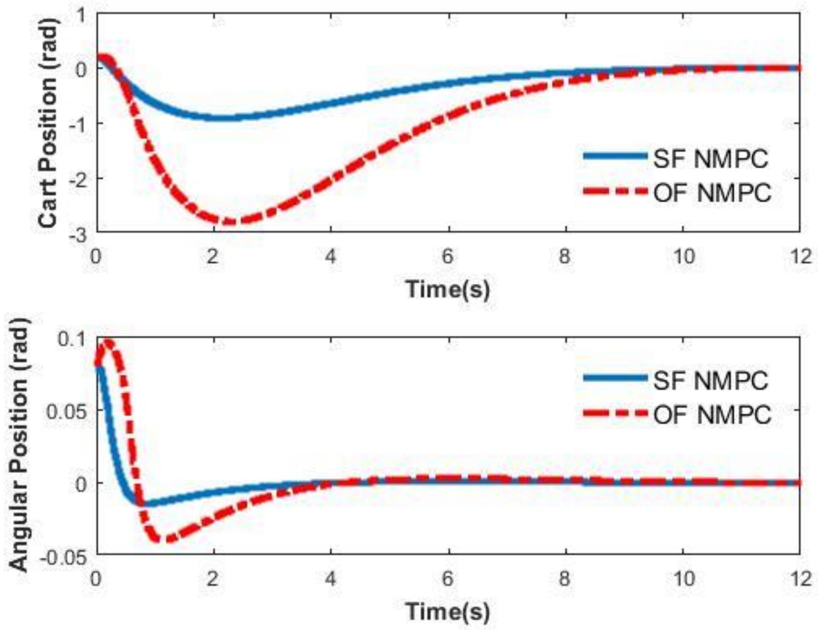

Figure 4-32 Cart and angular position of inverted pendulum on circular motion for regulation problem

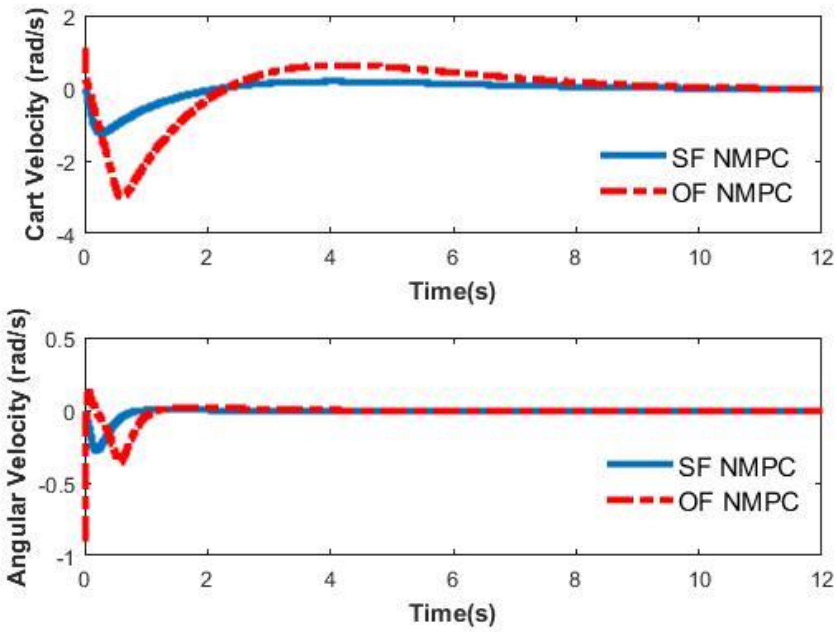

Figure 4-33 Cart and angular velocity of inverted pendulum on circular motion for regulation problem

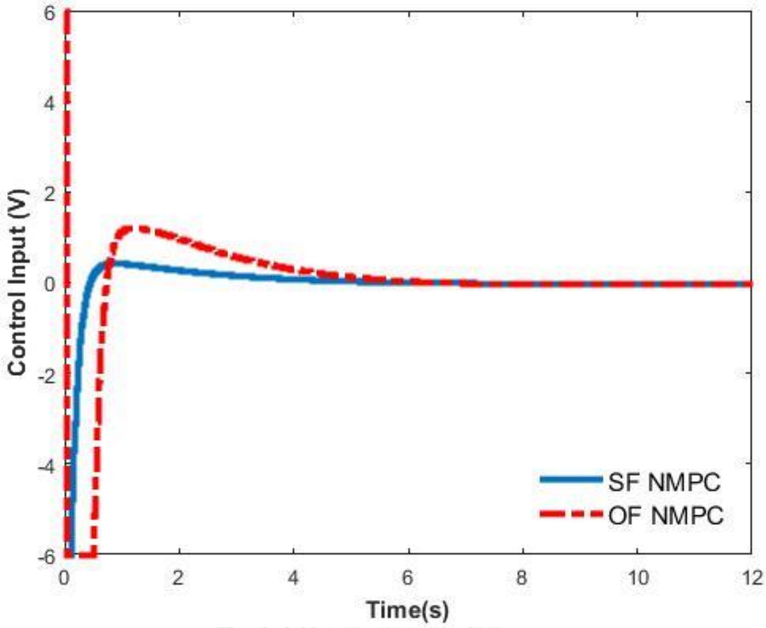

Figure 4-34 Control input of inverted pendulum on circular motion for regulation problem

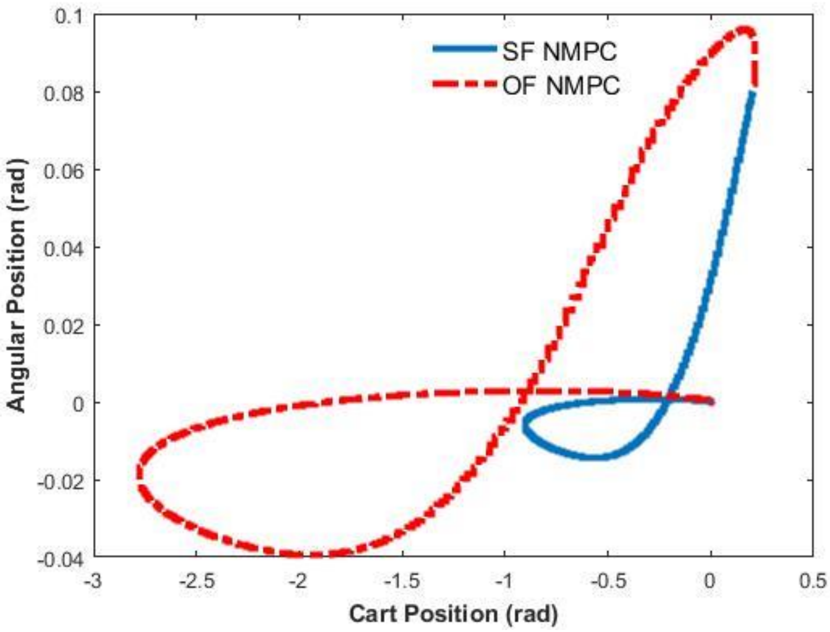

Figure 4-35 Phase plane trajectories of inverted pendulum on circular motion for regulation problem

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$, and $t_f = 7.2, T_s = 0.01, x_0 = [0.2 \quad 0.02 \quad 0.08 \quad 0]^T$. Measured output is cart and angular position so we want to estimate cart and angular velocity to be output feedback in the next step. On the estimation process, let initial cart and angular velocity be -1 rad/s and let w_k and v_k be expressed in terms of error of output as

$w_k = (\gamma(\tilde{e}_k)^T \tilde{e}_k + \varepsilon) I_{n-p}, v_k = \zeta H_k P_k {H_k}^T + \varepsilon I_{n-p}, \gamma = 10^{10}, \varepsilon = 1, \zeta = 10^{-2}$

The objective is to minimize the performance measure (48) and to control the angular position and angular velocity to converge to zero. Control input for output feedback NMPC is not similar to NMPC full state feedback because initial estimated states are different from real points.

Output feedback NMPC can stabilize pendulum into vertically upright position while cart moves in circle track. Angular position starts at 0.08 rad. Control input forces cart to moves in negative position to stabilize pendulum. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position is close to zero, control input forces cart to move in the opposite direction to keep pendulum into vertically upright position. Figure 4-34 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 4-35 shows phase plane trajectories between angular position and cart position. Control input for output feedback NMPC has the same trajectory from state feedback NMPC because we assume that angular velocity starts at -1 rad/s and cart velocity at 1 rad/s which are have the same sign with the actual state.

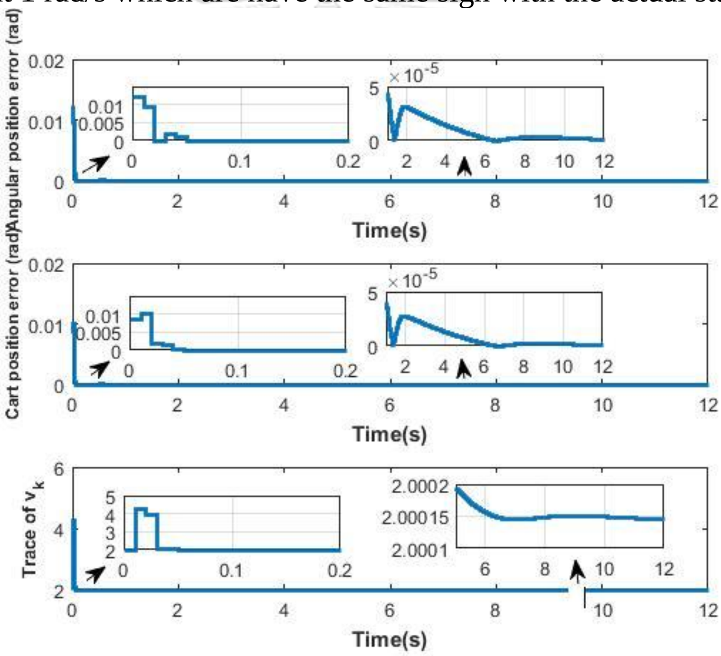

Figure 4-36 Cart and angular position error, trace of matrix $v_k$of inverted pendulum on circular motion cart for regulation problem

Figure 4-36 shows error of cart and angular position and trace value of matrix v_k. Errors of cart and angular position are very small value when we consider $w_k$in terms of error of cart and angular position and covariance error of system. It means w_k, v_k converge to 2.

Comparison state and output feedback NMPC with inverted pendulum on circular motion cart for tracking problem

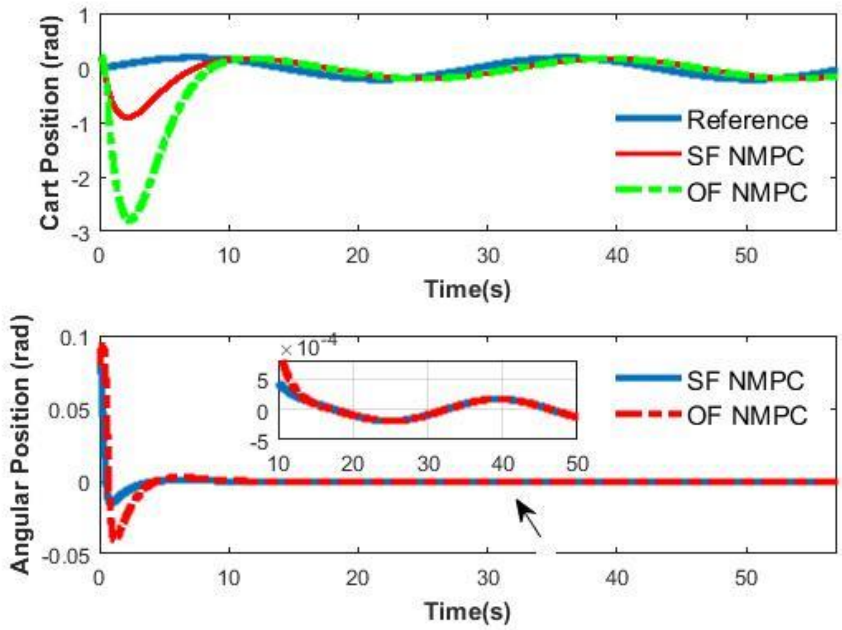

Figure 4-37 Cart and angular position of inverted pendulum on circular motion for tracking sinusoid signal

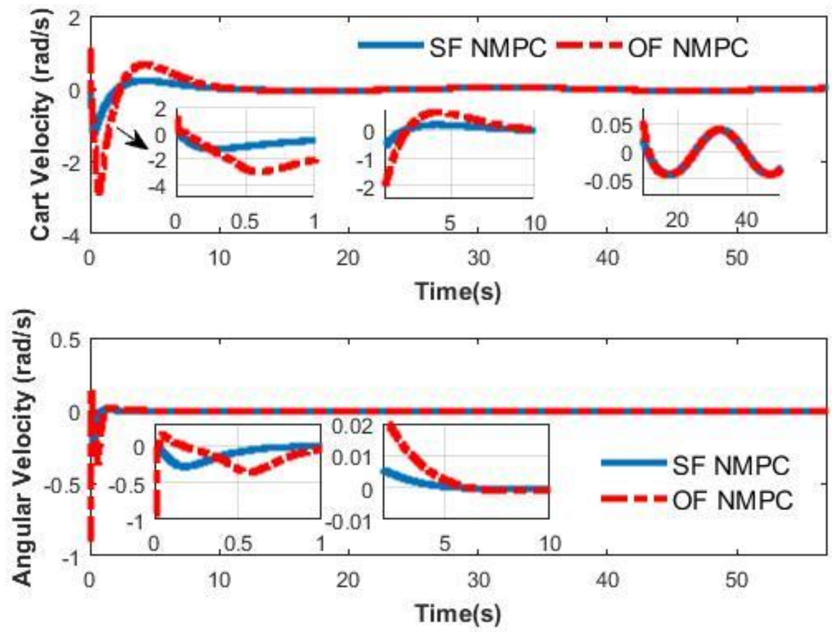

Figure 4-38 Cart and angular velocity of inverted pendulum on circular motion for tracking sinusoid signal

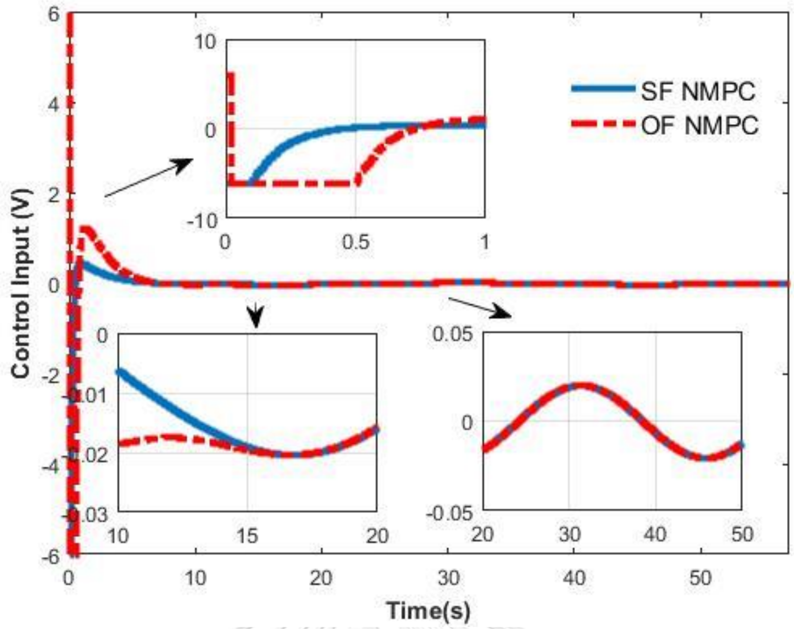

Figure 4-39 Control input of inverted pendulum on circular motion for tracking sinusoid signal

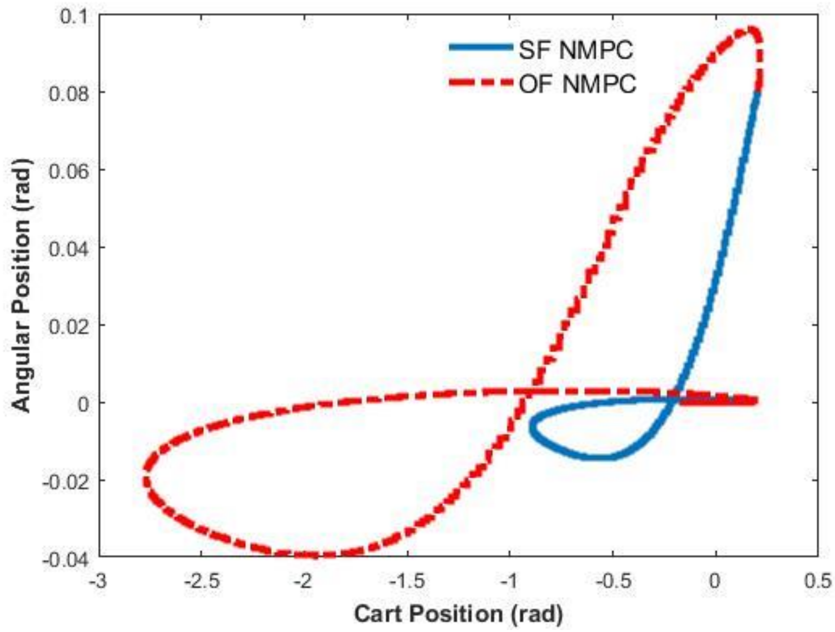

Figure 4-40 Phase plane trajectories of inverted pendulum on circular motion for tracking sinusoid signal

Let design parameters be chosen as $Q = 10 \times I_4$, $R = 100$, and $t_f = 7.2, T_s = 0.01, x_0 = [0.2 \quad 0.02 \quad 0.08 \quad 0]^T$. Measured output are cart and angular position so we want to estimate cart and angular velocity to be output feedback in the next step. On the estimation process, let initial cart and angular velocity be -1 rad/s and let w_k and v_k be expressed in terms of error of output as

$w_k = (\gamma(\tilde{e}_k)^T \tilde{e}_k + \varepsilon) I_{n-p}$, $v_k = \zeta H_k P_k {H_k}^T + \varepsilon I_{n-p}, \gamma = 10^{10}, \varepsilon = 1, \zeta = 10^{-2}$

The objective is to minimize the performance measure (49) and track the reference signal which is sin wave. As the figure 4-36, 4-37 shown, responses of the system can track sinusoid signal. Control input for output feedback NMPC is not similar to NMPC full state feedback because the initial estimated states are different from real states.

Output feedback NMPC can stabilize pendulum into vertically upright position while cart moves in circle track. Angular position starts at 0.08 rad. Control input forces cart to moves in negative position to stabilize pendulum. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position is close to zero, control input forces cart to track sinusoid signal while moving cart still keep pendulum into vertically upright position. Figure 4-38 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 4-39 shows phase plane trajectories between angular position and cart position. Output feedback NMPC has the same trajectory from state feedback NMPC because we assume that angular velocity starts at -1 rad/s and cart velocity at 1 rad/s which are have the same sign with the actual state.

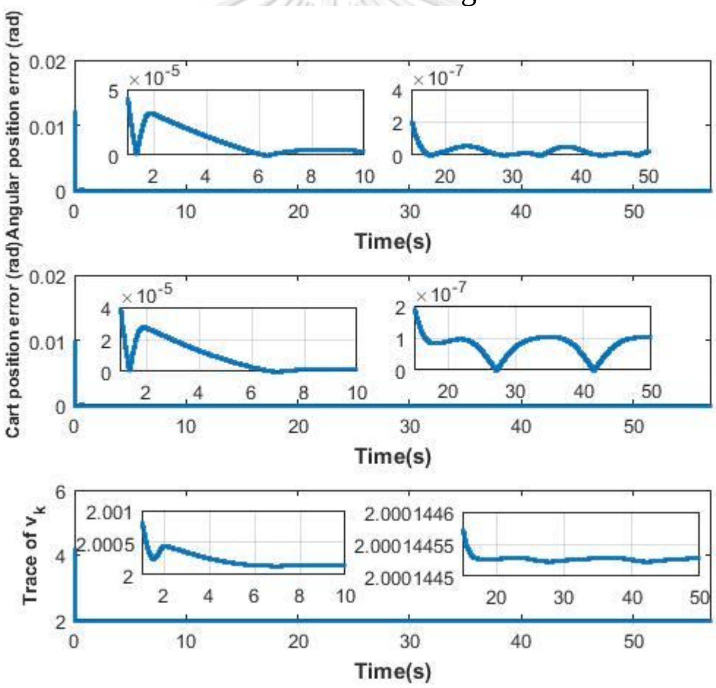

Figure 4-41 Cart and angular position error, trace of matrix $v_k$of inverted pendulum on circular motion cart for tracking sinusoid signal

Figure 4-41 shows error of cart and angular position and trace value of matrix v_k. Error of cart and angular position are very small value when we consider $w_k$in terms of error of cart and angular position and covariance error of system. It means w_k, v_k converge to 2. Although trace of v_k is close to 2, it has deviation because of sinusoid signal. As soon as cart moves trace of v_k and error of cart and angular position change with sinusoid function.

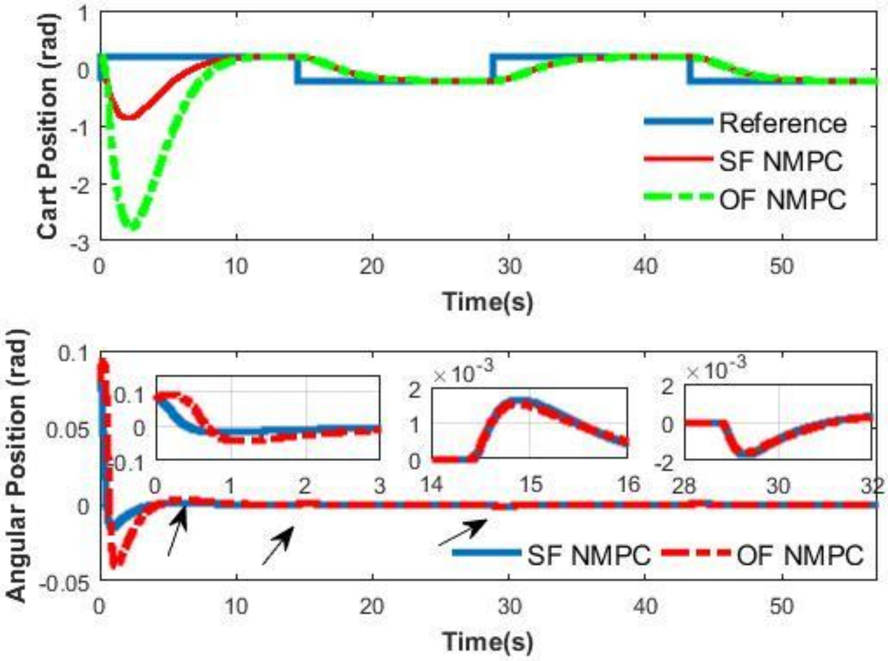

Figure 4-42 Cart and angular position of inverted pendulum on circular motion for tracking square wave signal

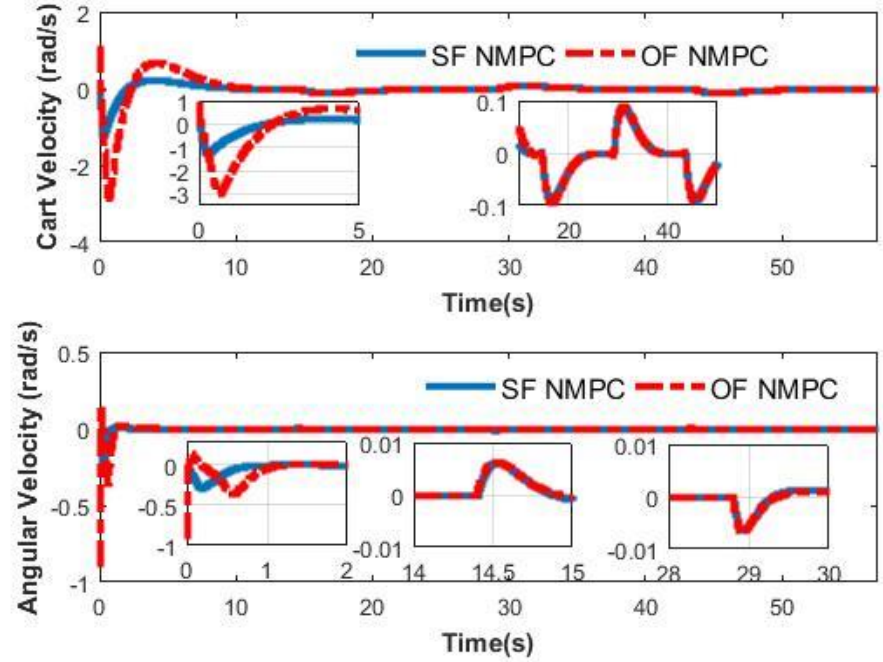

Figure 4-43 Cart and angular velocity of inverted pendulum on circular motion for tracking square wave signal

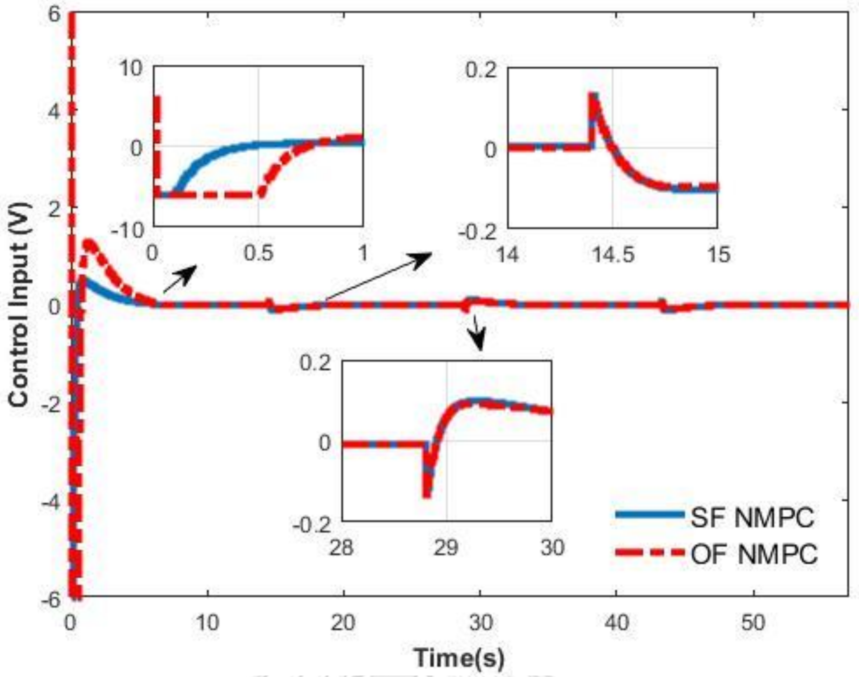

Figure 4-44 Control input of inverted pendulum on circular motion for tracking square wave signal

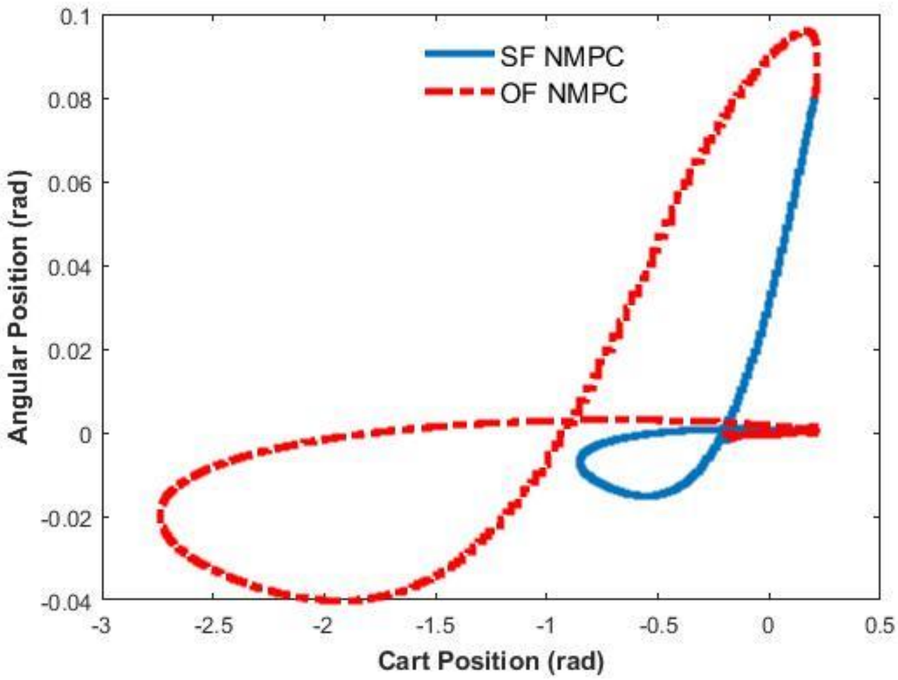

Figure 4-45 Phase plane trajectories of inverted pendulum on circular motion for tracking square wave signal

The objective is to minimize the performance measure (49) and track the reference signal which is square wave. As shown in the figure 4-42, 4-43, responses of the system can track square wave signal. Control input for output feedback NMPC is not similar to NMPC full state feedback because the initial estimated states are different from real states.

Output feedback NMPC can stabilize pendulum into vertically upright position while cart moves in circle track. Angular position starts at 0.08 rad. Control input forces cart to moves in negative position to stabilize pendulum. When cart moves, it has force at the bottom of pendulum. Angular velocity should be negative value because of control input force. Negative value of angular velocity drives pendulum back into origin. As soon as angular position is close to zero, control input forces cart to track square wave signal while moving cart keeps pendulum into vertically upright position. Figure 4-44 shows control input which drives cart move to stabilize pendulum into vertically upright position and figure 4-45 shows phase plane trajectories between angular position and cart position. Output feedback NMPC has the same trajectory from state feedback NMPC because we assume that angular velocity starts at -1 rad/s and cart velocity at 1 rad/s which have the same sign with the actual state.

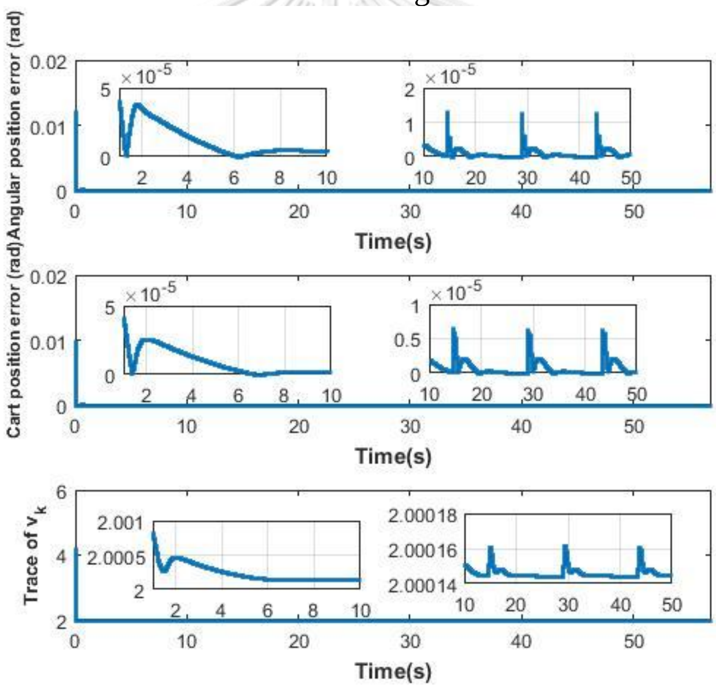

Figure 4-46 Cart and angular position error, trace of matrix v_k of inverted pendulum on circular motion cart for tracking square wave signal

Figure 4-46 shows error of cart and angular position and trace value of matrix v_k. Error of cart and angular position are very small value when we consider $w_k$in terms of error of cart and angular position and covariance error of system. It means w_k, v_k converge to 2. Although trace of v_k is close to 2, it has deviation because of square wave signal. As soon as cart moves suddenly from 0.2 rad to -0.2 rad or -0.2 rad to 0.2 rad, trace of v_k and error of cart and angular position change.

Conclusion

Output feedback NMPC can solve the optimal control problem with estimated output feedback very well. Observer can be applied to the tracking problem with inverted pendulum, inverted pendulum on cart by assuming that we can only measure some output. Initial estimated state has an impact on the control input for NMPC. If initial estimated state is close to the real state, control input for output feedback NMPC is similar to state feedback NMPC. State trajectories of output feedback NMPC are close to that of state feedback NMPC. On the other hand, if initial estimated state is far from the real state, control input for output feedback NMPC has larger magnitude than state feedback NMPC. State trajectories of output feedback NMPC is different from state feedback NMPC as the results of phase plane trajectories. Although state trajectories are different, the objective still be achieved.

Chapter 5

Conclusion and future work

Optimal control problem for nonlinear systems is solving the HJB equation which is difficult to find the exact solution so the TPBV problem which is the necessary condition for the optimal nonlinear control problem is applied to approximate the solution.

This thesis proposes two methods for solving the optimal control problem, that is, the steepest descent method and SDRE. They have different advantage and disadvantages. It depends on how much information you have. If you know trend of optimal control, it is suitable to use the steepest descent, If the system can be expressed in state-dependent linear like model, it is suitable to use SDRE.

NMPC strategy proposes to use the first control input to apply into the systems so we can see NMPC as the optimal control problem for nonlinear systems at every sampling time. All above we have achieved is based on the knowledge of full state of the systems. Unfortunately, we cannot measure all the state with the sensor in real life so this problem becomes a challenge. Using the output feedback is the way to overcome this problem.

Observer becomes useful to estimate the state and give it to be feedback. Many observers were proposed such as high-gain observer. High gain observer uses some parameters to estimate state of system. Peak phenomenon is important thing to be careful. When peak phenomenon occurs, the observed state may be outside the region. In this thesis, we use extended Kalman filter as observer for nonlinear systems to avoid peak phenomenon. Generally, Kalman filter is the optimal estimator for linear systems. Observer was applied by first-order approximation. Observer estimate state in real-time by using the information of output at real-time while NMPC is consider as optimal control problem at every sampling. Observer cannot guarantee the stability of system but state feedback NMPC can. Output feedback NMPC is developed in this thesis. As the numerical results are shown, observer can estimate state very well. The stability of the observer was proved on chapter 4.

Output feedback NMPC is applied to inverted pendulum, inverted pendulum on linear motion cart, inverted pendulum circular motion on cart. It can control all the states converge to zero in regulation problem. For tracking problem, we use observer based NMPC to track cart position to sin wave or square wave signal.

In future work, this thesis can be extended to real experiment with the real inverted pendulum to compare the results between simulation and real experiment. Another chemical system which can be expressed in state dependent linear like model is suitable to use output feedback NMPC.

REFERENCES

[1] D. E. Kirk, *Optimal control theory: an introduction*. Courier Corporation, 2012.

[2] A. E. Bryson, *Applied optimal control: optimization, estimation and control*. Routledge, 2018.

[3] T. Çimen and S. P. Banks, "Global optimal feedback control for general nonlinear systems with nonquadratic performance criteria," *Systems & Control Letters,* vol. 53, no. 5, pp. 327-346, 2004.

[4] S. Beeler, H. Tran, and H. Banks, "Feedback control methodologies for nonlinear systems," *Journal of Optimization Theory and Applications,* vol. 107, no. 1, pp. 1-33, 2000.

[5] R. W. Beard, G. N. Saridis, and J. T. Wen, "Galerkin approximations of the generalized Hamilton-Jacobi-Bellman equation," *Automatica,* vol. 33, no. 12, pp. 2159-2177, 1997.

[6] F. Biral, E. Bertolazzi, and P. Bosetti, "Notes on numerical methods for solving optimal control problems," *IEEJ Journal of Industry Applications,* vol. 5, no. 2, pp. 154-166, 2016.

[7] N. Sakamoto, "Case studies on the application of the stable manifold approach for nonlinear optimal control design," *Automatica,* vol. 49, no. 2, pp. 568-576, 2013.

[8] T. Dierks and S. Jagannathan, "Optimal tracking control of affine nonlinear discrete-time systems with unknown internal dynamics," in *Proc. of the 48th IEEE Conference on Decision and Control, held jointly with the 2009 28th Chinese Control Conference,* 2009, pp. 6750-6755.

[9] D. Gu and H. Hu, "Receding horizon tracking control of wheeled mobile robots," *IEEE Transactions on Control Systems Technology,* vol. 14, no. 4, pp. 743-749, 2006.

[10] H. J. Kim, D. H. Shim, and S. Sastry, "Nonlinear model predictive tracking control for rotorcraft-based unmanned aerial vehicles," in *Proc. of the 2002,American Control Conference,* 2002, vol. 5, pp. 3576-3581.

[11] T. Ohtsuka and H. A. Fujii, "Real-time optimization algorithm for nonlinear receding-horizon control," *Automatica,* vol. 33, no. 6, pp. 1147-1154, 1997.

[12] F. A. Fontes, "A general framework to design stabilizing nonlinear model predictive controllers," *Systems & Control Letters,* vol. 42, no. 2, pp. 127-143, 2001.

[13] V. Suphatsatienkul, D. Banjerdpongchai, and M. Wongsaisuwan, "Design of reduced-order observer and linear quadratic regulator for inverted pendulum on cart," in Proc. of *2017 56th Annual Conference of the Society of Instrument and Control Engineers of Japan,* 2017, pp. 1039-1044.

[14] R. Findeisen, L. Imsland, F. Allgower, and B. A. Foss, "State and output feedback nonlinear model predictive control: An overview," *European Journal of Control,* vol. 9, no. 2-3, pp. 190-206, 2003.

[15] R. Findeisen, L. Imsland, F. Allgöwer, and B. Foss, "Output-feedback nonlinear model predictive control using high-gain observers in original coordinates," in *Proc. of European Control Conference,* 2003, pp. 2014-2019.

[16] M. Boutayeb and M. Darouach, "A reduced-order observer for non-linear discrete-time systems," *Systems & Control Letters*, vol. 39, no. 2, pp. 141-151, 2000.

[17] F. Allgöwer and A. Zheng, *Nonlinear model predictive control*. Birkhäuser, 2012.

[18] H. Chen and F. Allgower, "A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability," in *Proc. of 1997 European Control Conference*, 1997, pp. 1421-1426.

[19] Y. Song and J. W. Grizzle, "The extended Kalman filter as a local asymptotic observer for nonlinear discrete-time systems," in *Proc. of 1992 American Control Conference*, 1992, pp. 3365-3369.

[20] K. Belarbi, H. Boumaza, and B. Boutamina, "Nonlinear model predictive control based on state dependent Riccati equation," *Sciences and Techniques of Automatic Control and Computer Engineering (STA), 2014 15th International Conference on*, 2014, pp. 65-69: IEEE.

[21] R. R. Bitmead, M. Gevers, and V. Wertz, *Adaptive optimal control: The thinking man's GPC*. Prentice Hall New York, 1990.

[22] M. Boutayeb and D. Aubry, "A strong tracking extended Kalman observer for nonlinear discrete-time systems," *IEEE Transactions on Automatic Control*, vol. 44, no. 8, pp. 1550-1556, 1999.

[23] M. Boutayeb and M. Darouach, "Observers design for nonlinear descriptor systems," in *Proc. of the 34th IEEE Conference on Decision and Control*, 1995, vol. 3, pp. 2369-2374.

[24] M. Boutayeb, H. Rafaralahy, and M. Darouach, "Convergence analysis of the extended Kalman filter as an observer for nonlinear discrete-time systems," in *Proc. of the 34th IEEE Conference on Decision and Control*, 1995, vol. 2, pp. 1555-1560.

[25] M. Boutayeb, H. Rafaralahy, and M. Darouach, "Convergence analysis of the extended Kalman filter used as an observer for nonlinear deterministic discrete-time systems," *IEEE Trans. on Automatic Control*, vol. 42, no. 4, pp. 581-586, 1997.

[26] I. Chang and J. Bentsman, "Constrained discrete-time state-dependent Riccati equation technique: A model predictive control approach," in *Proc. of IEEE 52nd Annual Conference on Decision and Control*, 2013, pp. 5125-5130.

[27] W.-H. Chen, D. J. Ballance, and P. J. Gawthrop, "Optimal control of nonlinear systems: a predictive control approach," *Automatica*, vol. 39, no. 4, pp. 633-641, 2003.

[28] A. S. Dutka, A. W. Ordys, and M. J. Grimble, "Optimized discrete-time state dependent Riccati equation regulator," in *Proc. of 2005 American Control Conference, Proceedings of the 2005*, 2005, pp. 2293-2298.

[29] R. M. Freund, "The steepest descent algorithm for unconstrained optimization and a bisection line-search method," *Lecture Notes, MIT Open Courseware, Massachusetts Institute of Technology. USA*, 2004.

[30] S. Gros, M. Zanon, R. Quirynen, A. Bemporad, and M. Diehl, "From linear to nonlinear MPC: bridging the gap via the real-time iteration," *International Journal of Control*, pp. 1-19, 2016.

[31] A. Heydari and S. N. Balakrishnan, "Approximate closed-form solutions to finite-horizon optimal control of nonlinear systems," in *Proc. of 2012 American Control Conference*, 2012, pp. 2657-2662.

[32] H. K. Khalil, "High-gain observers in nonlinear feedback control," in *Proc. of 2008. International Conference on Control, Automation and Systems*, 2008, pp. 47-52.

[33] A. Khamis and D. Naidu, "Nonlinear optimal tracking using finite horizon state dependent Riccati equation (SDRE)," in *Proc. of the 4th International Conference on Circuits, Systems, Control, Signals*, 2013, pp. 37-42.

[34] A. Khamis and D. S. Naidu, "Nonlinear optimal tracking with incomplete state information using finite-horizon State Dependent Riccati Equation (SDRE)," in *Proc. 2014 American Control Conference*, 2014, pp. 2420-2425.

[35] J. J. Lima, M. M. Santos, A. M. Tusset, D. G. Bassinello, F. C. Janzen, and J. M. Balthazar, "Nonlinear control strategy based on discrete-time state dependent Riccati equation for electronic throttle control," in *2016 12th IEEE International Conference on Industry Applications*, 2016, pp. 1-7.

[36] N. Singh, J. Dubey, and G. Laddha, "Control of pendulum on a cart with state dependent Riccati equations," *Int. Journal of Computer, Information, and Systems Science, and Engineering*, pp. 92-96, 2009.

[37] D. Slaughter, "Difference Equations to Differential Equations," ed: Gainesville, FL, USA: Univ. Press Florida, 2009.

[38] E. Todorov and Y. Tassa, "Iterative local dynamic programming," in *Proc. of IEEE Symposium on Adaptive Dynamic Programming and Reinforcement Learning*, 2009, pp. 90-95.

VITA

Biography

Petchakrit Pinyopawasutthi was born on September 25, 1993 in Samutsakhon, Thailand. He graduated from Department of Electrical Engineering, Faculty of Engineering Chulalongkorn University in the academic year 2015. He was awarded the honors scholarship to study in Master's degree from Department of Electrical Engineering. Faculty of Engineering Chulalongkorn University. He is interested in advanced control and optimization and worked at Control Systems Research Laboratory, Department of Electrical Engineering, Faculty of Engineering Chulalongkorn University.

List of publication

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