CHAPTER V

LINEAR OPERATORS AND MEASURE THEORY

This chapter extends the results in [9] which were proven for positive measures or complex measures to quaternion measures.

5.1 <u>Definition</u> Let V be a left vector space over $\mathbb{H}$. Then a map $\cdot : V \times V \longrightarrow \mathbb{H}$ is said to be a <u>left sympletic product</u> (LSP) on V if and only if

(i) $x.y = \overline{y.x}$ for all $x,y \in V$.

(ii) $(x+y).z = x.z+y.z$ for all $x,y,z \in V$.

(iii) $(\alpha x).y = \alpha(x.y)$ for all $x,y \in V$ for all $\alpha \in \mathbb{H}$.

(iv) $\forall x \in V$, $x.x \geqslant 0$ and $x.x = 0$ iff $x = 0$.

The consequences of these axioms are:

(a) $0.x = 0 = x.0$ for all $x \in V$.

(b) $\forall y \in V$, the map $x \mapsto x.y$ is a left linear function on V

(c) $x.(\alpha y) = (x.y)\bar{\alpha}$ for all $x,y \in V$, $\alpha \in \mathbb{H}$.

(d) $x.(y+z) = x.y+x.z$ for all $x,y,z \in V$.

If $\cdot$ is a left sympletic product space on V, then the pair $(V,\cdot)$ is called a left sympletic product space (LSPS).

5.2 <u>Definition</u> Let V be a right vector space over $\mathbb{H}$. Then a map $\cdot : V \times V \longrightarrow \mathbb{H}$ is said to be a <u>right sympletic product</u> (RSP) on V if and only if

(i) $x.y = \overline{y.x}$ for all $x,y \in V$.

(ii) $x.(y+z) = x.y+x.z$ for all $x,y,z \in V$.

(iii) $x.(y\alpha) = (x.y)\alpha$ for all $x,y \in V$ and for all $\alpha \in \mathbb{H}$.

(iv) For each $x \in V$, $x.x \geq 0$ and $x.x = 0$ if and only if $x = 0$.

The consequences of these axioms are:

(a) $0.x = 0 = x.0$ for all $x \in V$.

(b) For each $y \in V$, the map $x \mapsto y.x$ is a right linear function on V.

(c) $(x\alpha).y = \bar{\alpha}(x.y)$ for all $x,y \in V$, $\alpha \in \mathbb{H}$.

(d) $(x+y).z = x.z+y.z$ for all $x,y,z \in V$.

If $\cdot$ is a right sympletic product space on V, then the pair $(V,\cdot)$ is called a right sympletic product space(RSPS).

5.3 Definition Let V be a LSPS(RSPS) and let $x \in V$, define $\|x\|$, the norm of x to be $\sqrt{x.x}$.

5.4 Theorem Let V be a LSPS(RSPS). Then $|x.y| \leq \|x\| \|y\|$ for all $x,y \in V$.

Proof[11] Let $\alpha = \begin{cases} 1 & \text{if } y.x = 0, \\ \dfrac{|x.y|}{y.x} & \text{if } y.x \neq 0. \end{cases}$

Then $|\alpha| = 1$ and $\alpha(y.x) = |x.y| \geq 0$, hence $\overline{\alpha(y.x)} = \alpha(y.x) = |x.y|$. For $r \in \mathbb{R}$, we have

$$(1) \quad 0 \leq (x-r\alpha y).(x-r\alpha y) = x.x-x.(r\alpha y)-(r\alpha y).x+r^2(\alpha y).(\alpha y)$$
$$= x.x-r(x.y)\bar{\alpha}-r\alpha(y.x)+r^2|\alpha|^2(y.y)$$
$$= x.x-r(\overline{y.x})\bar{\alpha}-r\alpha(y.x)+r^2(y.y)$$
$$= x.x-r\overline{\alpha(y.x)}-r\alpha(y.x)+r^2(y.y)$$
$$= \|x\|^2-2r|x.y|+r^2\|y\|^2 .$$

<u>Case $\|y\|^2 = 0$</u> Then $y = 0$ and so $(x.y) = 0$. Hence $|x.y| \leq \|x\| \, \|y\|$.

<u>Case $\|y\|^2 \neq 0$</u> Let $r = \dfrac{|(x.y)|}{\|y\|^2}$. From (1), we have

$$0 \leqslant \|x\|^2 - \frac{|x.y|^2}{\|y\|^2}.$$

Hence $\dfrac{|x.y|^2}{\|y\|^2} \leq \|x\|^2$, so $|x.y|^2 \leq \|x\|^2 \|y\|^2$. Hence

$$|x.y| \leqslant \|x\| \, \|y\| \quad . \#$$

5.5 <u>The Triangle inequality</u> Let V be a LSPS(RSPS). Then

$$\|x + y\| \leq \|x\| + \|y\|$$

for all $x, y \in V$.

<u>Proof</u> Follows from Theorem 5.4 . #

<u>Remark</u>: Let V be a LSPS(RSPS). $\| \ \|$ is a map $\| \ \| : V \longrightarrow \mathbb{R}$ such that

(1) $\|x\| \geqslant 0$ for all $x \in V$ and $\|x\| = 0 \longleftrightarrow x = 0$.

(2) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in V$ for all $\alpha \in \mathbb{H}$.

(3) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Let V be a LSPS(RSPS). Define $d : V \times V \longrightarrow \mathbb{R}$ by

$$d(x,y) = \|x-y\| .$$

Then d is a metric on V, hence V is a topological space.

5.6 <u>Definition</u> Let V be a LSPS(RSPS). Then V is called a <u>left(right) Hilbert space</u> if and only if V is a complete metric space.

Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. $L^2(\mu) = \{f: X \to \mathbb{H} \,/\, f$ is measurable and $\|f\|_2 < \infty\}$. Define $\cdot : L^2(\mu) \times L^2(\mu) \longrightarrow \mathbb{H}$ by

$$f\cdot g = \int_X f\bar{g}\,d\mu$$

for all $f, g \in L^2(\mu)$. The integrand on the right is in $L^1(\mu)$, by Theorem 4.85, so that $f \cdot g$ is well-defined. Claim that $L^2(\mu)$ is a LSPS. If $f \in L^1(\mu)$ and $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions f_i, $i \leq 4$, then $\int_X \bar{f}\,d\mu =$

$$\int_X (f_1 - if_2 - jf_3 - kf_4)\,d\mu = \int_X f_1\,d\mu - i\int_X f_2\,d\mu - j\int_X f_3\,d\mu - k\int_X f_4\,d\mu$$

$$= \overline{\int_X f_1\,d\mu + i\int_X f_2\,d\mu + j\int_X f_3\,d\mu + k\int_X f_4\,d\mu} = \overline{\int_X f\,d\mu}\ .$$

(i) $f \cdot g = \int_X f\bar{g}\,d\mu = \int_X \overline{g \cdot \bar{f}}\,d\mu = \overline{\int_X g \cdot \bar{f}\,d\mu} = \overline{g \cdot f}$ for all $f, g \in L^2(\mu)$.

(ii) $(f+g)\cdot h = \int_X (f+g)\bar{h}\,d\mu = \int_X (f\bar{h} + g\bar{h})\,d\mu = \int_X f\bar{h}\,d\mu + \int_X g\bar{h}\,d\mu = f\cdot h + g\cdot h$ for all $f, g, h \in L^2(\mu)$.

(iii) $(\alpha f)\cdot g = \int_X (\alpha f)\bar{g}\,d\mu = \int_X \alpha(f\bar{g})\,d\mu = \alpha\int_X f\bar{g}\,d\mu = \alpha(f\cdot g)$ for all $f, g \in L^2(\mu)$ and for all $\alpha \in \mathbb{H}$.

(iv) $f\cdot f = \int_X f\bar{f}\,d\mu = \int_X |f|^2\,d\mu \geq 0$ for all $f \in L^2(\mu)$. For $f \in L^2(\mu)$. If $f \equiv 0$, then $f\cdot f = \int_X |f|^2\,d\mu = 0$. If $f\cdot f = 0$, then $\int_X |f|^2\,d\mu = 0$, so $|f|^2 = 0$ a.e. which implies that $f = 0$ a.e. Hence we have the claim. Note that

$$\|f\| = (f\cdot f)^{\frac{1}{2}} = \left\{\int_X |f|^2\,d\mu\right\}^{\frac{1}{2}} = \|f\|_2\ .$$

Since $(L^2(\mu), \| \|_2)$ is complete by Theorem 4.90, hence $(L^2(\mu), \| \|)$ is complete. Thus $L^2(\mu)$ is a left Hilbert space

<u>Remark</u>: If we define $\cdot : L^2(\mu) \times L^2(\mu) \longrightarrow \mathbb{H}$ by

$$f.g = \int_X \bar{f} g \, d\mu$$

for all $f, g \in L^2(\mu)$. Then we have $L^2(\mu)$ is a RSPS, so that $L^2(\mu)$ is a right Hilbert space.

5.7 <u>Theorem</u>(Riesz Representation Theorem for Hilbert Space) Let V be a LSPS(RSPS) which is also a left(right) Hilbert space and let $L : V \longrightarrow \mathbb{H}$ be a continuous left(right) linear function. Then there exists a unique $y \in V$ such that $L(x) = x.y$ ($L(x) = y.x$) for all $x \in V$.

<u>Proof</u> See [11] . #

5.8 <u>Definition</u> Let V be a left vector space over $\mathbb{H}$. A map $\| \| : V \longrightarrow \mathbb{R}$ is said to be <u>left</u> <u>norm</u> on V if and only if

(i) $\|x\| \geqslant 0$ for all $x \in V$ and $\|x\| = 0 \Longleftrightarrow x = 0$.

(ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in V$ and for all $\alpha \in \mathbb{H}$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

If $\| \|$ is a left norm on V, then the pair $(V, \| \|)$ is called a <u>left</u> <u>normed</u> <u>linear</u> <u>space</u>.

5.9 <u>Definition</u> Let V be a right vector space over $\mathbb{H}$. A map $\| \| : V \longrightarrow \mathbb{R}$ is said to be <u>right</u> <u>norm</u> on V if and only if

(i) $\|x\| \geqslant 0$ for all $x \in V$ and $\|x\| = 0 \Longleftrightarrow x = 0$.

(ii) $\|x\alpha\| = \|x\| |\alpha|$ for all $x \in V$ and for all $\alpha \in \mathbb{H}$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

If $\| \ \|$ is a right norm on V, then the pair $(V, \| \ \|)$ is called a right normed linear space.

5.10 Definition Let V be a vector space over $\mathbb{H}$. Then $(V, \| \ \|)$ is called a normed linear space if $\| \ \|$ is both a left norm and right norm.

Example Let X be a locally compact Hausdorff space. Then $C_0(X)$ with the supremum norm is a left(right) normed linear space. Also, $C_0(X)$ is a normed linear space with respect to the supremum norm.

5.11 Theorem Let V, W be left(right) normed linear spaces and $F: V \longrightarrow W$ is a left(right) linear map. If F is continuous at one point, then F is continuous everywhere.

Proof See [11]. #

5.12 Definition Let V, W be left(right) normed linear spaces and $F: V \longrightarrow W$ a left(right) linear map. Define the norm of F by

$$\|F\| = \sup_{x \neq 0} \left\{ \frac{\|F(x)\|}{\|x\|} \right\}.$$

Observe that $\|F(x)\| \leq \|F\| \|x\|$ for all $x \in X$. If $\|F\| < \infty$, then F is said to be bounded left(right) linear map.

5.13 Theorem Let V, W be left(right) normed linear spaces and $F: V \longrightarrow W$ a left(right) linear map. Then F is continuous if and only if F is bounded.

Proof See [11] . #

5.14 Theorem Let V,W be left(right) normed linear spaces and $F:V\longrightarrow W$ a left(right) linear map. Then

$$\|F\| = \sup_{x\neq 0}\left\{\frac{\|F(x)\|}{\|x\|}\right\} = \sup_{\|x\|=1}\{\|F(x)\|\} = \sup_{\|x\|\leq 1}\{\|F(x)\|\} .$$

Proof See [11] . #

5.15 Theorem(Hahn-Banach) Let V be a left(right) normed linear space and W is a left(right) linear subspace of V. Let F: $W \longrightarrow \mathbb{H}$ be a bounded left(right) linear functional on W. Then there exists a bounded left(right) linear functional F on V such that $F|_W = f$ and $\|f\| = \|F\|$.

Proof See [11] . #

5.16 Theorem(Lebesgue-Radon-Nikodym Theorem for a Quaternion Measure) Let μ be a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X. Suppose λ is a quaternion measure on $\mathfrak{M}$. Then

(a) There is a unique pair of quaternion measures λ_a, λ_s on $\mathfrak{M}$ such that

$$\lambda = \lambda_a + \lambda_s\ , \quad \lambda_a \ll \mu\ , \quad \lambda_s \perp \mu .$$

(b) There is a unique $h \in L^1(\mu)$ such that

$$\lambda_a(E) = \int_E h\,d\mu \qquad (E \in \mathfrak{M}) .$$

Remark: (1) The pair λ_a and λ_s is called the Lebesgue decomposition of λ relative to μ.

(2) Assertion (b) is known as the Radon-Nikodym Theorem.

Proof Uniqueness of λ_a and λ_s Let λ'_a and λ'_s

be quaternion measures such that

$$\lambda = \lambda'_a + \lambda'_s , \quad \lambda'_a \ll \mu , \quad \lambda'_s \perp \mu .$$

Then $\lambda'_a - \lambda_a = \lambda_s - \lambda'_s$. Since $-\lambda_a \ll \mu$ and $\lambda'_a \ll \mu$, $\lambda'_a - \lambda_a \ll \mu$ by Theorem 2.51 (d). Since $\lambda_s \perp \mu$ and $-\lambda'_s \perp \mu$, $\lambda_s - \lambda'_s \perp \mu$ by Theorem 2.51 (c). Hence $\lambda'_a - \lambda_a = 0 = \lambda_s - \lambda'_s$ by Theorem 2.51 (g). Then $\lambda'_a = \lambda_a$ and $\lambda_s = \lambda'_s$.

Uniqueness of h Suppose there exists $h_1 \in L^1(\mu)$ such that

$$\lambda_a(E) = \int_E h_1 d\mu \qquad (E \in \mathfrak{M}) .$$

Then $h - h_1 \in L^1(\mu)$ and $\int_E (h-h_1) d\mu = 0$ for all $E \in \mathfrak{M}$. Hence $h - h_1 = 0$ a.e. $[\mu]$ on X by Theorem 4.74 (a), so $h_1 = h$ a.e. $[\mu]$ on X.

Step I Assume μ and λ are finite positive measures on $\mathfrak{M}$. Put $\varphi = \lambda + \mu$. Then φ is a finite positive measure on $\mathfrak{M}$. Then for all $E \in \mathfrak{M}$,

$$\int_X \chi_E d\lambda + \int_X \chi_E d\mu = \lambda(E) + \mu(E) = \varphi(E) = \int_X \chi_E d\varphi .$$

Hence $\int_X s d\lambda + \int_X s d\mu = \int_X s d\varphi$ for all simple measurable functions s. Let f be a non negative measurable function. Then there exists a sequence of simple measurable functions $(s_n)_{n \in \mathbb{N}}$ on X such that

$$0 \le s_1 \le s_2 \le \dots \text{ and } \lim_{n \to \infty} s_n(x) = f(x) \text{ for all } x \in X.$$

By Lebesgue's Momotone Convergence Theorem,

$$\lim_{n\to\infty} \int_X s_n d\varphi = \int_X f d\varphi , \quad \lim_{n\to\infty} \int_X s_n d\lambda = \int_X f d\lambda \quad \text{and}$$

$\lim_{n\to\infty}\int_X s_n d\mu = \int_X f d\mu$. Hence

$$\int_X f d\varphi = \lim_{n\to\infty}\int_X s_n d\varphi = \lim_{n\to\infty}\left(\int_X s_n d\lambda + \int_X s_n d\mu\right)$$

$$= \lim_{n\to\infty}\int_X s_n d\lambda + \lim_{n\to\infty}\int_X s_n d\mu$$

$$= \int_X f d\lambda + \int_X f d\mu .$$

It follows that $\int_X f d\varphi = \int_X f d\lambda + \int_X f d\mu$ for all $f \in L^1(\varphi)$.

If $f \in L^2(\varphi)$, then $f \in L^1(\mu)$ (since $1 \in L^2(\varphi)$, $f.1 \in L^1(\varphi)$) so $f \in L^1(\lambda)$. If $f \in L^2(\varphi)$, then $\left|\int_X f d\lambda\right| \leq \int_X |f| d\lambda \leq \int_X |f| d\varphi$ (since $\varphi = \lambda + \mu$) $\leq \left\{\int_X |f|^2 d\varphi\right\}^{\frac{1}{2}}\left\{\int_X 1^2 d\varphi\right\}^{\frac{1}{2}}$ (by Hölder's inequality) $= \|f\|_{2,\varphi}(\varphi(X))^{\frac{1}{2}} < \infty$, so $f \longmapsto \int_X f d\lambda$ is a bounded left(right) linear functional on $L^2(\varphi)$. By Theorem 5.13. $f \longmapsto \int_X f d\lambda$ is continuous on $L^2(\varphi)$

By Riesz representation theorem for Hilbert space, there exists a unique $g \in L^2(\varphi)$ such that

(1) $$\int_X f d\lambda = \int_X f g d\varphi$$

for all $f \in L^2(\varphi)$.

If $\varphi(E) = 0$ for all $E \in \mathcal{M}$ then let $\lambda_a = \lambda_s = 0$ and $h \equiv 0$ and we have the theorem.

For $E \in \mathcal{M}$ such that $\varphi(E) > 0$, we have

$$\varphi(E) \geqslant \lambda(E) = \int_X \chi_E d\lambda = \int_X \chi_E g d\varphi = \int_E g d\varphi \geqslant 0 ,$$

so $0 \leq \frac{1}{\varphi(E)}\int_E g d\varphi \leq 1$. By Theorem 4.75, we have $g(x) \in [0,1]$

a.e. $[\varphi]$ on X. We may therefore assume that $0 \leq g(x) \leq 1$ for all $x \in X$, without affecting (1). From (1), we have

(2) $$\int_X f(1-g)\,d\lambda = \int_X fg\,d\mu$$

for all $f \in L^2(\varphi)$. Put $A = \{x \in X / \ g(x) \in [0,1)\}$, $B = \{x \in X / \ g(x) = 1\}$, and define

$$\lambda_a(E) = \lambda(E \cap A), \quad \lambda_s(E) = \lambda(E \cap B)$$

for all $E \in \mathfrak{M}$. Then λ_a and λ_s are finite positive measures, $\lambda = \lambda_a + \lambda_s$ and $\lambda_a \perp \lambda_s$. From (2), $\mu(B) = \int_B 1\,d\mu = \int_B g\,d\mu = \int_X \chi_B g\,d\mu = \int_X \chi_B(1-g)\,d\lambda = \int_B (1-g)\,d\lambda = 0$. Thus $\lambda_s \perp \mu$ (since λ_s is concentrated on B and μ is concentrated on B^c)

Since g is bounded and φ is finite, $(1+g+g^2+\ldots+g^n)\chi_E \in L^2(\varphi)$ for all $n = 1,2,3,\ldots$, $E \in \mathfrak{M}$; and from (2), we have

$$\int_E (1+g+g^2+\ldots+g^n)(1-g)\,d\lambda = \int_E (1+g+g^2+\ldots+g^n)g\,d\mu$$

so

$$\int_E (1-g^{n+1})\,d\lambda = \int_E (1+g+g^2+\ldots+g^n)g\,d\mu .$$

Since $E = (E \cap A) \cup (E \cap B)$ and $g(x) = 1$ for all $x \in B$, we have

(3) $$\int_{E\cap A} (1-g^{n+1})\,d\lambda = \int_E (1+g+g^2+\ldots+g^n)g\,d\mu$$

for all $n \in \mathbb{N}$ and for all $E \in \mathfrak{M}$. If $x \in A$, $g^{n+1}(x) \longrightarrow 0$ monotonically, so $\lim_{n\to\infty}(1-g^{n+1})(x) = 1$ and $|(1-g^{n+1})(x)| < 1$ for all $x \in A$ and for all $n \in \mathbb{N}$. By Lebesgue's Dominated Convergence Theorem,

(4) $$\lim_{n\to\infty} \int_{E\cap A} (1-g^{n+1})\,d\lambda = \int_{E\cap A} 1\,d\lambda = \lambda(E \cap A) = \lambda_a(E)$$

for all $E \in \mathfrak{M}$.

Let $h(x) = \lim_{n\to\infty}(1+g+g^2+\ldots+g^n)g(x)$ for all $x \in X$.

Then h is a non negative measurable function and $0 \leq g \leq (1+g)g \leq (1+g+g^2)g \leq \ldots \leq \infty$. By Lebesgue's Monotone Convergence Theorem,

$$(5) \qquad \lim_{n\to\infty} \int_E (1+g+g^2+\ldots+g^n)g\,d\mu = \int_E h\,d\mu$$

for all $E \in \mathfrak{M}$. By (3),(4) and (5),

$$\lambda_a(E) = \int_E h\,d\mu$$

for all $E \in \mathfrak{M}$. Hence $\lambda_a \ll \mu$. Since $\int_X |h|\,d\mu = \int_X h\,d\mu = \lambda_a(X) \leq \lambda(X) < \infty$, $h \in L^1(\mu)$.

<u>Step II</u> Assume μ is a σ-finite positive measure on $\mathfrak{M}$ and λ is a finite positive measure on $\mathfrak{M}$. Then there exists $X_1, X_2, \ldots \in \mathfrak{M}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n, \quad \mu(X_n) < \infty, \quad n \in \mathbb{N}.$$

Let $Y_1 = X_1$, $Y_n = X_n \setminus (Y_1 \cup Y_2 \cup \ldots \cup Y_{n-1})$ if $n \geq 2$. Then $X = \bigcup_{n=1}^{\infty} Y_n$, $Y_i \cap Y_j = \emptyset$ if $i \neq j$, $\mu(Y_n) < \infty$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathfrak{M}_n = \{E \cap Y_n / E \in \mathfrak{M}\}$ and let $\mu_n = \mu|_{\mathfrak{M}_n}$ and $\lambda_n = \lambda|_{\mathfrak{M}_n}$. By StepI, for each $n \in \mathbb{N}$ there exist unique positive measures $\lambda_a^{(n)}$, $\lambda_s^{(n)}$ on $\mathfrak{M}_n$ such that

$$\lambda_n = \lambda_a^{(n)} + \lambda_s^{(n)}, \quad \lambda_a^{(n)} \ll \mu_n, \quad \lambda_s^{(n)} \perp \mu_n$$

and there exist unique $h_n \in L^1(\mu_n)$ such that

$$\lambda_a^{(n)}(E) = \int_E h_n\,d\mu_n$$

for all $E \in \mathfrak{M}_n$. Note that h_n is positive (by StepI) for all $n \in \mathbb{N}$. Define λ_a, λ_s, h by

$$\lambda_a(E) = \sum_{n=1}^{\infty} \lambda_a^{(n)}(E \cap Y_n),$$

$$\lambda_s(E) = \sum_{n=1}^{\infty} \lambda_s^{(n)}(E \cap Y_n),$$

$$h(x) = h_n(x) \text{ if } x \in Y_n.$$

Since $\infty > \lambda(X) = \sum_{n=1}^{\infty} \lambda(Y_n)$ and $0 \le \lambda_a^{(n)}(Y_n) \le \lambda_n(Y_n) = \lambda(Y_n)$, $\lambda_a(X) = \sum_{n=1}^{\infty} \lambda_a^{(n)}(Y_n) \le \sum_{n=1}^{\infty} \lambda(Y_n) < \infty$. Then $\infty > \lambda_a(X) = \sum_{n=1}^{\infty} \lambda_a^{(n)}(Y_n) = \sum_{n=1}^{\infty} \int_{Y_n} h_n d\mu_n = \sum_{n=1}^{\infty} \int_{Y_n} h d\mu = \int_X h d\mu$, hence $h \in L^1(\mu)$. For each n, $\lambda_a^{(n)}$ and $\lambda_s^{(n)}$ are positive measures, so λ_a and λ_s are positive measures.

Claim that $\lambda(E) = \lambda_a(E) + \lambda_s(E)$ for all $E \in \mathfrak{M}$. To prove this, let $E \in \mathfrak{M}$. Then $\lambda_a(E) + \lambda_s(E) = \sum_{n=1}^{\infty} \lambda_a^{(n)}(E \cap Y_n) + \sum_{n=1}^{\infty} \lambda_s^{(n)}(E \cap Y_n) = \sum_{n=1}^{\infty} (\lambda_a^{(n)} + \lambda_s^{(n)})(E \cap Y_n) = \sum_{n=1}^{\infty} \lambda_n(E \cap Y_n) = \sum_{n=1}^{\infty} \lambda(E \cap Y_n) = \lambda(\bigcup_{n=1}^{\infty}(E \cap Y_n)) = \lambda(E \cap (\bigcup_{n=1}^{\infty} Y_n)) = \lambda(E \cap X) = \lambda(E)$.

Claim that $\lambda_a \ll \mu$. To prove this, let $E \in \mathfrak{M}$ be such that $\mu(E) = 0$. Then $0 = \mu(E) = \mu(E \cap (\bigcup_{n=1}^{\infty} Y_n)) = \sum_{n=1}^{\infty} \mu(E \cap Y_n) = \sum_{n=1}^{\infty} \mu_n(E \cap Y_n)$, hence $\mu_n(E \cap Y_n) = 0$ for all $n \in \mathbb{N}$. Since $\lambda_a^{(n)} \ll \mu_n$ for all $n \in \mathbb{N}$, $\lambda_a^{(n)}(E \cap Y_n) = 0$ for all $n \in \mathbb{N}$. Hence $\lambda_a(E) = 0$. So we have the claim.

Claim that $\lambda_s \perp \mu$. We have that for each $n \in \mathbb{N}$, $\lambda_s^{(n)} \perp \mu_n$, so there exist $A_n, B_n \in \mathfrak{M}_n$ such that $A_n \cap B_n = \emptyset$, $\lambda_s^{(n)}$ is concentrated on A and μ_n is concentrated on B_n. Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then $A, B \in \mathfrak{M}$. Since $Y_i \cap Y_j = \emptyset$ if $i \neq j$, it follows that $A \cap B = \emptyset$. Let $E \in \mathfrak{M}$ be such that $E \cap A = \emptyset$. Then $E \cap A_n = \emptyset$ for all $n \in \mathbb{N}$, so $(E \cap Y_n) \cap A_n = \emptyset$ for all $n \in \mathbb{N}$. Hence $\lambda_s^{(n)}(E \cap Y_n) = 0$ for all $n \in \mathbb{N}$.

Hence $\lambda_s(E) = \sum_{n=1}^{\infty} \lambda_s^{(n)}(E \cap Y_n) = 0$. Thus λ_s is concentrated on A. Next, let $F \in \mathfrak{M}$ be such that $F \cap B = \phi$, so $F \cap B_n = \phi$ for all $n \in \mathbb{N}$. Hence $(E \cap Y_n) \cap B_n = \phi$ for all $n \in \mathbb{N}$, hence $\mu_n(E \cap Y_n) = 0$ for all $n \in \mathbb{N}$. Thus $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E \cap Y_n)$ $= 0$. Hence μ is cocentrated on B. This proves that $\lambda_s \perp \mu$.

For $E \in \mathfrak{M}$, $\lambda_a(E) = \sum_{n=1}^{\infty} \lambda_a^{(n)}(E \cap Y_n) = \sum_{n=1}^{\infty} \int_{E \cap Y_n} h_n d\mu_n$ $= \sum_{n=1}^{\infty} \int_{E \cap Y_n} h d\mu = \int_E h d\mu$.

<u>Step III</u> Assume μ is a σ-finite positive measure on $\mathfrak{M}$ and λ is a quaternion measure on $\mathfrak{M}$. Then

$$\lambda = \lambda_1 + i\lambda_2 + j\lambda_3 + k\lambda_4$$

for some real meaures $\lambda_{1'}$, $1' \leq 4$. Then $\lambda_{1'}^{+}$, $\lambda_{1'}^{-}$ are finite positive measures for $1' = 1,2,3,4$. By Step II, there exist unique positive measures $\lambda_a^{1'}$, $\lambda_s^{1'}$, $\lambda_a^{-1'}$, $\lambda_s^{-1'}$, $1' = 1,2,3,4$ such that

$$\lambda_{1'}^{+} = \lambda_a^{1'} + \lambda_s^{1'}, \quad \lambda_a^{1'} \ll \mu, \quad \lambda_s^{1'} \perp \mu,$$

$$\lambda_{1'}^{-} = \lambda_a^{-1'} + \lambda_s^{-1'}, \quad \lambda_a^{-1'} \ll \mu, \quad \lambda_s^{-1'} \perp \mu,$$

for $1' = 1,2,3,4$, and there exist unique $h_1, h_2, h_3, \ldots, h_8$ positive measurable functions in $L^1(\mu)$ such that

$$\lambda_a^1(E) = \int_E h_1 d\mu, \quad \lambda_a^{-1}(E) = \int_E h_2 d\mu, \quad \lambda_a^2(E) = \int_E h_3 d\mu,$$

$$\lambda_a^{-2}(E) = \int_E h_4 d\mu, \quad \lambda_a^3(E) = \int_E h_5 d\mu, \quad \lambda_a^{-3}(E) = \int_E h_6 d\mu,$$

$$\lambda_a^4(E) = \int_E h_7 d\mu, \quad \lambda_a^{-4}(E) = \int_E h_8 d\mu,$$

for all $E \in \mathfrak{M}$. Define λ_a, λ_s, h by

$$\lambda_a = (\lambda_a^1 - \lambda_a^{-1}) + i(\lambda_a^2 - \lambda_a^{-2}) + j(\lambda_a^3 - \lambda_a^{-3}) + k(\lambda_a^4 - \lambda_a^{-4}),$$

$$\lambda_s = (\lambda_s^1-\lambda_s^{-1}) + i(\lambda_s^2-\lambda_s^{-2}) + j(\lambda_s^3-\lambda_s^{-3}) + k(\lambda_s^4-\lambda_s^{-4}),$$

$$h = (h_1-h_2) + i(h_3-h_4) + j(h_5-h_6) + k(h_7-h_8).$$

Then λ_a and λ_s are quaternion measures. $\lambda_a + \lambda_s =$ $(\lambda_a^1+\lambda_s^1) - (\lambda_a^{-1}+\lambda_s^{-1}) + i((\lambda_a^2+\lambda_s^2) - (\lambda_a^{-2}+\lambda_s^{-2})) +$ $j((\lambda_a^3+\lambda_s^3) - (\lambda_a^{-3}+\lambda_s^{-3})) + k((\lambda_a^4+\lambda_s^4) - (\lambda_a^{-4}+\lambda_s^{-4})) = \lambda_1^+-\lambda_1^-$ $+ i(\lambda_2^+-\lambda_2^-) + j(\lambda_3^+-\lambda_3^-) + k(\lambda_4^+-\lambda_4^-) = \lambda$. Hence $\lambda_a + \lambda_s = \lambda$. Let $E \in \mathfrak{M}$ be such that $\mu(E) = 0$. Then $\lambda_a^{l'}(E) = \lambda_a^{-l'}(E) = 0$ for $l' = 1,2,3,4$. Hence $\lambda_a(E) = 0$. Thus $\lambda_a \ll \mu$. Since $\lambda_s^1 \perp \mu$, $-\lambda_s^{-1} \perp \mu$, $i\lambda_s^2 \perp \mu$, $-i\lambda_s^{-2} \perp \mu$, $j\lambda_s^3 \perp \mu$, $-j\lambda_s^{-3} \perp \mu$, $k\lambda_s^4 \perp \mu$ and $-k\lambda_s^{-4} \perp \mu$, by Theorem 2.51 (c), we have

$$((\lambda_s^1-\lambda_s^{-1}) + i(\lambda_s^2-\lambda_s^{-2}) + j(\lambda_s^3-\lambda_s^{-3}) + k(\lambda_s^4-\lambda_s^{-4})) \perp \mu,$$

that is $\lambda_s \perp \mu$. By Theorem 4.70, we have

$$h = (h_1-h_2) + i(h_3-h_4) + j(h_5-h_6) + k(h_7-h_8) \in L^1(\mu)$$

and

$$\begin{aligned}
\lambda_a(E) &= (\lambda_a^1(E) - \lambda_a^{-1}(E)) + i(\lambda_a^2(E) - \lambda_a^{-2}(E)) + \\
&\quad j(\lambda_a^3(E) - \lambda_a^{-3}(E)) + k(\lambda_a^4(E) - \lambda_a^{-4}(E)) \\
&= \int_E h_1 d\mu - \int_E h_2 d\mu + i\left(\int_E h_3 d\mu - \int_E h_4 d\mu\right) + \\
&\quad j\left(\int_E h_5 d\mu - \int_E h_6 d\mu\right) + k\left(\int_E h_7 d\mu - \int_E h_8 d\mu\right) \\
&= \int_E ((h_1-h_2)+i(h_3-h_4)+j(h_5-h_6)+k(h_7-h_8))d\mu \\
&= \int_E h d\mu
\end{aligned}$$

for all $E \in \mathfrak{M}$. #

5.17 Theorem Let μ be a quaternion measure on a σ-algebra $\mathfrak{M}$ in X. Then there is a qauternion measurable function h such that $|h(x)| = 1$ for all $x \in X$ and such that

$$\mu(E) = \int_E h\,d|\mu| \qquad (E \in \mathfrak{M}) .$$

Proof Suppose that $\mu \equiv 0$, then $|\mu| \equiv 0$. Let $h \equiv 1$. Then $\mu(E) = \int_E h\,d|\mu|$ for all $E \in \mathfrak{M}$, so done.

Hence we may assume that $\mu \not\equiv 0$. Clearly, $\mu \ll |\mu|$. Hence by the Lebesgue-Radon-Nikodym Theorem, there exists $h \in L^1(|\mu|)$ such that

$$\mu(E) = \int_E h\,d|\mu|$$

for all $E \in \mathfrak{M}$.

Let $r \in [0,1)$ and $A_r = \{x \in X / \ |h(x)| \le r\}$. Let $(E_j)_{j \in N}$ be a partition of A_r. Then $\sum_{j=1}^{\infty} |\mu(E_j)| = \sum_{j=1}^{\infty} \left|\int_{E_j} h\,d|\mu|\right| \le \sum_{j=1}^{\infty} \int_{E_j} |h|\,d|\mu| \le \sum_{j=1}^{\infty} r|\mu|(E_j) = r|\mu|(A_r)$. Then $|\mu|(A_r) \le r|\mu|(A_r)$ (by the definition of $|\mu|$), so $|\mu|(A_r) = 0$ for all $r \in [0,1)$. Hence $|h| \ge 1$ a.e. $[|\mu|]$. Then $|\mu|(\{x \in X / \ |h(x)| < 1\}) = 0$. Since $\mu \not\equiv 0$, there exists $E \in \mathfrak{M}$ such that $\mu(E) \ne 0$, so $|\mu|(E) > 0$ and

$$\left|\frac{1}{|\mu|(E)} \int_E h\,d|\mu|\right| \le \left|\frac{\mu(E)}{|\mu|(E)}\right| \le 1 .$$

By Theorem 4.75, we have

$$|h| \le 1 \text{ a.e. } [|\mu|].$$

Then $|\mu|(\{x \in X / \ |h(x)| > 1\}) = 0$. Let

$$B = \{x \in X / \ |h(x)| \ne 1\} = \{x \in X / \ |h(x)| > 1\} \cup \{x \in X / \ |h(x)| < 1\}.$$

Thus $|\mu|(B) = 0$. Define $\hat{h}: X \longrightarrow \mathbb{H}$ by

$$\hat{h}(x) = \begin{cases} h(x) & \text{if } x \in B^c, \\ 1 & \text{if } x \in B. \end{cases}$$

Then $\hat{h}$ is measurable and $|\hat{h}| = 1$ and for all $E \in \mathfrak{M}$

$$\mu(E) = \int_E h\,d|\mu| = \int_{E\cap B} h\,d|\mu| + \int_{E\cap B^c} h\,d|\mu|$$

$$= \int_{E\cap B^c} h\,d|\mu| = \int_{E\cap B^c} \hat{h}\,d|\mu|$$

$$= \int_{E\cap B} \hat{h}\,d|\mu| + \int_{E\cap B^c} \hat{h}\,d|\mu| = \int_E \hat{h}\,d|\mu| \quad . \#$$

5.18 Theorem Assume μ is a σ-finite positive measure on a σ-algebra $\mathfrak{M}$ in X, $g \in L^1(\mu)$ and $\lambda(E) = \int_E g\,d\mu \quad (E \in \mathfrak{M})$.

Then

$$|\lambda|(E) = \int_E |g|\,d\mu \qquad (E \in \mathfrak{M}).$$

Proof By Theorem 4.69, λ is a quaternion measure on $\mathfrak{M}$, hence by Theorem 5.17, there exists a measurable function h such that $|h| = 1$ on X and $\lambda(E) = \int_E h\,d|\lambda|$ for all $E \in \mathfrak{M}$. Then $\lambda(E) = \int_E h\,d|\lambda| = \int_E g\,d\mu$ for all $E \in \mathfrak{M}$. Hence $\int_E \bar{h}\,d\lambda = \int_E \bar{h}h\,d|\lambda| = \int_E \bar{h}g\,d\mu$ for all $E \in \mathfrak{M}$ (by Theorem 4.69). Since $\bar{h}h = |h|^2 = 1$, $\int_E d|\lambda| = \int_E \bar{h}g\,d\mu$, so $|\lambda|(E) = \int_E \bar{h}g\,d\mu$ for all $E \in \mathfrak{M}$. Claim that $\bar{h}g \geq 0$ a.e. $[\mu]$. To prove this, assume $\bar{h}g = u_1 + iu_2 + ju_3 + ku_4$ for some real measurable functions u_i, $i \leq 4$. Then $|\lambda|(E) = \int_E \bar{h}g\,d\mu = \int_E u_1\,d\mu + i\int_E u_2\,d\mu + j\int_E u_3\,d\mu + k\int_E u_4\,d\mu$ for all $E \in \mathfrak{M}$. Since $|\lambda|(E) \geq 0$, $\int_E u_2\,d\mu = \int_E u_3\,d\mu = \int_E u_4\,d\mu = 0$ for all $E \in \mathfrak{M}$, hence, by Theorem 4.74 (a),

$u_2 = u_3 = u_4 = 0$ a.e. $[\mu]$. Hence $\bar{h}g$ is real a.e. $[\mu]$. Then there exists $B \in \mathcal{M}$ such that $\bar{h}g$ is real on B and $\mu(B^c) = 0$. Let $E = \{x \in B / (\bar{h}g)(x) < 0\}$. For each n, let

$$E_n = \{x \in E / \bar{h}g(x) < -\tfrac{1}{n}\}.$$

Then $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ and $\bigcup_{n=1}^{\infty} E_n = E$. For each n,

$$0 \leq |\lambda|(E_n) = \int_{E_n} \bar{h}g\,d\mu = -\int_{E_n} -\bar{h}g\,d\mu \leq -\int_{E_n} \tfrac{1}{n}\,d\mu = -\tfrac{1}{n}\mu(E_n)$$

≤ 0. Hence $\mu(E_n) = 0$ for all n, so $\mu(E) = 0$. Then $\bar{h}g \geq 0$ on $B \setminus E$ and $\mu((B \setminus E)^c) = 0$, so $\bar{h}g \geq 0$ a.e. $[\mu]$. Since $|\bar{h}| = 1$, we see that $|g| = |\bar{h}g| = \bar{h}g$ a.e. $[\mu]$. Hence $\int_E |g|\,d\mu = \int_E \bar{h}g\,d\mu$ for all $E \in \mathcal{M}$, so $|\lambda|(E) = \int_E |g|\,d\mu$ for all $E \in \mathcal{M}$. #

Let μ be a σ-finite positive measure on a σ-algebra $\mathcal{M}$ in X and $1 \leq p \leq \infty$. Let q be the exponent conjugate to p. Let $g \in L^q(\mu)$. Define $\phi_g : L^p(\mu) \longrightarrow \mathbb{H}$ by

$$\phi_g(f) = \int_X fg\,d\mu \quad \left(\int_X gf\,d\mu\right)$$

Then ϕ_g is a left(right) linear functional on $L^p(\mu)$ and

$$\begin{aligned} \|\phi_g\| &= \sup\left\{\frac{|\phi_g(f)|}{\|f\|_p} / \|f\|_p > 0\right\} \\ &\leq \sup\left\{\frac{\int_X |fg|\,d\mu}{\|f\|_p} / \|f\|_p > 0\right\} \\ &\leq \sup\left\{\frac{\|f\|_p \|g\|_q}{\|f\|_p} / \|f\|_p > 0\right\} \\ &= \|g\|_q < \infty . \end{aligned}$$

Hence ϕ_g is a bounded left(right) linear functional on $L^p(\mu)$

5.19 Theorem Suppose $1 \leq p \leq \infty$. Let μ be a non trivial finite positive measure on a σ-algebra $\mathcal{M}$ in X, and ϕ a bounded left(right) linear functional on $L^p(\mu)$. Then there exists a unique function $g \in L^q(\mu)$ where q is the exponent conjugate to p, such that

$$(1) \qquad \phi(f) = \int_X fg\,d\mu \quad \left(\int_X gf\,d\mu\right) \qquad (f \in L^p(\mu)).$$

Moreover, if ϕ and g are related as in (1), we have

$$(2) \qquad \|\phi\| = \|g\|_q .$$

Proof Uniqueness of g Suppose there exists $g_1 \in L^q(\mu)$ such that

$$\int_X fg\,d\mu = \int_X fg_1\,d\mu$$

for all $f \in L^p(\mu)$. Since $\mu(X) < \infty$, $1 \in L^1(\mu)$. Hence $\int_X g\,d\mu = \int_X g_1\,d\mu$, so $\int_X (g-g_1)\,d\mu = 0$ which implies that $g = g_1$ a.e. $[\mu]$.

We have shown that $\|\phi\| \leq \|g\|_q$. If $\|\phi\| = 0$, then $\phi \equiv 0$, so (1) and (2) hold with $g \equiv 0$. Assume $\|\phi\| > 0$. Define $\lambda : \mathcal{M} \longrightarrow \mathbb{H}$ by

$$\lambda(E) = \phi(\chi_E).$$

If $E, F \in \mathcal{M}$ is such that $E \cap F = \phi$, then $\lambda(E \cup F) = \phi(\chi_{E\cup F}) = \phi(\chi_E + \chi_F) = \phi(\chi_E) + \phi(\chi_F) = \lambda(E) + \lambda(F)$. Let E be the union of countable many disjoint measurable sets E_i. For each k, let

$$A_k = E_1 \cup \ldots \cup E_k .$$

Then $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and $\bigcup_{k=1}^{\infty} A_k = E$, so $E \setminus A_1 \supseteq E \setminus A_2 \supseteq \ldots$, hence for each k, we see that

$$\|\chi_E - \chi_{A_k}\|_p = \left\{\int_X |\chi_E - \chi_{A_k}|^p d\mu\right\}^{\frac{1}{p}} = \left\{\int_X |\chi_{E \setminus A_k}|^p d\mu\right\}^{\frac{1}{p}}$$

$$= \left\{\int_{E \setminus A_k} 1 d\mu\right\}^{\frac{1}{p}} = (\mu(E \setminus A_k))^{\frac{1}{p}} \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

(Because $\lim_{k\to\infty} \mu(E \setminus A_k) = \mu(\bigcap_{k=1}^{\infty}(E \setminus A_k)) = \mu(\phi) = 0$).

Since ϕ is bounded, by Theorem 5.13, ϕ is continuous. Then $\phi(\chi_{A_k}) \longrightarrow \phi(\chi_E)$ as $k \longrightarrow \infty$, so $\lambda(A_k) \longrightarrow \lambda(E)$ as $k \longrightarrow \infty$

Hence $\lambda(E) = \lim_{k\to\infty} \lambda(A_k) = \lim_{k\to\infty} \lambda(\bigcup_{i=1}^{k} E_i) = \lim_{k\to\infty} \sum_{i=1}^{k} \lambda(E_i) = \sum_{i=1}^{\infty} \lambda(E_i)$. Therefore λ is a quaternion measure. Claim that $\lambda << \mu$. To prove this, let $E \in \mathfrak{M}$ be such that $\mu(E) = 0$. Then $\|\chi_E\|_p = \left\{\int_X |\chi_E|^p d\mu\right\}^{\frac{1}{p}} = (\mu(E))^{\frac{1}{p}} = 0$.

Since $|\phi(\chi_E)| \leq \|\phi\| \, \|\chi_E\|_p = 0$, we have $\lambda(E) = 0$. So we the claim. By Lebesgue-Radon-Nikodym Theorem, there exists $g \in L^1(\mu)$ such that

$$\lambda(E) = \int_E g d\mu$$

for all $E \in \mathfrak{M}$. Then $\phi(\chi_E) = \int_E g d\mu = \int_X \chi_E g d\mu$ for all $E \in \mathfrak{M}$. By linearity, it follows that

$$\phi(s) = \int_X s g d\mu$$

for every simple measurable function s.

Let $f \in L^{\infty}(\mu)$. Then $|f(x)| \leq \|f\|_\infty$ for almost all x, so there exists $N \in \mathfrak{M}$ such that $\mu(N) = 0$ and $|f(x)| \leq \|f\|_\infty$ for all $x \in N^c$. Consider $f \geq 0$. By the proof of Theorem 3.15 there exists a sequence $(s_n)_{n \in \mathbb{N}}$ of simple measurable functions such that $s_n \longrightarrow f$ uniformly on N^c. Then $\|f - s_n\|_p \longrightarrow 0$ as $n \longrightarrow \infty$. Since ϕ is conitinuous, we have

(3) $$\lim_{n\to\infty} \phi(s_n) = \phi(f).$$

Claim that $\lim_{n\to\infty}\int_X s_n g d\mu = \int_X fgd\mu$. To prove this, we can choose $M > 0$ such that $|s_n| \le M$ on N^c for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} s_n g(x) = fg(x)$ for all $x \in N^c$, $|s_n g| \le |Mg|$ on N^c for all $n \in \mathbb{N}$ and $Mg \in L^1(\mu)$ (since $g \in L^1(\mu)$), by Lebesgue's Dominated Convergence Theorem,

(4) $$\lim_{n\to\infty}\int_X s_n g d\mu = \int_X fgd\mu ,$$

so we have the claim. By (3) and (4), we see that

$$\phi(f) = \lim_{n\to\infty} \phi(s_n) = \lim_{n\to\infty}\int_X s_n gd\mu = \int_X fgd\mu .$$

Next, consider f is real. Then $f = f^+ - f^-$, so

$$\begin{aligned}\phi(f) &= \phi(f^+-f^-) = \phi(f^+)-\phi(f^-)\\ &= \int_X f^+gd\mu - \int_X f^-gd\mu = \int_X (f^+-f^-)gd\mu = \int_X fgd\mu .\end{aligned}$$

Finally, consider f is quaternion. Then $f = f_1+if_2+jf_3+kf_4$ for some real measurable functions f_{i}', $i \le 4$. Then

$$\begin{aligned}\phi(f) &= \phi(f_1)+i\phi(f_2)+j\phi(f_3)+k\phi(f_4)\\ &= \int_X f_1gd\mu + i\int_X f_2gd\mu + j\int_X f_3gd\mu + k\int_X f_4gd\mu\\ &= \int_X (f_1+if_2+jf_3+kf_4)gd\mu = \int_X fgd\mu .\end{aligned}$$

Hence $\phi(f) = \int_X fgd\mu$ for all $f \in L^\infty(\mu)$.

We want to prove that $g \in L^q(\mu)$ and that (2) holds.

<u>Case I</u> $p = 1$. Then for all $E \in \mathfrak{M}$,

$$\left|\int_E gd\mu\right| = \left|\int_X \chi_E gd\mu\right| = |\phi(\chi_E)| \le \|\phi\|\|\chi_E\|_1 = \|\phi\|\mu(E).$$

Hence $\left|\frac{1}{\mu(E)}\int_E gd\mu\right| \le \|\phi\|$ for all $E \in \mathfrak{M}$ such that $\mu(E) > 0$.

By Theorem 4.75, $|g(x)| \le \|\phi\|$ a.e. $[\mu]$, hence $\|g\|_\infty \le \|\phi\| < \infty$.

Therefore $g \in L^{\infty}(\mu)$ and $\|g\|_{\infty} = \|\phi\|$.

Case II $1 < p < \infty$. Since g is quaternion measurable, similar to Corollary 3.5 (e), there exists a quaternion measurable function α such that $|\alpha| = 1$ and $|g| = \alpha g$. For each n, let

$$E_n = \{x \in X / \ |g(x)| \leq n\}$$

and put $f_n = \chi_{E_n} |g|^{q-1}\alpha$. Since q is the exponent conjugate to p, $|f_n|^p = |g|^q$ on E_n. Since $|f_n|^{-1}(n^{q-1}, \infty] = (\chi_{E_n}|g|^{q-1})^{-1}(n^{q-1}, \infty] = \phi$, we have $\|f_n\|_{\infty} = \inf\{\beta \in [0,\infty) / \ \mu(|f_n|^{-1}(\beta, \infty]) = 0\} \leq n^{q-1} < \infty$, hence $f_n \in L^{\infty}(\mu)$. Also, $f_n \in L^p(\mu)$ since f_n is bounded.

$$\int_{E_n} |g|^q d\mu = \int_{E_n} |g|^{q-1}|g| d\mu = \int_X \chi_{E_n}|g|^{q-1}\alpha g d\mu = \int_X f_n g d\mu$$

$= \phi(f_n) \leq \|\phi\| \, \|f_n\|_p = \|\phi\| \left\{\int_{E_n} |g|^q d\mu\right\}^{\frac{1}{p}}$ for all $n \in \mathbb{N}$. If $\int_{E_n} |g|^q d\mu = 0$ for all $n \in \mathbb{N}$. Then $|g|^q = 0$ a.e. $[\mu]$ on X, hence $\int_X |g|^q d\mu = 0$, hence $g \in L^q(\mu)$ and $\|g\|_q = 0 \leq \|\phi\|$. Since $\|\phi\| \leq \|g\|_q$, we see that $\|g\|_q = \|\phi\|$. If there exists $n_o \in \mathbb{N}$ such that $\int_{E_{n_o}} |g|^q d\mu > 0$. Since $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$ we have $\int_{E_n} |g|^q d\mu > 0$ for all $n \geq n_o$. Since $\int_{E_n} |g|^q d\mu \leq \|\phi\| \left\{\int_{E_n} |g|^q d\mu\right\}^{\frac{1}{p}}$ for all $n \in \mathbb{N}$, $\left\{\int_{E_n} |g|^q d\mu\right\}^{\frac{1}{p}} \leq \|\phi\|$ for all $n \geq n_o$. Hence $\left\{\int_X \chi_{E_n} |g|^q d\mu\right\}^{\frac{1}{p}} \leq \|\phi\|$ for all $n \geq n_o$. Since

$0 \leq \chi_{E_1}|g|^q \leq \chi_{E_2}|g|^q \leq \ldots$ and $\lim\limits_{n\to\infty}\chi_{E_n}|g|^q(x) = |g|^q(x)$ for all $x \in X$, by Lebesgue's Monotone Convergence Theorem, we have

$$\lim_{n\to\infty}\int_X \chi_{E_n}|g|^q d\mu = \int_X |g|^q d\mu ,$$

hence

$$\lim_{n\to\infty}\left\{\int_X \chi_{E_n}|g|^q d\mu\right\}^{\frac{1}{q}} = \left\{\int_X |g|^q d\mu\right\}^{\frac{1}{q}}.$$

Therefore

$$\left\{\int_X |g|^q d\mu\right\}^{\frac{1}{q}} \leq \|\phi\| ,$$

that is $\|g\|_q \leq \|\phi\| < \infty$. Hence $g \in L^q(\mu)$ and (2) holds.

For all $f_1, f \in L^p(\mu)$, $\left|\int_X fg d\mu - \int_X f_1 g d\mu\right| =$ $\left|\int_X (f-f_1)g d\mu\right| \leq \int_X |f-f_1||g|\, d\mu \leq \|f-f_1\|_p \|g\|_q$ by Hölder's inequality. Hence the map $f \longmapsto \int_X fg d\mu$ is continuous on $L^p(\mu)$

Now $L^\infty(\mu)$ contains $\mathcal{S} = \{s$ is a quaternion simple measurable function$\}$. By Theorem 4.92, $\mathcal{S}$ is dense in $L^p(\mu)$, hence $L^\infty(\mu)$ is dense in $L^p(\mu)$.

To show that $\phi(f) = \int_X fg d\mu$ for all $f \in L^p(\mu)$, let $f \in L^p(\mu)$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^\infty(\mu)$ such that $\lim\limits_{n\to\infty} f_n = f$. Since ϕ is continuous, $\lim\limits_{n\to\infty}\phi(f_n) =$ $\phi(f)$. Hence $\lim\limits_{n\to\infty}\int_X f_n g d\mu = \phi(f)$. Since the map $f' \longmapsto \int_X f' g d\mu$ is continuous on $L^p(\mu)$, $\lim\limits_{n\to\infty}\int_X f_n g d\mu = \int_X fg d\mu$. Hence $\phi(f) = \int_X fg d\mu$. #.

5.20 <u>Theorem</u> Let μ be a quaternion(Borel) measure on a σ-algebra $\mathfrak{M}$ in a topological space X. If $f \in L^1(\mu)$, then

$$\int_X f d\mu = \int_X f h d|\mu|$$

$$\left(\left[\int_X (d\mu) f\right] = \int_X h f d|\mu| \right)$$

for some quaternion(Borel) measurable function h such that $|h| = 1$ on X.

<u>Proof</u> By Theorem 5.17, there is a quaternion(Borel) measurable function h such that $|h| = 1$ on X and $\mu(E) = \int_E h d|\mu|$ for all $E \in \mathfrak{M}$. By Theorem 4.69, we have the theorem.#

5.21 <u>Theorem</u> Let μ and λ be quaternion(Borel) measures on a σ-algebra $\mathfrak{M}$ in a topological space X. If $f \in L^1(\mu + \lambda)$ then

$$\int_X f d(\mu + \lambda) = \int_X f d\mu + \int_X f d\lambda$$

$$\left(\left[\int_X (d(\mu + \lambda)) f\right] = \left[\int_X (d\mu) f\right] + \left[\int_X (d\lambda) f\right] \right).$$

<u>Proof</u> <u>Case I</u> $f = \chi_E$ for some $E \in \mathfrak{M}$. Since $\mu + \lambda$ is a quaternion measure, we have $\int_X \chi_E d(\mu + \lambda) = (\mu + \lambda)(E)$ $= \mu(E) + \lambda(E) = \int_X \chi_E d\mu + \int_X \chi_E d\lambda$.

<u>Case II</u> f is simple. Then

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

where $\alpha_1, \ldots, \alpha_n$ are distinct values of f and $E_i = f^{-1}(\alpha_i)$ for all $i = 1, 2, \ldots, n$. Then

$$\int_X f d(\mu + \lambda) = \int_X \sum_{i=1}^{n} \alpha_i \chi_{E_i} d(\mu + \lambda) = \sum_{i=1}^{n} \alpha_i \int_X \chi_{E_i} d(\mu + \lambda)$$

$$= \int_X \sum_{i=1}^{n} \alpha_i \chi_{E_i} d\mu + \int_X \sum_{i=1}^{n} \alpha_i \chi_{E_i} d\lambda$$

$$= \int_X f d\mu + \int_X f d\lambda \quad .$$

<u>Case III</u> $f \geq 0$. Then there exists a sequence $(s_n)_{n\in N}$ of simple measurable functions such that $0 \leq s_1 \leq s_2 \leq \ldots$ and $\lim_{n\to\infty} s_n(x) = f(x)$ for all $x \in X$. By Lebesgue's Monotone Convergence Theorem, we have

$$\lim_{n\to\infty} \int_X s_n d\mu = \int_X f d\mu$$

and

$$\lim_{n\to\infty} \int_X s_n d\lambda = \int_X f d\lambda \quad .$$

Also, $\lim_{n\to\infty} \int_X s_n d(\mu+\lambda) = \int_X f d(\mu+\lambda)$. But $\lim_{n\to\infty} \int_X s_n d(\mu+\lambda)$ $= \lim_{n\to\infty} \int_X s_n d\mu + \lim_{n\to\infty} \int_X s_n d\lambda = \int_X f d\mu + \int_X f d\lambda$. Hence

$$\int_X f d(\mu+\lambda) = \int_X f d\mu + \int_X f d\lambda \quad .$$

<u>Case IV</u> f is real. Then $f = f^+ - f^-$. By Case III, we see that

$$\begin{aligned}
\int_X f d(\mu+\lambda) &= \int_X (f^+ - f^-) d(\mu+\lambda) \\
&= \int_X f^+ d(\mu+\lambda) - \int_X f^- d(\mu+\lambda) \\
&= \int_X f^+ d\mu + \int_X f^+ d\lambda - (\int_X f^- d\mu - \int_X f^- d\lambda) \\
&= \int_X (f^+ - f^-) d\mu + \int_X (f^+ - f^-) d\lambda \\
&= \int_X f d\mu + \int_X f d\lambda \quad .
\end{aligned}$$

<u>Case V</u> f is quaternion. Then $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions f_i, $i \leq 4$. By Case IV, we see that

$$\int_X fd(\mu+\lambda) = \int_X f_1 d(\mu+\lambda) + i\int_X f_2 d(\mu+\lambda) + j\int_X f_3 d(\mu+\lambda) +$$
$$k\int_X f_4 d(\mu+\lambda)$$
$$= \int_X f_1 d\mu + \int_X f_1 d\lambda + i(\int_X f_2 d\mu + \int_X f_2 d\lambda) +$$
$$j(\int_X f_3 d\mu + \int_X f_3 d\lambda) + k(\int_X f_4 d\mu + \int_X f_4 d\lambda)$$
$$= \int_X fd\mu + \int_X fd\lambda \quad . \#$$

5.22 <u>Definition</u> Let μ be a quaternion Borel measure on a σ-algebra $\mathfrak{M}$ in a topological space X. μ is called <u>regular</u> if $|\mu|$ is regular.

The map

$$f \longmapsto \int_X fd\mu \quad ([\int_X (d\mu)f])$$

is a bounded left(right) linear functional on $C_o(X)$ whose norm is no larger than $|\mu|(X)$.

5.23 <u>The Riesz Representation Theorem</u> Let X be a locally compact, σ-compact Hausdorff space. To each bounded left (right) linear functional ϕ on $C_o(X)$, there corresponds a unique quaternion regular Borel measure μ such that

(1) $\qquad \phi(f) = \int_X fd\mu \quad ([\int_X (d\mu)f])$

for all $f \in C_o(X)$. Moreover, if ϕ and μ are related as in (1), then

(2) $\qquad \|\phi\| = |\mu|(X)$.

<u>Proof</u> <u>Uniqueness of μ</u> Suppose μ and μ' are quaternion regular Borel measures such that

$$\int_X f\,d\mu = \int_X f\,d\mu'$$

for all $f \in C_o(X)$. Note that $f \in C_o(X) \Longrightarrow f$ is bounded $\Longrightarrow$ $f \in L^1(\mu)$. Then

(3) $$\int_X f\,d(\mu - \mu') = 0$$

for all $f \in C_o(X)$. Let $\mu'' = \mu - \mu'$. By Theorem 5.17, there exists a quaternion Borel function h such that $|h| = 1$ on X and $\forall E \in \mathfrak{M}, \int_E d\mu'' = \int_E h\,d|\mu''|$. For any sequence $(f_n)_{n\in \mathbb{N}}$ of $C_o(X)$,

$$|\mu''|(X) = |\mu''|(X) - \int_X f_n\,d\mu'' \quad (\text{since } \int_X f_n\,d\mu'' = 0 \text{ by } (3))$$

$$= \int_X \bar{h}h\,d|\mu''| - \int_X f_n h\,d|\mu''| = \int_X (\bar{h} - f_n)h\,d|\mu''|$$

$$\leq \int_X |\bar{h} - f_n|\,d|\mu''| .$$

Since $C_c(X)$ is dense in $L^1(|\mu''|)$ and $\bar{h} \in L^1(|\mu''|)$, there exists a sequence $(f_n)_{n\in \mathbb{N}}$ in $C_c(X)$ such that $f_n \longrightarrow \bar{h}$ in $L^1(|\mu''|)$, so

$$\int_X |\bar{h} - f_n|\,d|\mu''| = \|\bar{h} - f_n\|_1 \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Hence $|\mu''|(X) = 0$. Since $|\mu''(E)| \leq |\mu''|(E) \leq |\mu''|(X) = 0$ for all $E \in \mathfrak{M}$, $\mu'' \equiv 0$ which implies that $\mu = \mu'$.

If $\|\phi\| \equiv 0$, then $\phi \equiv 0$, so put $\mu \equiv 0$. Assume without loss of generality that $\|\phi\| = 1$ (If $0 < \|\phi\| \neq 1$, then we put $\phi_1 = \frac{\phi}{\|\phi\|}$). Let

$$C_c^+(X) = \{f \in C_c(X) / f \text{ is non negative real}\}.$$

For $f \in C_c^+(X)$, define

$$\Lambda f = \sup\{|\phi(h)| / h \in C_c(X),\ |h| \leq f\} .$$

Then $\Lambda f \geqslant 0$ for all $f \in C_c^+(X)$, $0 \leq f_1 \leq f_2$ in $C_c^+(X) \Longrightarrow \Lambda f_1 \leq \Lambda f_2$, and for $c \in [0, \infty)$, $\Lambda(cf) = c\Lambda f$. Also, for $f \in C_c(X)$,

$$\Lambda(|f|) = \sup\{|\phi(h)| / h \in C_c(X),\ |h| \leq |f|\}$$

$$\leq \sup\{\|\phi\| \|h\| / h \in C_c(X), |h| \leq |f|\}$$
$$= \sup\{\|h\| / h \in C_c(X), |h| \leq |f|\} \leq \|f\| ,$$

hence

(4) $\quad |\phi(f)| \leq \Lambda(|f|) \leq \|f\|$.

Let $f, g \in C_c^+(X)$. To show that $\Lambda(f+g) = \Lambda f + \Lambda g$. To prove this let $\varepsilon > 0$ be given. Then there exist $h_1, h_2 \in C_c(X)$ such that $|h_1| \leq f$, $|h_2| \leq g$ and

$$\Lambda f \leq |\phi(h_1)| + \varepsilon \quad , \quad \Lambda g \leq |\phi(h_2)| + \varepsilon \ .$$

Let $\alpha_1, \alpha_2 \in \mathbb{K}$ be such that $|\alpha_1| = |\alpha_2| = 1$ and

$$\alpha_1\phi(h_1) = |\phi(h_1)| , \quad \alpha_2\phi(h_2) = |\phi(h_2)|.$$

Then

$$\begin{aligned}
\Lambda f + \Lambda g &\leq |\phi(h_1)| + |\phi(h_2)| + 2\varepsilon \\
&= \alpha_1\phi(h_1) + \alpha_2\phi(h_2) + 2\varepsilon \\
&= \phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\varepsilon \\
&\leq \Lambda(|h_1| + |h_2|) + 2\varepsilon \text{ (by the definition of } \Lambda \text{)} \\
&\leq \Lambda(f+g) + 2\varepsilon \quad \text{(since } |h_1| + |h_2| \leq f+g \text{)}
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\Lambda f + \Lambda g \leq \Lambda(f+g)$.

Let $h \in C_c(X)$ such that $|h| \leq f+g$. Let $V = \{x \in X / f(x)+g(x) > 0\}$, and define

$$h_1(x) = \frac{f(x)h(x)}{f(x)+g(x)} , \quad h_2(x) = \frac{g(x)h(x)}{f(x)+g(x)} \quad \text{if } x \in V,$$

$$h_1(x) = h_2(x) = 0 \text{ if } x \notin V.$$

Then h_1 and h_2 are continuous on V. Let $x_o \notin V$. Then $h(x_o) = 0$ since $|h| \leq f+g$. By the definition of h_1, $|h_1(x)| \leq |h(x)|$ for all $x \in X$. Let $\varepsilon > 0$ be given. Since h is continuous at x_o, there exists a nbhd N of x_o such that $|h(x)-h(x_o)| < \varepsilon$ for all $x \in N$, so $|h_1(x)-h_1(x_o)| = |h_1(x)| \leq$

$|h(x)| = |h(x)-h(x_0)| < \varepsilon$ for all $x \in N$. Thus h_1 is continuous at x_0. This shows that h_1 is continuous on X. Since $|h_1| \leq |h|$ and $h \in C_c(X)$, it follows that $h_1 \in C_c(X)$. Similarly, $h_2 \in C_c(X)$.

Because $h_1+h_2 = h$ and $|h_1| \leq f$, $|h_2| \leq g$, we have $|\phi(h)| = |\phi(h_1+h_2)| = |\phi(h_1)+\phi(h_2)| \leq |\phi(h_1)| + |\phi(h_2)| \leq \Lambda f + \Lambda g$ (since $|h_1| \leq f$ and $\Lambda f = \sup\{|\phi(h)| / h \in C_c(X), |h| \leq f\}$). Hence $\Lambda(f+g) \leq \Lambda f + \Lambda g$. Thus $\Lambda(f+g) = \Lambda f + \Lambda g$ for all $f, g \in C_c^+(X)$.

Let f be a real function, $f \in C_c(X)$. Since $2f^+ = |f|+f$ and $2f^- = |f|-f$, we have $f^+, f^- \in C_c^+(X)$. Define

$$\Lambda f = \Lambda f^+ - \Lambda f^-.$$

If $f = f_1+if_2+jf_3+kf_4 \in C_c(X)$ for some real measurable functions f_l, $l \leq 4$, we define

$$\Lambda f = \Lambda f_1+i\Lambda f_2+j\Lambda f_3+k\Lambda f_4.$$

From the proof of Theorem 4.70, we have Λ is a positive left linear functional on $C_c(X)$. By Theorem 4.76, there exists a σ-finite positive Borel measure λ such that

$$\Lambda f = \int_X f d\lambda$$

for all $f \in C_c(X)$ and λ is regular if $\lambda(X) < \infty$. From the proof of Theorem 4.76, since X is open in X,

$$\lambda(X) = \sup\{\Lambda f / f \in C_c(X), 0 \leq f \leq 1\}.$$

If $f \in C_c(X)$ is such that $0 \leq f \leq 1$, $\Lambda f = \Lambda(|f|) \leq \|f\| \leq 1$ by (4), so $\lambda(X) \leq 1$. By (4) again, $|\phi(f)| \leq \Lambda(|f|) = \int_X |f| d\lambda = \|f\|_1$ ($\| \, \|_1$ in $L^1(\lambda)$) for all $f \in C_c(X)$. Hence

$\phi : C_c(X) \longrightarrow \mathbb{H}$ is left linear with $\|\phi\| \le 1$, with respect to $L^1(\lambda)$-norm on $C_c(X)$ ($C_c(X) \subseteq L^1(\lambda)$). By Theorem 5.15 (Hahn-Banach), there is a norm-preserving extension of ϕ to a left linear functional on $L^1(\lambda)$. By Theorem 5.19 (the case $p = 1$), there exists a Borel function $g \in L^\infty(\lambda)$ such that

$$\phi(f) = \int_X fg d\lambda$$

for all $f \in C_c(X)$ and $|g| \le 1$ a.e. $[\lambda]$ on X (because $\|g\|_\infty = \|\phi\| \le 1 \Longrightarrow |g(x)| \le 1$ for almost all $x \in X$). By Theorem 1.43, $C_c(X)$ is dense in $C_o(X)$. Let $f \in C_o(X)$. Then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $C_c(X)$ such that $f_n \longrightarrow f$ as $n \longrightarrow \infty$. Since ϕ is continuous, $\lim_{n \longrightarrow \infty} \phi(f_n) = \phi(f)$, hence $\lim_{n \longrightarrow \infty} \int_X f_n g d\lambda = \phi(f)$. Since the map $f' \longmapsto \int_X f' g d\lambda$ is continuous on $C_o(X)$, we have $\lim_{n \longrightarrow \infty} \int_X f_n g d\lambda = \int_X fg d\lambda$. Hence

$$(5) \qquad \phi(f) = \int_X fg d\lambda$$

for all $f \in C_o(X)$. For $E \in \mathfrak{M}$, define $\mu(E) = \int_E g d\lambda$. Then μ is a quaternion measure on $\mathfrak{M}$, and $\int_X f d\mu = \int_X fg d\lambda = \phi(f)$ for all $f \in C_o(X)$. Hence (1) holds.

Since $\|\phi\| = 1$ and from (5), we have

$$\int_X |g| d\lambda \ge \sup\{|\phi(f)| \,/\, f \in C_o(X),\ \|f\| \le 1\}$$
$$= \|\phi\| \text{ (by Theorem 5.14) } = 1.$$

Since $|g| \le 1$ a.e. $[\lambda]$ on X and $\lambda(X) \le 1$, $\int_X |g| d\lambda \le \lambda(X) \le 1$, hence $\lambda(X) = 1$ and $\int_X |g| d\lambda = 1$. Thus $\int_X |g| d\lambda = \lambda(X) = 1$.

Since $\mu(E) = \int_E g\,d\lambda$ for all $E \in \mathcal{M}$, by Theorem 5.18, we have

$$|\mu|(E) = \int_E |g|\,d\lambda$$

for all $E \in \mathcal{M}$. Hence

$$|\mu|(X) = \int_X |g|\,d\lambda = 1 = \|\phi\| . \ \#$$

5.24 <u>Definition</u> Suppose $\mathcal{S}$ is a σ-algebra in a set X and $\mathcal{T}$ is a σ-algebra in a set Y. A <u>measurable</u> <u>rectangle</u> of $X \times Y$ is a set of the form $A \times B$ where $A \in \mathcal{S}$, $B \in \mathcal{T}$.

If $Q = R_1 \cup \ldots \cup R_n$ where each R_i is a measurable rectangle and $R_i \cap R_j = \phi$ if $i \neq j$, we say that $Q \in \mathcal{E}$, the class of all <u>elementary</u> <u>sets</u>. Then every measurable rectangle is an elementary set.

5.25 <u>Definition</u> $\mathcal{S} \times \mathcal{T}$ is defined to be the smallest σ-algebra in $X \times Y$ containing all measurable rectangles.

5.26 <u>Definition</u> Let $E \subseteq X \times Y$, and $x \in X$, $y \in Y$, we define

$$E_x = \{y \in Y / \ (x,y) \in E\}, \quad E^y = \{x \in X / \ (x,y) \in E\} .$$

E_x and E^y are called the <u>x-section</u> of E and the <u>y-section</u> of E, respectively. Note that $E_x \subseteq Y$ and $E^y \subseteq X$.

5.27 <u>Theorem</u> If $E \in \mathcal{S} \times \mathcal{T}$, then $E_x \in \mathcal{T}$ for all $x \in X$ and $E^y \in \mathcal{S}$ for all $y \in Y$.

<u>Proof</u> See [9]. #

5.28 <u>Theorem</u> If $P, Q \in \mathcal{E}$, then $P \cap Q$, $P \sim Q$ and $P \cup Q$ belong to $\mathcal{E}$.

<u>Proof</u> See [9]. #

5.29 Theorem $\mathcal{S}\times\mathcal{T}$ is the smallest monotone class which contains all elementary sets.

Proof See [9]. #

5.30 Definition Let f be a function on XxY. For each $x\in X$, let f_x be the function on Y defined by

$$f_x(y) = f(x,y)$$

and for each $y\in Y$, let f^y be the function on X defined by

$$f^y(x) = f(x,y).$$

5.31 Theorem Let f be an $(\mathcal{S}\times\mathcal{T})$ - measurable function on XxY. Then

(a) for all $x\in X$, f_x is a $\mathcal{T}$ - measurable function,

(b) for all $y\in Y$, f^y is an $\mathcal{S}$ - measurable function.

Proof[9]Let V be an open set in $\mathbb{H}$. Then $f^{-1}(V) \in \mathcal{S}\times\mathcal{T}$ ($f^{-1}(V) = \{(x,y)\in X\times Y/\ f(x,y)\in V\}$). Let $x\in X$, $y\in Y$. Then $(f^{-1}(V))_x\in\mathcal{T}$ and $(f^{-1}(V))^y\in\mathcal{S}$ by Theorem 5.27. But $(f^{-1}(V))_x = \{y'\in Y/\ f(x,y')\in V\} = \{y'\in Y/\ f_x(y')\in V\} = f_x^{-1}(V) \in\mathcal{T}$ and $(f^{-1}(V))^y = \{x'\in X/\ f(x',y)\in V\} = \{x'\in X/\ f^y(x')\in V\} = (f^y)^{-1}(V)\in\mathcal{S}$. #

5.32 Theorem Suppose that $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\lambda)$ are quaternion measure spaces, $Q\in\mathcal{S}\times\mathcal{T}$ and $|\lambda|(B)\mu(A) = |\mu|(A)\lambda(B)$ for all $A\in\mathcal{S}$, $B\in\mathcal{T}$. Let

$$\varphi(x) = |\lambda|(Q_x), \qquad \psi(y) = |\mu|(Q^y)$$

for all $x \in X$, $y \in Y$. Then φ is $\mathcal{S}$-measurable, ψ is $\mathcal{T}$-measurable and $\int_X \varphi \, d\mu = \int_Y \psi \, d\lambda$. Note that if $\mathcal{S} = \{\emptyset, X\}$, $\mathcal{T} = \{\emptyset, Y\}$, $\mu(X) = i$ and $\lambda(Y) = j$, then $|\lambda|(Y)\mu(X) \neq |\mu|(X)\lambda(Y)$.

<u>Proof</u> The definition of φ and ψ make sense by Theorem 5.27.

Let Ω be the class of all $Q \in \mathcal{S} \times \mathcal{T}$ for which the conclusion of the theorem holds. We claim that Ω has the following properties:

(a) Ω contains all measurable rectangles.

(b) If $Q_1, Q_2, \ldots \in \Omega$ such that $Q_1 \subseteq Q_2 \subseteq \ldots$, then $\bigcup_{i=1}^{\infty} Q_i \in \Omega$.

(c) If $(Q_i)_{i \in \mathbb{N}}$ is a disjoint collection of members of Ω, then $\bigcup_{i=1}^{\infty} Q_i \in \Omega$.

(d) If $A \times B \in \mathcal{S} \times \mathcal{T}$, $Q_1, Q_2, \ldots \in \Omega$ and $A \times B \supseteq Q_1 \supseteq Q_2 \supseteq \ldots$, then $\bigcap_{i=1}^{\infty} Q_i \in \Omega$.

For each $Q \in \mathcal{S} \times \mathcal{T}$, let

$$\varphi_Q(x) = |\lambda|(Q_x), \quad \psi_Q(y) = |\mu|(Q^y)$$

for all $x \in X$, $y \in Y$.

To prove (a), let $A \in \mathcal{S}$, $B \in \mathcal{T}$. Then $\varphi_{A \times B}(x) = |\lambda|((A \times B)_x) = |\lambda|(B)\chi_A(x)$ for all $x \in X$ and $\psi_{A \times B}(y) = |\mu|((A \times B)^y) = |\mu|(A)\chi_B(y)$ for all $y \in Y$. Hence $\varphi_{A \times B}$ is $\mathcal{S}$-measurable, $\psi_{A \times B}$ is $\mathcal{T}$-measurable, $\int_X \varphi_{A \times B} \, d\mu = \int_X |\lambda|(B)\chi_A \, d\mu = |\lambda|(B)\mu(A)$ and $\int_Y \psi_{A \times B} \, d\lambda = \int_Y |\mu|(A)\chi_B \, d\lambda$

$= |\mu|(A)\lambda(B)$. But $|\lambda|(B)\mu(A) = |\mu|(A)\lambda(B)$, so $\int_X \varphi_{A\times B}\, d\mu = \int_Y \psi_{A\times B}\, d\lambda$.

To prove (b), for all $x \in X$ for all $y \in Y$,

$$(Q_1)_x \subseteq (Q_2)_x \subseteq \ldots; \quad Q_1^y \subseteq Q_2^y \subseteq \ldots .$$

Then for all $x \in X$, $\lim_{n\to\infty} |\lambda|((Q_n)_x) = |\lambda|(\bigcup_{i=1}^{\infty}(Q_i)_x) = |\lambda|((\bigcup_{i=1}^{\infty} Q_i)_x)$ and for all $y \in Y$, $\lim_{n\to\infty} |\mu|(Q_n^y) = |\mu|(\bigcup_{i=1}^{\infty} Q_i^y) = |\mu|((\bigcup_{i=1}^{\infty} Q_i)^y)$. Since $|\lambda|(Q_1)_x \le |\lambda|(Q_2)_x \le \ldots$ and $|\mu|(Q_1^y) \le |\mu|(Q_2^y) \le \ldots$, $\varphi_{Q_1}(x) \le \varphi_{Q_2}(x) \le \ldots$ and $\psi_{Q_1}(y) \le \psi_{Q_2}(y) \le \ldots$ and $\lim_{n\to\infty} \varphi_{Q_n}(x) = \varphi_{\bigcup_{i=1}^{\infty} Q_i}(x)$ and $\lim_{n\to\infty} \psi_{Q_n}(y) = \psi_{\bigcup_{i=1}^{\infty} Q_i}(y)$ for all $x \in X$ for all $y \in Y$. By Lebesgue's Monotone Convergence Theorem, we see that $\varphi_{\bigcup_{i=1}^{\infty} Q_i}$ is $\mathcal{S}$-measurable, $\psi_{\bigcup_{i=1}^{\infty} Q_i}$ is $\mathcal{T}$-measurable, $\lim_{n\to\infty} \int_X \varphi_{Q_n}\, d\mu = \int_X \varphi_{\bigcup_{i=1}^{\infty} Q_i}\, d\mu$ and $\lim_{n\to\infty} \int_Y \psi_{Q_n}\, d\lambda = \int_Y \psi_{\bigcup_{i=1}^{\infty} Q_i}\, d\lambda$. Since $Q_n \in \Omega$ for all $n \in \mathbb{N}$, $\int_X \varphi_{Q_n}\, d\mu = \int_Y \psi_{Q_n}\, d\lambda$ for all $n \in \mathbb{N}$. It follows that

$$\int_X \varphi_{\bigcup_{i=1}^{\infty} Q_i}\, d\mu = \int_Y \psi_{\bigcup_{i=1}^{\infty} Q_i}\, d\lambda .$$

To prove (c), let $Q_1, Q_2 \in \Omega$ be such that $Q_1 \cap Q_2 = \emptyset$. For $x \in X$, $\varphi_{Q_1 \cup Q_2}(x) = |\lambda|(Q_1 \cup Q_2)_x = |\lambda|((Q_1)_x \cup (Q_2)_x) = |\lambda|(Q_1)_x + |\lambda|(Q_2)_x$. But $|\lambda|(Q_1)_x + |\lambda|(Q_2)_x =$

$\varphi_{Q_1}(x) + \varphi_{Q_2}(x)$ for all $x \in X$. Then $\varphi_{Q_1 \cup Q_2}(x) = \varphi_{Q_1}(x) + \varphi_{Q_2}(x)$ for all $x \in X$. Since φ_{Q_1} and φ_{Q_2} are δ-measurable $\varphi_{Q_1 \cup Q_2}$ is δ-measurable. Similarly, $\psi_{Q_1 \cup Q_2}$ is $\mathcal{J}$-measurable

$\int_X \varphi_{Q_1 \cup Q_2} d\mu = \int_X (\varphi_{Q_1} + \varphi_{Q_2}) d\mu = \int_X \varphi_{Q_1} d\mu + \int_X \varphi_{Q_2} d\mu =$

$\int_Y \psi_{Q_1} d\lambda + \int_Y \psi_{Q_2} d\lambda = \int_Y (\psi_{Q_1} + \psi_{Q_2}) d\lambda = \int_Y \psi_{Q_1 \cup Q_2} d\lambda$ (since

$\psi_{Q_1} + \psi_{Q_2} = \psi_{Q_1 \cup Q_2}$ as proof similar to $\varphi_{Q_1 \cup Q_2} = \varphi_{Q_1} + \varphi_{Q_2}$). Hence $Q_1 \cup Q_2 \in \Omega$. So if $Q_1, Q_2, \ldots, Q_n \in \Omega$ for all $n \in \mathbb{N}$ and $Q_i \cap Q_j = \emptyset$ if $i \neq j$, then $\bigcup_{i=1}^{n} Q_i \in \Omega$ for all $n \in \mathbb{N}$ Let $P_n = \bigcup_{i=1}^{n} Q_i$ for all $n \in \mathbb{N}$. Then $P_n \in \Omega$ for all $n \in \mathbb{N}$ and $P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots$. By (b), we have $\bigcup_{i=1}^{\infty} P_i \in \Omega$. But $\bigcup_{i=1}^{\infty} P_i = \bigcup_{i=1}^{\infty} Q_i$, so $\bigcup_{i=1}^{\infty} Q_i \in \Omega$.

To prove (d), let $x \in X$, $y \in Y$. Then

$$(A \times B)_x \supseteq (Q_1)_x \supseteq (Q_2)_x \supseteq \ldots \text{ and } (A \times B)^y \supseteq Q_1^y \supseteq Q_2^y \supseteq \ldots .$$

Then $\lim_{n \to \infty} |\lambda|((Q_n)_x) = |\lambda|(\bigcap_{i=1}^{\infty} (Q_i)_x) = |\lambda|((\bigcap_{i=1}^{\infty} Q_i)_x)$ and $\lim_{n \to \infty} |\mu|(Q_n^y) = |\mu|((\bigcap_{i=1}^{\infty} Q_i)^y)$. Thus $\lim_{n \to \infty} \varphi_{Q_n}(x) = \varphi_{\bigcap_{i=1}^{\infty} Q_i}(x)$ and $\lim_{n \to \infty} \psi_{Q_n}(y) = \psi_{\bigcap_{i=1}^{\infty} Q_i}(y)$. Since $Q_n \in \Omega$ for all $n \in \mathbb{N}$, $\varphi_{\bigcap_{i=1}^{\infty} Q_i}$ is δ-measurable and $\psi_{\bigcap_{i=1}^{\infty} Q_i}$ is $\mathcal{J}$-measurable. From

$\varphi_{Q_n}(x) = |\lambda|(Q_n)_x \leq |\lambda|(A \times B)_x = |\lambda|(B) \chi_A(x)$ for all $n \in \mathbb{N}$

for all $x \in X$, $\int_X |\lambda|(B)\chi_A d|\mu| = |\lambda|(B)|\mu|(A) < \infty$, so $|\lambda|(B)\chi_A \in L^1(\mu)$. Similarly, $\psi_{Q_n}(y) \leq |\mu|(A)\chi_B(y)$ for all $n \in \mathbb{N}$ for all $y \in Y$ and $|\mu|(A)\chi_B \in L^1(\lambda)$. By Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_X \varphi_{Q_n} d\mu = \int_X \varphi_{\bigcap_{i=1}^{\infty} Q_i} d\mu, \quad \lim_{n\to\infty}\int_Y \psi_{Q_n} d\lambda = \int_Y \psi_{\bigcap_{i=1}^{\infty} Q_i} d\lambda .$$

Since for all $n \in \mathbb{N}$, $Q_n \in \Omega$, $\int_X \varphi_{Q_n} d\mu = \int_Y \psi_{Q_n} d\lambda$ for all $n \in \mathbb{N}$. Hence $\int_X \varphi_{\bigcap_{i=1}^{\infty} Q_i} d\mu = \int_Y \psi_{\bigcap_{i=1}^{\infty} Q_i} d\lambda$. So we have (d).

Now, let $X = \bigcup_{i=1}^{\infty} X_i$ and $Y = \bigcup_{j=1}^{\infty} Y_j$ be disjoint unions, where $X_i \in \mathcal{S}$ for all $i \in \mathbb{N}$ and $Y_j \in \mathcal{T}$ for all $j \in \mathbb{N}$. For $m, n \in \mathbb{N}$, $Q \subseteq X \times Y$, define

$$Q_{mn} = Q \cap (X_n \times Y_m) .$$

Let $\mathcal{M} = \{Q \in \mathcal{S} \times \mathcal{T} / Q_{mn} \in \Omega \text{ for all } m, n \in \mathbb{N}\}$. Then (b) and (d) shows that $\mathcal{M}$ is a monotone class.

To show $\mathcal{E} \subseteq \mathcal{M}$, let $A \in \mathcal{S}$, $B \in \mathcal{T}$. Then for all $m, n \in \mathbb{N}$, $(A\times B)\cap(X_n \times Y_m) = (A \cap X_n)\times(B \cap Y_m) \in \Omega$ by (a). Then $A \times B \in \mathcal{M}$. Hence $\mathcal{M}$ contains all measurable rectangle. By (c), $\mathcal{E} \subseteq \mathcal{M}$. Since $\mathcal{S} \times \mathcal{T}$ is the smallest monotone class which contains $\mathcal{E}$, $\mathcal{S} \times \mathcal{T} \subseteq \mathcal{M}$. By the definition of $\mathcal{M}$, $\mathcal{M} \subseteq \mathcal{S} \times \mathcal{T}$. Hence $\mathcal{M} = \mathcal{S} \times \mathcal{T}$.

Let $Q \in \mathcal{S} \times \mathcal{T}$, so $Q \in \mathcal{M}$. Then $Q_{mn} \in \Omega$ for all $m, n \in \mathbb{N}$. $Q = Q \cap (X \times Y) = Q \cap (\bigcup_{m,n\in\mathbb{N}}(X_n \times Y_m)) = \bigcup_{m,n\in\mathbb{N}}(Q \cap (X_n \times Y_m))$

$= \bigcup_{m,n\in\mathbb{N}} Q_{mn}$ which is a disjoint union. By (c), we have $Q \in \Omega$. #

<u>Remark</u>: Since $\varphi(x) = |\lambda|(Q_x) = \int_Y \chi_Q(x,y)\,d|\lambda|(y)$ and $\psi(y) = |\mu|(Q^y) = \int_X \chi_Q(x,y)\,d|\mu|(x)$, the conclusion $\int_X \varphi\,d\mu = \int_Y \psi\,d\lambda$ can be written in the form

$$\int_X d\mu(x)\int_Y \chi_Q(x,y)\,d|\lambda|(y) = \int_Y d\lambda(y)\int_X \chi_Q(x,y)\,d|\mu|(x)$$

$$\left(\int_X\left(\int_Y \chi_Q(x,y)\,d|\lambda|(y)\right)d\mu(x) = \int_Y\left(\int_X \chi_Q(x,y)\,d|\mu|(x)\right)d\lambda(y)\right).$$

5.33 <u>Definition</u> Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be quaternion measure spaces such that $|\lambda|(B)\mu(A) = |\mu|(A)\lambda(B)$ for all $A \in \mathcal{S}$, $B \in \mathcal{T}$. If $Q \in \mathcal{S} \times \mathcal{T}$, we define

$$(\mu\times\lambda)(Q) = \int_X |\lambda|(Q_x)\,d\mu(x) = \int_Y |\mu|(Q^y)\,d\lambda(y) .$$

We call $\mu \times \lambda$ the <u>product</u> of quaternion measures μ and λ. $\mu \times \lambda$ is really a quaternion measure on $\mathcal{S} \times \mathcal{T}$ follows immediately from Theorem 4.67.

5.34 <u>The Fubini Theorem</u> Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be quaternion measure spaces such that $|\lambda|(B)\mu(A) = |\mu|(A)\lambda(B)$ for all $A \in \mathcal{S}$, $B \in \mathcal{T}$. Suppose f is an $\mathcal{S} \times \mathcal{T}$-measurable function on $X \times Y$.

(a) If $0 \le f \le \infty$, and if

(1) $\varphi(x) = \int_Y f_x\,d|\lambda|$, $\psi(y) = \int_X f^y\,d|\mu|$ $\quad (x \in X,\ y \in Y)$,

then φ is $\mathcal{S}$-measurable and ψ is $\mathcal{T}$-measurable, and

(2) $\int_X \varphi\,d\mu = \int_{X\times Y} f\,d(\mu\times\lambda) = \int_Y \psi\,d\lambda$.

(b) If f is quaternion and $|f| < \infty$, then the functions φ and ψ, defined by (1), are in $L^1(\mu)$ and $L^1(\lambda)$, respectively, and (2) holds.

<u>Proof of (a)</u> Let $0 \leq f \leq \infty$, and let $\varphi(x) = \int_Y f_x d|\lambda|$ $\psi(y) = \int_X f^y d|\mu|$ for all $x \in X$, $y \in Y$. By Theorem 5.31, f_x is $\mathcal{T}$-measurable and f^y is δ-measurable. Then the definition of φ and ψ make sense.

Suppose $Q \in \delta \times \mathcal{T}$ and $f = \chi_Q$. Then $\varphi(x) = \int_Y (\chi_Q)_x d|\lambda|$ $= \int_Y (\chi_{Q_x}) d|\lambda| = |\lambda|(Q_x)$ for all $x \in X$ and $\psi(y) = \int_X (\chi_Q)^y d|\mu|$ $= \int_X (\chi_{Q^y}) d|\mu| = |\mu|(Q^y)$ for all $y \in Y$. By Theorem 5.32, φ is δ-measurable, ψ is $\mathcal{T}$-measurable and $\int_X \varphi d\mu = \int_Y \psi d\lambda$. But $\int_X \varphi d\mu = \int_X |\lambda|(Q_x) d\mu = (\mu \times \lambda)(Q) = \int_{X \times Y} \chi_Q d(\mu \times \lambda)$ and $\int_Y \psi d\lambda = \int_Y |\mu|(Q^y) d\lambda = (\mu \times \lambda)(Q) = \int_{X \times Y} \chi_Q d(\mu \times \lambda)$. Hence (a) holds for all non negative simple $\delta \times \mathcal{T}$-measurable functions.

Let f be such that $0 \leq f \leq \infty$. Then there exists a sequence of simple measurable functions $(s_n)_{n \in \mathbb{N}}$ such that $0 \leq s_1 \leq s_2 \leq \ldots$ and $\lim_{n \to \infty} s_n(x,y) = f(x,y)$ for all $x \in X$ for all $y \in Y$. For each $n \in \mathbb{N}$, let $\varphi_n(x) = \int_Y (s_n)_x d|\lambda|$, $\psi_n(y) =$ $\int_X (s_n)^y d|\mu|$ for all $x \in X$, $y \in Y$. Then we have φ_n is

δ-measurable, Ψ_n is $\mathcal{T}$-measurable and $\int_X \varphi_n d\mu = \int_{X\times Y} s_n d(\mu\times\lambda) = \int_Y \Psi_n d\lambda$ for all $n\in\mathbb{N}$. By Lebesgue's Monotone Convergence Theorem, $\lim_{n\to\infty}\int_{X\times Y} s_n d(\mu\times\lambda) = \int_{X\times Y} f d(\mu\times\lambda)$. Since for all $x\in X$, $0\le (s_1)_x \le (s_2)_x \le \ldots$ and $\lim_{n\to\infty}(s_n)_x(y) = f_x(y)$ for all $y\in Y$, by Lebesgue's Monotone Convergence Theorem, $\lim_{n\to\infty}\int_Y (s_n)_x d|\lambda| = \int_Y f_x d|\lambda|$ and $0\le \int_Y (s_1)_x d|\lambda| \le \int_Y (s_2)_x d|\lambda| \le \ldots$. Then $\lim_{n\to\infty}\varphi_n(x) = \varphi(x)$ and $0\le \varphi_1(x)\le\varphi_2(x)\le\ldots$. By Lebesgue's Monotone Convergence Theorem, φ is δ-measurable and $\lim_{n\to\infty}\int_X \varphi_n d\mu = \int_X \varphi d\mu$. Similarly, Ψ is $\mathcal{T}$-measurable and $\lim_{n\to\infty}\int_Y \Psi_n d\lambda = \int_Y \Psi d\lambda$. But for all $n\in\mathbb{N}$, $\int_X \varphi_n d\mu = \int_{X\times Y} s_n d(\mu\times\lambda) = \int_Y \Psi_n d\lambda$. Hence

$$\int_X \varphi d\mu = \int_{X\times Y} f d(\mu\times\lambda) = \int_Y \Psi d\lambda .$$

Proof of (b) Let f be a quaternion measurable function and $|f|<\infty$. Since $|\mu\times\lambda|(X\times Y)<\infty$, $\int_{X\times Y} f\, d|\mu\times\lambda| < \infty$, so $f\in L^1(\mu\times\lambda)$.

Step I f is real. Then $f = f^+ - f^-$. Let

$$\varphi_1(x) = \int_Y (f^+)_x d|\lambda| , \quad \varphi_2(x) = \int_Y (f^-)_x d|\lambda| ,$$

$$\Psi_1(y) = \int_X (f^+)^y d|\mu| , \quad \Psi_2(y) = \int_X (f^-)^y d|\mu| ,$$

for all $x \in X$ for all $y \in Y$. By (a), we have

$$\int_X \varphi_1 d\mu = \int_{X \times Y} f^+ d(\mu \times \lambda) = \int_Y \psi_1 d\lambda ,$$

$$\int_X \varphi_2 d\mu = \int_{X \times Y} f^- d(\mu \times \lambda) = \int_Y \psi_2 d\lambda .$$

Since $(f^+)_x = |(f^+)_x| \le |f| < \infty$ and $|\lambda|(Y) < \infty$. $\varphi_1(x) < \infty$ for all $x \in X$. Similarly, $\varphi_2(x) < \infty$ for all $x \in X$, $\psi_1(y) < \infty$ for all $y \in Y$ and $\psi_2(y) < \infty$ for all $y \in Y$. Since $|\lambda|(Y) < \infty$ and $|\mu|(X) < \infty$, $\varphi_1, \varphi_2 \in L^1(\mu)$ and $\psi_1, \psi_2 \in L^1(\lambda)$. Since $f_x = (f^+)_x - (f^-)_x$ and $f^y = (f^+)^y - (f^-)^y$, $\varphi(x) = \int_Y f_x d|\lambda|$ $= \int_Y (f^+)_x d|\lambda| - \int_Y (f^-)_x d|\lambda| = \varphi_1(x) - \varphi_2(x)$ and $\psi(y) =$ $\int_X f^y d|\mu| = \int_X (f^+)^y d|\mu| - \int_X (f^-)^y d|\mu| = \psi_1(y) - \psi_2(y)$. Hence $\varphi \in L^1(\mu)$ and $\psi \in L^1(\lambda)$. Thus $\int_X \varphi d\mu = \int_X \varphi_1 d\mu -$ $\int_X \varphi_2 d\mu = \int_{X \times Y} f^+ d(\mu \times \lambda) - \int_{X \times Y} f^- d(\mu \times \lambda) = \int_Y \psi_1 d\lambda -$ $\int_Y \psi_2 d\lambda = \int_Y \psi d\lambda$. Hence

$$\int_X \varphi d\mu = \int_{X \times Y} f d(\mu \times \lambda) = \int_Y \psi d\lambda .$$

<u>Step II</u> f is quaternion. Then $f = f_1 + if_2 + jf_3 + kf_4$ for some real measurable functions f_{i}', $i \le 4$. Since $f_x = (f_1)_x + i(f_2)_x$ $+ j(f_3)_x + k(f_4)_x$, $\varphi(x) = \int_Y f_x d|\lambda| = \int_Y (f_1)_x d|\lambda| + i\int_Y (f_2)_x d|\lambda|$ $+ j\int_Y (f_3)_x d|\lambda| + k\int_Y (f_4)_x d|\lambda|$. Since $f^y = (f_1)^y + i(f_2)^y + j(f_3)^y$ $+ k(f_4)^y$, $\psi(y) = \int_X f^y d|\mu| =$

$\int_X (f_1)^y d|\mu| + i\int_X (f_2)^y d|\mu| + j\int_X (f_3)^y d|\mu| + k\int_X (f_4)^y d|\mu|$. Let $\varphi_{i'}(x) = \int_Y (f_{i'})_x d|\lambda|$ and let $\psi_{i'}(y) = \int_X (f_{i'})^y d|\mu|$, $i' = 1, 2,3,4$. By Step I, $\varphi_{i'} \in L^1(\mu)$ and $\psi_{i'} \in L^1(\lambda)$, $i' = 1,2,3,4$. and $\int_X \varphi_{i'} d\mu = \int_{X\times Y} f_{i'} d(\mu\times\lambda) = \int_Y \psi_{i'} d\lambda$, $i' = 1,2,3,4$.

Since $\varphi(x) = \varphi_1(x) + i\,\varphi_2(x) + j\,\varphi_3(x) + k\,\varphi_4(x)$ and $\psi(y) = \psi_1(y) + i\,\psi_2(y) + j\,\psi_3(y) + k\,\psi_4(y)$, $\varphi \in L^1(\mu)$ and $\psi \in L^1(\lambda)$ and

$$\begin{aligned}
\int_X \varphi\, d\mu &= \int_X \varphi_1 d\mu + i\int_X \varphi_2 d\mu + j\int_X \varphi_3 d\mu + k\int_X \varphi_4 d\mu \\
&= \int_{X\times Y} f_1 d(\mu\times\lambda) + i\int_{X\times Y} f_2 d(\mu\times\lambda) + \\
&\quad j\int_{X\times Y} f_3 d(\mu\times\lambda) + k\int_{X\times Y} f_4 d(\mu\times\lambda) \\
&= \int_Y \psi_1 d\lambda + i\int_Y \psi_2 d\lambda + j\int_Y \psi_3 d\lambda + k\int_Y \psi_4 d\lambda \\
&= \int_Y \psi\, d\lambda .
\end{aligned}$$

Hence $\int_X \varphi\, d\mu = \int_{X\times Y} f\, d(\mu\times\lambda) = \int_Y \psi\, d\lambda$. #