CHAPTER II

SEMIMODULES OVER SEMIRINGS

This chapter covers basic results about semimodules. It is noteworthy that most of the results are analogous to standard results concerning modules over rings. Because of this, and the fact that much of this material is discussed in *[1]*, very few proofs will be presented in detail. I will divide this chapter into five sections. Section 2.1 introduces semimodules and morphisms. Sections 2.2 and 2.3 discuss congruence relations and group semimodules, respectively, which are concepts not found in the theory of modules over rings. The two last sections return to familiar topics, namely direct products and direct sums, and free semimodules.

2.1. Generalities

This section introduces the basic definitions concerning semimodules over semirings, including those pertaining to morphisms, and proves a few elementary results. The most important propositions are probably the last two; they will be very useful in Chapter III. All definitions and results in this section either may be found in Chapter 13 of *[1]*, or are analogs of results about modules over rings (see, for example, Chapter 1 of *[2]*), except for Definition 2.1.8, the last part of Lemma 2.1.11, and Proposition 2.1.13.

Definition 2.1.1. *Let S be a semiring (not necessarily commutative). A* **left S-semimodule** *is a commutative monoid $(A, +)$, with additive identity 0_A, for which there is a function $S \times A \to A$, denoted by $(s, a) \mapsto sa$ and called* **scalar multiplication,** *which satisfies the following conditions for all elements s and s' of S and all elements a and a' of A:*

(i) $(ss')a = s(s'a)$;

(ii) $s(a + a') = sa + sa'$;

(iii) $(s + s')a = sa + s'a$;

(iv) $1_S a = a$; and

(v) $s0_A = 0_A = 0_S a$.

Right semimodules over S are defined in an analogous manner. In what follows, I will generally work with left semimodules, with the corresponding results for right semimodules being assumed without explicit mention. Occasionally, I will use "semimodule" instead of "S-semimodule" if I don't want to specify the semiring.

A simple but important example of an S-semimodule is S itself, where the scalar multiplication coincides with the normal multiplication in S.

Definition 2.1.2. *Let S be a semiring and A an S-semimodule. Then a nonempty subset B of A is a* **subsemimodule** *of A iff B is closed under addition and scalar multiplication. (Thus B is an S-semimodule with the same addition and scalar multiplication as A; note also that this implies $0_A \in B$.)*

Subsemimodules of right semimodules are defined analogously.

Lemma 2.1.3. *Let* $(B_i)_{i\in I}$ *be a family of subsemimodules of an* S-*semimodule* A. *Then* $\bigcap_{i\in I} B_i$ *is a subsemimodule of* A.

Proof. Obvious. #

The previous lemma suggests the following definition.

Definition 2.1.4. *Let* B *be a subset of an* S-*semimodule* A. *Then the* **subsemimodule generated by** B, *denoted by* $[B]$, *is the intersection of all subsemimodules* C *of* A *such that* $B \subseteq C$ *(i.e.,* $[B] = \bigcap_{i\in I} C_i$, *where* $\{C_i \mid i \in I\}$ *is the set of all subsemimodules of* A *containing* B*).*

Remarks.

(i) $[B]$ is a subsemimodule of A because of Lemma 2.1.3.

(ii) In fact, $[B]$ is the smallest subsemimodule of A containing B.

(iii) If $[B] = A$, then we say that B **generates** A.

The following proposition describes what $[B]$ looks like.

Proposition 2.1.5. *Let* B *be a nonempty subset of an* S-*semimodule* A. *Then*

$$[B] = \left\{ \sum_{i=1}^{n} s_i b_i \;\middle|\; n \geq 1, \text{ and } s_i \in S,\ b_i \in B \text{ for all } i \in \bar{n} \right\}$$

Proof. Clear. #

The rest of this section will be about morphisms.

Definition 2.1.6. *Let S be a semiring and A and B S-semimodules. Then a function f from A to B is an* **S-homomorphism** *iff the following conditions are satisfied:*

(i) $f(a+a') = f(a) + f(a')$ *for all a and a' in A; and*

(ii) $f(sa) = sf(a)$ *for all s in S and all a in A.*

Homomorphisms of right semimodules are defined similarly. Also, observe that if $f: A \to B$ is an S-homomorphism, then $f(0_A) = f(0_S 0_A) = 0_S f(0_A) = 0_B$.

Proposition 2.1.7. *Let $f: A \to B$ be an S-homomorphism. Then Imf is a subsemimodule of B.*

Proof. Clear. #

Definition 2.1.8. *An S-homomorphism $f: A \to B$, where A and B are S-semimodules, is called an* **epimorphism** *iff $Imf = B$; f is called a* **monomorphism** *iff f is injective; and f is called an* **isomorphism** *iff f is both injective and surjective. (Note that my definitions of these terms are different from those in [1] —see page 154.)*

If $f: A \to B$ is an S-isomorphism, observe that f^{-1} is also an S-isomorphism.

Lemma 2.1.9. *Let B be a subsemimodule of an S-semimodule A. Then the inclusion map $1_{B,A}: B \to A$ is an S-monomorphism.*

Proof. Obvious. #

The set of all S-homomorphisms mapping A into B will be denoted by $\mathrm{Hom_S}(\mathbf{A}, \mathbf{B})$, or simply $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$. If $B = A$, then the elements of $\mathrm{Hom}_S(A, A)$ will be called **S-endomorphisms.** I will denote $\mathrm{Hom}_S(A, A)$ by $\mathrm{End}(A)$.

Lemma 2.1.10. *Let A and B be S-semimodules. Then $\mathrm{Hom}(A, B)$ is a commutative monoid under the operation*

$$(f_1 + f_2)(a) = f_1(a) + f_2(a) \qquad \text{for all } a \in A.$$

Proof. Clear. #

Lemma 2.1.11. *If A, B, and C are S-semimodules and $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, then the mapping $g \circ f$ is an S-homomorphism of A to C, i.e., $g \circ f \in \mathrm{Hom}(A, C)$. The following properties hold, with the obvious notations:*

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f,$$

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2,$$

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

$$1_B \circ f = f \circ 1_A = f.$$

Moreover, if f and g are S-monomorphisms (or S-epimorphisms or S-isomorphisms), then so is $g \circ f$.

Finally, if A and B are commutative monoids and $\varphi\colon A \to B$ is a homomorphism, then

(i) *the action*

$$0x = 0$$

$$kx = \underbrace{x + \cdots + x}_{k \text{ times}}\ ; \quad k \in \mathbb{Z}^+$$

makes A and B into $\mathbb{Z}_0^+$-semimodules; and

(ii) *φ is a $\mathbb{Z}_0^+$-homomorphism.*

Proof. Clear. #

Proposition 2.1.12. *Let A be an abelian group and V a right S-semimodule. Then $\mathrm{Hom}_{\mathbb{Z}_0^+}(V, A)$ is an abelian group, and in addition is a left S-semimodule using the scalar multiplication defined by $(s\lambda)(v) = \lambda(vs)$ for all $\lambda \in \mathrm{Hom}_{\mathbb{Z}_0^+}(V, A)$, all $s \in S$, and all $v \in V$.*

Proof. Easy. #

Note. An analogous result holds in the case where V is a left S-semimodule.

Proposition 2.1.13. *Let A, B and C be S-semimodules with $B \subseteq C$ and $\varphi: A \to B$ an S-isomorphism. Then there exists an S-semimodule C^* with $A \subseteq C^*$ and an S-isomorphism $\varphi^*: C^* \to C$ such that $\varphi^*|_A = \varphi$.*

$$\begin{array}{ccc} \exists C^* & \overset{\varphi^*}{\dashrightarrow} & C \\ \cup\!| & & \cup\!| \\ A & \overset{\varphi}{\longrightarrow} & B \end{array}$$

Proof. WLOG, I can assume that $C \cap A = \varnothing$. Let $C^* = (C \setminus B) \cup A$. Then $A \subseteq C^*$. Define $\varphi^*: C^* \to C$ by

$$\varphi^*(c) = \begin{cases} c & \text{if } c \in C \setminus B, \\ \varphi(c) & \text{if } c \in A. \end{cases}$$

Clearly φ^* is a bijection and $\varphi^*|_A = \varphi$.

Now define $+^*: C^* \times C^* \to C^*$ and $\cdot^*: S \times C^* \to C^*$ by $c_1 +^* c_2 = (\varphi^*)^{-1}(\varphi^*(c_1) + \varphi^*(c_2))$ and $s \cdot^* c = (\varphi^*)^{-1}(s\varphi^*(c))$ for all $c_1, c_2, c \in C^*$ and all $s \in S$. It is straightforward to check that these operations make C^* into an S-semimodule, and that φ^* is an S-homomorphism. #

2.2. Congruence relations on semimodules

As I mentioned before, this is a concept not found explicitly in the theory of modules over rings, although it is very important in the theory of semigroups and in universal algebra. Congruence relations allow us to define quotient semimodules and kernels of homomorphisms in such a way that many important results in module theory hold for semimodules as well. In particular, the extremely useful Homomorphism Theorem and universal mapping property of quotients are still true in the more general setting. As usual most of the material in this section either may be found in Chapter 13 of *[1]* or is an analog of a result in module theory. The exception is Definition 2.2.9, which will be commented on later.

Definition 2.2.1. *Let A and B be S-semimodules and $\varphi: A \to B$ an S-homomorphism. Define a binary relation $\equiv_\varphi$ on A by saying that for all a, $b \in A$, $a \equiv_\varphi b$ iff $\varphi(a) = \varphi(b)$.*

It is easily seen that $\equiv_\varphi$ is an equivalence relation on A. Moreover, it satisfies the following: for every $a, b \in A$, if $a \equiv_\varphi b$, then $a + c \equiv_\varphi b + c$ and $sa \equiv_\varphi sb$ for all $c \in A$ and all $s \in S$.

Definition 2.2.2. *An equivalence relation $\sim$ on an S-semimodule A will be called an* S-**congruence** *on A iff it satisfies the following properties:*

(i) *for all a, b and c in A, $a \sim b$ implies that $a + c \sim b + c$; and*

(ii) *for all a, b in A and all $s \in S$, $a \sim b$ implies that $sa \sim sb$.*

Lemma 2.2.3. *If $\varphi: A \to B$ is an S-homomorphism of S-semimodules, then $\equiv_\varphi$ is an S-congruence on A.*

Proof. Obvious from the above comments. #

Suppose that $\sim$ is an S-congruence on an S-semimodule A. Then the interesting question of whether we can define an S-semimodule B and an S-homomorphism $\psi: A \to B$ such that $\equiv_\psi$ is the same as $\sim$ arises. The answer will come after the following definition.

Definition 2.2.4. *Let $\sim$ be an S-congruence on an S-semimodule A. For each element a of A, let $[a]_\sim$ denote the equivalence class of a with respect to this relation. Set $A/\sim$ equal to $\{[a]_\sim \mid a \in A\}$. In addition, define addition and scalar multiplication on $A/\sim$ by setting $[a]_\sim + [b]_\sim = [a+b]_\sim$ and $s[a]_\sim = [sa]_\sim$ for all a, b in A and all s in S.*

Proposition 2.2.5. *Using the above addition and scalar multiplication, $A/\sim$ is an S-semimodule.*

Proof. Standard. #

Definition 2.2.6. *An S-semimodule C is called a* **quotient semimodule** *of A iff there exists an S-congruence $\sim$ on A such that $C = A/\sim$.*

Proposition 2.2.7. *Let $\sim$ be an S-congruence on an S-semimodule A. Then there exists an S-semimodule B and a surjective S-homomorphism $\varphi\colon A \to B$ such that $\equiv_\varphi$ is the same as $\sim$.*

Proof. Let $B = A/{\sim}$ and define $\varphi\colon A \to B$ by $\varphi(a) = [a]_\sim$ for all $a \in A$. Then it is easy to show that φ is a surjective S-homomorphism and $\equiv_\varphi$ is the same as $\sim$. #

Definition 2.2.8. *Let $\sim$ be an S-congruence on an S-semimodule A. Then the S-homomorphism $\pi\colon A \to A/{\sim}$ defined by $\pi(a) = [a]_\sim$ for all $a \in A$ is called the* **natural** *or* **canonical surjection** *of A onto $A/{\sim}$.*

Now, I am ready to define kernels of homomorphisms. In a sense I will define two kinds of kernels. One kind, actually called a zero set, is the unmodified definition of kernel taken from the theory of modules over rings. While still useful, it is deficient in an important way: there is no direct way to test a homomorphism for injectivity using its zero set. What I will call the kernel is actually a congruence relation. This kernel does provide a simple condition for determining whether or not a homomorphism is injective. I should point out here that my definitions are different from those found in *[1]*.

Definition 2.2.9. *Let $\varphi\colon A \to B$ be an S-homomorphism of S-semimodules. Then the* **kernel of** *φ, denoted by $\ker\varphi$, is the relation $\equiv_\varphi$. That is, $\ker\varphi = \{(a,b) \in A \times A \mid \varphi(a) = \varphi(b)\}$. Also, define the* **zero set of** *φ, denoted by $\operatorname{zs}\varphi$, to be $\{a \in A \mid \varphi(a) = 0\}$.*

Lemma 2.2.10. *Let $\varphi: A \to B$ be an S-homomorphism of S-semimodules. Then φ is injective iff $\ker\varphi = \{(a,a) \mid a \in A\}$.*

Proof. Trivial. #

Lemma 2.2.11. *Let $\varphi: A \to B$ be an S-homomorphism of S-semimodules. Then $\operatorname{zs}\varphi$ is a subsemimodule of A.*

Proof. Easy. #

The following two propositions are very important basic results on semimodules.

Proposition 2.2.12. The Homomorphism Theorem

Let $\varphi: A \to B$ be an S-homomorphism of S-semimodules. Then $A/\ker\varphi \cong \operatorname{Im}\varphi$.

Proof. Define $\psi: A/\ker\varphi \to \operatorname{Im}\varphi$ by

$$\psi\left([a]_{\ker\varphi}\right) = \varphi(a) \qquad \text{for all} \quad a \in A.$$

Then it is easy to check that ψ is well-defined and an S-isomorphism. #

Proposition 2.2.13. The universal mapping property of quotients

Let $\varphi: A \to B$ be an S-homomorphism of S-semimodules, and $\sim$ an S-congruence on A such that $\sim\ \subseteq \ker\varphi$. Then there exists a unique S-homomorphism $\varphi': A/\!\sim\ \to B$ such that $\varphi = \varphi' \circ \pi$, where $\pi: A \to A/\!\sim$ is the canonical surjection. Moreover, $\operatorname{Im}\varphi' = \operatorname{Im}\varphi$.

Proof. Define $\varphi' \colon A/{\sim} \to B$ by

$$\varphi'([a]_\sim) = \varphi(a) \qquad \text{for all} \quad a \in A.$$

Then $\sim \subseteq \ker\varphi$ implies φ' is well-defined, and a simple argument proves φ' is the desired S-homomorphism and is unique. It is clear from the definition of φ' that $\operatorname{Im}\varphi' = \operatorname{Im}\varphi$. #

2.3. Group semimodules

This section is probably the most important one in this chapter. The results on group semimodules, especially Proposition 2.3.3 and Proposition 2.3.6, will be crucial for the proofs of many essential propositions in the next chapter. Before defining group semimodules, however, I will define the more general notion of a cancellative semimodule. As will soon become clear, there is a close connection between cancellative semimodules and group semimodules. Moreover, it is the category of cancellative semimodules that will be of primary interest in the next chapter.

Definition 2.3.1. *An S-semimodule A will be called* a **cancellative semimodule** *iff $(A, +)$ is a cancellative monoid.*

Definition 2.3.2. *An S-semimodule A will be called* a **group semimodule** *iff $(A, +)$ is a group.*

Proposition 2.3.3. *Every cancellative S-semimodule can be embedded in a group S-semimodule.*

Proof. Let A be a cancellative S-semimodule. Set $X = A \times A$ and define an operation $+: X \times X \to X$ by

$$(a_1, a_2) + (a'_1, a'_2) = (a_1 + a'_1, a_2 + a'_2) \qquad \text{for all} \quad (a_1, a_2), (a'_1, a'_2) \in X.$$

Then $(X, +)$ is a commutative monoid with identity $(0, 0)$. For each $s \in S$, define $s(a_1, a_2) = (sa_1, sa_2)$ for all $(a_1, a_2) \in X$. It is easily checked that this makes X into an S-semimodule.

Next define the relation $\sim$ on X by $(a_1, a_2) \sim (a'_1, a'_2)$ iff $a_1 + a'_2 = a'_1 + a_2$. It is straightforward to show that $\sim$ is an S-congruence on X. Hence $X/\sim$ is an S-semimodule. To show $X/\sim$ is a group, it only remains to show every element has an inverse. Let $[(a_1, a_2)]_\sim$ be an element of $X/\sim$. Then $[(a_1, a_2)]_\sim + [(a_2, a_1)]_\sim = [(a_1 + a_2, a_1 + a_2)]_\sim = [(0, 0)]_\sim$. This shows $[(a_2, a_1)]_\sim$ is the inverse of $[(a_1, a_2)]_\sim$.

Finally, I claim the map $\varphi: A \to X/\sim$ defined by $\varphi(a) = [(a, 0)]_\sim$ for all $a \in A$ is an embedding. Verifying that φ is an S-homomorphism is easy. To prove φ is injective, suppose $a, a' \in A$ are such that $\varphi(a) = \varphi(a')$. Then $(a, 0) \sim (a', 0)$, which implies $a + 0 = a' + 0$, i.e., $a = a'$. Hence φ is an embedding. #

Definition 2.3.4. *For each cancellative S-semimodule A I will denote the semimodule $X/\sim$ in the proof of Proposition 2.3.3 by A^Δ and call A^Δ the* **group S-semimodule of differences of A**. *In addition, I will denote the map $a \mapsto [(a, 0)]_\sim$ by $i^\Delta: A \to A^\Delta$ and call it the* **standard embedding of A** *into A^Δ.*

In *[1]*, A^{Δ} is defined for an arbitrary S-semimodule A, and is given the name the **S-module of differences of A**. It is proved there that i^{Δ} is injective iff A is cancellative.

Because $i^{\Delta}\colon A \to A^{\Delta}$ is injective, it follows that $A \cong i^{\Delta}(A)$. It is also of interest to note that $[(a_1, a_2)]_{\sim} = i^{\Delta}(a_1) - i^{\Delta}(a_2)$ for all $a_1, a_2 \in A$, and thus $A^{\Delta} = \{i^{\Delta}(a_1) - i^{\Delta}(a_2) \mid a_1, a_2 \in A\}$.

The following proposition is similar to Proposition 14.1 in *[1]*.

Proposition 2.3.5. *Let A be a cancellative S-semimodule and G a group S-semimodule. If $\varphi\colon A \to G$ is an S-homomorphism, then there exists a unique S-homomorphism $\bar{\varphi}\colon A^{\Delta} \to G$ such that $\bar{\varphi} \circ i^{\Delta} = \varphi$.*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & G \\ {\scriptstyle i^{\Delta}}\downarrow & \nearrow {\scriptstyle \bar{\varphi}} & \\ A^{\Delta} & & \end{array}$$

Proof. Define $\bar{\varphi}\colon A^{\Delta} \to G$ by

$$\bar{\varphi}\left(i^{\Delta}(a_1) - i^{\Delta}(a_2)\right) = \varphi(a_1) - \varphi(a_2) \qquad \text{for all} \quad a_1, a_2 \in A.$$

The proof will be done once the four steps below are proved.

Step 1. $\bar{\varphi}$ is well-defined.

Let $i^{\Delta}(a_1) - i^{\Delta}(a_2), i^{\Delta}(a_1') - i^{\Delta}(a_2') \in A^{\Delta}$ be such that $i^{\Delta}(a_1) - i^{\Delta}(a_2) = i^{\Delta}(a_1') - i^{\Delta}(a_2')$. Then the following sequence of equations may be derived.

$$i^{\Delta}(a_1) + i^{\Delta}(a_2') = i^{\Delta}(a_1') + i^{\Delta}(a_2)$$

$$i^{\Delta}(a_1 + a_2') = i^{\Delta}(a_1' + a_2)$$

$$a_1 + a_2' = a_1' + a_2$$

$$\varphi(a_1 + a_2') = \varphi(a_1' + a_2)$$

$$\varphi(a_1) + \varphi(a_2') = \varphi(a_1') + \varphi(a_2)$$

$$\varphi(a_1) - \varphi(a_2) = \varphi(a_1') - \varphi(a_2').$$

Hence $\bar{\varphi}$ is well-defined.

Step 2. $\bar{\varphi}$ is an S-homomorphism.

Step 3. $\bar{\varphi} \circ i^{\Delta} = \varphi$.

Step 4. $\bar{\varphi}$ is unique.

Steps 2–4 are very easy, so I will leave their proofs out. #

Proposition 2.3.6. *Let M be a submonoid of an abelian group G, $\varphi: M \to D$ a homomorphism of M into an abelian group D. Let $\langle M \rangle$ denote the subgroup of G generated by M. Then there exists a unique homomorphism $\varphi^{\Delta}: \langle M \rangle \to D$ such that $\varphi^{\Delta}|_M = \varphi$.*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & D \\ \cap\!\!| & \nearrow_{\varphi^{\Delta}} & \\ \langle M \rangle & & \end{array}$$

Proof. I will divide the proof into four steps.

Step 1. $\langle M \rangle = \{x - y \mid x, y \in M\}$.

It is clear that $\{x - y \mid x, y \in M\}$ is a subgroup of G containing M, so $\langle M \rangle \subseteq \{x - y \mid x, y \in M\}$. Conversely, if $x, y \in M$, then $x, y \in \langle M \rangle$, which is a group, and thus $x - y \in \langle M \rangle$. Therefore $\langle M \rangle = \{x - y \mid x, y \in M\}$.

Step 2. Define $\varphi^{\Delta}: \langle M \rangle \to D$ by $\varphi^{\Delta}(x-y) = \varphi(x) - \varphi(y)$ for all $x, y \in M$. Then φ^{Δ} is a well-defined group homomorphism.

Recall that D is a group, so $-\varphi(y)$ is defined. The rest of the proof of this step is similar to Steps 1 and 2 in the previous proof.

Step 3. $\varphi^{\Delta}|_M = \varphi$.

Step 4. φ^{Δ} is unique.

Steps 3 and 4 are proved by arguments similar to those used for the corresponding steps in the previous proof. #

Proposition 2.3.7. *Let A be a group S-semimodule, B an S-semimodule and $\varphi: A \to B$ an S-homomorphism. Then φ is injective iff $\operatorname{zs}\varphi = \{0\}$.*

Proof. It is obvious that $\operatorname{zs}\varphi = \{0\}$ if φ is injective. Thus assume that $\operatorname{zs}\varphi = \{0\}$. Let $x, y \in A$ be such that $\varphi(x) = \varphi(y)$. Then $\varphi(x-y) = \varphi(x) + \varphi(-y) = \varphi(y) + \varphi(-y) = \varphi(y-y) = \varphi(0) = 0$. Thus $x - y \in \operatorname{zs}\varphi$, so $x = y$. Hence φ is injective. #

2.4. Direct products and direct sums

Fortunately, we can define direct products and direct sums in the same way as for modules over rings. We also get that most of their basic properties are the same. Again, the material in this section may be found (in an abbreviated form) in *[1]*.

Definition 2.4.1. *Let $(A_i)_{i\in I}$ be a family of S-semimodules. Consider the set $A = \prod_{i\in I} A_i$ with operations:*

$$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I},$$

$$s(x_i)_{i\in I} = (sx_i)_{i\in I}.$$

Then A is an S-semimodule called the **direct product** *of the family $(A_i)_{i\in I}$.*

For each $i_0 \in I$ the mapping $\pi_{i_0}: \prod_{i\in I} A_i \to A_{i_0}$ defined by $\pi_{i_0}\big((x_i)_{i\in I}\big) = x_{i_0}$ is called the **natural** *or* **canonical projection** *of $\prod_{i\in I} A_i$ onto A_{i_0}, and is clearly an S-epimorphism.*

For every $i_0 \in I$ the mapping $j_{i_0}: A_{i_0} \to \prod_{i\in I} A_i$ defined by $j_{i_0}(t) = (x_i)_{i\in I}$, where $x_i = 0$ for all $i \neq i_0$ and $x_{i_0} = t$, is called the **natural** *or* **canonical injection** *of A_{i_0} into $\prod_{i\in I} A_i$, and is clearly an S-monomorphism.*

With the obvious modification this definition can also be used to define the direct product of a family of right S-semimodules.

Proposition 2.4.2. The universal mapping property of direct products

Let $A = \prod_{i\in I} A_i$ be the direct product of the family of S-semimodules $(A_i)_{i\in I}$ and let $\pi_i: A \to A_i$ be the canonical projection for all $i \in I$. If B is an S-semimodule and $q_i: B \to A_i$ are S-homomorphisms for all $i \in I$, then there exists a unique S-homomorphism $f: B \to A$ such that $q_i = \pi_i \circ f$ for all $i \in I$; in other words, the following diagram is commutative for every $i \in I$:

$$\begin{array}{ccc} B & \xrightarrow{f} & A = \prod_{i\in I} A_i \\ & {\scriptstyle q_i}\searrow & \downarrow{\scriptstyle \pi_i} \\ & & A_i \end{array}$$

Proof. Let $f: B \to A$ be defined by $f(x) = \big(q_i(x)\big)_{i\in I}$ for all $x \in B$. Then it is easy to check that f is the desired S-homomorphism and that f is unique. #

Lemma 2.4.3. *The canonical projections and injections satisfy the following properties:*

$$\pi_i \circ j_i = 1_{A_i} \qquad \text{for all} \quad i \in I,$$

$$\pi_i \circ j_{i'} = 0 \text{ (the zero mapping)} \quad \text{for all} \quad i, i' \in I, i \neq i'.$$

Proof. Obvious. #

Notation. If $I = \{1,2\}$, I will denote $\prod_{i\in I} A_i$ by $A_1 \times A_2$.

Before stating the next definition, let me introduce a bit of terminology. If B is a set, then I will say that a particular property holds for **almost all** elements of B iff there is a finite subset F of B such that the property holds for every element in $B \smallsetminus F$.

Definition 2.4.4. *Let $(A_i)_{i\in I}$ be a family of S-semimodules. Consider the subset*

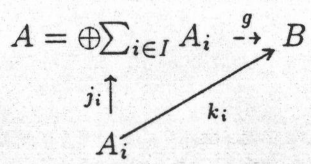
$$\oplus\sum_{i\in I} A_i = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} A_i \;\middle|\; x_i = 0 \quad \text{for almost all indicies } i \in I \right\}$$

of $\prod_{i\in I} A_i$. Then $A = \oplus\sum_{i\in I} A_i$ is clearly a subsemimodule of $\prod_{i\in I} A_i$, called the **direct sum** *of the family $(A_i)_{i\in I}$.*

Remarks. Observe that the above definition applies equally well to families of left semimodules and families of right semimodules. Let $(A_i)_{i\in I}$ be a family of S-semimodules and fix $i_0 \in I$. The restriction of the canonical projection π_{i_0} to $\oplus\sum_{i\in I} A_i$ is still an S-epimorphism, which will be called the **canonical projection of** $\oplus\sum_{i\in I} A_i$ onto A_{i_0}; it will also be denoted by π_{i_0}, even though, strictly speaking, it is a different function. Similarly the canonical injection j_{i_0} actually maps A_{i_0} into $\oplus\sum_{i\in I} A_i$. Hence whenever I am discussing direct sums I will refer to it as the canonical injection of A_{i_0} into $\oplus\sum_{i\in I} A_i$.

Proposition 2.4.5. The universal mapping property of direct sums

Let $A = \oplus\sum_{i\in I} A_i$ be the direct sum of the family of S-semimodules $(A_i)_{i\in I}$ and let $j_i: A_i \to A$ be the canonical injections for all $i \in I$. If B is an S-semimodule and $k_i: A_i \to B$ are S-homomorphisms for all $i \in I$, then there exists a unique S-homomorphism $g: A \to B$ such that $k_i = g \circ j_i$ for all $i \in I$; that is, the following diagram commutes for all $i \in I$:



Proof. Let $g: A \to B$ be defined by

$$g(x) = \sum_{i\in I} k_i[\pi_i(x)] \quad \text{for all} \quad x \in A,$$

where $\pi_i: A \to A_i$ is the canonical projection for all $i \in I$. Note that this sum is well-defined since only finitely many summands $k_i[\pi_i(x)]$ are different from 0. It is easy to finish the proof from here. #

Notation. If $I = \{1, 2\}$, then $\oplus\sum_{i\in I} A_i$ will be denoted by $A_1 \oplus A_2$.

Note. $\oplus\sum_{i\in I} A_i = \prod_{i\in I} A_i$ iff I is a finite set.

2.5. Free semimodules

The idea for defining free semimodules is similar to that for defining free modules over rings. Perhaps surprisingly, most of the basic properties of free modules are still true in this more general setting. The approach I have taken follows that of *[2]* for the case of modules over rings. For a slightly different, but equivalent approach, see *[1]*.

Definition 2.5.1. *A subset B of an S-semimodule A is called a* **basis** *of A iff for every element a of A there exists a unique family $(s_b)_{b\in B}$ of elements of S such that $s_b = 0$ for almost all $b \in B$ and $a = \sum_{b\in B} s_b b$. Moreover, A is called a* **free (left)** *semimodule iff there exists a subset B of A such that B is a basis of A.*

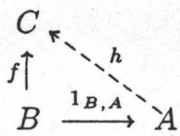
Note. If B is a basis of a free semimodule A, then B generates A and B is linearly independent (using the standard definition of linear independence). However, the converse is not true. For example, the set $\{1, 2\}$ is a linearly independent subset of the $\mathbb{Z}_0^+$-semimodule $\mathbb{Z}_0^+$, and it also generates $\mathbb{Z}_0^+$, but it is not a basis (observe that 2 does not have a unique representation).

Free right S-semimodules can be defined in an analogous manner. The next proposition is very important.

Proposition 2.5.2. Fundamental properties of free S-semimodules

(i) **(The universal mapping property)**

If A is a free S-semimodule with basis B and $f: B \to C$ is any mapping into an S-semimodule C, then there exists one and only one S-homomorphism $h: A \to C$ which extends f.



(ii) *Let B be a nonempty set, and let $M_b = S$ for all $b \in B$. Then $\oplus\sum_{b\in B} M_b$ is a free S-semimodule. Moreover, if for each element b of B, $f_b \in \oplus\sum_{b\in B} M_b$ is defined by*

$$f_b(b') = \begin{cases} 1 & \text{if } b' = b, \\ 0 & \text{if } b' \neq b; \end{cases}$$

then $\{f_b \mid b \in B\}$ is a basis of $\oplus\sum_{b\in B} M_b$ and the map $b \mapsto f_b$ is a bijection of B with $\{f_b \mid b \in B\}$.

(iii) *Let F be a free S-semimodule with basis B. For each $b \in B$ let $M_b = S$. Then $\oplus\sum_{b\in B} M_b \cong F$.*

(iv) *For every S-semimodule A there is a free S-semimodule F and a surjective S-homomorphism $\alpha: F \to A$. If A is finitely generated, it is possible to choose F with a finite basis.*

Note. All four parts of the above proposition are true for right semimodules as well as left semimodules, although I will only prove them in the case of left semimodules.

Proof. (i) By hypothesis, for every element a of A there is a unique family $(s_b)_{b\in B}$ of elements of S such that $s_b = 0$ for almost all $b \in B$ and $a = \sum_{b\in B} s_b b$. Define $h(a) = \sum_{b\in B} s_b f(b)$; it is an easy task to check that h satisfies the conditions of the statement.

(ii) It is straightforward to verify that $\{f_b \mid b \in B\}$ is a basis of $\oplus\sum_{b\in B} M_b$; thus by Definition 2.5.1 $\oplus\sum_{b\in B} M_b$ is free. It is obvious that the map $b \mapsto f_b$ is a bijection.

(iii) Let $G = \oplus\sum_{b\in B} M_b$. Then G is a free S-semimodule and $B^* = \{f_b \mid b \in B\}$ is a basis of G. Using the universal mapping property of free S-semimodules there exist S-homomorphisms $\varphi: F \to G$ and $\psi: G \to F$ such that $\varphi(b) = f_b$ and $\psi(f_b) = b$ for all $b \in B$. But then $\psi \circ \varphi: F \to F$ and $\psi \circ \varphi(b) = b$ for all $b \in B$, so by the uniqueness part of the universal mapping property, $\psi \circ \varphi = 1_F$. Likewise $\varphi \circ \psi = 1_G$. Thus φ is an isomorphism of F with G.

(iv) Let X be a set of generators for A, and for each $x \in X$ let $M_x = S$. Then $F = \oplus\sum_{x\in X} M_x$ is a free S-semimodule and $\{f_x \mid x \in X\}$ is a basis of F. By the universal mapping property there is a unique S-homomorphism $\alpha: F \to A$ such that $\alpha(f_x) = x$ for all $x \in X$. Since X generates A, α must be surjective. The last assertion is clear from the above proof. #