CHAPTER II

SOME USEFUL PROPERTIES OF SEQUENCE SPACES AND MATRIX TRANSFORMATIONS

The purpose of this chapter is to give some useful properties of sequence spaces and matrix transformations which will be used in this research. The first two propositions of this chapter give some properties of BK-spaces. In the last proposition of this chapter, we give some useful necessary conditions for a matrix to be each of the following: convergence preserving matrix transformations, limit preserving matrix transformations, summability preserving matrix transformations and sum preserving matrix transformations.

Proposition 2.1. If X is a BK-space with norms $\|.\|_1$ and $\|.\|_2$, then $\|.\|_1 \sim \|.\|_2$, that is, there exist postive real numbers a and b such that $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ for all $x \in X$.

Proof: Let i be the identity mapping on X. By Theorem 1.3, we have that $i : (X, \|.\|_2) \longrightarrow (X, \|.\|_1)$ is continuous, so it is bounded. Hence there exists $a > 0$ such that $\|x\|_1 = \|i(x)\|_1 \leq a\|x\|_2$ for all $x \in X$. By interchanging the role of $\|.\|_1$ and $\|.\|_2$, we obtain that there exists $b > 0$ such that $\|x\|_2 \leq b\|x\|_1$ for all $x \in X$. Hence $\frac{1}{a}\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ for all $x \in X$. #

Proposition 2.2. Let X and Y be normed sequence spaces. Then the following statements hold.

(i) If X and Y are BK-spaces, then $X \cap Y$ with norm $\max\{\|.\|_X, \|.\|_Y\}$ is a BK-space where $\|.\|_X$ and $\|.\|_Y$ are the norms of X and Y, respectively.

(ii) If X and Y have the AK property, then $X \cap Y$ with norm $\max\{\|.\|_X, \|.\|_Y\}$ has the AK property.

Proof: Let $\|.\| = \max\{\|.\|_X, \|.\|_Y\}$ on $X \cap Y$.

(i) Assume that X and Y are BK-spaces. Let $k \in \mathbb{N}$. Since the k^{th} coordinate mapping p_k is a continuous linear functional on X, there exists $\alpha > 0$ such that $|p_k(x)| \leq \alpha\|x\|_X$ for every $x \in X$. Then for $z \in X \cap Y$, $|p_k(z)| \leq \alpha\|z\|_X \leq \alpha \max\{\|z\|_X, \|z\|_Y\} = \alpha\|z\|$. Hence p_k is continuous on $X \cap Y$ with respect to $\|.\|$. Therefore $(X \cap Y, \|.\|)$ is a K-space.

To show $(X \cap Y, \|.\|)$ is a Banach space, let $(z^{(n)})$ be a Cauchy sequence in $(X \cap Y, \|.\|)$. Then $(z^{(n)})$ is a Cauchy sequence in both X and Y. This implies that there exists $x \in X$ and $y \in Y$ such that $\lim_{n\to\infty} z^{(n)} = x$ in X and $\lim_{n\to\infty} z^{(n)} = y$ in Y. Since X and Y are K-spaces, it follows that for every $k \in \mathbb{N}$, $\lim_{n\to\infty} z_k^{(n)} = x_k$ and $\lim_{n\to\infty} z_k^{(n)} = y_k$. Hence $x_k = y_k$ for every $k \in N$, so $x = y \in X \cap Y$. Therefore $\lim_{n\to\infty} \|z^n - x\|_X = 0$ $= \lim_{n\to\infty} \|z^n - x\|_Y$ which implies that $\lim_{n\to\infty} \|z^n - x\| = \lim_{n\to\infty} \max\{\|z^n - x\|_X, \|z^n - x\|_Y\}$ $= 0$. Hence $\lim_{n\to\infty} z^{(n)} = x$ in $(X \cap Y, \|.\|)$.

(ii) Assume that X and Y have the AK property. Then X and Y contain all finite sequences. Thus $X \cap Y$ contains all finite sequences.

Let $z \in X \cap Y$. Since X and Y have the AK property, $z = \lim_{n\to\infty} \sum_{k=1}^{n} z_k e^{(k)}$ in X and $z = \lim_{n\to\infty} \sum_{k=1}^{n} z_k e^{(k)}$ in Y. Then $\lim_{n\to\infty} \| \sum_{k=1}^{n} z_k e^{(k)} - z \|_X = 0$ and $\lim_{n\to\infty} \| \sum_{k=1}^{n} z_k e^{(k)} - z \|_Y = 0$, which implies that $\lim_{n\to\infty} \| \sum_{k=1}^{n} z_k e^{(k)} - z \| = 0$. Hence $z = \lim_{n\to\infty} \sum_{k=1}^{n} z_k e^{(k)}$ in $(X \cap Y, \|.\|)$. This proves that $(X \cap Y, \|.\|)$ has the AK property. #

<u>Proposition 2.3</u>. Let X be a sequence space and A an infinite matrix mapping X into a sequence space. Then the following statements hold.

(i) If A preserves convergence and $e^{(k)} \in X$ for every $k \in N$, then $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

(ii) If A preserves limits and $e^{(k)} \in X$ for every $k \in \mathbb{N}$,then $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

(iii) If A preserves convergence and $e \in X$, then $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists.

(iv) If A perserves limits and $e \in X$, then $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} = 1$.

(v) If A preserves summability and $e^{(k)} \in X$ for every $k \in X$, then $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

(vi) If A preserves sums and $e^{(k)} \in X$ for every $k \in N$, then $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in N$.

<u>Proof</u>: We obtain (i)-(vi) from the following facts: $\lim_{n\to\infty} e_n^{(k)} = 0$ for every $k \in N$, $\lim_{n\to\infty} e_n = 1$, $\sum_{n=1}^{\infty} e_n^{(k)} = 1$, $Ae^{(k)} = (A_{nk})_{n=1}^{\infty}$ for every $k \in N$ and $Ae = (\sum_{k=1}^{\infty} A_{nk})_{n=1}^{\infty}$. #

From Proposition 2.3, the following corollary is directly obtained.

Corollary 2.4. Let X be a sequence space and A an infinite matrix. Then the following statements hold:

(i) If $A:X \longrightarrow c$ and $e^{(k)} \in X$ for every $k \in N$, then $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

(ii) If $A:X \longrightarrow c$ and $e \in X$, then $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists.

(iii) If $A:X \longrightarrow cs$ and $e^{(k)} \in X$ for every $k \in N$, then $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.