CHAPTER III

THE SPACE OF ALL SEQUENCES AND THE SPACE OF ALL FINITE SEQUENCES

Among sequence spaces, W is the largest one. Most of the standard sequence spaces contain Φ as a subspace. The purpose of this chapter is to characterize any infinite matrix A such that $A : X \longrightarrow \ell_\infty$ where $X = W$ or Φ and A posesses one of the properties of convergence preservation, limit preservation, summability preservation and sum preservation.

3.1 THE SPACE OF ALL SEQUENCES

To characterize an infinite matrix A such that $A : W \longrightarrow \ell_\infty$ and A preserves convergence, the following lemma is required.

<u>Lemma 3.1.1</u>. For any infinite matrix A, $A : W \longrightarrow \ell_\infty$ and A preserves convergence if and only if $A : W \longrightarrow c$.

<u>Proof</u>: Assume that $A : W \longrightarrow \ell_\infty$ and A preserves convergence. By Theorem 1.10, A is row-bounded. Since $e^{(k)} \in W$ for every $k \in \mathbb{N}$, it follows from Proposition 2.3(i) that $\lim\limits_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$. Hence we have from Theorem 1.11 that $A : W \longrightarrow c$.

The converse is trivial. #

The following theorem is directly obtained from Lemma 3.1.1 and Theorem 1.11.

Theorem 3.1.2. For an infinite matrix A, $A: W \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) A is row-bounded and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

Suppose that an infinite matrix $A: W \longrightarrow \ell_\infty$ and $A(c) \subseteq c$. Then A satisfies the condition (i) and (ii) of Theorem 3.1.2. By (i), there exists $m \in \mathbb{N}$ such that $A_{nk} = o$ for all $n \in \mathbb{N}$ and for all $k > m$. Thus for every $n \in \mathbb{N}$,

$$(Ae)_n = \sum_{k=1}^{m} A_{nk} = \sum_{k=1}^{m} (Ae^{(k)})_n$$

and so

$$\lim_{n\to\infty} (Ae)_n = \sum_{k=1}^{m} (\lim_{n\to\infty} (Ae^{(k)})_n)$$

which implies that A is not limit preserving since $\lim_{n\to\infty} e_n = 1$ and $\lim_{n\to\infty} e_n^{(k)} = 0$ for every $k \in \mathbb{N}$. Hence we obtain the following theorem.

Theorem 3.1.3. There is no matrix transformation from W into ℓ_∞ which preserves limits.

Our characterization of a summability preserving matrix transformation from W into ℓ_∞ is obtained directly from the following two lemmas.

Lemma 3.1.4. For an infinite matrix A, A : W $\longrightarrow$ cs if and only if

(i) A is row-bounded and

(ii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

If this is the case, $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \sum_{n=1}^{\infty} A_{nk})$ for every sequence $x = (x_k)$.

Proof: If A:W $\longrightarrow$cs, (i) and (ii) hold respectively by Theorem 1.10 and Corollary 2.4 (iii).

For the converse, assume that (i) and (ii) hold. Then there exists $m \in \mathbb{N}$ such that $A_{nk} = 0$ for all $n \in \mathbb{N}$ and for all $k > m$. Let $x = (x_k) \in W$. Then $(Ax)_n = \sum_{k=1}^{m} A_{nk}x_k$ for all $n \in N$. Since $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$, we have that $\sum_{k=1}^{m} (x_k \sum_{n=1}^{\infty} A_{nk}) = \sum_{n=1}^{\infty} (\sum_{k=1}^{m} A_{nk}x_k) = \sum_{n=1}^{\infty} (Ax)_n$. Hence $Ax \in cs$. Since $A_{nk} = 0$ for all $n \in \mathbb{N}$ and for all $k > m$,it follows that $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{m} (x_k \sum_{n=1}^{\infty} A_{nk}) = \sum_{k=1}^{\infty} (x_k \sum_{n=1}^{\infty} A_{nk})$. #

Lemma 3.1.5. For an infinite matrix A, A : W $\longrightarrow \ell_\infty$ and A preserves summability if and only if A : W $\longrightarrow$ cs .

Proof: Assume that A:W$\longrightarrow\ell_\infty$ and A preserves summability. Then the conditions (i) and (ii) of Lemma 3.1.4 hold by Theorem 1.11 and Proposition 2.3 (v), respectively. Hence A: W$\longrightarrow$cs by Lemma 3.1.4.

The converse is trivial. #

Theorem 3.1.6. For an infinite matrix A, A : W $\longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) A is row-bounded and

(ii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

If this is the case, $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \sum_{n=1}^{\infty} A_{nk})$ for every sequence $x = (x_k)$.

Let an infinite matrix $A : W \longrightarrow \ell_\infty$ be such that $A(cs) \subseteq cs$. Then A is row-bounded, so there exists $m \in \mathbb{N}$ such that $A_{nm} = 0$ for all $n \in \mathbb{N}$, so $\sum_{n=1}^{\infty} A_{nm} = 0$. It follows from Proposition 2.3 (vi) that A is not sum preserving. Hence we have the following theorem:

Theorem 3.1.7. There is no matrix transformation from W into ℓ_∞ which preserves sums.

3.2 THE SPACE OF ALL FINITE SEQUENCES

Since $\Phi \subseteq c$, we have that for an infinite matrix A, $A : \Phi \longrightarrow \ell_\infty$ and A preserves convergence if and only if $A : \Phi \longrightarrow c$. The following lemma gives necessary and sufficient conditions for an infinite matrix A to map Φ into c .

Lemma 3.2.1. For an infinite matrix A, $A:\Phi \longrightarrow c$ if and only if $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

If this is the case, $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \lim_{n\to\infty} A_{nk})$ for every $x = (x_k) \in \Phi$.

Proof: The necessity of the condition is obtained from Corollary 2.4 (i)

To prove the condition is sufficient, suppose that $\lim_{n\to\infty} A_{nk}$ exists for all $k \in N$. Let $x = (x_k) \in \Phi$. Then there exists $m \in N$ such that $x_k = 0$ for all $k > m$. Hence $(Ax)_n = \sum_{k=1}^{m} A_{nk}x_k$ for every $n \in N$. Since $\lim_{n\to\infty} A_{nk}$ exists for every $k \in N$, we have that $Ax \in c$ and $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{m} (x_k \lim_{n\to\infty} A_{nk})$. Hence $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \lim_{n\to\infty} A_{nk})$ since $x_k = 0$ for all $k > m$. #

Hence we have the following theorem.

<u>Theorem 3.2.2</u>. For an infinite matrix A, $A : \Phi \longrightarrow \ell_\infty$ and A preserves convegence if and only if $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

If this is the case, $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \lim_{n\to\infty} A_{nk})$ for every $x = (x_k) \in \Phi$.

From the fact that $\Phi \subseteq c_o$, we have that for an infinite matrix A, $A : \Phi \longrightarrow \ell_\infty$ and A preserves limits if and only if $A : \Phi \longrightarrow c_o$. The necessary and sufficient conditions for an infinite matrix A to map Φ into c_o are as follows:

<u>Lemma 3.2.3</u>. For an infinite matrix A, $A:\Phi \longrightarrow c_o$ if and only if $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

<u>Proof</u>: The condition is necessary because $e^{(k)} \in \Phi \subseteq c_o$ and $(A_{nk})_{n=1}^{\infty} = Ae^{(k)}$ for every $k \in \mathbb{N}$.

To show the condition is sufficient, suppose that $\lim_{n\to\infty} A_{nk} = 0$

for every $k \in N$. By Lemma 3.2.1, we have that $Ax \in c$ and $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \lim_{n\to\infty} A_{nk})$ for every $x = (x_k) \in \Phi$. Since $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$, we get that $\lim_{n\to\infty} (Ax)_n = 0$ for every $x \in \Phi$. Therefore $A : \Phi \longrightarrow c_o$.#

Hence the following theorem is obtained.

<u>Theorem 3.2.4</u>. For an infinite matrix A, $A : \Phi \longrightarrow \ell_\infty$ and A presserves limits if and only if $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Since $\Phi \subseteq cs$, we have that for an infinite matrix A, $A:\Phi \longrightarrow \ell_\infty$ and A preserves summability if and only if $A:\Phi \longrightarrow cs$. The next lemma gives a characterization of an infinite matrix A such that $A:\Phi \longrightarrow cs$.

<u>Lemma 3.2.5</u>. For an infinite matrix A, $A:\Phi \longrightarrow cs$ if and only if $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

If this is the case, $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \sum_{n=1}^{\infty} A_{nk})$.

<u>Proof</u>: From Corollary 2.4 (iii), we have that $A : \Phi \longrightarrow cs$ implies $\sum_{n=1}^{\infty} A_{nk}$ converges for very $k \in \mathbb{N}$.

For the converse, assume that $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$. Let $x = (x_k) \in \Phi$. Then there exists $m \in \mathbb{N}$ such that $x_k = 0$ for all $k > m$. Hence $(Ax)_n = \sum_{k=1}^{m} A_{nk}x_k$ for every $n \in \mathbb{N}$, which implies that $\sum_{k=1}^{m} (\sum_{n=1}^{\infty} A_{nk}x_k) = \sum_{n=1}^{\infty} (\sum_{k=1}^{m} A_{nk}x_k) = \sum_{n=1}^{\infty} (Ax)_n$. But $x_k = 0$ for all $k > m$, so $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \sum_{n=1}^{\infty} A_{nk})$. #

Hence we have the following theorem.

Theorem 3.2.6. For an infinite matrix A, $A : \Phi \longrightarrow \ell_\infty$ and A preserves summability if and only if $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in N$.

If this is the case, $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \sum_{n=1}^{\infty} A_{nk})$ for every $x \in \Phi$.

The last theorem of this section shows that the condition $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$ of an infinite matrix A is necessary and sufficient for A to map Φ into ℓ_∞ and preserve sums.

Theorem 3.2.7. For an infinite matrix A, $A : \Phi \longrightarrow \ell_\infty$ and A preserves sums if and only if $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$.

Proof: The condition is necessary by Proposition 2.3 (vi).

To show the condition is sufficient, assume that $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$. By Theorem 3.2.6, $A:\Phi \longrightarrow \ell_\infty$ and A preserves summability and also $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} (x_k \sum_{n=1}^{\infty} A_{nk})$ for every $x = (x_k) \in \Phi$. From $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$, we have that $\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} x_k$. Hence A is sum preserving. #