CHAPTER III

EMBEDDING THEOREMS

In [4], we have already studied the embedding of a semiring in a ratio semiring, a semiring in a semifield of type I, a ratio semiring in a field and a semifield of type I in a field. We will develop some further embedding theorems in this chapter.

Theorem 3.1 Let K be a semifield of type I, then K can be embedded into a field iff K is additively cancellative and satisfies : for all $x, y \in K$, $1+xy = x+y \rightarrow x = 1$ or $y = 1$.

Proof See [4], page 43-44. #

Remark 3.2 Let $K = \{a,e\}$ with structure :

$\cdot$	a	e
a	a	e
e	e	e

and

+	a	e
a	e	a
e	a	e

Then K is a semifield of type II w.r.t.a. Define $\Phi : K \rightarrow \mathbb{Z}_2$ by $\Phi(a) = \bar{1}$ and $\Phi(e) = \bar{0}$. Then Φ is an isomorphism. Hence $K \cong \mathbb{Z}_2$. So we have K is a semifield of type II which can be embedded into a field.

Theorem 3.3 Let K be a semifield of type II. Then K can be embedded into a field iff $K \cong \mathbb{Z}_2$.

Proof Assume that K can be embedded into a field F. Then up to isomorphism we can consider that $K \subseteq F$. Let $a \in K$ be such that

$(K \setminus \{a\},\cdot)$ is a group, let e be the identity of $(K \setminus \{a\},\cdot)$, let 0 be the additive identity of F and let 1 be the multiplicative identity F. Since K is a type II semifield, we get that a and e are the only multiplicative dempotents of K. Since 0 and 1 are the only multiplicative idempotents of F, we obtain that $\{a,e\} = \{0,1\}$. Assume that $a = 0$. Let $x \in K \setminus \{a\}$. Then $x = ax = ox = o = a$, a contradiction. Hence $a \neq 0$. Thus $a = 1$, so $e = 0$. Suppose that $|K| > 2$. Let $x \in K \setminus \{a,e\}$. Then $x = xe = xo = o = e$, a contradiction. Hence $|K| = 2$. Then $K = \{a,e\} = \{0,1\}$. If $1+1 = 1$, then $1+1 = 1+0$. Since F is additively cancellative, we get that $1 = 0$, a contradiction. Thus $1+1 = 0$. Therefore $K = \{0,1\}$ is a field of order 2. Hence $K \cong \mathbb{Z}_2$.

Conversely assume that $K \cong \mathbb{Z}_2$. Then K is isomorphic to a field. Therefore K can be embedded into a field. #

Theorem 3.4 Let K be a semifield of type III. Then K cannot be embedded into any field.

Proof Let $a \in K$ be such that $(K \setminus \{a\},\cdot)$ is a group and e the identity of $(K \setminus \{a\},\cdot)$. Suppose that K can be embedded into a field F. Then up to isomorphism we can consider $K \subseteq F$. Let 0 be the additive identity of F and 1 the multiplicative identity of F. Then 0 and 1 are the only idempotents of F. Since K is a type III semifield, we get that e is the only multiplicative idempotent of K. Then $a \neq 0$ and $e = 0$ or $e = 1$. If $e = 1$, then $ae = a1 = a$, a contradiction. Hence $e = 0$. Then $a^2 = a^2e = a^20 = 0$. Thus $a^2 = 0$ which is a contradiction because a is a non-zero element in F. #

The next two theorems will show that every ratio semiring can be embedded into a semifield of zero type and a semifield of infinity type.

Theorem 3.5 Every ratio semiring can be embedded into a semifield of zero type.

Proof Let D be a ratio semiring. Let a be a symbol not representing any element of D. Extend + and • from D to D ∪ {a} by ax = xa = a for all x ε D ∪ {a} and a+x = x+a = x for all x ε D ∪ {a}. To show D ∪ {a} is a semifield, we must show that (a) (xy)z = x(yz) for all x,y, z ε D ∪ {a}, (b) (x+y)+z = x+(y+z) for all x,y,z ε D ∪ {a} and (c) (x+y)z = xz yz for all x,y,z ε D ∪ {a}. To show (a), we will consider the following cases. If one of x,y,z is a, then (xy)z = a = x(yz). If all of x,y,z are not a, then x,y,z ε D. Thus (xy)z = x(yz).

To show (b), we will consider the following cases :

Case 1 x = y = z = a.

(x+y)+z = (a+a)+a = a+a = a+(a+a) = x+(y+z) since a+a = a.

Case 2 x = y = a, z ≠ a.

(x+y)+z = (a+a)+z = a+z = a+(a+z) = x+(y+z).

Case 3 x = z = a, y ≠ a.

(x+y)+z = (a+y)+a = y+a = a+(y+a) = x+(y+z).

Case 4 y = z = a, x ≠ a.

(x+y)+z = (x+a)+a = x+a = x+(a+a) = x+(y+z).

Case 5 x = a, y ≠ a, z ≠ a.

(x+y)+z = (a+y)+z = y+z = a+(y+z) = x+(y+z).

Case 6 x ≠ a, y = a, z ≠ a.

(x+y)+z = (x+a)+z = x+z = x+(a+z) = x+(y+z).

Case 7 $x \neq a$, $y \neq a$, $z = a$.

$(x+y)+z = (x+y)+a = x+y = x+(y+a) = x+(y+z)$.

Case 8 $x \neq a$, $y \neq a$, $z \neq a$. Then $x,y,z \in D$. Thus $(x+y)+z = x+(y+z)$.

To show (c), we will consider the following cases :

Case 1 $z = a$.

$(x+y)z = (x+y)a = a = a+a = xa+xa = xz+yz$.

Case 2 $z \neq a$, $x = a$.

$(x+y)z = (a+y)z = yz = a+yz = az+yz = xz+yz$.

Case 3 $z \neq a$, $y = a$.

$(x+y)z = (x+a)z = xz = xz+a = xz+az = xz+yz$.

Case 4 $z \neq a$, $x \neq a$, $y \neq a$. Then $x,y,z \in D$. Thus $(x+y)z = xz+yz$. Therefore $D \cup \{a\}$ is a semifield. Clearly it is a semifield of zero type. Define $f: D \to D \cup \{a\}$ by $f(x) = x \; \forall x \in D$. Then f is a monomorphism. Hence D can be embedded into a semifield of zero type. #

Theorem 3.6 Every ratio semiring can be embedded into a semifield of infinity type.

Proof Let D be a ratio semiring. Let a be a symbol not representing any element of D. Extend + and $\cdot$ from D to $D \cup \{a\}$ by $ax = xa = a$ for all $x \in D \cup \{a\}$ and $a+x = x+a = a$ for all $x \in D \cup \{a\}$. To show $D \cup \{a\}$ is a semifield, we must show that (1) $(x+y)z = x(yz)$ for all $x,y,z \in D \cup \{a\}$, (2) $(x+y)+z = x+(y+z)$ for all $x,y,z \in D \cup \{a\}$ and (3) $(x+y)z = xz+yz$ for all $x,y,z \in D \cup \{a\}$. The Proof of (1) is the same as the proof of (a) in Theorem 3.5.

To show (2), we will consider the following cases. If one of x,y,z is a, then $(x+y)+z = a = x+(y+z)$. If all of x,y,z are not a, then $x,y,z \in D$. Thus $(x+y)+z = x+(y+z)$.

To show (3), we will consider the following cases. If one of x,y,z is a, then $(x+y)z = a = xz+yz$. If all of x,y,z are not a, then $x,y,z \in D$. Thus $(x+y)z = xz+yz$.

Therefore $D \cup \{a\}$ is a semifield and clearly it is a semifield of infinity type. Define $f: D \to D \cup \{a\}$ by $f(x) = x$ for all $x \in D$. Then f is a monomorphism. Hence D can be embedded into a semifield of infinity type. #

Remark 3.7 Since $\mathbb{Q}^+$ with the usual addition and multiplication is a ratio semiring, it follows from theorem 3.6 that $\mathbb{Q}^+ \cup \{\infty\}$ can be made into an infinity semifield by extending + and $\cdot$ by $x\cdot\infty = \infty\cdot x = \infty$ and $x+\infty = \infty+x = \infty$ for all $x \in \mathbb{Q}^+ \cup \{\infty\}$. Clearly ∞ is an additive zero and hence we see that $\mathbb{Q}^+$ can be embedded into $\mathbb{Q}^+ \cup \{\infty\}$ which is a semifield of infinity type.

Proposition 3.8 Every ratio semiring can be embedded into a semifield of type II.

Proof Let D be a ratio semiring. Let a be a symbol not representing any element in D. Extend + and $\cdot$ from D to $D \cup \{a\}$ by $ax = xa = x$ for all $x \in D \cup \{a\}$ and $a+x = x+a = 1+x$ for all $x \in D \cup \{a\}$. By Theorem 2.39, $D \cup \{a\}$ is a semifield of type II. Let $f: D \to D \cup \{a\}$ defined by $f(x) = x \ \forall x \in D$. Clearly f is a monomorphism. Hence we can embed D into a semifield of type II. #

Corollary 3.9 Let S be a semiring. If S is multiplicatively cancellative then S can be embedded into a semifield of type II.

Proof By Theorem 1.16, S can be embedded into a ratio semiring, say D. And by Proposition 3.8, D can be embedded into a semifield of type II. Therefore S can be embedded into a semifield of type II. #

Proposition 3.10 Every ratio semiring can be embedded into a semifield of type III.

Proof Let D be a ratio semiring, $d \in D$ and a a symbol not representing any element of D. Extend + and $\cdot$ from D to $D \cup \{a\}$ by $ax = xa = dx$ for all $x \in D \cup \{a\}$ and $a+x = x+a = d+x$ for all $x \in D \cup \{a\}$. By Theorem 2.51, $D \cup \{a\}$ is a semifield of type III. Let $f: D \to D \cup \{a\}$ defined by $f(x) = x$ for all $x \in D$. Clearly f is a monomorphism. Hence we can embed D into a semifield of type III. #

Corollary 3.11 Let S be a semiring. If S is multiplicatively cancellative then S can be embedded into a semifield of type III.

Proof By Theorem 1.16, S can be embedded into a ratio semiring, say D. And by Proposition 3.10, D can be embedded into a semifield of type III. Therefore S can be embedded into a semifield of type III. #

Proposition 3.12 Let K be a semifield. Then K cannot be embedded into a ratio semiring.

Proof Let $a \in K$ be such that $(K \setminus \{a\}, \cdot)$ is a group and e the identity of $(K \setminus \{a\}, \cdot)$.

Suppose that K can be embedded into a ratio semiring, say D. Then up to isomorphism we can consider that $K \subseteq D$. Let 1 be the multiplicative identity of D. Then 1 is the only multiplicative idempotent of D. If K is a semifield of type I or type II, then K contains two multiplicative idempotents. Thus K cannot be embedded into D. Suppose that K is a semifield of type III. Then e is the only multiplicative idempotent of K, so e = 1. Thus ae = a1 = a, a contradiction. Therefore K cannot be embedded into a ratio semiring. #

Remark 3.13 Let K = {a,e} be a semifield of type I. Then K must have one of the structures given below :

(1)

•	a	e
a	a	a
e	a	e

and

+	a	e
a	a	a
e	a	a

or

(2)

•	a	e
a	a	a
e	a	e

and

+	a	e
a	a	e
e	e	e

or

(3)

•	a	e
a	a	a
e	a	e

and

+	a	e
a	a	a
e	a	e

or

(4)

•	a	e
a	a	a
e	a	e

and

+	a	e
a	a	e
e	e	a

.

Let L = {b,f} be a semifield of type II. Then L must have one of the structures given below :

(i)

•	b	f
b	b	f
f	f	f

and

+	b	f
b	f	f
f	f	f

or

(ii)

•	b	f
b	b	f
f	f	f

and

+	b	f
b	b	b
f	b	f

or

(iii)

•	b	f
b	b	f
f	f	f

and

+	b	f
b	b	f
f	f	f

(iv)

$\cdot$	b	f
b	b	f
f	f	f

and

+	b	f
b	f	b
f	b	f

.

Define $\varphi: K \to L$ by $\varphi(a) = f$ and $\varphi(e) = b$. Then we can show that (1) $\cong$ (i), (2) $\cong$ (ii), (3) $\cong$ (iii) and (4) $\cong$ (iv). Thus K can be embedded into a semifield of type II and L can be embedded into a semifield of type I.

Theorem 3.14 Let K be a semifield of type I. Then K can be embedded into a semifield of type II iff $|K| = 2$.

Proof Assume that K can be embedded into a semifield of type II, say L. Then up to isomorphism, we can consider that $K \subseteq L$. Suppose that $|K| > 2$. Let $a \in K$ be such that $(K \setminus \{a\}, \cdot)$ is a group, let e be the identity of $(K \setminus \{a\}, \cdot)$, let $b \in L$ be such that $(L \setminus \{b\}, \cdot)$ is a group and let f be the identity of $(L \setminus \{b\}, \cdot)$. Since K is a semifield of type I and L a semifield of type II, we get that a and e are the only multiplicative idempotents of K and b and f are the only multiplicative idempotents of L. Thus $\{a,e\} = \{b,f\}$. If $a = b$, then $e = f$. Thus $b = a = ae = bf = f$, a contradiction. Hence $a = f$, so $e = b$. Let $x \in K \setminus \{a,e\}$. Then $a = ax = fx = x$, a contradiction. Therefore $|K| = 2$.

Conversely, assume that $|K| = 2$. Then K is a semifield of type I of order 2. Then we get that K is (1) or (2) or (3) or (4) given in remark 3.13. By Remark 3.13 we also get that K can be embedded into a semifield of type II. #

Theorem 3.15 Let K be a semifield of type II. Then K can be embedded into a semifield of type I iff $|K| = 2$.

Proof Assume that K can be embedded into a semifield of type I, say L. Then up to isomorphism, we can consider that $K \subseteq L$. Suppose that $|K| > 2$. Let $a \in K$ be such that $(K \setminus \{a\}, \cdot)$ is a group, let e be the identity of $(K \setminus \{a\}, \cdot)$, let $b \in L$ be such that $(L \setminus \{b\}, \cdot)$ is a group and let f be the identity of $(L \setminus \{b\}, \cdot)$. Since K is a semifield of type II and L a semifield of type I, we get that a and e are the only multiplicative idempotents of K and b and f are the only multiplicative idempotents of L. Thus $\{a,e\} = \{b,f\}$. If $a = b$, then $e = f$. Thus $a = b = bf = ae = e$, a contradiction. Thus $a = f$, so $e = b$. Let $x \in K \setminus \{a,e\}$. Then $x = xe = xb = b = e$, a contradiction. Hence $|K| = 2$.

Conversely, assume that $|K| = 2$. Then K is a semifield of type II of order 2. Then K is (i) or (ii) or (iii) or (iv) given in remark 3.13. By Remark 3.13 we also get that K can be embedded into a semifield of type I. #

Theorem 3.16 Let K be a semifield of type I or type II. Then K cannot be embedded into a semifield of type III.

Proof Suppose that K is a semifield of type I or II. Then K contains exactly two multiplicative idempotents. Since a type III semifield contains exactly one multiplicative idempotent, we get that K cannot be embedded into a semifield of type III. #

Corollary 3.17 Let K be a semifield of zero type or infinity type. Then K cannot be embedded into a semifield of type III.

Proof Follows directly from Theorem 3.16. #

Theorem 3.18 Let K be a semifield of type III. Then K cannot be embedded into a semifield of type I.

<u>Proof</u> Suppose that K can be embedded into a semifield of type I, say L. Then up to isomorphism we can consider that $K \subseteq L$. Let $a \varepsilon K$ be such that $(K \setminus \{a\}, \cdot)$ is a group, let e be the identity of $(K \setminus \{a\}, \cdot)$, let $b \varepsilon L$ be such that $(L \setminus \{b\}, \cdot)$ is a group and let f be the identity of $(L \setminus \{b\}, \cdot)$. Since K is a type III semifield, we get that e is the only multiplicative idempotent of K. Since L is a semifield of type I, we get that b and f are the only multiplicative idempotents of L. Thus $a \neq b$ and $e = b$ or $e = f$. If $e = b$, then $ae = ab = b$. Thus $ae = b$. Since $a \varepsilon L \setminus \{b\}$, $\exists\, y \varepsilon L \setminus \{b\}$ such that $ay = f$. Then $a = af = a(ay) = a^2 y$. Since K is a semifield of type III, we get that $(ae)^2 = a^2 e^2 = a^2 e = a^2$. Thus $a = (ae)^2 y = b^2 y = by = b$, a contradiction Hence $e = f$. Then $ae = af = a$, a contradiction. Therefore K cannot be embedded into a semifield of type I. #

<u>Theorem 3.19</u> Let K be a semifield of type III. Then K cannot be embedded into a semifield of type II.

<u>Proof</u> Suppose that K can be embedded into a semifield of type II, say L. Then up to isomorphism we can consider that $K \subseteq L$. Let $a \varepsilon K$ be such that $(K \setminus \{a\}, \cdot)$ is a group, let e be the identity of $(K \setminus \{a\}, \cdot)$, let $b \varepsilon L$ be such that $(L \setminus \{b\}, \cdot)$ is a group and let f be the identity of $(L \setminus \{b\}, \cdot)$. Since K is a semifield of type III, we get that e is the only multiplicative idempotent of K. Since L is a semifield of type II, we get that b and f are the only multiplicative idempotents. Thus $a \neq b$ and $e = b$ or $e = f$. Suppose that $e = b$. Then $ae = ab = a$, a contradiction. Thus $e = f$. Then $ae = af = a$, a contradiction. Therefore K cannot be embedded into a semifield of type II. #

<u>Theorem 3.20</u> Let K be a semifield of type II w.r.t.a and let e be the identity of $(K\setminus\{a\},\cdot)$. Then K can be embedded into an ∞-semifield iff K is of the form $\{a,e\}$ where $a+e = e$.

<u>Proof</u> Assume that K can be embedded into an ∞-semifield, say K_∞. Then up to isomorphism, we can consider that $K \subseteq K_\infty$. Since K_∞ is a semifield of type I, we get that $|K| = 2$ (Theorem 3.15). Thus $K = \{a,e\}$. Let 1 be the identity of K_∞. If $a+e \neq e$, then $a+e = a$. Suppose that $e = \infty$. Then $a = a+e = a+\infty = \infty = e$. Thus $a = e$, a contradiction. Hence $e \neq \infty$. Since $e\cdot e = e$, $e = 1$. Then $e = a\cdot e = a\cdot 1 = a$, so $a = e$ which is a contradiction. Hence $a+e = e$. So we obtain that K is of the form $\{a,e\}$ with $a+e = e$.

Conversely, assume that K is of the form $\{a,e\}$ with $a+e = e$. Since K is a semifield of type II w.r.t.a, $a\cdot a = a$, $a\cdot e = e\cdot a = e$ and $e\cdot e = e$. By Theorem 2.29, $K\setminus\{a\}$ is a ratio semiring. Thus $\{e\}$ is a ratio semiring, so $e+e = e$, Since $K = \{a,e\}$, so $a+a = a$ or $a+a = e$. So K has structure

(1)

+	a	e
a	a	e
e	e	e

or (2)

+	a	e
a	e	e
e	e	e

Let $K = \{\infty,1\}$ with $\infty\cdot\infty = \infty$, $\infty\cdot 1 = 1\cdot\infty = \infty$ and $1\cdot 1 = 1$ and + defined by

(a)

+	∞	1
∞	∞	∞
1	∞	1

or (b)

+	∞	1
∞	∞	∞
1	∞	∞

Then K_∞ is an ∞-semifield. Define $f\colon K \to K_\infty$ by $f(a) = 1$ and $f(e) = \infty$. It is easy to show that (1) $\cong$ (a) and (2) $\cong$ (b). Thus K can be embedded into an ∞-semifield. #

<u>Theorem 3.21</u> Let K be an ∞-semifield. Then K can be embedded into a semifield of type II iff $|K| = 2$.

<u>Proof</u> Suppose that K can be embedded into a semifield of type II. By Theorem 3.14, $|K| = 2$.

Conversely, assume that K is an ∞-semifield of order 2. Then $K \cong \{\infty, 1\}$ with the structure;

(1)

•	∞	1
∞	∞	∞
1	∞	1

and

+	∞	1
∞	∞	∞
1	∞	1

or

(2)

•	∞	1
∞	∞	∞
1	∞	1

and

+	∞	1
∞	∞	∞
1	∞	∞

Let L = {a,e} with structure;

(a)

•	a	e
a	a	e
e	e	e

and

+	a	e
a	a	e
e	e	e

or

(b)

•	a	e
a	a	e
e	e	e

and

+	a	e
a	e	e
e	e	e

Then L is a semifield of type II. Define $\varphi : K_\infty \to K$ by $\phi(\infty) = e$ and $\varphi(1) = a$. It is easy to show that (1) $\cong$ (a) and (2) $\cong$ (b). Thus K can be embedded into a semifield of type II. #

<u>Remark 3.22</u> Let $K_0 = \{0,1\}$ with the structure

•	0	1
0	0	0
1	0	1

+	0	1
0	0	1
1	1	1

Then K_0 is a semifield of zero type which is not a field. We shall call K_0 the Boolean semifield.

Theorem 3.23 Let K be a semifield of type II w.r.t.a and let e be the identity $(K \setminus \{a\}, \cdot)$. Then K can be embedded into a 0-semifield which is not a field iff K is isomorphic to the Boolean semifield.

Proof Assume that K can be embedded into a semifield of zero type which is not a field, say K_0. Then up to isomorphism we can consider that $K \subseteq K_0$. Let 1 be the identity of K_0. Claim that a = 1 and e = 0. Since K is a semifield of type II w.r.t.a, $a \cdot a = a$ and $a \cdot e = e \cdot a = e$. If a = 0, then $a = 0 = 0 \cdot e = a \cdot e = e$, a contradiction. Thus $a \neq 0$. Since $a \cdot a = a$, a = 1. Since $e \cdot e = e$, we see that if $e \neq 0$ then e = 1 so a = e, a contradiction. Thus e = 0 and hence we have the claim. Since K_0 is a semifield of type I, we get that $|K| = 2$ (Theorem 3.15) Thus $K = \{a,e\}$. Since K_0 is a 0-semifield which is not a field, by Proposition 2.26, we get that $K \setminus \{0\}$ is a ratio semiring. Thus $1+1 \neq 0$. Since a = 1 and e = 0, $a+a \neq e$. Thus a+a = a. Since e = 0, a+e = a+0 = a. Thus a+e = a. So we obtain that K is isomorphic to the Boolean semifield.

Conversely, assume that K is isomorphic to the Boolean semifield. Therefore K can be embedded into a 0-semifield which is not a field. #

Theorem 3.24 Let K_0 be a semifield of zero type which is not a field and 1 the identity of K_0. Then K_0 can be embedded into a semifield of type II iff K_0 is isomorphic to the Boolean semifield.

Proof Assume that K_0 can be embedded into a semifield of type II, say K. Then up to isomorphism we can consider that $K_0 \subseteq K$. Since K_0 is a semifield of type I, by Theorem 3.14 we get that $|K_0| = 2$. Thus

$K_0 = \{0,1\}$. Since K_0 is a 0-semifield which is not a field, by Proposition 2.26, $K_0 \setminus \{0\}$ is a ratio semiring. Thus $1+1 \neq 0$. Since $K_0 = \{0,1\}$, $1+1 = 1$. So we have that K_0 is isomorphic to the Boolean semifield.

Conversely, assume that K_0 is isomorphic to the Boolean semifield. Hence $K_0 = \{0,1\}$ and the additive and multiplicative structures of K_0 are given by :

•	0	1
0	0	0
1	0	1

and

+	0	1
0	0	1
1	1	1

.

Let K = {a,e} with structure

•	a	e
a	a	e
e	e	e

and

+	a	e
a	a	a
e	a	e

Then K is a semifield of type II. Defined $\Phi : K_0 \rightarrow K$ by $\Phi(0) = e$ and $\Phi(1) = a$. It is easy to show that Φ is an isomorphism. Hence $K_0 \cong K$. Then K_0 can be embedded into a semifield of type II. #