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(Maximum and minimum values of linear combination with interval linear equation)

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MAXIMUM AND MINIMUM VALUES OF LINEAR COMBINATION WITH INTERVAL LINEAR EQUATION

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In this project, we study and prove the method for finding the smallest and the largest expected values when a probability interval is given. Then we apply the method with the properties of probability interval to analyze the minimum and the maximum values of a linear combination with interval linear constraint. We further write a code for finding the minimum and the maximum values using Python language.

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Contents

CHAPTER I

INTRODUCTION

Let $X = \{x_1, x_2, \ldots, x_n\}$. For a given $d \in \mathbb{R}$ and $\alpha_i, \underline{c_i}, \overline{c_i} \in \mathbb{R}\ \forall i = 1, 2, \ldots, n$ where $\underline{c_i} \leq \overline{c_i}$, we call the equation $\sum_{i=1}^{n} \alpha_i c_i = d$ where the unknown $c_i \in [\underline{c}_i, \overline{c}_i]$ $\forall i = 1, 2, \ldots, n$ as an interval linear equation. And we call $\sum_{i=1}^{n} c_i x_i$ where the unknown $c_i \in [\underline{c}_i, \overline{c}_i]$ as an interval linear combination.

Let us consider an interval linear combination $\sum_{i=1}^{n} c_i x_i$ such that c_i's have the interval linear relation

$$\sum_{i=1}^{n} \alpha_i c_i = d$$

where c_i is an unknown parameter in the interval $[\underline{c}_i, \overline{c}_i]$, such that $\underline{c}_i \leq \overline{c}_i$,
x_i is an element in $X = \{x_1, x_2, \ldots, x_n\}$,
α_i is a known of c_i's,
d is a known parameter.

L. M. De Campos, et al. provided algorithms for finding the smallest and the largest expected values of a probability interval linear combination with probability interval in [**1**]. We study and prove the Algorithms in [**1**]. Then we try to apply these Algorithm to find the smallest and the largest values of a general interval linear combination with the linear relation $\sum_{i=1}^{n} \alpha_i c_i = d$ where $c_i \in [\underline{c}_i, \overline{c}_i]$ and $\underline{c}_i, \overline{c}_i \in \mathbb{R}$. Then we write a code for finding the minimum and the maximum values using Python language.

The interval linear combination can be adjusted to be a probability interval linear combination $\sum_{i=1}^{N} p_i t_i$ where p_i is an unknown probability in a probability interval and $t_i \in \mathbb{R}$. When we concern only on non-negative lower bound $\underline{c_i}$, $\forall i = 1, 2, \ldots, n$, it had been shown in [**2**]. However, the boundaries of c_i are not always greater than or equal to zero. So, we will provide a method for adjusting the interval linear combination to be a probability interval linear combination when $\underline{c_i}$ and $\overline{c_i}$ are not only non-negative values. In other word, in this project we concern the general case when c_i is an unknown parameter in the interval $[\underline{c}_i, \overline{c}_i]$ such that $\underline{c}_i \leq \overline{c}_i$ and $\underline{c}_i, \overline{c_i} \in \mathbb{R}$.

We first provide some properties of probability interval, a method finding the smallest and the largest expected values and a proof of an Algorithm for finding the smallest and the largest expected values in Chapter II. A method for adjusting an interval linear combination to probability interval linear combination is shown in Chapter III. Chapter IV serves the Algorithm to find the minimum and the maximum values of linear combination with interval linear equation. The final chapter is served for the conclusion of this project.

CHAPTER II

PROBABILITY INTERVAL

In this chapter, we present all basic knowledge needed to find the lowest and the largest values of an interval linear combination. We start with properties of probability interval. Some basic idea of extreme probabilities will provided. Furthermore, We prove that probability got from Algorithm 1 and 2 provide the smallest and the largest expected values when a probability interval is given in the last section of this chapter.

2.1 Definition of probability interval

Let $X = \{x_1, x_2, \dots, x_n\}$ be the set of all n realizations of a random variable x and $L = \{[l_i, u_i] \mid 0 \leq l_i \leq u_i \leq 1,\ i = 1, 2, \dots, n\}$ be a family of intervals bounded by 0 and 1. We can explain these intervals as a set of bounds of probabilities by defining the set $\mathscr{P}$ of probability distributions on X as

$$\mathscr{P} = \left\{ \boldsymbol{p} = (p_1, p_2, \dots, p_n) \mid l_i \leq p_i \leq u_i,\ \sum_{i=1}^{n} p_i = 1 \right\}. \tag{2.1}$$

Then the set L is called a set of probability intervals (or *probability interval*, in short), if there exist $p_i \in [l_i, u_i]$, for all $i = 1, 2, \dots, n$ such that $\sum_{i=1}^{n} p_i = 1$, and $\mathscr{P}$ is the set of all *possible probabilities* associated to L.

In order to avoid the emptiness of set $\mathscr{P}$, it is necessary to have more properties on the intervals which is called *proper*. A probability interval L is called a *proper probability interval* if

$$\sum_{i=1}^{n} l_i \leq 1 \leq \sum_{i=1}^{n} u_i. \tag{2.2}$$

In addition, we can also associate with the proper intervals $[l_i, u_i]$ by presenting a pair (l, u) of the lower and upper probabilities through $\mathscr{P}$ as follows.

$$l(A) = \inf_{\boldsymbol{p} \in \mathscr{P}} p(A) \text{ and } u(A) = \sup_{\boldsymbol{p} \in \mathscr{P}} p(A), \forall A \subseteq X. \tag{2.3}$$

Therefore, $l(\{x_i\}) = \inf_{\boldsymbol{p} \in \mathscr{P}} p_i \geq l_i$ and $u(\{x_i\}) = \sup_{\boldsymbol{p} \in \mathscr{P}} p_i \leq u_i$. We use these two properties to get the tight bound of each interval.

2.2 Properties of probability interval

A proper probability interval must also have the following properties to ensure that the lower bound l_i and the upper bound u_i of the probability interval can be reached by some probabilities in the set $\mathscr{P}$, so called a reachable probability interval.

Definition (Reachable). Let $L = \{[l_i, u_i] \mid 0 \le l_i \le u_i \le 1,\ i = 1, 2, \ldots, n\}$ be a probability interval. If there exist $p_j, q_j \in [l_j, u_j]$ for all $j \in \{1, 2, \ldots, n\}$ such that $\forall i$, $l_i + \sum_{j \ne i} p_j = 1$ and $u_i + \sum_{j \ne i} q_j = 1$, then L is called a reachable probability interval.

Theorem 1. Given a reachable probability interval

$$L = \{[l_i, u_i] \mid 0 \le l_i \le u_i \le 1,\ i = 1, 2, \ldots, n\},$$

we have

$$\sum_{j \ne i} l_j + u_i \le 1 \text{ and } \sum_{j \ne i} u_j + l_i \ge 1, \quad \forall i = 1, 2, \ldots, n. \tag{2.4}$$

The conditions (2.4) guarantee that the lower bound l_i and the upper bounds u_i can be reached by some probabilities in $\mathscr{P}$. Sometimes probability interval is not reachable, we must change it to become a reachable probability interval. Now, let us see through the series of theorems how to modify a probability interval to be reachable without changing the associated set of possible probabilities $\mathscr{P}$.

Lemma 1. Let $L = \{[l_i, u_i] \mid 0 \le l_i \le u_i \le 1,\ i = 1, 2, \ldots, n\}$ be a set of proper probability intervals and

$$l_i' = \max\left\{l_i, 1 - \sum_{j \ne i} u_j\right\} \text{ and } u_i' = \min\left\{u_i, 1 - \sum_{j \ne i} l_j\right\}, \quad \text{for all } i = 1, 2, \ldots, n \tag{2.5}$$

Then $l_i' \le u_i'$, for all $i = 1, 2, \ldots .n$.

Proof. We will show that $l_i' \leq u_i'$, for all $i = 1, 2, \ldots, n$.

Case I: $l_i \leq 1 - \sum_{j \neq i} u_j$ and $u_i \leq 1 - \sum_{j \neq i} l_j$.

$$\text{So, } l_i' = \max\{l_i, 1 - \sum_{j \neq i} u_j\} = 1 - \sum_{j \neq i} u_j \leq u_i = \min\{u_i, 1 - \sum_{j \neq i} l_j\} = u_i'.$$

Case II: $l_i \leq 1 - \sum_{j \neq i} u_j$ and $u_i > 1 - \sum_{j \neq i} l_j$.

$$\text{So, } l_i' = \max\{l_i, 1 - \sum_{j \neq i} u_j\} = 1 - \sum_{j \neq i} u_j \leq 1 - \sum_{j \neq i} l_j = \min\{u_i, 1 - \sum_{j \neq i} l_j\} = u_i'.$$

Case III: $l_i > 1 - \sum_{j \neq i} u_j$ and $u_i \leq 1 - \sum_{j \neq i} l_j$.

$$\text{So, } l_i' = \max\{l_i, 1 - \sum_{j \neq i} u_j\} = l_i \leq u_i = \min\{u_i, 1 - \sum_{j \neq i} l_j\} = u_i'.$$

Case IV: $l_i > 1 - \sum_{j \neq i} u_j$ and $u_i > 1 - \sum_{j \neq i} l_j$.

$$\text{So, } l_i' = \max\{l_i, 1 - \sum_{j \neq i} u_j\} = l_i \leq 1 - \sum_{j \neq i} l_j = \min\{u_i, 1 - \sum_{j \neq i} l_j\} = u_i'.$$

From Case I - IV, we have $l_i' \leq u_i'$, for all $i = 1, 2, \ldots, n$. □

Theorem 2. Let $L = \{[l_i, u_i] \mid 0 \leq l_i \leq u_i \leq 1,\ i = 1, 2, \ldots, n\}$ and $L' = \{[l_i', u_i'] \mid 0 \leq l_i' \leq u_i' \leq 1,\ i = 1, 2, \ldots, n$ where $l_i' = \max\left\{l_i, 1 - \sum_{j \neq i} u_j\right\}$ and $u_i' = \min\left\{u_i, 1 - \sum_{j \neq i} l_j\right\}$ for all $i = 1, 2, \ldots, n$. Then $\mathscr{P} = \mathscr{P}'$ where $\mathscr{P} = \left\{\boldsymbol{p} = (p_1, p_2, \ldots, p_n) \mid l_i \leq p_i \leq u_i,\ \sum_{i=1}^{n} p_i = 1\right\}$ and

$$\mathscr{P}' = \left\{\boldsymbol{p'} = (p_1', p_2', \ldots, p_n') \mid l_i' \leq p_i' \leq u_i',\ \sum_{i=1}^{n} p_i' = 1\right\}.$$

Proof. From Lemma 1, we have $l_i \leq l_i' \leq u_i' \leq u_i$ for all $i = 1, 2, \ldots, n$. Thus $\mathscr{P} \subseteq \mathscr{P}'$. On the other hand, let $\boldsymbol{p} = (p_1, p_2, \ldots, p_n) \in \mathscr{P}$ and we have $\sum_{i=1}^{n} p_i = 1$. We will show that $l_i' \leq p_i \leq u_i'$ $\forall i = 1, 2, \ldots, n$. Consider $l_i' \leq p_i$.

Case I: If $l'_i = l_i$, then $l'_i = l_i \le p_i$.

Case II: If $l'_i = 1 - \sum_{j \ne i} u_j$ and we have $p_j \le u_j, \forall j$.

$$\begin{aligned}
\sum_{j \ne i} p_j &\le \sum_{j \ne i} u_j \\
-1 + p_i + \sum_{j \ne i} p_j &\le -1 + p_i + \sum_{j \ne i} u_j \\
-1 + \sum_{i=1}^{n} p_i &\le -1 + p_i + \sum_{j \ne i} u_j \\
0 &\le -1 + p_i + \sum_{j \ne i} u_j \\
1 - \sum_{j \ne i} u_j &\le p_i \\
l'_i &\le p_i.
\end{aligned}$$

From Case I - II, we have $l'_i \le p_i$, for all $i = 1, 2, \ldots, n$. Consider $p_i \le u'_i$.

Case I: If $u'_i = u_i$, then $p_i \le u_i = u'_i$.

Case II: If $u'_i = 1 - \sum_{j \ne i} l_j$ and we have $l_j \le p_j, \forall j$.

$$\begin{aligned}
\sum_{j \ne i} l_j &\le \sum_{j \ne i} p_j \\
p_i - 1 + \sum_{j \ne i} l_j &\le p_i - 1 + \sum_{j \ne i} p_j \\
p_i - 1 + \sum_{j \ne i} l_j &\le \sum_{i=1}^{n} p_i - 1 = 1 - 1 = 0 \\
p_i - 1 + \sum_{j \ne i} l_j &\le 0 \\
p_i &\le 1 - \sum_{j \ne i} l_j \\
p_i &\le u'_i.
\end{aligned}$$

From Case I - II, we have $p_i \le u'_i$, for all $i = 1, 2, \ldots, n$.
Hence, $l'_i \le p_i \le u'_i, \forall i$. Then $\boldsymbol{p} \in \mathscr{P}'$ and thus $\mathscr{P} \subseteq \mathscr{P}'$. □

According to Lemma 1, we can replace the original set of probability intervals L to L' defined in (2.5) without affacting the set $\mathscr{P}$. This replacement permits us to refine the probability bounds that define $\mathscr{P}$ in such a way that these bounds can always be reached, as shown the next theorem.

Theorem 3. The probability interval L' defined in (2.5) are reachable.

Proof. We will prove that $\sum_{j\neq i} l'_j + u'_i \leq 1$, for all $i = 1, 2, \dots, n$.

Let $i = 1, 2, \dots, n$.

Case I: $\forall j \neq i$, and $l_j \geq 1 - \sum_{m\neq j} u_m$, then $l'_j = l_j, \forall j \neq i$.

From (2.5), we have $u'_i \leq 1 - \sum_{j\neq i} l_j$

$$\begin{aligned} \text{So, } \sum_{j\neq i} l'_j + u'_i &\leq \sum_{j\neq i} l'_j + (1 - \sum_{j\neq i} l_j) \\ &= \sum_{j\neq i} (l'_j - l_j) + 1 \\ &= 0 + 1 = 1 \end{aligned}$$

Case II: $\forall h \neq i$, and $l_h < 1 - \sum_{j\neq h} u_m$, then $l'_h = 1 - \sum_{j\neq h} u_j$.

$$\begin{aligned} \text{So, } \sum_{j\neq i} l'_j + u'_i &= \sum_{j\neq i,h} l'_j + l'_h + u'_i \\ &= \sum_{j\neq i,h} l'_j + (1 - \sum_{j\neq h} u_j) + u'_i \\ &= \sum_{j\neq i,h} l'_j - (\sum_{j\neq i,h} u_j + u_i) + u'_i + 1 \\ &= \sum_{j\neq i,h} (l'_j - u_j) + (u'_i - u_i) + 1 \\ &\leq 0 + 0 + 1 = 1. \end{aligned}$$

From Case I - II, we have $\sum_{j\neq i} l'_j + u'_i \leq 1$, for all $i = 1, 2, \dots, n$.

We will prove that $\sum_{j\neq i} u'_j + l'_i \geq 1$, for all $i = 1, 2, \dots, n$.

Case I: $\forall h \neq i$, and $u_h \geq 1 - \sum_{j\neq h} l_j$, then $u'_h = 1 - \sum_{j\neq h} l_j$.

$$\begin{aligned} \text{So, } \sum_{j\neq i} u'_j + l'_i &= \sum_{j\neq i,h} u'_j + u'_h + l'_i \\ &= \sum_{j\neq i,h} u'_j + (1 - \sum_{j\neq h} l_j) + l'_i \\ &= \sum_{j\neq i,h} u'_j - (\sum_{j\neq i,h} l_j + l_i) + l'_i + 1 \\ &= \sum_{j\neq i,h} (u'_j - l_j) + (l'_i - l_i) + 1 \\ &\geq 0 + 0 + 1 = 1 \end{aligned}$$

Case II: $\forall j \neq i$, and $u_j < 1 - \sum_{m \neq j} l_m$, then $u'_j = u_j$.

From (2.5), we have $l'_i \geq 1 - \sum_{j \neq i} u_j$

$$\begin{aligned} \text{So, } \sum_{j \neq i} u'_j + l'_i &\geq \sum_{j \neq i} u'_j + (1 - \sum_{j \neq i} u_j) \\ &= \sum_{j \neq i} (u'_j - u_j) + 1 \\ &= 0 + 1 = 1 \end{aligned}$$

From Case I - II, we have $\sum_{j \neq i} u'_j + l'_i \geq 1$, for all $i = 1, 2, \ldots, n$. □

When we have a reachable probability interval, we will find all of the possible probabilities $\boldsymbol{p} \in \mathscr{P}$ to obtain the smallest and largest expected values by Algorithms in Section 2.3.

2.3 Extreme probabilities

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all n realizations of a random variable x, where each x_i has its corresponding unknown probability p_i bounded by $[l_i, u_i]$. Without loss of generality, if $x_1 \leq x_2 \leq \cdots \leq x_n$, L.M. de Campos, et al.[**1**] provided probabilities $\underline{\boldsymbol{p}}$ and $\overline{\boldsymbol{q}}$ that give the smallest expected values $\underline{E}(x)$ and the largest expected values $\overline{E}(x)$, respectively.

Extreme probability is based on the assumption that when we want to have the smallest expected values. If the realization of x_i is less than or equal to x_j, then probability p_i should be closer to the upper bound than probability p_j for all $i, j = 1, 2, \ldots, n$; i.e.,

$$(p_1, p_2, \ldots, p_n) = (u_1, u_2, \ldots, u_{k-1}, 1 - \sum_{j=1}^{k-1} u_j - \sum_{j=k+1}^{n} l_j, l_{k+1}, \ldots, l_n),$$

where k is the index such that $l_k \leq 1 - \sum_{j=1}^{k-1} u_j - \sum_{j=k+1}^{n} l_j \leq u_k$. On the other hand, when we want to have the largest expected values, if the realization of $x_i$is less than or equal to x_j, then probability q_i should be closer to the lower bound than probability q_j for all $i, j = 1, 2, \ldots, n$; i.e.,

$$(q_1, q_2, \ldots, q_n) = (l_1, l_2, \ldots, l_{h-1}, 1 - \sum_{j=1}^{h-1} l_j - \sum_{j=h+1}^{n} u_j, u_{h+1}, \ldots, u_n),$$

where h is the index such that $l_h \leq 1 - \sum_{j=1}^{h-1} l_j - \sum_{j=h+1}^{n} u_j \leq u_h$.

The smallest expected values $\underline{E}(x) = \sum_{i=1}^{n} p_i x_i$, where p_i is a probability computed from the following Algorithm.

Algorithm 1 for $\underline{\boldsymbol{p}} = (p_1, p_2, \ldots, p_n)$
$S \leftarrow 0$
For $i = 0$ to $n-1$ do $S \leftarrow S + u_i$;
$S \leftarrow S + l_n$;
$k \leftarrow n$;
While $S \geq 1$ do $S \leftarrow S - u_{k-1} + l_{k-1}$; $p_k \leftarrow l_k$; $k \leftarrow k-1$;
For $i = 1$ to $k-1$ do $p_i \leftarrow u_i$;
$p_k \leftarrow 1 - S + l_k$;

The largest expected values $\overline{E}(x) = \sum_{i=1}^{n} q_i x_i$, where q_i is a probability computed from the following Algorithm.

Algorithm 2 for $\overline{\boldsymbol{q}} = (q_1, q_2, \ldots, q_n)$
$S \leftarrow 0$
For $i = 0$ to $n-1$ do $S \leftarrow S + l_i$;
$S \leftarrow S + u_n$;
$k \leftarrow n$;
While $S \leq 1$ do $S \leftarrow S + u_{k-1} - l_{k-1}$; $q_k \leftarrow u_k$; $k \leftarrow k-1$;
For $i = 1$ to $k-1$ do $q_i \leftarrow l_i$;
$q_k \leftarrow 1 - S + u_k$;

At this end, we provide the proof of Algorithm 1 and 2, since there are no proofs provided in [**1**].

Theorem 4. For a given reachable probability interval $L = \{[l_i, u_i] \mid 0 \leq l_i \leq u_i \leq 1,\ i = 1, 2, \ldots, n\}$, let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all n known realizations of a random variable x, where each realization x_i has its corresponding unknown probability p_i such that $p_i \in [l_i, u_i]$. If $x_1 \leq x_2 \leq \cdots \leq x_n$, and there exists an index k such that $p_k = 1 - \sum_{j=1}^{k-1} u_j - \sum_{j=k+1}^{n} l_j \in [l_k, u_k]$, then $(p_1, p_2, \ldots, p_n) =$

$(u_1, u_2, \ldots, u_{k-1}, p_k, l_{k+1}, \ldots, l_n)$ providing the smallest expected value. Similarly, if there exists an index h such that $q_h = 1 - \sum_{j=1}^{h-1} l_j - \sum_{j=h+1}^{n} u_j \in [l_h, u_h]$, then $(q_1, q_2, \ldots, q_n) = (l_1, l_2, \ldots, l_{h-1}, q_h, u_{h+1}, \ldots, u_n)$ providing the largest expected value.

Proof. Suppose there exists an index $k \in \{1, 2, \ldots, n\}$ such that

$$p_k = 1 - u_1 - u_2 - \cdots - u_{k-1} - l_{k+1} - l_{k+2} - \cdots - l_n \in [l_k, u_k].$$

We want to show that $(p_1, p_2, \ldots, p_k, \ldots, p_n) = (u_1, u_2, \ldots, u_{k-1}, p_k, l_{k+1}, \ldots, l_n)$ is providing the smallest expected value $\underline{E}(x)$, when realization of x are $\{x_1, x_2, \ldots, x_n\}$ where $x_1 \le x_2 \le \cdots \le x_n$.

Let $(p_1, p_2, \ldots, p_n)$ be any probability where $p_i \in [l_i, u_i]$ $i = 1, 2, \ldots, n$

Let $\delta_i = u_i - p_i \quad \forall i = 1, 2, \ldots, k-1 \qquad \Rightarrow u_i = p_i + \delta_i$

$\beta_i = p_i - l_i \quad \forall i = k+1, k+2, \ldots, n \quad \Rightarrow l_i = p_i - \beta_i$

$$\begin{aligned}
E(x) &= x_1p_1 + x_2p_2 + \cdots + x_np_n \\
&\ge x_1p_1 + x_2p_2 + \cdots + x_np_n + (x_1 - x_k)\delta_1 + (x_2 - x_k)\delta_2 + \cdots + (x_{k-1} - x_k)\delta_{k-1} \\
&\quad + (x_k - x_{k+1})\beta_{k+1} + (x_k - x_{k+2})\beta_{k+2} + \cdots + (x_k - x_n)\beta_n \\
&= x_1(p_1 + \delta_1) + x_2(p_2 + \delta_2) + \cdots + x_{k-1}(p_{k-1} + \delta_{k-1}) \\
&\quad + x_k(p_k - \delta_1 - \delta_2 - \cdots - \delta_{k-1} + \beta_{k+1} + \beta_{k+2} + \cdots + \beta_n) \\
&\quad + x_{k+1}(p_{k+1} - \beta_{k+1}) + x_{k+2}(p_{k+2} - \beta_{k+2}) + \cdots + x_n(p_n - \beta_n) \\
&= x_1u_1 + x_2u_2 + \cdots + x_{k-1}u_{k-1} + x_k\underline{p_k} + x_{k+1}l_{k+1} + x_{k+2}l_{k+2} + \cdots + x_nl_n
\end{aligned}$$

The proof of the largest expected value can be done in the same fashion. □

CHAPTER III

LINEAR COMBINATION WITH INTERVAL LINEAR EQUATION

In this chapter, we transform an interval linear equation to a probability interval then use probability interval properties to get the maximum and the minimum values of a linear combination with interval linear equation.

3.1 Adjusting $[\underline{c}_i, \overline{c}_i]$ to be non-negative interval

In this section, we will show how to adjust $\sum_{i=1}^{n} \alpha_i c_i = d$ where $c_i \in [\underline{c}_i, \overline{c}_i]$ for the given $\alpha_i, \underline{c_i}, \overline{c_i}, d \in \mathbb{R}$ to the form of an interval linear equation $\sum_{i=1}^{N} \alpha'_i c'_i = d$ where $c'_i \in [\underline{c'}_i, \overline{c'}_i]$ such that $\underline{c'_i} \geq 0$ for some $N \in \mathbb{N}$. It is reasonable to discard the case when $\alpha_i = 0$ and $\underline{c_i} = \overline{c_i} = 0$ out of our consideration. Note that we will use the notation $-[a, b]$ to represent $\{x \mid -b \leq x \leq -a\}$.

Lemma 2. For the given $\alpha_i \neq 0, \underline{c_i}, \overline{c_i}, d \in \mathbb{R}$ where $\underline{c_i} \leq \overline{c_i}$ and $\underline{c_i}, \overline{c_i}$ are not zero at the same time, $i = 1, 2, \ldots, n$, let $\sum_{i=1}^{n} \alpha_i c_i = d$, where $c_i \in [\underline{c_i}, \overline{c_i}]$. Then there exist $N \in \mathbb{N}$ and $\alpha'_i, \underline{c'_i}, \overline{c'_i} \in \mathbb{R}$ where $0 \leq \underline{c'_i} \leq \overline{c'_i}$, $i = 1, 2, \ldots, N$ such that $\sum_{i=1}^{N} \alpha'_i c'_i = d$ where $c'_i \in [\underline{c'}_i, \overline{c'}_i]$.

Proof. In order to get non-negative values of $\underline{c'}_i$'s , $\forall i = 1, 2, \ldots, N$, we split $[\underline{c}_i, \overline{c}_i]$ into three cases as follows :

- case $\underline{c_i} \geq 0$: $[\underline{c}_i, \overline{c}_i]$ stays the same,
- case $\overline{c_i} \leq 0$: $[\underline{c}_i, \overline{c}_i]$ = $-[|\overline{c_i}|, |\underline{c_i}|]$
- case $\underline{c_i} < 0$ and $\overline{c_i} > 0$: $[\underline{c}_i, \overline{c}_i]$ is splitted into $[\underline{c}_i, 0]$ and $[0, \overline{c}_i]$

Let $I_1 = \{i \mid \underline{c}_i \geq 0\}$, $I_2 = \{i \mid \underline{c}_i < 0$ and $\overline{c}_i \leq 0\}$ and $I_3 = \{i \mid \underline{c}_i < 0$ and $\overline{c}_i > 0\}$ and $|I_1| = n_1, |I_2| = n_2$ and $|I_3| = n_3$.

$$\sum_{i=1}^{n} \alpha_i c_i = d$$
$$\sum_{i\in I_1} \alpha_i c_i + \sum_{i\in I_2} \alpha_i c_i + \sum_{i\in I_3} \alpha_i c_i = d$$

We reorder the indices of α_i and c_i by using the first n_i indices as the indices of α_i and c_i in I_1, then the next n_2 indices as the indices of α_i and c_i in I_2 and the last n_3 indices as the indices of α_i and c_i in I_3.

$$\sum_{j=1}^{n_1} \alpha_j c_j + \sum_{j=n_1+1}^{n_1+n_2} \alpha_j c_j + \sum_{j=n_1+n_2+1}^{n} \alpha_j c_j = d$$

Since, if $\underline{c_i} < 0$ and $\overline{c_i} > 0$, the interval $[\underline{c}_i, \overline{c}_i]$ can be splitted into $[\underline{c}_i, 0]$ and $[0, \overline{c}_i]$ Therefore,

$$\underbrace{\sum_{j=1}^{n_1} \alpha_j c_j}_{(1)} + \underbrace{\sum_{j=n_1+1}^{n_1+n_2} \alpha_j c_j}_{(2)} + \underbrace{\sum_{j=n_1+n_2+1}^{n} \alpha_j c_j}_{(3)} + \underbrace{\sum_{j=n+1}^{n+n_3} \alpha_j c_j}_{(4)} = d$$

where,

- in (1) : $\alpha_j = \alpha_j$ and $c_j \in [\underline{c}_j, \overline{c}_j]$ when $j = 1, 2, \ldots, n_1$
- in (2) : $\alpha_j = \alpha_j$ and $c_j \in [\underline{c}_j, \overline{c}_j]$ when $j = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2$
- in (3) : $\alpha_j = \alpha_j$ and $c_j \in [\underline{c}_j, 0]$ when $j = n_1 + n_2 + 1, n_1 + n_2 + 2, \ldots, n$
- in (4) : $\alpha_j = \alpha_{j-n_3}$ and $c_j \in [0, \overline{c}_{j-n_3}]$ when $j = n + 1, n + 2, \ldots, n + n_3$

In (1), since $\underline{c_j} \geq 0$, we use $\alpha'_j = \alpha_j$ and $c'_j \in [\underline{c}_j, \overline{c}_j]$ when $j = 1, 2, \ldots, n_1$
In (2), since $\overline{c_j} \geq 0$, we use $\alpha'_j = -\alpha_j$ and $c'_j \in [|\overline{c}_j|, |\underline{c}_j|]$ when $j = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2$
In (3), since $\underline{c_j} \leq 0$, we use $\alpha'_j = -\alpha_j$ and $c'_j \in [0, |\underline{c}_j|]$ when $j = n_1 + n_2 + 1, n_1 + n_2 + 2, \ldots, n$
In (4), since $\underline{c_i} \geq 0$, we use $\alpha'_j = \alpha_j$ and $c'_j \in [0, \overline{c}_j]$ when $j = n + 1, n + 2, \ldots, n + n_3$

Thus,

$$\sum_{j=1}^{n_1} \alpha'_j c'_j + \sum_{j=n_1+1}^{n_1+n_2} \alpha'_j c'_j + \sum_{j=n_1+n_2+1}^{n} \alpha'_j c'_j + \sum_{j=n+1}^{n+n_3} \alpha'_j c'_j = d$$

$$\sum_{j=1}^{n+n_3} \alpha'_j c'_j = d$$

Now, the interval linear equation $\sum_{i=1}^{n} \alpha_i c_i = d$ where c_i is an unknown parameter in the interval $[\underline{c}_i, \overline{c}_i]$ was adjusted to be an interval linear equation $\sum_{i=1}^{N} \alpha'_i c'_i = d$, where $N = n + n_3$, with non-negative interval $[\underline{c'}_i, \overline{c'}_i]$. In other word, $\underline{c'}_i \geq 0$. □

3.2 Transformation of an interval linear combination

An interval linear combination $\sum_{i=1}^{n} c_i x_i$ such that c_i's have the linear relation $\sum_{i=1}^{n} \alpha_i c_i = d$ when c_i is an unknown parameter in a given interval $[\underline{c_i}, \overline{c_i}]$ such that $\underline{c_i} \leq \overline{c_i}$ where $\underline{c_i} \geq 0$ and $\underline{c_i}, \overline{c_i}$ are not zero at the same time for the given $\alpha_i \neq 0, d \in \mathbb{R}$ can be adjusted to be a probability interval linear combination $\sum_{i=1}^{N} p_i t_i$ such that $\sum_{i=1}^{N} p_i = 1$ where $p_i \in [l_i, u_i]$ and $t_i \in \mathbb{R}$ for some integer N, by the method presented in this section. We first convert an interval linear equation $\sum_{i=1}^{n} \alpha_i c_i = 1$ to $\sum_{i=1}^{N} p_i = 1$ for some $N \in \mathbb{N}$. Then we convert $\sum_{i=1}^{n} c_i x_i$ to $\sum_{i=1}^{N} p_i t_i$.

3.2.1 Transformation of interval linear equation

After we can adjust all boundaries $\underline{c_i}, \overline{c_i}$ to the non-negative $\underline{c'_j}, \overline{c'_j}$ $\forall j = 1, 2, \ldots, N$, in this subsection, we assume that all boundaries are non-negative. We will show how to convert $\sum_{i=1}^{n} \alpha_i c_i = d$ where $c_i \in [\underline{c_i}, \overline{c}_i]$ to become a $\sum_{i=1}^{N} p_i = 1$, $\exists N \in \mathbb{N}$.

Consider $\sum_{i=1}^{n} \alpha_i c_i = d$. Since $c_i \in [\underline{c}_i, \overline{c}_i]$, we must check first that there exists $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in (\underline{\boldsymbol{c}}, \overline{\boldsymbol{c}})$ such that $\sum_{i=1}^{n} \alpha_i c_i = d$, or not. In other words, we have to check that

$$\min_{c_i\in[\underline{c}_i,\overline{c}_i]} \sum_{i=1}^{n} \alpha_i c_i \leq d \leq \max_{c_i\in[\underline{c}_i,\overline{c}_i]} \sum_{i=1}^{n} \alpha_i c_i, \tag{3.1}$$

where

$$\min_{c_i\in[\underline{c}_i,\overline{c}_i]} \sum_{i=1}^{n} \alpha_i c_i = \sum_{j\in J_1} \alpha_j \overline{c}_j + \sum_{j\in J_2} \alpha_j \underline{c}_j \text{ and } \max_{c_i\in[\underline{c}_i,\overline{c}_i]} \sum_{i=1}^{n} \alpha_i c_i = \sum_{j\in J_1} \alpha_j \underline{c}_j + \sum_{j\in J_2} \alpha_j \overline{c}_j,$$

such that $J_1 = \{j \mid \alpha_j < 0\}$ and $J_2 = \{j \mid \alpha_j > 0\}$.

After we check the validity of the interval linear equation, we now try to transform $\sum_{i=1}^{n} \alpha_i c_i = d$ where $c_i \in [\underline{c}_i, \overline{c}_i]$ to become a proper probability interval as stated in Lemma 3.

Lemma 3. Let $d \in \mathbb{R}$ and $\alpha_i \in \mathbb{R} \setminus \{0\}$ for all $i = 1, 2, \ldots, n$. Then the system $\sum_{i=1}^{n} \alpha_i c_i = d$ where $c_i \in [\underline{c}_i, \overline{c}_i]$, $\underline{c_i} \geq 0$ and $\overline{c_i} \neq 0$ can be transform to the corresponding system $\sum_{i=1}^{n+n_1} p_i = 1$ where $p_i \in [l_i, u_i]$, $n_1 = |J_1|$ and $J_1 = \{j \mid \alpha_j < 0\}$.

Proof. We will consider only the case $d \geq 0$. In case of negative value of d, we can multiply both side of the equation by -1. Let $J_1 = \{j \mid \alpha_j < 0\}$, $J_2 = \{j \mid \alpha_j > 0\}$ where $|J_1| = n_1$ and $|J_2| = n_2$. Define $c_{jnew} = \overline{c}_j - c_j$, for each $j \in J_1$.
Then for any unknown $c_i \in [\underline{c}_i, \overline{c}_i] \quad i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n} \alpha_i c_i = d$$

$$\sum_{j\in J_1} \alpha_j c_j + \sum_{j\in J_2} \alpha_j c_j = d$$

$$\sum_{j\in J_1} \alpha_j c_j + \sum_{j\in J_2} \alpha_j c_j + 2\left(\sum_{j\in J_1} |\alpha_j| \overline{c}_j\right) = 2\left(\sum_{j\in J_1} |\alpha_j| \overline{c}_j\right) + d$$

$$\sum_{j\in J_1} |\alpha_j| (\overline{c}_j - c_j) + \sum_{j\in J_2} \alpha_j c_j + \sum_{j\in J_1} |\alpha_j| \overline{c}_j = 2\left(\sum_{j\in J_1} |\alpha_j| \overline{c_j}\right) + d$$

$$\sum_{j\in J_1} |\alpha_j| c_{jnew} + \sum_{j\in J_2} |\alpha_j| c_j + \sum_{j\in J_1} |\alpha_j| \overline{c}_j = 2\left(\sum_{j\in J_1} |\alpha_j| \overline{c}_j\right) + d.$$

Let $D = 2\left(\sum_{j\in J_1}|\alpha_j|\overline{c}_j\right) + d$. Since $D > 0$, we have

$$\frac{\sum_{j\in J_1}|\alpha_j|c_{jnew} + \sum_{j\in J_2}|\alpha_j|c_j + \sum_{j\in J_1}|\alpha_j|\overline{c}_j}{D} = 1 \tag{3.2}$$

$$\underbrace{\frac{|\alpha_1|c_{1new}}{D}}_{p_1\geq 0} + \cdots + \underbrace{\frac{|\alpha_{n_1}|c_{n_1new}}{D}}_{p_{n_1}\geq 0} + \underbrace{\frac{|\alpha_{n_1+1}|c_{n_1+1}}{D}}_{p_{n_1+1}} + \cdots + \underbrace{\frac{|\alpha_n|c_n}{D}}_{p_n\geq 0} + \underbrace{\frac{|\alpha_1|\overline{c}_1}{D}}_{p_{n+1}} + \cdots + \underbrace{\frac{|\alpha_{n_1}|\overline{c}_{n_1}}{D}}_{p_{n+n_1}\geq 0} = 1$$

$$p_1 + p_2 + \cdots + p_{n+n_1} = 1$$

As c_{jnew} in (3.2) is an arbitrary value where $c_{jnew} = \overline{c}_j - c_j \in \left[0, \overline{c}_j - \underline{c}_j\right]$, so $|\alpha_j|c_{jnew}$ is in $\left[0, |\alpha_j|(\overline{c}_j - \underline{c}_j)\right]$. Since p_i's depends on $c_i \in [\underline{c}_i, \overline{c}_i]$ and $c_{jnew} \in \left[0, \overline{c}_j - \underline{c}_j\right]$, the boundary $[l_i, u_i]$ for p_i's can be represented as

- $l_i = 0$ and $u_i = \min\left\{\frac{|\alpha_i|c_{inew}}{D}, 1\right\}$, if $i = 1, 2, \ldots, n_1$
- $l_i = \frac{|\alpha_i|\underline{c}_i}{D}$ and $u_i = \min\left\{\frac{|\alpha_i|\overline{c}_i}{D}, 1\right\}$, if $i = n_1+1, n_1+2, \ldots, n$
- $l_i = \frac{|\alpha_{i-n}|\overline{c}_{i-n}}{D}$ and $u_i = \frac{|\alpha_{i-n}|\overline{c}_{i-n}}{D}$, if $i = n+1, n+2, \ldots, n+n_1$

□

3.2.2 Transformation of interval linear combination

Lemma 4. $\sum_{i=1}^{n} c_i x_i$ where $\sum_{i=1}^{n}\alpha_i c_i = d$, $c_i \in [\underline{c_i}, \overline{c_i}]$, $\underline{c_i} \geq 0$ and $\overline{c_i} > 0$ can be transformed to $\sum_{i=1}^{n+n_1} p_i t_i$ where $p_i \in [l_i, u_i]$ and $t_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n+n_1$.

Proof. By Lemma 2, we can transform $\sum_{i=1}^{n}\alpha_i c_i = d$ to $\sum_{i=1}^{n+n_1} p_i = 1$ where $p_i \in [l_i, u_i]$, for all $i = 1, 2, \dots, n + n_1$.

$$\begin{aligned}
\sum_{i=1}^{n} c_i x_i &= \sum_{i=1}^{n} c_i x_i + \sum_{j\in J_1} \overline{c}_j x_i - \sum_{j\in J_1} \overline{c}_j x_i \\
&= \sum_{j\in J_1} (\overline{c}_j - c_j) y_j + \sum_{j\in J_2} c_j x_j + \sum_{j\in J_1} \overline{c}_j x_j \qquad ;\ y_j = -x_j \\
&= \sum_{j\in J_1} c_{jnew} y_j + \sum_{j\in J_2} c_j x_j + \sum_{j\in J_1} \overline{c}_j x_j \\
&= \sum_{j=1}^{n_1} (\frac{|\alpha_j| c_{jnew}}{D})(\frac{D y_j}{|\alpha_j|}) + \sum_{j=n_1+1}^{n} (\frac{|\alpha_j| c_j}{D})(\frac{D x_j}{|\alpha_j|}) + \sum_{j=n+1}^{n+n_1} (\frac{|\alpha_j| \overline{c}_j}{D})(\frac{D x_j}{|\alpha_j|}) \\
&= \sum_{i=1}^{n+n_1} p_i t_i.
\end{aligned}$$

where
$$\begin{aligned}
p_i &= \frac{|\alpha_i| c_{inew}}{D}, & t_i &= \frac{D y_i}{|\alpha_i|} \le 0 & &\forall\, i = 1, 2, \dots, n_1, \\
p_i &= \frac{|\alpha_i| c_i}{D}, & t_i &= \frac{D x_i}{|\alpha_i|} \ge 0 & &\forall\, i = n_1 + 1, n_1 + 2, \dots, n, \\
p_i &= \frac{|\alpha_{i-n}| \overline{c}_{i-n}}{D}, & t_i &= \frac{D x_{i-n}}{|\alpha_{i-n}|} \ge 0 & &\forall\, i = n + 1, n + 2, \dots, n + n_1. \qquad \square
\end{aligned}$$

Now we transformed the interval linear combination $\sum_{i=1}^{n} c_i x_i$ such that c_i's have the linear relation $\sum_{i=1}^{n} \alpha_i c_i = d$ when c_i is an unknown parameter in a given interval $[\underline{c_i}, \overline{c_i}]$ such that $\underline{c_i} \le \overline{c_i}$ for the given $\alpha_i \neq 0, d \in \mathbb{R}$ to be a probability interval linear combination $\sum_{i=1}^{N} p_i t_i$ such that $\sum_{i=1}^{N} p_i = 1$ where $p_i \in [l_i, u_i]$ and $t_i \in \mathbb{R}$ for some integer N. So, we can get the smallest and the largest expected values of interval linear combination by using Algorithms from the last section of chapter II. Because the interval linear combination $\sum_{i=1}^{n} c_i x_i$ is equivalent to the probability interval linear combination $\sum_{i=1}^{N} p_i t_i$. Then, The minimum and the maximum values of linear combination with interval linear equation be the same as the smallest and the largest expected values of the probability interval linear combination with probability interval, respectively.

CHAPTER IV

EXAMPLE AND ALGORITHM

In this chapter, we provide an example and Algorithm for finding the minimum and the maximum values of linear combination with interval equation by using Python language.

Example Find the largest and the smallest values of $5c_1 - 3c_2$ when $c_1 + c_2 = 7$ and $c_1 \in [-1, 6],\ c_2 \in [1, 5]$.

Solution We first adjust interval to be non-negative interval as follows :

- $[-1, 6]$ is splitted in to $[-1, 0]$ and $[0, 6]$
- $[1, 5]$ stays the same

So, the interval linear equation $c_1 + c_2 = 7$ is adjusted to be $\alpha'_1 c'_1 + \alpha'_2 c'_2 + \alpha'_3 c'_3 = 7$ where $c'_1 \in [1, 5]$ and $\alpha'_1 = \alpha_1 = 1$,

$c'_2 \in [0, 1]$ and $\alpha'_2 = -\alpha_2 = -1$,

$c'_3 \in [0, 6]$ and $\alpha'_3 = \alpha_3 = 1$.

Then, we must check that there exists $\boldsymbol{c} = (c'_1, c'_2, c'_3)$ such that $\sum_{i=1}^{3} \alpha'_i c'_i = 7$, or not. In other words, we have to check that

$$\min_{c'_i \in [\underline{c'}_i, \overline{c'}_i]} \sum_{i=1}^{3} \alpha'_i c'_i \leq d \leq \max_{c'_i \in [\underline{c'}_i, \overline{c'}_i]} \sum_{i=1}^{3} \alpha'_i c'_i$$

where $J_1 = \{j \mid \alpha'_j < 0\}$ and $J_2 = \{j \mid \alpha'_j \geq 0\}$,

$$\min_{c'_i \in [\underline{c'}_i, \overline{c'}_i]} \sum_{i=1}^{3} \alpha'_i c'_i = \sum_{j \in J_1} \alpha'_j \overline{c'}_j + \sum_{j \in J_2} \alpha'_j \underline{c'}_j = (-1)(1) + (1)(1) + (1)(0) = 0 \text{ and}$$

$$\max_{c'_i \in [\underline{c'}_i, \overline{c'}_i]} \sum_{i=1}^{3} \alpha'_i c'_i = \sum_{j \in J_1} \alpha'_j \underline{c'}_j + \sum_{j \in J_2} \alpha'_j \overline{c'}_j = (-1)(0) + (1)(5) + (1)(6) = 11$$

After we check the validity of the interval linear equation, we try to convert the interval linear equation to a probability interval.

Since $\alpha'_2 = -1 \leq 0$, therefore

$$\alpha'_2 c'_2 + \alpha'_1 c'_1 + \alpha'_3 c'_3 = d \Leftrightarrow p_1 + p_2 + p_3 + p_4 = 1.$$

We have $-c'_2 + c'_1 + c'_3 = 7$ Then

$$\begin{aligned}
-c'_2 + c'_1 + c'_3 &= 7 \\
-c'_2 + c'_1 + c'_3 + 2(|-1|\overline{c'}_2) &= 7 + 2(|-1|\overline{c'}_2) \\
|-1|(\overline{c'}_2 - c'_2) + |1|c'_1 + |1|c'_3 + |-1|\overline{c'_2} &= 7 + 2(1) \\
c'_{2new} + c'_1 + c'_3 + \overline{c'_2} &= 9 \\
\frac{c'_{2new}}{9} + \frac{c'_1}{9} + \frac{c'_3}{9} + \frac{1}{9} &= 1 \\
p_1 + p_2 + p_3 + p_4 &= 1
\end{aligned}$$

where $p_1 = \dfrac{c'_{2new}}{9} \in \left[\dfrac{0}{9}, \dfrac{1}{9}\right]$,

$p_2 = \dfrac{c'_1}{9} \in \left[\dfrac{1}{9}, \dfrac{5}{9}\right]$,

$p_3 = \dfrac{c'_3}{9} \in \left[\dfrac{0}{9}, \dfrac{6}{9}\right]$,

$p_4 = \dfrac{\overline{c'_2}}{9} \in \left[\dfrac{1}{9}, \dfrac{1}{9}\right]$.

Next, we try to transform the interval linear combination to a probability interval linear combination. Since c_1 was splitted, so we now have the interval linear combination as; $\sum_{i=1}^{3} c'_i x'_i = c'_1 x'_1 + c'_2 x'_2 + c'_3 x'_3$

where $c'_1 \in [1, 5]$ and $x'_1 = x_2 = -3$,

$c'_2 \in [0, 1]$ and $x'_2 = -x_1 = -5$,

$c'_3 \in [0, 6]$ and $x'_3 = x_1 = 5$.

Since, $\alpha'_2 = -1 \leq 0$, then

$$\begin{aligned}
\sum_{i=1}^{3} c'_i x'_i &= -3c'_1 + 5c'_2 + 5c'_3 \\
&= 5c'_2 - 3c'_1 + 5c'_3 \\
&= (\overline{c'}_2 - c'_2)(-5) - 3c'_1 + 5c'_3 + 5\overline{c'_2} \\
&= (\frac{|-1|c'_{2new}}{9})(\frac{(9)(-5)}{|1|}) + (\frac{|1|c'_1}{9})(\frac{9(-3)}{|1|}) + (\frac{|1|c'_3}{9})(\frac{9(5)}{|1|}) + (\frac{|-1|\overline{c'}_2}{9})(\frac{9(5)}{|-1|}) \\
&= p_1 t_1 + p_2 t_2 + p_3 t_3 + p_4 t_4
\end{aligned}$$

where
$$\begin{aligned}
p_1 &= \frac{c'_{2new}}{9} \in \left[\frac{0}{9}, \frac{1}{9}\right], & t_1 &= \frac{9(-x'_2)}{|\alpha'_2|} = \frac{(9)(5)}{|-1|} = 45, \\
p_2 &= \frac{c'_1}{9} \in \left[\frac{1}{9}, \frac{5}{9}\right], & t_2 &= \frac{9x'_1}{|\alpha'_1|} = \frac{(9)(-3)}{|1|} = -27, \\
p_3 &= \frac{c'_3}{9} \in \left[\frac{0}{9}, \frac{6}{9}\right], & t_3 &= \frac{9x'_3}{|\alpha'_3|} = \frac{(9)(5)}{|1|} = 45, \\
p_4 &= \frac{\overline{c'_2}}{9} \in \left[\frac{1}{9}, \frac{1}{9}\right], & t_4 &= \frac{9x'_2}{|\alpha'_2|} = \frac{(9)(-5)}{|-1|} = -45.
\end{aligned}$$

Next, we need to check that each interval of p_i has reachable probability property according to Theorem 1. If the probability interval is not reachable, we can adjust it to be reachable by Theorem 1.

First, check the proper properties that sum of the lower bounds must be less than or equal to one and the sum of the upper bounds must be greater than or equal to one:

$$\sum_{i=1}^{4} l_i = \frac{0+1+0+1}{9} = \frac{2}{9} \leq 1$$
$$\sum_{i=1}^{4} u_i = \frac{1+5+6+1}{9} = \frac{13}{9} \geq 1.$$

After that check the reachable probability properties.

$$\sum_{j \neq 1} l_j + u_1 = \frac{1+0+1+1}{9} = \frac{3}{9} \leq 1,$$

$$\begin{aligned}
\sum_{j\neq 1} u_j + l_1 &= \frac{5+6+1+0}{9} = \frac{12}{9} \geq 1,\\
\sum_{j\neq 2} l_j + u_2 &= \frac{0+0+1+5}{9} = \frac{6}{9} \leq 1,\\
\sum_{j\neq 2} u_j + l_2 &= \frac{1+6+1+1}{9} = \frac{9}{9} \geq 1,\\
\sum_{j\neq 3} l_j + u_3 &= \frac{0+1+1+6}{9} = \frac{8}{9} \leq 1,\\
\sum_{j\neq 3} u_j + l_3 &= \frac{1+5+1+0}{9} = \frac{7}{9} \ngeq 1,\\
\sum_{j\neq 4} l_j + u_4 &= \frac{0+1+0+1}{9} = \frac{2}{9} \leq 1,\\
\sum_{j\neq 4} u_j + l_4 &= \frac{1+5+6+1}{9} = \frac{13}{9} \geq 1.
\end{aligned}$$

We can see that the probability interval is not reachable. So we must change it to become a reachable probability interval shown below.

$$\begin{aligned}
l'_1 &= \max\left\{0, 1-\frac{12}{9}\right\} = 0 & u'_1 &= \min\left\{\frac{1}{9}, 1-\frac{7}{9}\right\} = \frac{1}{9}\\
l'_2 &= \max\left\{\frac{1}{9}, 1-\frac{8}{9}\right\} = \frac{1}{9} & u'_2 &= \min\left\{\frac{5}{9}, 1-\frac{1}{9}\right\} = \frac{5}{9}\\
l'_3 &= \max\left\{0, 1-\frac{7}{9}\right\} = \frac{2}{9} & u'_3 &= \min\left\{\frac{6}{9}, 1-\frac{2}{9}\right\} = \frac{6}{9}\\
l'_4 &= \max\left\{\frac{1}{9}, 1-\frac{12}{9}\right\} = \frac{1}{9} & u'_4 &= \min\left\{\frac{1}{9}, 1-\frac{1}{9}\right\} = \frac{1}{9}
\end{aligned}$$

Now, we transform interval linear combination with interval linear equation to a probability interval linear combination with probability interval. Then we get the minimum and the maximum values of the interval linear combination by using Theorem 4 in Chapter II. The minimum value is -5 and the maximum value is 27.

Next, we provide an Algorithm for finding the minimum and the maximum values of linear combination with interval linear constraint in Python language.

```
# -*- coding: utf-8 -*-
"""
Created on Tue Jan 15 21:06:05 2019
```

```
@author: siratcha
"""
i=0
A=[]
Aold=[]
Aupdate=[]
L=[]
Lupdate=[]
Lprob=[]
U=[]
Uupdate=[]
Uprob=[]
Y=[]
Yupdate=[]
T=[]

y=input("Please enter all coefficients: ")
Y=y.split()
C=input("Please enter all coefficients of condition: ")
A=C.split()
d=float(input("Please enter d: "))
n=len(A)
n0=0
n1=0
m=0
n2=0
D=d

# Adjusting general interval to be a non-negative interval
while i<n:
I=input("Please enter an interval in form lower,upper : ")
l,u=I.split(',')
l=float(l)
u=float(u)
A[i]=float(A[i])
```

```
Y[i]=float(Y[i])
L.append(l)
U.append(u)
if L[i] >=0: #[+,+]
Lupdate.append(L[i])
Uupdate.append(U[i])
Aupdate.append(A[i])
Yupdate.append(Y[i])
Aold.append(A[i])
n0=n0+1
if L[i] < 0:
if U[i] < 0: #A[i][-,-]--> -A[i][+,+]
Lupdate.append(abs(U[i]))
Uupdate.append(abs(L[i]))
Aupdate.append(-A[i])
Yupdate.append(-Y[i])
Aold.append(A[i])
n1=n1+1 #the number of negative interval [-,-]
if U[i] >= 0: #[-l,+u]--> -A[i][0,abs(l)] and A[i][0,u]
Lupdate.append(0)
Uupdate.append(abs(L[i]))
Aupdate.append(-A[i])
Yupdate.append(-Y[i])
Aold.append(A[i])
n2=n2+1 #the number of interval [-l,+u]
i=i+1
i=0
while i < n:
if L[i] < 0 and U[i] >= 0:
Lupdate.append(0)
Uupdate.append(U[i])
Aupdate.append(A[i])
Yupdate.append(Y[i])
i=i+1
```

```
i=0
while i<(n+n2):
if Aupdate[i] < 0:
m=m+1 #the number of negative alpha_i
D=D+2*((abs(Aupdate[i])*(Uupdate[i])))
i=i+1
i=0
while i<(n+n2):
if Aupdate[i] < 0:
Lupdate.append(Uupdate[i])
Uupdate.append(Uupdate[i])
Aupdate.append(abs(Aupdate[i]))
Yupdate.append(Yupdate[i])
i=i+1


#Adjusting linear equation of non-negative interval
to be probability interval
i=0
while i<n+n2:
print("adjusted to probability intervals: ")
if Aupdate[i] < 0:
T.append((D*(-Yupdate[i]))/abs(Aupdate[i]))
upC=(abs(Aupdate[i])*(Uupdate[i]-Lupdate[i]))/D
Lprob.append(0)
if upC < 1:
Uprob.append(upC)
else:
Uprob.append(1)
else:
T.append((D*(Yupdate[i]))/abs(Aupdate[i]))
Lprob.append((abs(Aupdate[i])*Lupdate[i])/D)
upC=(abs(Aupdate[i])*(Uupdate[i]))/D
if upC <1:
Uprob.append(upC)
```

```
else:
Uprob.append(1)

print([Lprob[i],Uprob[i]],"with coefficient t=",T[i])
i=i+1

while  i <n+n2+m:
T.append((D*(Yupdate[i]))/abs(Aupdate[i]))
Lprob.append((abs(Aupdate[i])*abs(Uupdate[i]))/D)
Uprob.append((abs(Aupdate[i])*abs(Uupdate[i]))/D)
print([Lprob[i],Uprob[i]],"with coefficient t=",T[i])
i=i+1


#proper check
i=0
l=0
u=0
N=n+n2+m
while i<N:
l=l+Lprob[i] #calculate the sum of lower bounds
u=u+Uprob[i] #calculate the sum of upper bounds
i=i+1
if l <= 1 and u >= 1 :
#if the sum of lower bounds is lower or equal to 1 and
the sum of upper bounds is greater or equal to 1, A set
of probability intervals is a Proper otherwise is not.
print("A set of probability intervals is a Proper")

#reachable check if not then adjust to reachable
Xr=[]
Yr=[]
Xcheck=[]
Ycheck=[]
Lprobreach=[]
```

```
Uprobreach=[]
j=0
while j<N:
x=sum(Lprob)-Lprob[j]+Uprob[j] #calculate following thm
y=sum(Uprob)-Uprob[j]+Lprob[j] #calculate following thm
Xr.append(x)
Yr.append(y)
j=j+1

for i in Xr:
if i <=1:
Xcheck.append(i)
for i in Yr:
if i >=1:
Ycheck.append(i)
if len(Xcheck) ==N and len(Ycheck)==N:
print(”A set of probability intervals is reachable.”)
#if a set of probability interval L is not a reachable,
Now we adjust it to become a reachable by calculated
following theorem
else:
print(”A set of probability intervals is not reachable.” )
print(”A reachable set of probability intervals is ”)
j=0
while j<N:
a=1-sum(Uprob)+Uprob[j]
b=1-sum(Lprob)+Lprob[j]
if Lprob[j]<a:
Lprob[j]=a
if Uprob[j]>b:
Uprob[j]=b
Lprobreach.append(Lprob[j])
Uprobreach.append(Uprob[j])
print(Lprobreach[j],Uprobreach[j])
j=j+1
```

```
#Ascending order follow x
i=0
S=0
N=n+n2+m
P=[]
Q=[]
Acal=[]
Lcal=[]
Ucal=[]
Tcal=[]
Aresp=[]
Lresp=[]
Uresp=[]
Tresp=[]
Acal.extend(Aupdate)
Lcal.extend(Lprobreach)
Ucal.extend(Uprobreach)
Tcal.extend(T)
while i < N:
P.append(1)
Q.append(1)
i=i+1

i=0
while i < len(Tcal):
if min(Tcal)==Tcal[i]:
Aresp.append(Acal[i])
Lresp.append(Lcal[i])
Uresp.append(Ucal[i])
Tresp.append(Tcal[i])
del Acal[i]
del Lcal[i]
```

```
del Ucal[i]
del Tcal[i]
i=-1

i=i+1


#Algorithm1 finding probability that will provide samllest
expected value
i=0
while i<N-1:
S=S+Uresp[i]
i=i+1
i=0
S=S+Lresp[N-1]
k=N-1
while S >= 1 and k>=0:
if k==0:
P[k]=Lresp[k]
else:
S=S-Uresp[k-1]+Lresp[k-1]
P[k]=Lresp[k]
k=k-1
while i < k:
P[i]=Uresp[i]
i=i+1
P[k]=1-S+Lresp[k]


#Algorithm2 finding probability that will provide largest
expected value
i=0
S=0
while i<N-1:
S=S+Lresp[i]
```

```
i=i+1
i=0
S=S+Uresp[N-1]
k=N-1
while S <= 1 and k>=0:
if k==0:
Q[k]=Uresp[k]
else:
S=S+Uresp[k-1]-Lresp[k-1]
Q[k]=Uresp[k]
k=k-1
while i < k:
Q[i]=Lresp[i]
i=i+1
Q[k]=1-S+Uresp[k]


#calculate the smallest and thelargest expected values
i=1
minE=P[0]*Tresp[0]
maxE=Q[0]*Tresp[0]
while i<N:
minE=minE+(P[i]*Tresp[i])
maxE=maxE+(Q[i]*Tresp[i])
i=i+1
print("The smallest expected value is",minE)
print("The largest expected value is",maxE)
else:
print("A set of probability intervals is not a Proper")
# A set of probability intervals is a proper if and only
if the sum of the lower bounds is less than or equal to 1,
and the sum of upper bounds is greater than or equal to 1.
```

CHAPTER V

CONCLUSION

We prove that probability got from Algorithm 1 and 2 in [1] provide the smallest and the largest expected values of linear combination with probability interval linear constraint when probability interval is given in Chapter II. We transform an interval linear combination $\sum_{i=1}^{n} c_i x_i$ such that c_i's have the linear relation $\sum_{i=1}^{n} \alpha_i c_i = d$ when c_i is an unknown parameter in a given interval $[\underline{c_i}, \overline{c_i}]$ such that $\underline{c_i} \leq \overline{c_i}$ for the given $\alpha_i \neq 0, d \in \mathbb{R}$ to be a probability interval linear combination $\sum_{i=1}^{N} p_i t_i$ such that $\sum_{i=1}^{N} p_i = 1$ where probability $p_i \in [l_i, u_i]$ and $t_i \in \mathbb{R}$ for some integer N and then we get the minimum and the maximum values of the interval linear combination with interval linear constraint by the method as explained in Chapter III. Then we provide an example and Algorithm for finding the minimum and the maximum values of linear combination with interval equation by using Python language.

Bibliography

[1] L. M. De Campos, J. F. Huete and S. Moral, Probability Interval; A Tool for Uncertain Reasoning, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, Vol. 2, No. 2, 167 - 196, 1994.

[2] N. Burana, *Interval Price Objective Coefficient Linear Programming Problem*, Project Report, Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, 2017.

[3] J. Rohn, Systems of Interval Linear Equations, *Linear Algebra and Its Applications*, No. 126, 39 - 78, 1989.

Appendix

The Project Proposal of Course 2301399 Project Proposal
Academic Year 2018

Project Tittle (Thai)	ค่าสูงสุดต่ำสุดของฟังก์ชันแบบช่วงที่มีเงื่อนไขเชิงเส้น
Project Tittle (English)	Maximum and minimum of interval function with linear condition.
Project Advisor	Associate Professor Phantipa Thipwiwatpotjana, Ph.D.
By	Miss Siratcha Ruanruen ID 5833542423 Mathematics, Department of Mathematics and Computer Science Faculty of Science, Chulalongkorn University

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Background and Rationale

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all n realizations of a random variable x. All we know is that we can apply the idea of the lowest and the largest expected values of the set of all expected values : $\{\sum_{i=1}^{n} p_i x_i \mid p_i \in [l_i, u_i]\}$ where $L = \{[l_i, u_i] \mid 0 \leq l_i \leq u_i \leq 1,\ i = 1, 2, \ldots, n\}$ is a reachable set of proper probability intervals to find the lowest and the largest values of $\{\sum_{i=1}^{n} c_i x_i\}$ when c_i is an unknown in a given interval $[\underline{t_i}, \overline{v_i}]$ where $0 \leq t_i < v_i$ and $c_i's$ have the linear relation $\sum_{i=1}^{n} \alpha_i c_i = d$ for the given $\alpha_i, d \in \mathbb{R}$, **[2]**. In general, the bound of c_i is not always greater than or equal to zero, so we apply these ideas to find the lowest and the largest values of $\{\sum_{i=1}^{n} c_i x_i\}$ when c_i is an unknown in a given interval $[\underline{t_i}, \overline{v_i}]$ where $t_i < v_i$ and $\sum_{i=1}^{n} \alpha_i c_i = d$ for the fixed $\alpha_i, d \in \mathbb{R}$.

Objective

1. Prove that $\boldsymbol{p} = (u_1, \ldots, u_{i-1}, p_i, l_{i+1}, \ldots, l_n)$ for some index i such that $p_i = 1 - \sum_{j=1}^{i-1} u_j - \sum_{j=i+1}^{n} l_j$ providing the smallest expected value and

   $\boldsymbol{q} = (l_1, \ldots, l_{i-1}, q_i, u_{i+1}, \ldots, u_n)$ for some index i such that $q_i = 1 - \sum_{j=1}^{i-1} l_j -$

$\sum_{j=i+1}^{n} u_j$ providing largest expected value.

2. Find the lowest and the largest values of $\{\sum_{i=1}^{n} c_i x_i\}$ when c_i is an unknown in a given interval $[\underline{t_i}, \overline{v_i}]$ where $t_i < v_i$ and $\sum_{i=1}^{n} \alpha_i c_i = d$ for the fixed $\alpha_i, d \in \mathbb{R}$ and $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all n realizations of a random variable x.

Project Activity

1. Study probability interval in **[1]**.
   - Proper probability interval
   - Reachable probability interval
   - Extreme probabilities
2. Prove that probabilities $\boldsymbol{p}$, $\boldsymbol{q}$ got from the method in **[1]** provide the smallest and largest expected values, respectively.
3. Provide an algorithm to find the lowest and largest values of $\{\sum_{i=1}^{n} c_i x_i\}$ when c_i is an unknown in a given interval $[\underline{t_i}, \overline{v_i}]$ where $t_i < v_i$ and $\sum_{i=1}^{n} \alpha_i c_i = d$ for the fixed $\alpha_i, d \in \mathbb{R}$ and $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all n realizations of a random variable x.
4. Recheck the process.
5. Conclude all results and write a report.

Duration

Procedue	Month								
	Aug.	Sep.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.
1.Study probability interval in [1].	■	■							
2.Prove that probabilities $\underline{\boldsymbol{p}}$, $\overline{\boldsymbol{q}}$ got from the method in [1] provide the smallest and largest expected values, respectively.			■	■					
3.Provide an algorithm to find the lowest and largest values of $\boldsymbol{c}^T\boldsymbol{x}$.					■	■	■		
4.Recheck the process.							■	■	
5.Conclude all results and write a report.								■	■

Benefits

1. The benefits to the student who implements this project.
    - Apply the basic knowledge in mathematics and the knowledge gained from our learning to a related application problem.
    - Know how to work systematically.
2. The benefits for users of the project.
    - Find the lowest and largest values of $\{\sum_{i=1}^{n} c_i x_i\}$ when c_i is an unknown in a given interval $[\underline{t_i}, \overline{v_i}]$ where $t_i < v_i$ and $\sum_{i=1}^{n} \alpha_i c_i = d$ for the fixed $\alpha_i, d \in \mathbb{R}$ and $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all n realizations of a random variable x.
    - Apply the idea got from this project to a big data set.

Equipment

1. Hardware
    - Notebook computer Intel core i5-6200U

- Printer
- Thumb drive

2. Software

- Microsoft Word 365 ProPlus
- TeXstudio 2.12.8
- Spyder (Python 3.6)

References

[1] L. M. De Campos, J. F. Huete and S. Moral, Probability Interval; A Tool for Uncertain Reasoning, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, Vol. 2, No. 2, 167 - 196, 1994.
[2] N. Burana, *Interval Price Objective Coefficient Linear Programming Problem*, Project Report, Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, 2017.
[3] J. Rohn, Systems of Interval Linear Equations, *Linear Algebra and Its Applications*, No. 126, 39 - 78, 1989.

Appendix

1. Definition of probability interval

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all n realizations of a random variable x and $L = \{[l_i, u_i] \mid 0 \le l_i \le u_i \le 1,\ i = 1, 2, \ldots, n\}$ be a family of intervals bounded by 0 and 1. We can explain these intervals as a set of bounds of probabilities by defining the set $\mathscr{P}$ of probability distributions on X as

$$\mathscr{P} = \left\{ \boldsymbol{p} = (p_1, p_2, \ldots, p_n) \mid l_i \le p_i \le u_i,\ \sum_{i=1}^{n} p_i = 1 \text{ and } p_i = p(\{x_i\}),\ \forall i \right\}.$$

Then the set L is called a set of probability intervals (or *probability interval*, in short), if there exist $p_i \in [l_i, u_i]$, for all $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} p_i = 1$, and $\mathscr{P}$ is the set of all *possible probabilities* associated to L.

In order to avoid the emptiness of set $\mathscr{P}$, it is necessary to have more properties on the intervals which is called *proper*. A probability interval L is called a *proper probability interval* if

$$\sum_{i=1}^{n} l_i \le 1 \le \sum_{i=1}^{n} u_i.$$

In addition, we can also associate with the proper intervals $[l_i, u_i]$ by presenting a pair (l, u) of the lower and upper probabilities through $\mathscr{P}$ as follows.

$$l(A) = \inf_{\boldsymbol{p} \in \mathscr{P}} p(A) \text{ and } u(A) = \sup_{\boldsymbol{p} \in \mathscr{P}} p(A), \forall A \subseteq X.$$

Therefore, $l(\{x_i\}) = \inf_{\boldsymbol{p} \in \mathscr{P}} p_i \ge l_i$ and $u(\{x_i\}) = \sup_{\boldsymbol{p} \in \mathscr{P}} p_i \le u_i$. We use these two properties to get the tight bound of each interval.

2. Properties of probability interval

A proper probability interval must also have the following properties to ensure that the lower bound l_i and the upper bound u_i of the probability interval can be reached by some probabilities in the set $\mathscr{P}$, so called a reachable probability interval.

[**Reachable**] Let $L = \{[l_i, u_i] \mid 0 \le l_i \le u_i \le 1,\ i = 1, 2, \ldots, n\}$ be a probability interval. If there exist $p_j, q_j \in [l_j, u_j]$ for all $j \in \{1, 2, \ldots, n\}$ such that $\forall i,\ l_i + \sum_{j \ne i} p_j = 1$ and $u_i + \sum_{j \ne i} q_j = 1$, then L is called a reachable probability interval.

Theorem 5. Given a reachable probability interval

$$L = \{[l_i, u_i] \mid 0 \le l_i \le u_i \le 1,\ i = 1, 2, \ldots, n\},$$

we have

$$\sum_{j \ne i} l_j + u_i \le 1 \text{ and } \sum_{j \ne i} u_j + l_i \ge 1, \quad \forall i = 1, 2, \ldots, n.$$

These conditions guarantee that the lower bound l_i and the upper bounds u_i can be reached by some probabilities in $\mathscr{P}$. Sometimes probability interval is not reachable, we must change it to become a reachable probability interval. Now, let us see through the series of theorems how to modify a probability interval to be reachable without changing the associated set of possible probabilities $\mathscr{P}$.

3. Extreme probabilities

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of all n realizations of a random variable x, where each x_i has its corresponding unknown probability p_i bounded by $[l_i, u_i]$. Without loss of generality, if $x_1 \le x_2 \le \cdots \le x_n$, L.M. de Campos, et al. [1]provided probabilities $\underline{\boldsymbol{p}}$ and $\overline{\boldsymbol{q}}$ that give the smallest expected values $\underline{E}(x)$ and the largest expected values $\overline{E}(x)$, respectively.

Extreme probability is based on the assumption that when we want to have smallest expected values. If the realization of x_i is less than or equal to x_j, then probability p_i should be closer to the upper bound than probability p_j for all $i, j = 1, 2, \ldots, n$; i.e.,

$$(p_1, p_2, \ldots, p_n) = (u_1, u_2, \ldots, u_{k-1}, 1 - \sum_{j=1}^{k-1} u_j - \sum_{j=k+1}^{n} l_j, l_{k+1}, \ldots, l_n),$$

where k is the index such that $l_k \le 1 - \sum_{j=1}^{k-1} u_j - \sum_{j=k+1}^{n} l_j \le u_k$. On the other hand, when we want to have largest expected values, if the realization of $x_i$is less than or equal to x_j, then probability q_i should be closer to the lower bound than probability q_j for all$i, j = 1, 2, \ldots, n$; i.e.,

$$(q_1, q_2, \ldots, q_n) = (l_1, l_2, \ldots, l_{h-1}, 1 - \sum_{j=1}^{h-1} l_j - \sum_{j=h+1}^{n} u_j, u_{h+1}, \ldots, u_n),$$

where h is the index such that $l_h \leq 1 - \sum_{j=1}^{h-1} l_j - \sum_{j=h+1}^{n} u_j$, $u_{h+1} \leq u_h$.

The smallest expected values $\underline{E}(x) = \sum_{i=1}^{n} p_i x_i$, where p_i is a probability computed from the following algorithm.

Algorithm 1 for $\underline{\boldsymbol{p}} = (p_1, p_2, \ldots, p_n)$
$S \leftarrow 0$
For $i = 0$ to $n-1$ do $S \leftarrow S + u_i$;
$S \leftarrow S + l_n$;
$k \leftarrow n$;
While $S \geq 1$ do $S \leftarrow S - u_{k-1} + l_{k-1}$; $p_k \leftarrow l_k$; $k \leftarrow k-1$;
For $i = 1$ to $k-1$ do $p_i \leftarrow u_i$;
$p_k \leftarrow 1 - S + l_k$;

The largest expected values $\overline{E}(x) = \sum_{i=1}^{n} q_i t_i$, where q_i is a probability computed from the following algorithm.

Algorithm 2 for $\overline{\boldsymbol{q}} = (q_1, q_2, \ldots, q_n)$
$S \leftarrow 0$
For $i = 0$ to $n-1$ do $S \leftarrow S + l_i$;
$S \leftarrow S + u_n$;
$k \leftarrow n$;
While $S \leq 1$ do $S \leftarrow S + u_{k-1} - l_{k-1}$; $q_k \leftarrow u_k$; $k \leftarrow k-1$;
For $i = 1$ to $k-1$ do $q_i \leftarrow l_i$;
$q_k \leftarrow 1 - S + u_k$;

Biography

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