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A study on pricing barrier geometric average Asian options with Heston model

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A study on pricing barrier geometric average Asian options with Heston model

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แบบจำลองเฮสตันเป็นแบบจำลองที่จำลองความแปรปรวนแบบสุ่ม ที่เป็นระบบของสมการเชิงอนุพันธ์สโตแคสติกที่ใช้เพื่ออธิบายวิวัฒนาการของความแปรปรวนของสินทรัพย์อ้างอิง โดยในงานนี้เราได้ศึกษาการกำหนดราคาตราสารสิทธิในการซื้อแบบเอเชียเฉลี่ยราคาเรขาคณิตระดับราคาของสินทรัพย์อ้างอิงที่เป็นไปตามแบบจำลองเฮสตัน ซึ่งเราได้ใช้วิธีการมอนติคาร์โลในการกำหนดราคาของตราสารสิทธิด้วยการจำลองเส้นทางตัวอย่างของราคาสินทรัพย์อ้างอิง จากนั้นคำนวนหามูลค่าปัจจุบันของผลตอบแทนที่คาดหวังของตราสารสิทธิ นอกจากนี้ยังมีการวิเคราะห์ความอ่อนไหวของพารามิเตอร์บางตัวในแบบจำลอง

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The Heston model is a stochastic volatility model, which is a system of of stochastic differential equations, used to describe the evolution of the volatility of the underlying asset. In this work, we study how to price the knock-out barrier geometric average Asian call options whose underlying asset follow the Heston model. We use the Monte Carlo approach to price the option by simulating sample paths of the underlying asset price process and calculating the present value of the expected payoff of the option. A sensitivity analysis of some parameters in the model is also provided.

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Chapter 1

Introduction

In mathematical finance, stochastic differential equations (SDEs) are used to model various phenomena including unstable stock prices and interest rates. A stochastic volatility model is a system of SDEs in which the variance of a main stochastic process is another stochastic process. The Heston model is one of the stochastic volatility models that are easily and popularly used to describe the evolution of the volatility of the underlying asset. The Heston model have the form

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^s, \tag{1}$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t} dW_t^v, \tag{2}$$

where S_t is the price process of the underlying asset,

V_t is instantaneous variance process,

μ is the rate of return of the asset price,

θ is the long run average price variance,

κ is the rate at which V_t reverts to θ,

ξ is the volatility of the instantaneous variance,

and W_t^s and W_t^v are Wiener processes with correlation ρ.

The knock-out barrier geometric average Asian call option is an "exotic option" that is a mix between a knock-out barrier option and an Asian call option that have a payoff function with geometric average given by

$$max\{(\prod_{i=1}^{n} S_{t_i})^{\frac{1}{n}} - K, 0\},$$

where S_{t_i} is the asset price at the pre-specified time and K is the strike price.

In this work, we study how to determine the premium price of a knock-out barrier geometric average Asian call option when the underlying asset of the option is assumed to follow the Heston model. The knock-out barrier option considered in this work has the upper and lower barriers. We will use the Monte Carlo approach to price the option by simulating sample paths of the underlying asset price process, which is assumed to follow the Heston model, using the Euler-Maruyama (EM) method and the Milstein (MS) method and calculating the present value of the expected payoff of the option.

The organization of this project is as follows. In chapter 2, we give some background knowledge in this work such as introduction to stochastic differential equation, the numerical methods that we use, and the meaning of geometric average Asian option and barrier option. In chapter 3, we talk about our methodology in this work. We state the assumption of the model, show schemes for the numerical methods, give our MATLAB code for simulating sample paths of the price process and calculating the premium price of the barrier geometric average Asian option, and talk about our sensitivity analysis of some parameters in the model. In chapter 4, we will show graphs of the simulated sample paths of the underlying asset price process using Euler-Maruyama method and Milstein method, premium of option, and run time. In the last chapter, we give the conclusion and recommendations from the results obtained from this work.

Chapter 2

Preliminaries

2.1 Introduction to Probability Theory

2.1.1 Probability Spaces

A ***probability space*** $(\Omega, \mathcal{F}, P)$ consists of the following three components.

1. A **sample space** Ω is the set of all possible outcomes.

2. The σ**-algebra** $\mathcal{F}$ is a set of subsets of Ω, called "events", such that:

    - $\mathcal{F}$ contains the sample space: $\Omega \in \mathcal{F}$,
    - $\mathcal{F}$ is closed under complements: if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and
    - $\mathcal{F}$ is closed under countable unions: if $A_1, A_2, ... \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

3. The ***probability measure*** P: $\mathcal{F} \to [0, 1]$ is a function on $\mathcal{F}$ such that:

    - P is countably additive: if $\{A\}_{i=1}^{\infty} \subseteq \mathcal{F}$ is a countable sequence of disjoint set, then
    $$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i),$$
    - the measure of the entire sample space is equal to one: $P(\Omega) = 1$.

2.1.2 Random Variables and Distribution Functions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real-valued function $X : \Omega \to \mathbb{R}$ is called a ***random variable***, if $\forall x \in \mathbb{R}$, $\{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F}$.

For a random variable X, we say that $F_X : \mathbb{R} \to [0,1]$ is a ***distribution function*** of the random variable X, if F_X corresponds to

$$F_X(x) = P(\{\omega \in \Omega : X(\omega) \leq x\}) = P(X \leq x), \forall x \in \mathbb{R}.$$

A random variable X is said to be a continuous random variable, if there exists a real-valued function $f : \mathbb{R} \to [0,1]$ such that

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt, \forall x \in \mathbb{R},$$

and we said that f is the ***probability density function (PDF)*** of X.

2.1.3 Joint Distribution Function and Independence of Random Variables

Let $X_1, X_2, \ldots, X_m$ be random variables on probability space $(\Omega, \mathcal{F}, P)$. The ***joint distribution function*** of them is the function $F_{X_1,\ldots,X_m} : \mathbb{R}^m \to [0,1]$ defined by

$$F_{X_1,\ldots,X_m}(x_1, \ldots, x_m) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_m \leq x_m).$$

If $P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_m \leq x_m) = P(X_1 \leq x_1)P(X_2 \leq x_2) \ldots P(X_m \leq x_m)$, $\forall x_1, x_2, \ldots, x_m \in \mathbb{R}$, we will say that random variables $X_1, X_2, \ldots, X_m$ are ***independent***.

2.1.4 Expected Values and Variances

For a continuous random variable X on a probability space $(\Omega, \mathcal{F}, P)$ with a probability density function $f(x)$, the ***expected value*** of X is defined as

$$E(X) = \int_{\mathbb{R}} xf(x)\, dx.$$

For a positive integer k and a random variable X such that $E(X^k) < \infty$, $E(X^k)$ is called the k^{th} ***moment*** of X. If $E(X)$ and $E(X^2)$ exist, then the ***variance*** of X is defined as

$$\begin{aligned} Var(X) &= E[(X - E(X))^2] \\ &= E(X^2) - [E(X)]^2. \end{aligned}$$

2.1.5 Normal Distribution

A random variable X has a ***normal distribution*** with parameters μ and σ^2 denoted by $X \sim \mathcal{N}(\mu, \sigma)$, if the probability density function of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

If $X \sim \mathcal{N}(0, 1)$, we say that X is a ***standard normal distribution*** and its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

2.2 Introduction to Stochastic Differential Equation

2.2.1 Standard Brownian Motion

A ***stochastic process*** is defined as a collection of random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Let $[0, T]$ be an interval. A collection of random variables $\{X(t) : t \in [0, T]\}$ is called a ***stochastic process***.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_t\}_{t\in[0,T]}$ be a stochastic process. If we fix $\omega \in \Omega$, a function $X.(\omega) : [0, T] \to \mathbb{R}$ is called a ***sample path***. We say that the process $\{X_t\}_{t\in[0,T]}$ has ***continuous sample paths***, if for almost all $\omega \in \Omega, X.(\omega)$ is a continuous function.

A stochastic process $\{W_t\}_{t\in[0,T]}$ is a scalar ***standard Brownian motion*** or standard ***Wiener process***, if the following conditions hold.

1. $W_0 = 0$ with probability 1.

2. For $0 \leq s < t \leq T$, $W_t - W_s \sim \mathcal{N}(0, t - s)$.

3. For $0 \leq s < t \leq u < v \leq T$, the increments $W_t - W_s$ and $W_v - W_u$ are independent.

4. $\{W_t\}_{t\in[0,T]}$ has continuous sample paths.

2.2.2 Stochastic Differential Equations

A stochastic differential equation (SDE) is a form of an integral equation in which one or more of the terms is a stochastic integral. SDEs are used to model various phenomena such as unstable stock prices and interest rates. Typically, an SDE has the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$

where X_t is the target process,

W_t is the Wiener process,

$\mu(X_t, t)$ is a drift function of X_t and t,

and $\sigma(X_t, t)$ is a diffusion function of X_t and t.

This equation should be understood as the stochastic integral equation

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s.$$

Here, the integral $\int_0^t \mu(X_s, s)ds$ is interpreted as the Riemann integral and the integral $\int_0^t \sigma(X_s, s)dW_s$ is interpreted as the Itô stochastic integral.

2.3 Numerical Method

2.3.1 Euler-Maruyama Method

To apply the Euler-Maruyama (EM) method to an SDE

$$dY_t = a(Y_t)dt + b(Y_t)dW_t$$

with initial condition that Y_0 is a constant in $\mathbb{R}$, where W_t stands for the Wiener process on [0, T], we discretize the interval [0, T] into N equidistant sub-intervals of width $\Delta t = \frac{T}{N}$. Define $t_n = n\Delta t$ for $n = 0, 1, 2, \ldots, N$ and denote the numerical solution of Y_{t_n} by y_n. The Euler-Maruyama scheme takes the form

$$y_0 = Y_0$$
$$y_{n+1} = y_n + a(y_n)\Delta t + b(y_n)\Delta W_n$$

for $n = 0, 1, 2, \ldots, N-1$, where $\Delta W_n = W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta t)$. Figure 2.1 shows an example of a simulated sample path for the EM scheme.

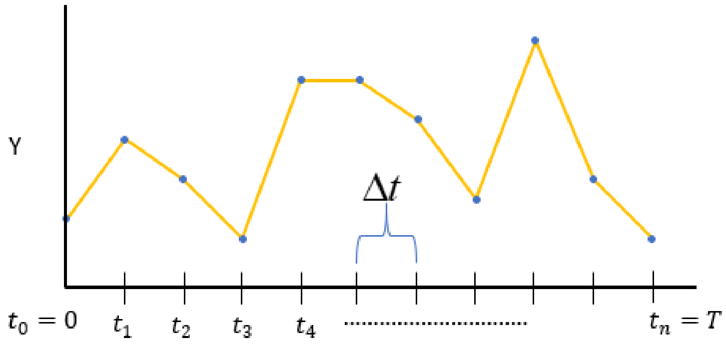

Figure 2.1: A simulated sample path for the Euler-Maruyama scheme.

2.3.2 Milstein Method

The Milstein (MS) method is a technique to approximate numerical solutions of an SDE. The Milstein scheme adds one additional term to the Euler scheme and has the form

$$y_0 = Y_0$$
$$y_{n+1} = y_n + a(y_n)\Delta t + b(y_n)\Delta W_n + \frac{1}{2}b(y_n)b'(y_n)((\Delta W_n)^2 - \Delta t)$$

for $n = 0, 1, 2, \ldots, N-1$, where $\Delta W_n = W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta t)$.

2.4 Option

Options are a financial derivative sold by an option writer to an option buyer. The contract offers the buyer the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price prior to or on a specified date. There are 2 general types of options: call and put.

A ***call option*** gives the option buyer the right to buy the underlying asset. The most basic type of options is a European call option which allows the option buyer to buy the asset only at the expiration date. If the price of the underlying asset at the expiry date S_T is higher than the strike price K, the option buyer will exercise (use the right to buy) the option and buy the asset with the lower price K. If S_T is lower than K, the option buyer will do nothing. Thus, a European call option has

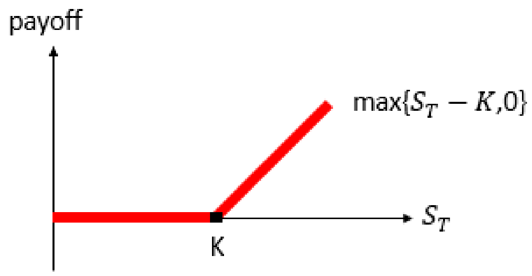

Figure 2.2: A payoff function of a European call option.

the payoff function given by

$$max\{S_T - K, 0\}.$$

Figure 2.2 shows a graph of a payoff function of a European call option.

A ***put option*** gives the option buyer the right to sell the underlying asset. A European put option has the payoff function given by

$$max\{K - S_T, 0\},$$

where S_T is the price of the underlying asset at the expiry time T and K is the strike price. Figure 2.3 shows a graph of a payoff function of a European put option.

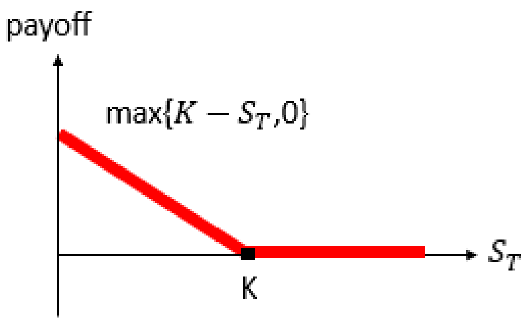

Figure 2.3: A payoff function of a European put option.

2.4.1 Geometric Average Asian Option

An ***Asian option*** is a special type of an option contract. The payoff of an Asian option is determined by the average underlying stock price over some pre-specified period of time. Thus, it is a path-dependent option. A ***geometric average Asian call option*** is an Asian option whose payoff is given by

$$max\{(\prod_{i=1}^{n} S_{t_i})^{\frac{1}{n}} - K, 0\}$$

where $t_1, t_2, ..., t_n$ are the pre-specified time,

$S_{(t_i)}$ is the price of the underlying asset at time t_i ,

and K is the strike price.

2.4.2 Barrier Option

A ***barrier option*** is also another path-dependent option. It is activated or inactivated only if the price of the underlying asset reaches a barrier of the predetermined region. Typically, barrier options are classified as knock-in and knock-out. A ***knock-in*** barrier option is activated and remains in existence until it expires only when the underlying asset price process reaches the predetermined barrier. A ***knock-out*** barrier option cease to exist, if the underlying asset price process reaches the predetermined barrier during the life of the option.

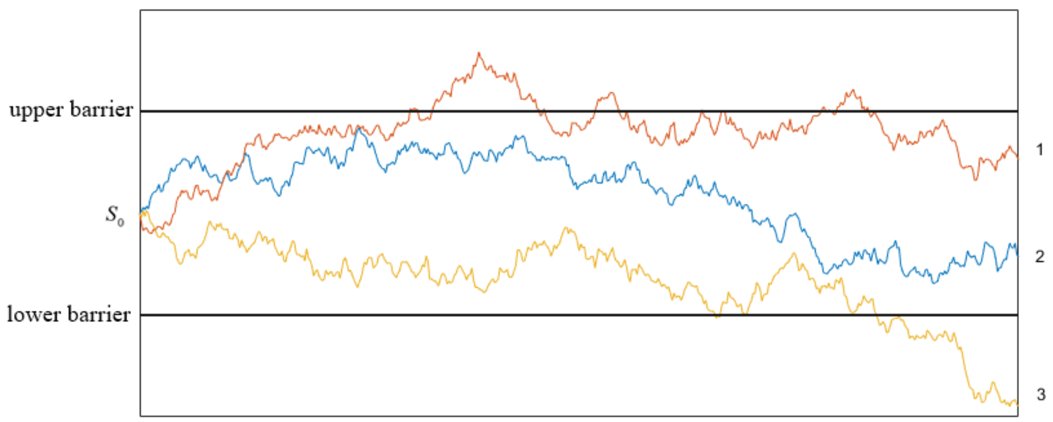

Figure 2.4: An example for a barrier option.

Figure 2.4 shows an example for a barrier option. The barrier option has upper and lower barriers shown in the figure. Also, there are three sample paths for the underlying asset price process. For a knock-in barrier option, the sample paths **1** and **3** make the option activated because the underlying asset price process in these paths reach the upper barrier or the lower barrier. Thus, the payoff of the option will be calculated as usual. However, the sample path **2** makes the option inactivated because the underlying asset price process in this path never reaches the barriers. For sample path **2**, the option is worthless and the option payoff is zero. As for a knock-out barrier option, it will be in the opposite direction. The sample paths **1** and **3** make the option inactivated, and the sample path **2** makes the option activated.

Chapter 3

Methodology

In this chapter, we will use the Monte Carlo approach to simulate sample paths of the underlying asset price process, which is assumed to follow the Heston model, using the EM and MS schemes and calculating the present value of a knock-out barrier geometric average Asian call option. Also, a sensitivity analysis for some parameters in the model is given.

3.1 The Assumption of the Model

Before we talk about our methodology in this work, we would like to discuss about the assumption of the model. We consider the underlying asset price process, which is assumed to follow the Heston model given by equations (1) and (2) in chapter 1. We consider 2 methods for the numerical methods that we use to simulate sample paths. Method I is to use the EM method for both equations (1) and (2), and Method II is to use the EM method for equation (1) and the MS method for equation (2). Unless otherwise specified, the following assumptions for the underlying asset and the option are used throughout this work.

1. The underlying asset is assumed to follow the Heston model with the starting value $S_0 = 100$. The starting value of instantaneous variance is $V_0 = 0.04$. The parameters in equations (1) and (2) are assumed to be $\mu = 0.02$, $\theta = 0.06$, $\kappa = 1.5$, $\xi = 0.2$, $\rho = 0.3$. The risk-free continuously compounded interest rate is assumed to be $r = 1.6\%$.

2. The option is a knock-out barrier geometric average Asian call option that has the expiry date $T = 1$ year. The strike price is $K = 90$. The upper barrier and the lower barrier are 170 and 60, respectively. The underlying asset price at the end of every month will be used to calculate the payoff of the Asian option. For simplicity, we assume that every month has the same number of days.

3.2 Schemes for the Model

In this section, we give the schemes of the numerical methods that we use for the Heston model. From the SDE (1), $dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^s$, we have that

$$a(S_t) = \mu S_t \qquad \text{and} \qquad b(S_t, V_t) = \sqrt{V_t}\, S_t.$$

The Euler-Maruyama (EM) scheme for S_t takes the form

$$\begin{aligned} s_0 &= S_0, \\ s_{n+1} &= s_n + a(s_n)\Delta t + b(s_n, v_n)\Delta W_n^s \\ &= s_n + \mu s_n \Delta t + \sqrt{v_n} s_n \Delta W_n^s, \end{aligned}$$

for $n = 0, 1, 2, \ldots, N-1$. To update s_{n+1}, we need to find v_n first. From the SDE (2), $dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t} dW_t^v$, we have that

$$a(V_t) = \kappa(\theta - V_t), \qquad b(V_t) = \xi\sqrt{V_t} \qquad \text{and} \qquad b'(V_t) = \frac{1}{2\sqrt{V_t}}\xi.$$

The Euler-Maruyama (EM) scheme for V_t takes the form

$$\begin{aligned} v_0 &= V_0, \\ v_{n+1} &= v_n + a(v_n)\Delta t + b(v_n)\Delta W_n^v \\ &= v_n + \kappa(\theta - v_n)\Delta t + \xi\sqrt{v_n}\Delta W_n^v, \end{aligned}$$

for $n = 0, 1, 2, \ldots, N-1$. The Milstein (MS) scheme for V_t takes the form

$$\begin{aligned} v_0 &= V_0, \\ v_{n+1} &= v_n + a(v_n)\Delta t + b(v_n)\Delta W_n^v + \frac{1}{2} b(v_n) b'(v_n)((\Delta W_n^v)^2 - \Delta t) \\ &= v_n + \kappa(\theta - v_n)\Delta t + \xi\sqrt{v_n}\Delta W_n^v + \frac{1}{2}\xi\sqrt{v_n}\frac{1}{2\sqrt{v_n}}\xi((\Delta W_n^v)^2 - \Delta t) \end{aligned}$$

$$= v_n + \kappa(\theta - v_n)\Delta t + \xi\sqrt{v_n}\Delta W_n^v + \frac{1}{4}\xi^2((\Delta W_n^v)^2 - \Delta t),$$

for $n = 0, 1, 2, \ldots, N-1.$

3.3 Option Pricing

In this work, we use MATLAB program to numerically price the knock-out barrier geometric average Asian call option. Figure 3.1 shows the MATLAB code for simulation of method II.

```
T = 1 ;
N = 12000 ;
Delta  = T/N;
M = 10000 ;
K=100;
up = 130;
down = 70;
S0 = 100; mu = 0.02 ;
rho = 0.3;
V0 = 0.04; theta = 0.06 ; kappa = 1.5 ; xi = 0.2 ;
r = 0.016;
```
1

```
V = [V0*ones(M,1), zeros(M,N)];
S = [S0*ones(M,1), zeros(M,N)];
Wv = randn(M,N)*sqrt(Delta) ;
Ws = rho*Wv + sqrt(1-rho^2)*randn(M,N)*sqrt(Delta) ;
```
2

```
for j = 1:N
  V(:,j+1) = V(:,j) + kappa*(theta-V(:,j))*Delta +
             xi*sqrt(V(:,j)).*Wv(:,j) + (1/4)*(xi^2).*((Wv(:,j).^2)-Delta)
  S(:,j+1) = S(:,j) + mu*S(:,j)*Delta + sqrt(V(:,j)).*S(:,j).*Ws(:,j)
end
```
3

```
check = all(S<up&S>down,2) ;
pay = max(geomean(S(:,N/12:N/12:N),2).*check-K,0) ;
Cavg = sum(pay)/M;
C = exp(-r*T)*Cavg;
```
4

Figure 3.1: MATLAB code for simulation of method II

This code is divided into 4 parts. The first part is to set all parameters for the simulation which are the expiration time (T), the number of time-steps (N), the time-step size (Delta), the number of sample paths (M), the strike price (K), the upper barrier (up), the lower barrier (down), continuously compounded interest rate (r) and the parameters $S_0, \mu, V_0, \theta, \kappa, \xi$ and ρ in the Heston model (1) - (2).

The second part is to preallocate 2 matrices **V** and **S** with dimension $M \times (N+1)$ for collecting values of V_t and S_t at each time-step for every path. Each row of the matrices represents one sample path, and there are $N+1$ steps, $n = 0, 1, 2, \ldots, N$, for each sample path. Also, we create the matrix **Wv** with dimension $M \times N$ for collecting Brownian increments at each time-step of W_t^v where each element in this matrix is generated from a normal distribution with mean 0 and variance $Delta$. Then, we create the matrix **Ws** with dimension $M \times N$ where each element in this matrix is generated from a normal distribution with mean 0 and variance $Delta$ in a way that **Wv** and **Ws** have correlation ρ.

The next part is to generate sample paths for V_t and S_t. For method II, we use a **for loop** to update each column of the matrices **V** and **S** using the scheme

$$v_{n+1} = v_n + \kappa(\theta - v_n)\Delta t + \xi\sqrt{v_n}\Delta W_n^v + \frac{1}{4}\xi^2\left((\Delta W_n^v)^2 - \Delta t\right),$$
$$s_{n+1} = s_n + \mu s_n \Delta t + \sqrt{v_n} s_n \Delta W_n^s.$$

For method I, we just change the MS scheme for V_t to the EM scheme instead. We use the scheme

$$v_{n+1} = v_n + \kappa(\theta - v_n)\Delta t + \xi\sqrt{v_n}\Delta W_n^v,$$
$$s_{n+1} = s_n + \mu s_n \Delta t + \sqrt{v_n} s_n \Delta W_n^s.$$

In the last part, we create the boolean matrix **check** with dimension $M \times 1$ to check whether each sample path live within the barrier (**down**,**up**). An entry in the matrix **check** is set to 1, if the corresponding sample path lives in the barrier, or 0, if the corresponding sample path is once outside the barrier. Next, we compute the payoff of the option for each path. We use the sample path of the price process at the end of every month to calculate the payoff of the option

$$max\{(\prod_{i=1}^{12} s_{1000i})^{\frac{1}{12}} - K, 0\} \cdot (\prod_{j=1}^{12000} \mathbb{1}_{(down,up)}(s_j)).$$

Since this option is a knock-out barrier option, we calculate the payoff for the path that never has element reaches the barrier. Then, **Cavg** is the average of the payoffs for all paths. We use it to calculate the present value of the expected payoff of the option $e^{-rT}\cdot$**Cavg**.

3.4 Sensitivity Analysis

In this section, we will study the effect of changing some parameters. The parameters considered in this section are only ρ, K and the level of barriers because the effect of varying those parameters is interesting. We use method II, which is the EM method for S_t and the MS method for V_t, to simulate sample paths for the underlying asset price process and use the parameter setup $S_0 = 100, V_0 = 0.04, \mu = 0.02, \theta = 0.06, \kappa = 1.5, \xi = 0.2, \rho = 0.3, K = 100, r = 1.6\%, M = 10000, N = 12000$, upper barrier = 130 and lower barrier = 70. With this parameter setup, we simulate the sample path using different values of the certain considered parameter. For the parameter ρ, we use $\rho = 0,\ \rho = 0.3$ and $\rho = 0.6$. For the stike price K, we use $K = 95,\ K = 100$ and $K = 105$. For the level of the barriers, we use the range of the level of the barriers to be $\pm 10\%, \pm 30\%$, and $\pm 50\%$ from the starting price S_0.

Chapter 4

Results and Discussion

In this chapter, we show graphs of the simulated sample paths of the underlying asset price process using EM method and MS method, the approximated premium of the option, and run time. Also, we present the sensitivity of the option premium when the parameter ρ, K, and the level of the barriers are changed.

4.1 Sample Paths of the Price Process

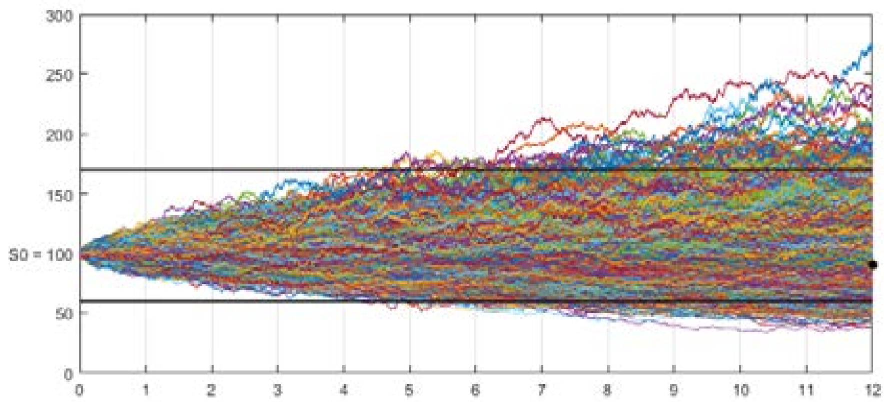

Figure 4.1: Graphs for sample paths of method I

Figure 4.1 and 4.2 show 10,000 sample paths of the price process that we simulate in section 3.3 using methods I and II, respectively. From the figure 4.1 and 4.2, the upper barrier and the lower barrier are the black thick lines shown in figures. All of the sample paths start at 100. At the expiry date, we calculate the payoff of the option for every sample path. For the sample paths that once reach the upper barrier or lower barrier, their payoff at the expiry date will be set to 0. Then, we calculate the average of the payoff for every path and discount back to the present time to get the value of the option premium.

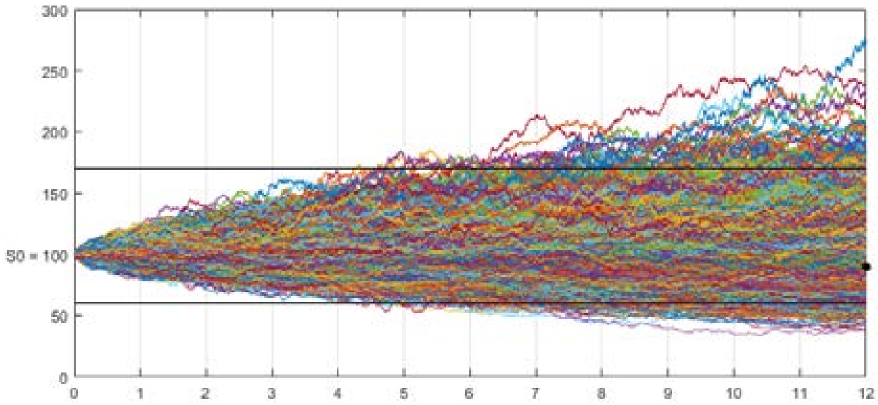

Figure 4.2: Graphs for sample paths of method II

4.2 Premium of the Option and Run Time

In the same experiment that we simulate sample paths of the underlying asset price process and numerically find the option premium in section 3.3, we show the price of the knock-out barrier geometric average Asian call option. Here, we use two sizes for the number of simulated sample paths: M = 10000 and M = 75000. The results with M = 10000 and M = 75000 are presented in Table 4.1 and 4.2, respectively .

	S_t	V_t	Premium of Option
Scheme	EM	EM	11.024603091680783
	EM	MS	11.020464309395157

Table 4.1: Premium of option with M = 10000.

From the table 4.1, if we simulate sample paths with M = 10000, the approximated premium of the option for method I and method II are equal up to two decimal places. Although the sample paths is figure 4.1 and 4.2 look similar, the corresponding option premiums are different. From the table 4.2, if we simulate sample paths with M = 75000, the approximated premium of the option for method I and method II are equal up to four decimal places.

	S_t	V_t	Premium of Option
Scheme	EM	EM	11.064457234404511
	EM	MS	11.064463972522715

Table 4.2: Premium of option with M = 75000.

In this experiment, we use a computer with Intel(R) Core(TM) i5-6200U CPU @ 2.30GHz 2.40GHz and RAM 12 GB. The run time with M = 10000 and M = 75000 are presented in Table 4.3 and 4.4, respectively.

	S_t	V_t	Run Time (seconds)
Scheme	EM	EM	10.434104
	EM	MS	12.755577

Table 4.3: Run Time with M = 10000.

	S_t	V_t	Run Time (seconds)
Scheme	EM	EM	452.221241
	EM	MS	473.262705

Table 4.4: Run Time with M = 75000.

From the table 4.3 and 4.4, the run times using the MS methods are longer than those using the EM methods for both cases M = 10000 and M = 75000. This is because the number of updating terms in the MS scheme is more than those in the EM scheme.

4.3 The Effect of Varying Parameters

In this section, we use the parameter setup $S_0 = 100, V_0 = 0.04, \mu = 0.02, \theta = 0.06, \kappa = 1.5, \xi = 0.2, \rho = 0.3, K = 100, r = 1.6\%, M = 10000, N = 12000$ and the level of barriers is $\pm 30\%$ from S_0. Then, we study the effect of changing the parameter ρ, K, and the level of barriers.

4.3.1 The effect of changing values of ρ

Figure 4.3 shows sample paths for different correlation ρ between W_t^s and W_t^v. The price processes simulated with different correlation have different sample paths. Therefore, they also yield different option premiums. Table 4.5 shows the option premiums when we use ρ = 0, 0.3 and 0.6. From the table 4.5, we can see that the more ρ increases, the less the option premium decreases.

ρ	Premium of Option
0	1.520329455663076
0.3	1.290747477950326
0.6	1.110312300481883

Table 4.5: Premium of option when changing values of ρ.

4.3.2 The effect of changing values of K

Figure 4.4 shows sample paths with different strike prices shown in the figure. The sample paths in the three graphs are the same, but there are different strike prices K = 95, 100 and 105. Therefore, the option premiums are different. Table 4.6 shows the option premiums when we use K = 95, 100 and 105. From the table 4.6, we can see that the more K increases, the less the option premium decreases.

K	Premium of Option
95	2.851576959744568
100	1.290747477950326
105	0.420741537742028

Table 4.6: Premium of option when changing values of K.

4.3.3 The effect of changing range of the level of the barriers

Figure 4.5 shows sample paths with different level of the barriers shown in the figure. The sample paths in the three graphs are the same, but there are different level of the barriers $\pm 10\%, \pm 30\%$, and $\pm 50\%$ from S_0. Therefore, the option premiums are different. Table 4.7 shows the option premiums when we use the level of the barriers = $\pm 10\%, \pm 30\%$, and $\pm 50\%$ from S_0. From the table 4.7, we can see that the more range of the level of the barriers increases, the more the option premium increases.

range of the level of the barriers	Premium of Option
$\pm 10\%$	0.004844363835729
$\pm 30\%$	1.290747477950326
$\pm 50\%$	3.374743804160928

Table 4.7: Premium of option when changing range of the level of the barriers.

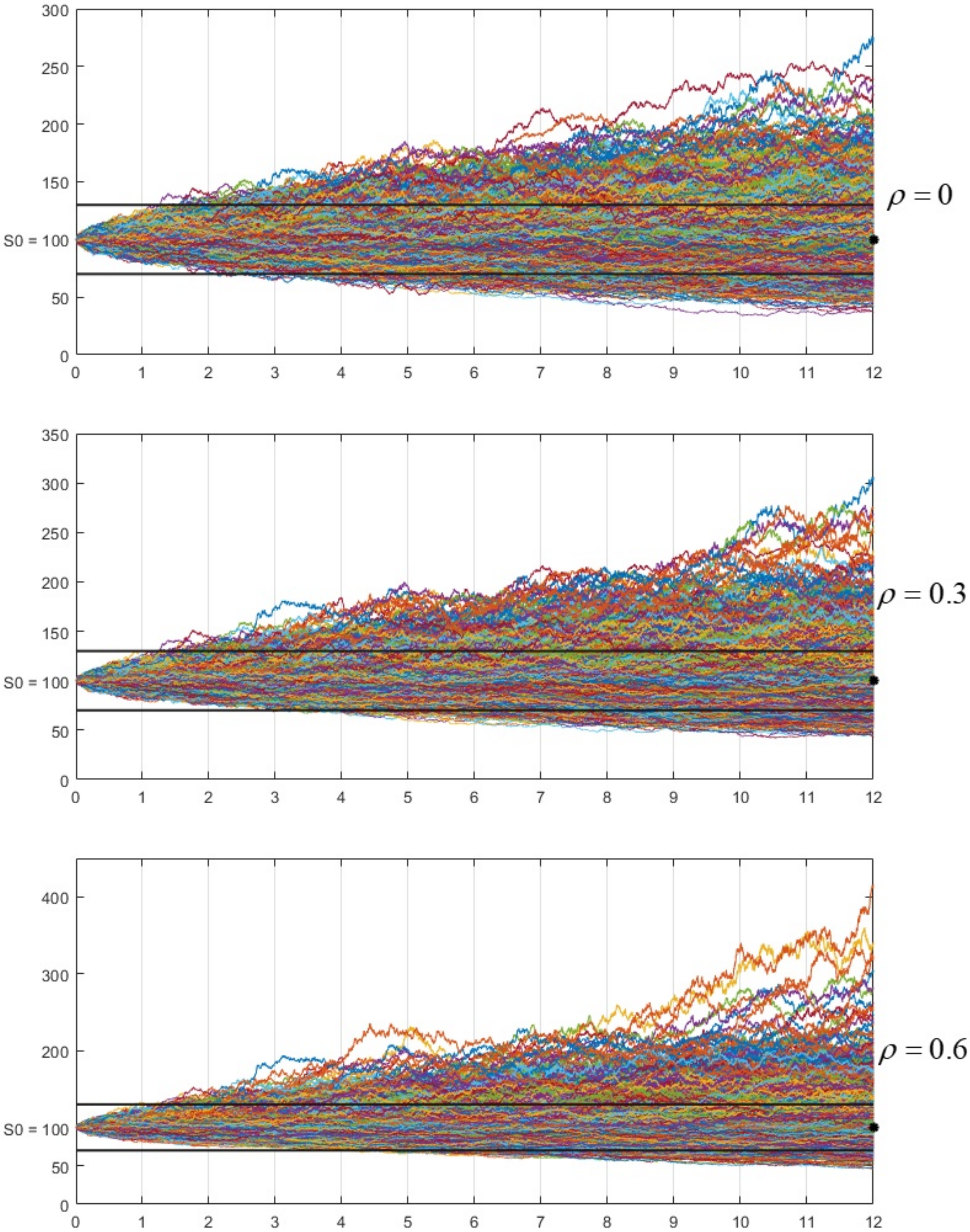

Figure 4.3: Graphs for the effect of changing values of ρ.

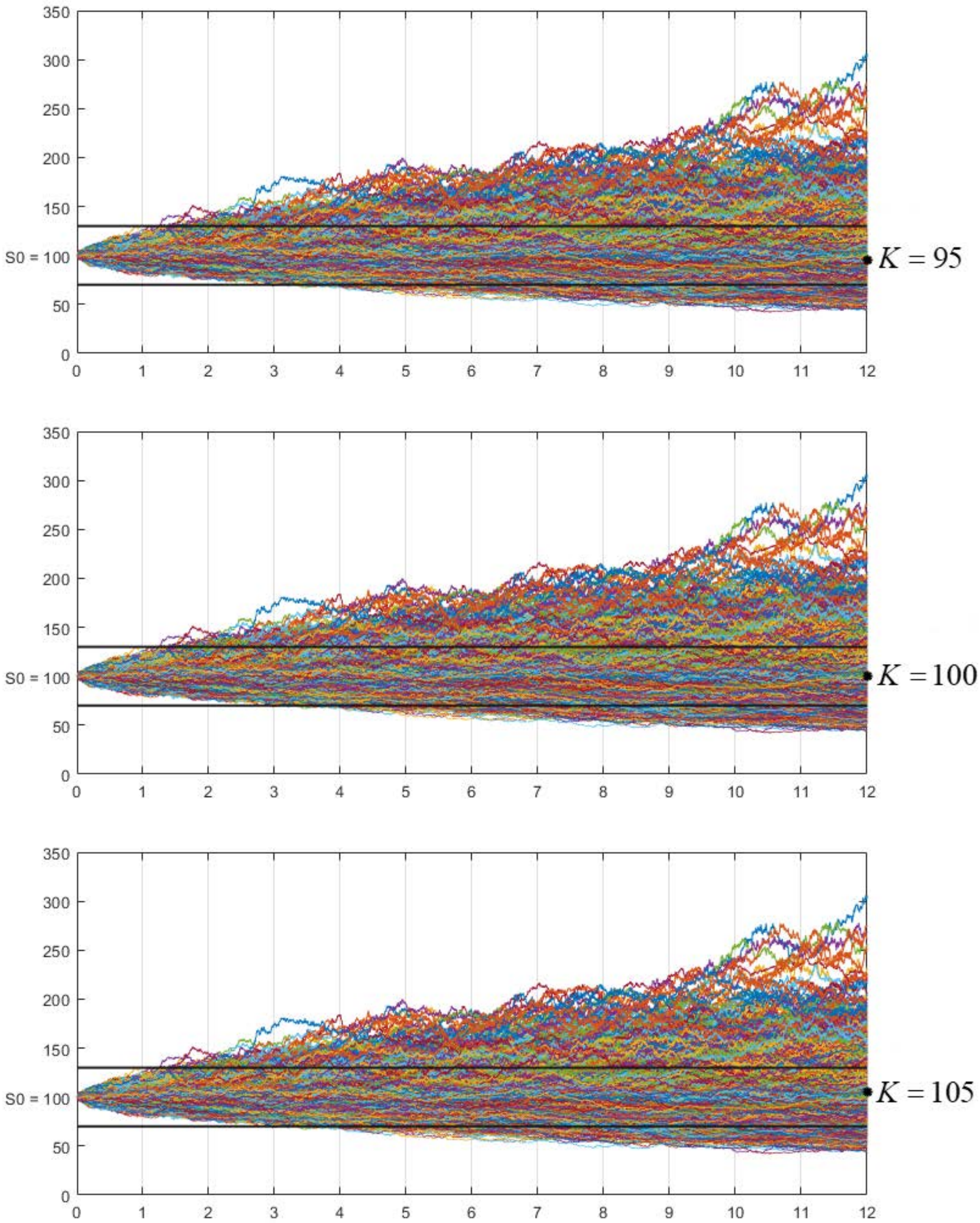

Figure 4.4: Graphs for the effect of changing values of K.

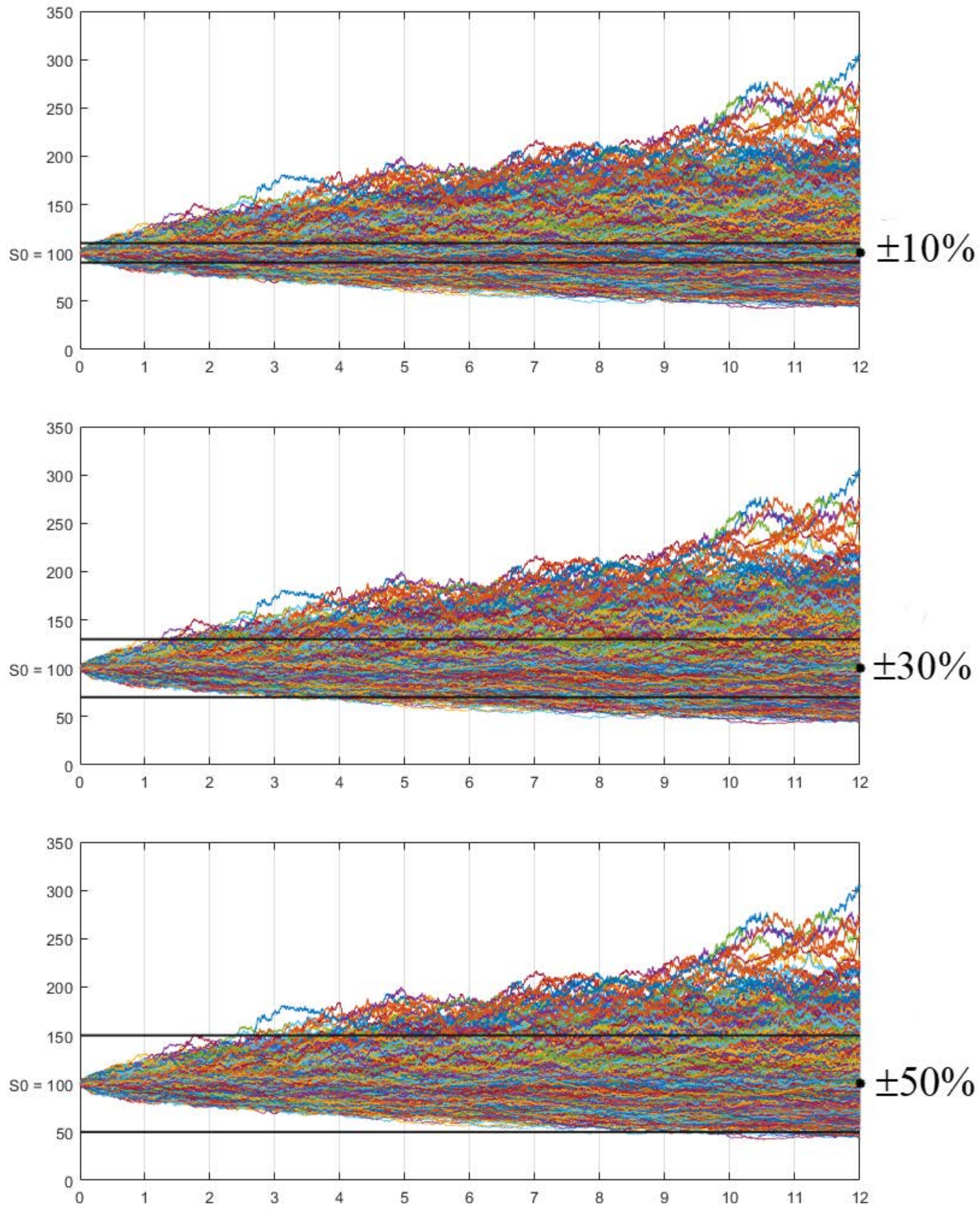

Figure 4.5: Graphs for the effect of changing range of the level of the barriers.

Chapter 5

Conclusions

In this work, we study how to price a knock-out barrier geometric average Asian call option. We use the Monte Carlo approach to simulate sample paths of the underlying asset price process, which is assumed to follow the Heston model using the EM and MS methods, and calculate the present value of the expected payoff of the option. In addition, the author gains new knowledge about stochastic differential equations, Euler-Maruyama method, Milstein method and using MATLAB which can be useful in the future. Also, the author knows how to simulate the price process of the underlying asset, how to calculate the present value of the average payoff of the option.

From the results in chapter 4, the run time using the MS method is more than the run time using the EM method because there is an additional calculation in the MS scheme. However, the premium of the option obtained from the MS method may be close to the actual price of the premium than that obtained from the EM method. As for sensitivity analysis, the more ρ increases, the less option premium will be. Similarly, the increase in the strike price yields the lower premium. On the contrary, the increase in the level of the barriers from the starting price give in the higher premium.

References

[1] Mikhailov, E. *Programming with MATLAB for Scientists: A Beginner's Introduction*. CRC Press, 2017.

[2] Kijima, M. *Stochastic Processes with Applications to Finance*, 2nd Edition. Chapman and Hall/CRC, 2013.

[3] Oksendal, B. *Stochastic Differential Equations: An Introduction with Applications*. Berlin: Springer, 2003.

The Project Proposal of Course 2301399 Project Proposal

First Semester, Academic Year 2018

Title (Thai)	การศึกษาการกำหนดราคาตราสารสิทธิแบบเอเชียเฉลี่ยราคาเรขาคณิตระดับราคาโดยแบบจำลองเฮสตัน
Title (English)	A study on pricing barrier geometric average Asian options with Heston model
Advisor	Raywat Tanadkithirun, Ph.D.
By	Mr. Anantaphom Boonsri ID 5833556223 Mathematics, Department of Mathematics and Computer Science

Background and Rationale

A stochastic differential equation (SDE) is a form of an integral equation in which one or more of the terms is a stochastic integral. SDEs are used to model various phenomena such as unstable stock prices or physical systems subject [2,3]. Typically, an SDE has the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

where X_t is the target process,

W_t is the Wiener process,

$\mu(X_t, t)$ is a drift function of X_t and t,

and $\sigma(X_t, t)$ is a diffusion function of X_t and t.

The Heston model is a stochastic volatility model that is popularly used to describe the evolution of the volatility of the underlying asset [2]. The basic Heston can be represented by

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^s \quad (1)$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t} dW_t^v \quad (2)$$

where S_t is the price process of the underlying asset,

V_t is instantaneous variance process,

μ is the rate of return of the asset price,

θ is the long run average price variance,

κ is the rate at which V_t reverts to θ,

ξ is the volatility of the instantaneous variance,

and W_t^s, W_t^v are Wiener processes or Brownian motion with correlation ρ.

An option is a contract which gives the buyer the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price prior to or on a specified date. There are 2 general types of options: call and put. A call option gives the option buyer the right to buy, and a put option gives the option buyer the right to sell.

An Asian option is a special type of option contract. The payoff of an Asian option is determined by the average underlying price over some pre-specified period of time. Thus, it is a path-dependent option. A geometric average Asian call option is an Asian option whose payoff is given by

$$max\{(\prod_{i=1}^{n} S_{t_i})^{\frac{1}{n}} - K, 0\}$$

where $t_1, t_2, ..., t_n$are the pre-specified time,

$S_{(t_i)}$ is the price of the underlying asset at time t_i

and K is the strike price.

A barrier option is also another path-dependent option. It is activated or inactivated only if the price of the underlying asset reaches a barrier of the predetermined region. Typically, Barrier options are classified as knock-in and knock-out. A knock-in barrier option is activated and remains in existence until it expires only when the underlying asset price process reaches a barrier. A knock-out barrier option cease to exist, if the underlying asset price process reaches a barrier during the life of the option.

In this work, we study how to price a barrier geometric average Asian option. We will use the Monte Carlo approach by simulating sample paths of the underlying asset price process, which is assumed to follow the Heston model (1) – (2), and calculating the present value of the expected payoff of the option.

Objectives

To study how to price barrier geometric average Asian options using Heston model.

Scope

In this work, we will consider the Heston model (1)-(2) for the underlying asset price process. The option that we will price is a knock-out barrier geometric average Asian call option with predetermined upper and lower barriers. We will use the Euler-Maruyama (EM) method for both (1) and (2), and then use the EM method for (1) and the Milstein (MS) method for (2).

Project Activities

1. Determine the topic and scope of the project through the feedback from the advisor.
2. Study contents used in the project.
    - Geometric average Asian option
    - The Heston model
    - The Euler-Maruyama and Milstein methods
    - Matlab [1]
3. Write Matlab codes to estimate numerical solutions.
4. Analyze the results.
5. Check for accuracy.
6. Conclude all results and write a report.
7. Prepare for the presentation.

Duration

Procedure	August 2017 - April 2018								
	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
1. Determine the topic and scope of the project through the feed-back from the advisor.	■	■							
2. Study contents used in this project.		■	■						
3. Write Matlab codes to estimate numerical solutions.				■	■				
4. Analyze the results.				■	■				
5. Check for accuracy.					■	■			
6. Conclude all results and write a report.						■	■		
7. Prepare for the presentation.								■	■

Benefit

1. To develop knowledge in stochastic differential equations and MATLAB program.
2. To understand the Euler–Maruyama and the Milstein methods and to know the procedure for estimating solutions of a stochastic differential equation.
3. Other researchers may benefit from this project and improve it in the future.

Equipment

1. Hardware
   - A notebook computer
   - A printer
   - Thumb drives
2. Software
   - Microsoft Word
   - MATHLAB R2018a
   - Overleaf ver. 2

Budget

1. Photocopying
2. Solid State Drive (SSD)
3. Thumb drives
4. Random-access memory (Ram) 8 Gb

References

[1] Mikhailov, E. *Programming with MATLAB for Scientists: A Beginner's Introduction*, 2017.

[2] Kijima, M. *Stochastic Processes with Applications to Finance*, 2nd Edition, 2013.

[3] Oksendal, B. *Stochastic differential equations: An introduction with applica tions*. Berlin: Springer, 2003.

Biography

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