เฟอร์มิออนมีประจุในรูหนอน 2 มิติกับสนามแม่เหล็กสมมาตรรอบแกน

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CHARGED FERMION IN TWO-DIMENSIONAL WORMHOLE WITH AXIAL MAGNETIC FIELD

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A Thesis Submitted in Partial Fulfillment of the Requirements

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ไตรทศ รจนาสันห์ : เฟอร์มิออนมีประจุในรูหนอน 2 มิติกับสนามแม่เหล็กสมมาตรรอบแกน. (CHARGED FERMION IN TWO-DIMENSIONAL WORMHOLE WITH AXIAL MAGNETIC FIELD) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : รศ. ดร.ปิยบุตร บุรีคำ, 47 หน้า.

เราศึกษาผลกระทบจากสนามแม่เหล็กบนเฟอร์มิออนมีปประจุในกาลอวกาศ $1+2$ มิติรูป “รูหนอน” โดยที่การใส่สนามแม่เหล็กภายนอกตลอดแนวสมมาตรรอบแกนของรูหนอนสามารถแก้หาผลเฉลยของสมการดิแรคได้ในสองสถานการณ์คือ ฟลักซ์แม่เหล็กคงที่และสนามแม่เหล็กคงที่ตลอดแนวท่อของรูหนอน สำหรับกรณีฟลักซ์แม่เหล็กคงที่นั้นสามารถทำการหาผลเฉลยทั่วไปได้อย่างแน่นอน แต่ในส่วนของกรณีสนามแม่เหล็กคงที่นั้นเราจะได้ผลเฉลยด้วยการประมาณรูหนอนแบบสั้น ซึ่งระบบทั้งสองกรณีจะแสดงให้เห็นถึงทั้งอันตรกิริยาของสปินกับออร์บิท และแลนดาวควอนไทเซชันสำหรับสถานะคงที่ ในระบบทั้งสองจะมีพลังงานจินตภาพที่ถูกสร้างโดย อันตรกิริยาของสปินกับออร์บิทและอันตรกิริยาของสปินกับแลนดาวแม่เหล็ก ซึ่งเกิดมาจากความโค้งที่แท้จริงของพื้นผิว พลังงานจินตภาพนี้สามารถตีความได้ถึงการกระจัดกระจายออกและความไม่เสถียรของสถานะ โดยทั่วไปแล้วสถานะของเฟอร์มิออนที่มีประจุในรูหนอนนี้จะเป็นโหมดกึ่งปกติ ที่อาจสื่อถึงการไม่เสถียรสำหรับพลังงานจินตภาพเชิงบวก และสื่อถึงการสลายตัวสำหรับพลังงานจินตภาพเชิงลบ สำหรับเฟอร์มิออนที่อยู่ในรูหนอนในกรณีฟลักซ์แม่เหล็กคงที่สามารถประพฤติตัวคล้ายโบซอน และมีรูปแบบสถิติใดๆ ขึ้นกับฟลักซ์แม่เหล็กจากสนามแม่เหล็กในระบบ นอกจากนี้เราจะทำการวิเคราะห์ผลที่ได้ในระบบแกรฟีนรูหนอนอีกด้วย

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TRITHOS ROJJANASON : CHARGED FERMION IN TWO-DIMENSIONAL WORMHOLE WITH AXIAL MAGNETIC FIELD. ADVISOR : PIYABUT BURIKHAM, Ph.D., 47 pp.

We investigate the effects of magnetic field on a charged fermion in a $(1+2)$-dimensional wormhole. Applying external magnetic field along the axis direction of the wormhole, the Dirac equation is set up and analytically solved in two scenarios, constant magnetic flux and constant magnetic field through the throat of the wormhole. For the constant magnetic flux scenario, the system can be solved analytically and exact solutions are found. For the constant magnetic field scenario, with the short wormhole approximation, the quantized energies and eigenstates are obtained. The system exhibits both the spin-orbit coupling and the Landau quantization for the stationary states in both scenarios. The intrinsic curvature of the surface induces the spin-orbit and spin-magnetic Landau couplings that generate imaginary energy. Imaginary energy can be interpreted as the energy dissipation and instability of the states. Generically, the states of charged fermion in wormhole are quasinormal modes (QNMs) that could be unstable for positive imaginary frequencies and decaying for negative imaginary ones. For the constant flux scenario, the fermions in the wormhole can behave like bosons and have arbitrary statistics depending on the flux. We also discuss the implications of our results in the graphene wormhole system.

Department : Physics Student's Signature ....

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Contents

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List of Symbols

h Planck's constant.

$\hbar$ Reduced Planck constant.

$\hbar \equiv h/2\pi$

e Electric charge.

c The speed of light.

$\vec{B}$ The external magnetic field.

Φ The magnetic flux.

Φ_0 The magnetic flux quantum .

$\Phi_0 \equiv hc/e$

L The magnetic length .

$L \equiv \sqrt{\hbar c/eB}$

η_{ab} Minkowski metric is diagonal with the signature $(-1, +1, ..., +1)$.

ε_{ijk} Levi-Civita symbol.

$$\varepsilon_{ijk} \equiv \begin{cases} +1 & \text{if} \quad ijk = 123, 231, 312, \\ -1 & \text{if} \quad ijk = 132, 213, 321, \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

δ_{ij} Kronecker delta.

$$\delta_{ij} \equiv \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

σ^k Pauli matrices.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

CHAPTER I

INTRODUCTION

Non-relativistic quantum mechanics in the presence of extrinsic and intrinsic curvature was already described 40 years ago [?, ?]. For the relativistic property of the half-spin particle, Dirac equation reveals interesting physical consequences of the particle constraints in $(2+1)-$dimensional spacetime system. A quantum effect of a particle is determined by the curvature of space. Notably, an electron in graphene [?, ?] near the Dirac points can be described as fermionic quasiparticle obeying massless Dirac-like equation [?, ?, ?, ?, ?].

When charged particles are subjected to a magnetic field perpendicular to a surface, the orbits will be quantized [?]. The Aharonov-Bohm(AB) effect is a quantum mechanical phenomenon, It gives a phase shift in which a charged particle is affected by a gauge potential. When a particle travelling in the region with zero magnetic fields, it can still acquire a phase shift [?]. For example, the charged particles are confined to carbon nanotubes in the presence of magnetic flux [?]. The quantum hall effects [?, ?] are notable phenomena in the constrained charged fermions with gauge fields. In the quantum system, a strain is equivalent to an effective gauge field, e.g. electrons in deformed nanotube and graphene experience deformed potential generated from the strain tensors [?, ?, ?]. Freshly, there is a number of investigation of fermions confined to a curved surface [?, ?, ?, ?, ?, ?] as well as the implications to carbon nanotubes properties [?, ?], and applications in curved graphene [?, ?, ?, ?, ?, ?]. Graphene is an ideal place to study behaviour of confined charged fermions such as electrons in a two-dimensional surface since its thickness is only roughly one-carbon-atom diameter. A sheet of graphene can be curved, rolled, stretched, twisted and deformed or even punctured holes into. The holes can be connected to a nanotube and become a wormhole bridging two graphene sheets. Multiple graphene sheets can be connected with one another by multiple wormholes forming a network of entangled electronic structure. Wormholes can even be built into a cage structure of schwarzite with many promising properties [?].

There have been many studies concerning the behaviour of electron on curved graphene surfaces. González et al. [?] consider a wormhole attached to two graphene sheets via 12 heptagonal defects, the defects act like effective non-Abelian gauge flux that swaps two Dirac points on the graphene lattice [?]. Garcia et al. [?] investigate the charged fermion in two-dimensional spherical space in a rotating frame, study the change in the spectrum of the C_{60} molecule when it is crossed by a magnetic flux tube in the z-direction, and the appearance of an analogue of the Aharonov-Carmi phase in the system [?]. Cariglia et al. [?] consider Dirac fermions on an essentially smooth simplified spacetime, namely a Bronnikov-Ellis wormhole. In Ref.[?], the surface of the graphene wormhole is realized by a two-dimensional axially-symmetric curved space of constant Gaussian curvature. The effective action of the fermion in the graphene wormhole is then derived in the $(1+2)$-dimensional spacetime. The similarities (in the long wormhole limit) and differences of Hilbert and event horizon are discussed. In Ref.[**?**], a charged fermion in the curved surface subject to an external electric field is analyzed in the stationary optical metric conformal to the BTZ black hole. Firstly, the condensed matter wormhole is basically spatially curved 2-dimensional surface, so it is $(1+2)$-dimensional. Secondly, there is no time dilatation like in the astrophysical wormhole, it is assumed that time is not affected by the curved surface.

In this work, we study the physical properties of a charged fermion confined on the surface of $(1+2)$-dimensional wormhole in the presence of the external magnetic field along the axis direction of the wormhole. In chapter II, basic geometric and gauge setup are established. In chapter III, we try to make sense of the Dirac fermion where the Gaussian curvature is zero and positive constant. To solve for the energy and wave function, in constant magnetic flux scenario is considered. Analysis in special a cylinder is one of the most basic curved geometric shapes to identify the crucial role of surface curvature. A simple study of the results is given in terms of the angle between the spin and orbital angular momentum of the surface-confined fermion. The special cases of hyperbolic, Beltrami, and elliptic pseudo-sphere, to solve for the energy and wave function, two scenarios of constant flux and constant field are considered. In chapter IV. Implications of the graphene wormhole system are discussed in chapter V. We summarize and discuss our results in chapter VI.

CHAPTER II

THEORETICAL BACKGROUNDS

2.1 Geometrical condition of curved space, and the metric tensor, and the dreibein field

Line element in the curved spacetime is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.1}$$

For orthogonality axial symmetric coordinates the metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R(u) \end{pmatrix}, \tag{2.2}$$

where $R(u)$ is a radius function of the polar coordinate system. The $x^\mu = \{ct, u, \theta\}$ are $(2+1)$-dimensional coordinates on the spacetime, than is dual of the tangent vector in the curved surface. In this work, we consider static and axial spacetime.

Embedding (2 + 1)-dimensional curved spacetime into the (3 + 1)-dimensional Minkowski spacetime coordinates $x^{\mu'} = \{ct, x, y, z\}$. The transformation matrix between the two coordinates is then

$$\frac{\partial x^{\mu'}}{\partial x^{\nu}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R'(u)\cos\theta & R'(u)\sin\theta & \sqrt{1-(R'(u))^2} \\ 0 & -R(u)\sin\theta & R(u)\cos\theta & 0 \end{pmatrix}. \tag{2.3}$$

The Christoffel symbols $\Gamma_{\beta\mu\nu}$ are defined as

$$\Gamma_{\beta\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right). \tag{2.4}$$

Therefore, we have

$$-\Gamma_{u\theta\theta} = \Gamma_{\theta u\theta} = \Gamma_{\theta\theta u} = R'(u)R(u), \tag{2.5}$$

and zero otherwise.

The dreibein e^a_μ is then defined as

$$e^a_\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R(u) \end{pmatrix}, \tag{2.6}$$

where $g_{\mu\nu} \equiv e^a_\mu e^b_\nu \eta_{ab}$, and $\eta_{ab} = diag(-1, 1, 1)$ in flat (1+2)-dimensions and $a, b \in \{0, 1, 2\}$.

The tangent vector in the meridian direction on the curved surface is

$$\hat{u} = \frac{\partial_u \vec{\mathbf{r}}}{\|\partial_u \vec{\mathbf{r}}\|} = R'(u)\cos\theta\,\hat{x} + R'(u)\sin\theta\,\hat{y} + \sqrt{1-(R'(u))^2}\,\hat{z}, \tag{2.7}$$

and the tangent vector in the circumference direction is

$$\hat{\theta} = \frac{\partial_\theta \vec{\mathbf{r}}}{\|\partial_\theta \vec{\mathbf{r}}\|} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}. \tag{2.8}$$

The normal vector for the curved surface which is defined by

$$\hat{\theta} \times \hat{u} = \sqrt{1-(R'(u))^2}\cos\theta\,\hat{x} + \sqrt{1-(R'(u))^2}\sin\theta\,\hat{y} - R'(u)\,\hat{z}, \tag{2.9}$$

where $\hat{u}$ and $\hat{\theta}$ are the tangent vectors to the curved surface. The constraint on z follows from the relation $d|\vec{\mathbf{r}}|^2 = dx^2 + dy^2 + dz^2 = du^2 + R^2 d\theta^2$. It gives the Hilbert horizons at $R'(u_H) = \pm 1$.

2.2 Dirac equation in curved space with gauge field

The Dirac equation for a charged fermion in curved space with an electromagnetic field can be written as

$$\left[\gamma^a e_a^\mu \left(-\hbar\nabla_\mu + i\frac{e}{c}A_\mu\right) - Mc\right]\Psi = 0. \tag{2.10}$$

$\Psi = \Psi(t, u, \theta)$ represents the Dirac spinor field, M represents the rest mass of the particle, and A_μ is the electromagnetic four-potential. The γ^a are the Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & i\sigma^k \\ -i\sigma^k & 0 \end{pmatrix}. \tag{2.11}$$

They obey the anti-commutation relations between the Clifford algebra operators

$$\{\gamma^a, \gamma^b\} \equiv \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathbb{1}_{4\times 4}. \tag{2.12}$$

The covariant derivative of the spinor interaction with gauge field in the curved space is defined as follows

$$\nabla_\mu \equiv \partial_\mu - \Gamma_\mu, \tag{2.13}$$

where the spin connection Γ_μ [?] is

$$\Gamma_\mu = -\frac{1}{4}\gamma^a\gamma^b e^\nu_a\Big(\partial_\mu(g_{\nu\beta}e^\beta_b) - e^\beta_b\Gamma_{\beta\mu\nu}\Big). \tag{2.14}$$

From the Christoffel symbols equation (??), it is then easy to show

$$\begin{aligned}
\Gamma_t &= -\frac{1}{4}\gamma^a\gamma^b e^\nu_a\left(\partial_t(g_{\nu\beta}e^\beta_b) - e^\beta_b\Gamma_{\beta\nu t}\right) = 0,\\
\Gamma_u &= -\frac{1}{4}\gamma^a\gamma^b e^\nu_a\left(\partial_u(g_{\nu\beta}e^\beta_b) - e^\beta_b\Gamma_{\beta\nu u}\right)\\
&= -\frac{1}{4}\gamma^2\gamma^2 e^\theta_2\partial_u\left(g_{\theta\theta}e^\theta_2\right) + \frac{1}{4}\gamma^2\gamma^2 e^\theta_2 e^\theta_2\Gamma_{\theta\theta u}\\
&= -\frac{1}{4}\left(\frac{1}{R}\right)\partial_u\left(R^2\frac{1}{R}\right) + \frac{1}{4}\left(\frac{1}{R^2}\right)(R'R)\\
&= 0, \qquad\qquad (2.15)\\
\Gamma_\theta &= -\frac{1}{4}\gamma^a\gamma^b e^\nu_a\left(\partial_v(g_{\nu\beta}e^\beta_b) - e^\beta_b\Gamma_{\beta\nu\theta}\right)\\
&= \frac{1}{4}\gamma^1\gamma^2 e^u_1 e^\theta_2\Gamma_{\theta u\theta} + \frac{1}{4}\gamma^2\gamma^1 e^\theta_2 e^u_1\Gamma_{u\theta\theta}\\
&= \frac{1}{4}\left(\gamma^1\gamma^2 - \gamma^2\gamma^1\right)\left(\frac{1}{R}\right)(R'R)\\
&= \frac{1}{2}\gamma^1\gamma^2 R'.
\end{aligned}$$

In this work, we will apply an external magnetic field such that the z-component $B_z = B(z)$ is uniform with respect to the plane (x, y) in two different ways: a.) the magnetic *flux* through the circular area enclosed by the wormhole at a fixed z is constant, namely $B_z \sim 1/R^2$ and b.) the magnetic *field* is uniform and constant. Due to the axial symmetry, the electromagnetic four-potential can be expressed in the axial gauge as

$$A_{\mu'}\left(t,x,y,z\right)=\left(0,-\frac{y}{2}B,\frac{x}{2}B,0\right). \tag{2.16}$$

Under coordinate transformations

$$\begin{aligned}A_u &= \frac{\partial x}{\partial u}A_x+\frac{\partial y}{\partial u}A_y+\frac{\partial z}{\partial u}A_z\\ A_u &= \left(R'\cos\theta\right)\left(-\frac{B}{2}R\sin\theta\right)+\left(R'\sin\theta\right)\left(\frac{B}{2}R\cos\theta\right)=0,\end{aligned}$$

$$\begin{aligned}A_\theta &= \frac{\partial x}{\partial \theta}A_x+\frac{\partial y}{\partial \theta}A_y+\frac{\partial z}{\partial \theta}A_z\\ A_\theta &= \left(-R\sin\theta\right)\left(-\frac{B}{2}R\sin\theta\right)+\left(R\cos\theta\right)\left(\frac{B}{2}R\cos\theta\right)=\frac{1}{2}BR^2,\end{aligned}$$

the electromagnetic four-potential in the conformally flat (2 + 1)-dimensional spacetimes as

$$A_\mu(t,u,\theta)=\frac{\partial x^{\nu'}}{\partial x^\mu}A_{\nu'}(t,x,y,z)=\left(0,0,\frac{1}{2}BR^2\right). \tag{2.17}$$

The magnetic field takes the form

$$\vec{B}=\vec{\nabla}\times\vec{A}=\left(-\frac{x}{2}\partial_z B,-\frac{y}{2}\partial_z B,B\right). \tag{2.18}$$

Now, the magnetic field having all x, y, z components for the constant magnetic flux case. And x, y component are vanishing for the constant magnetic field case. From the Dirac equation (**??**), we will expand

$$\begin{aligned}\left[\gamma^0 e_0^t\partial_{ct}+\gamma^1 e_1^u\partial_u+\gamma^2 e_2^\theta\left(\partial_\theta-\Gamma_\theta-i\frac{e}{\hbar c}A_\theta\right)+\frac{Mc}{\hbar}\right]\Psi=0,\\ \left[\gamma^0\partial_{ct}+\gamma^1\partial_u+\gamma^2\frac{1}{R}\left(\partial_\theta-\frac{1}{2}\gamma^1\gamma^2R'-i\frac{eB}{2\hbar c}R^2\right)+\frac{Mc}{\hbar}\right]\Psi=0,\end{aligned}$$

we have used the anti-commutation relations (**??**), so that

$$\left[\gamma^0\partial_{ct}+\gamma^1\left(\partial_u+\frac{R'}{2R}\right)+\gamma^2\frac{1}{R}\left(\partial_\theta-i\frac{eB}{2\hbar c}R^2\right)+\frac{Mc}{\hbar}\right]\Psi=0. \tag{2.19}$$

The Dirac equation (**??**) can be written in the form

$$\begin{pmatrix} \frac{Mc}{\hbar} + i\partial_{ct} & i\mathbf{D} \\ -i\mathbf{D} & \frac{Mc}{\hbar} - i\partial_{ct} \end{pmatrix} \Psi = 0, \tag{2.20}$$

the differential operator $\mathbf{D}$ is then defined as

$$\mathbf{D} \equiv \sigma^1 \left(\partial_u + \frac{R'}{2R} \right) + \frac{\sigma^2}{R} \left(\partial_\theta - i\frac{\Phi}{\Phi_0} \right), \tag{2.21}$$

where $\Phi = \int \vec{B} \cdot d\vec{a} = \pi R^2 B$ and the magnetic flux quantum is defined as $\Phi_0 \equiv hc/e$.

The first term is equivalent to the Dirac equation with the pseudo gauge potential $A_{\tilde{u}}(u) \equiv i\hbar c R'/2eR$ in the u-direction. It is generated by the curvature along the θ-direction, Γ_θ. In this sense, the intrinsic gravity connection can be interpreted as the effective (*imaginary*) gauge connection (in the locally *perpendicular* direction) that leads to the complexity of the energy and the emergence of the QNMs and unstable modes on a surface with the negative Gaussian curvature. The second term is similar to a spin-orbit-curvature coupling potential [**?**].

In the presence of external magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ along the z-direction, the charged fermion moving in θ-direction is expected to form a stationary state with quantized angular momentum and energy, i.e. the Landau levels in the curved space with hole.

The Dirac spinor $\Psi(t, u, \theta)$ should be stationary state

$$\Psi(t, u, \theta) = e^{-\frac{i}{\hbar}Et} e^{im\theta} \begin{pmatrix} \chi(u) \\ \varphi(u) \end{pmatrix}, \tag{2.22}$$

where $\chi(u), \varphi(u)$ are two-component spinors, for stationary states, the wave function needs to be single-valued at every point in spacetime, $\Psi(t,u,\theta)$ must be a periodic function in $\theta \in [0, 2\pi]$, the orbital angular momentum quantum number $m = 0, \pm1, \pm2, \ldots$. Now, the equation takes the following form

$$\left(E + Mc^2\right)\chi(u) + i\hbar c\mathbf{D}\varphi(u) = 0, \tag{2.23}$$

$$\left(E - Mc^2\right)\varphi(u) + i\hbar c\mathbf{D}\chi(u) = 0. \tag{2.24}$$

Using equation (**??**) in (**??**), leads to

$$\begin{aligned} 0 =& \mathbf{D}^2\varphi(u) + \frac{\left(E - Mc^2\right)}{\hbar c}\frac{\left(E + Mc^2\right)}{\hbar c}\varphi(u), \\ 0 =& \mathbf{D}^2\varphi(u) + \frac{E^2 - M^2c^2}{\hbar^2c^2}\varphi(u), \end{aligned}$$

then

$$\begin{aligned} \mathbf{D}^2\varphi(u) =& \Bigg[\sigma^1\left(\partial_u + \frac{R'}{2R}\right)\sigma^1\left(\partial_u + \frac{R'}{2R}\right) + \frac{i\sigma^2}{R}\left(m - \frac{\Phi}{\Phi_0}\right)\frac{i\sigma^2}{R}\left(m - \frac{\Phi}{\Phi_0}\right) \\ &+ \sigma^1\left(\partial_u + \frac{R'}{2R}\right)\frac{i\sigma^2}{R}\left(m - \frac{\Phi}{\Phi_0}\right) + \frac{i\sigma^2}{R}\left(m - \frac{\Phi}{\Phi_0}\right)\sigma^1\left(\partial_u + \frac{R'}{2R}\right)\Bigg]\varphi(u) \\ =& \Bigg[\partial_u^2 + \frac{R'}{R}\partial_u + \left(\frac{R'}{2R}\right)^2 + \frac{R''}{2R} - \frac{1}{2}\left(\frac{R'}{R}\right)^2 - \frac{1}{R^2}\left(m - \frac{\Phi}{\Phi_0}\right)^2 \\ &- i\sigma^1\sigma^2\frac{R'}{R^2}\left(m - \frac{\Phi}{\Phi_0}\right) - \frac{i\sigma^1\sigma^2}{R}\frac{\Phi'}{\Phi_0}\Bigg]\varphi(u). \end{aligned}$$

The Pauli matrices satisfy a useful identity $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, we have Klein-Gordon-like equation

$$0 = \varphi''(u) + \frac{R'}{R}\varphi'(u) + \left[\frac{R''}{2R} + \frac{\acute{m}\sigma^3R' - \acute{m}^2 - \left(R'/2\right)^2}{R^2} + \frac{\sigma^3}{R}\frac{\Phi'}{\Phi_0} + k^2\right]\varphi(u), \tag{2.25}$$

where the new orbital angular momentum in the presence of magnetic flux $\acute{m} = m - \frac{\Phi}{\Phi_0}$ [**?**]. We have used the momentum parameter $k^2 \equiv (E^2 - M^2c^4)/\hbar^2c^2$ and σ^3 is a spin-state index corresponding to spin up or down of the fermion.

In this work, we consider the wave function to be regular in the constant Gaussian curved space and will not specify the boundary condition at the Hilbert horizons. The Gaussian curvature is given by the simple expression

$$\mathcal{K} \equiv -\frac{R''(u)}{R(u)} \begin{cases} > 0 & \text{(closed space)}, \\ = 0 & \text{(flat space)}, \\ < 0 & \text{(open space)}. \end{cases} \tag{2.26}$$

CHAPTER III

FERMION FIELD IN POSITIVE GAUSSIAN CURVATURE

3.1 Flat Space

First of all, we will start to consider the Dirac fermion in a flat place when $R(u) = u$. It's easy to show the Gaussian curvature (**??**) becomes zero, the effects of "gravity" is vanish. In this section, we investigate the Landau quantization for the constant magnetic field $\vec{B} = B_0\hat{z}$, uniform magnetic field is perpendicular to the place. Klein-Gordon-like equation (**??**) takes the following form

$$\begin{aligned} 0 =& \varphi''(u) + \frac{1}{u}\varphi'(u) + \left[k^2 - \frac{\left(\acute{m} - \frac{\sigma^3}{2} \right)^2}{u^2} \right] \varphi(u), \\ 0 =& \varphi''(u) + \frac{1}{u}\varphi'(u) + \left[k^2 - \frac{\left(m - \frac{\Phi}{\Phi_0} - \frac{\sigma^3}{2} \right)^2}{u^2} \right] \varphi(u). \end{aligned} \tag{3.1}$$

The solution of fermion in the flat place of Eq.(**??**) are a Bessel function of haft-integer order (Hankel functions), It's defined by

$$\varphi_\kappa(u) = A_\kappa J_j\left(ku\right) + B_\kappa Y_j\left(ku\right), \tag{3.2}$$

where $j = (m - \Phi/\Phi_0 - \kappa/2)$ is the total angular momentum quantum number, and κ is a spin-state index corresponding to spin up ($\kappa = +1$) or down ($\kappa = -1$) of the fermion for each eigenvalue of σ^3. Energies of the fermion in this case depend on choices boundary condition.

Example, if the wave solution is vanish at $u = u_0$ and $\varphi_\kappa(0)$ are finite $\to B_\kappa = 0$. We assume that the condition $J_j(\alpha_{n_0m}) \equiv 0$ then requires that ku_0 is equal to one of the zeros of the Bessel function, we have $ku_0 = \alpha_{n_0m}$ where n_0 is the radial quantum numbers

$$E_{n_0m\kappa} = \pm\sqrt{M^2c^4 + \frac{\hbar^2c^2}{u_0^2}\alpha^2_{n_0m}}$$

3.2 Cylindrical Geometry

To understand essential physics of the magnetized charged fermion in the curved spaces, consider a simple case when $R(u)$ is constant, i.e. a cylindrical tube. In this case, the intrinsic (Gaussian) curvature is zero, so we can identify which effects are induced by the "gravity". Both cases of the magnetic field and flux become same constant value for $\vec{B}$. Klein-Gordon-like equation (**??**) its obtained by $R'(u) \to 0$, and the axial direction $z = u$. We then have

$$\begin{aligned}
0 =& \varphi''(z) + \left[k^2 - \left(\frac{\acute{m}}{R}\right)^2\right]\varphi(z) \\
0 =& \varphi''(z) + \left[k^2 - \left(\frac{m - \Phi/\Phi_0}{R}\right)^2\right]\varphi(z) \qquad (3.3)\\
0 =& \varphi''(z) + \left[k^2 - \left(\frac{m}{R}\right)^2 + \frac{eB_0}{\hbar c}m - \left(\frac{eB_0}{2\hbar c}\right)^2 R^2\right]\varphi(z).
\end{aligned}$$

Assuming the solution in the form $\varphi(u) \sim \exp\left[i\mathtt{k}_z z\right]$ to obtain the energy eigenvalue,

$$k^2 = \frac{E^2 - M^2c^4}{\hbar^2 c^2} = \mathtt{k}_z^2 + \left(\frac{m - \Phi/\Phi_0}{R}\right)^2,$$

then

$$E\left(\mathtt{k}_z, m\right) = \pm\sqrt{M^2c^4 + \hbar^2c^2\mathtt{k}_z^2 + \left(\frac{\hbar c}{R}\right)^2\left(m - \frac{\Phi}{\Phi_0}\right)^2}. \qquad (3.4)$$

3.3 Spherical Geometry

In this section, we focus on a surface of positive constant Gaussian curvature when $R(u) = d\cos(\phi)$ where the zenith angle is defined by $\phi \equiv u/r$. With this choice of $R(u)$, we perform the following

$$X(u) \equiv \frac{r}{d}R'(u) = -\sin\phi, \qquad \frac{dX}{du} = -\frac{R(u)}{rd}, \qquad \frac{d^2X}{du^2} = -\frac{R'(u)}{rd} = -\frac{X}{r^2}. \qquad (3.5)$$

From the Pythagorean theorem, we get

$$r^2\left(\frac{dX}{du}\right)^2 = 1 - X^2. \qquad (3.6)$$

Eq.(**??**) becomes

$$\begin{aligned}
0 =& \left(\frac{dX}{du}\right)^2 \varphi''(X) + \frac{d^2X}{du^2}\varphi'(X) + \frac{R'}{R}\frac{dX}{du}\varphi'(X) + \left(k^2 + \frac{R''}{2R}\right)\varphi(X) \\
&+ \left[\frac{\acute{m}\sigma^3 R' - \acute{m}^2 - (R'/2)^2}{R^2} + \frac{\sigma^3}{R}\frac{\Phi'(u)}{\Phi_0}\right]\varphi(X), \\
0 =& \left(\frac{1-X^2}{r^2}\right)\varphi''(X) + \left(-\frac{X}{r^2} + \frac{\frac{d}{r}X}{-rd\frac{dX}{du}}\frac{dX}{du}\right)\varphi'(X) + \left(k^2 + \frac{\frac{d}{r}\frac{dX}{du}}{-2rd\frac{dX}{du}}\right)\varphi(X) \\
&+ \left[\frac{\acute{m}\sigma^3\frac{d}{r}X - \acute{m}^2 - \left(\frac{d}{2r}X\right)^2}{\left(-rd\frac{dX}{du}\right)^2} + \frac{\sigma^3}{-rd\frac{dX}{du}}\frac{\Phi'(u)}{\Phi_0}\right]\varphi(X) \\
0 =& \left(1 - X^2\right)\varphi''(X) - 2X\varphi'(X) + \left(k^2r^2 - \frac{1}{2}\right)\varphi(X) \\
&+ \left[\frac{\acute{m}\sigma^3 dX - \acute{m}^2r^2 - \left(\frac{d}{2}X\right)^2}{(1-X^2)\,d^2} - \frac{\sigma^3 r}{d}\frac{\Phi'(X)}{\Phi_0}\right]\varphi(X).
\end{aligned} \tag{3.7}$$

Consider the electron confined on the spherical surface with the constant magnetic flux scenario; $\Phi'(u) = 0$, Eq.(**??**) takes the form

$$0 =(1-X^2)\varphi''(X) - 2X\varphi'(X) + \left[k^2r^2 - \frac{1}{4} + \frac{\acute{m}\sigma^3 X\frac{r}{d} - \left(\acute{m}\frac{r}{d}\right)^2 - \frac{1}{4}}{1-X^2}\right]\varphi(X). \tag{3.8}$$

Now, we assume the solution $\varphi^{(\kappa)}(X) = (1-X)^{\acute{\alpha}}(1+X)^{\acute{\beta}}P^{(\kappa)}(X)$ where $P^{(\kappa)}(X)$ is a component corresponding to the state for spin up or down ($\kappa = +1$ or -1),

$$\sigma^3\varphi(X) = (1-X)^{\acute{\alpha}}(1+X)^{\acute{\beta}}\begin{pmatrix} P^{(+)}(X) \\ P^{(-)}(X)\end{pmatrix}. \tag{3.9}$$

We then see that

$$\begin{aligned}
\frac{\varphi^{(\kappa)\prime}(X)}{(1-X)^{\acute{\alpha}}(1+X)^{\acute{\beta}}} =& P^{(\kappa)\prime}(X) + \frac{\acute{\beta}-\acute{\alpha}-\left(\acute{\beta}+\acute{\alpha}\right)X}{1-X^2}P^{(\kappa)}(X),\\
\frac{\varphi^{(\kappa)\prime\prime}(X)}{(1-X)^{\acute{\alpha}}(1+X)^{\acute{\beta}}} =& P^{(\kappa)\prime\prime}(X) + 2\frac{\acute{\beta}-\acute{\alpha}-\left(\acute{\beta}+\acute{\alpha}\right)X}{1-X^2}P^{(\kappa)\prime}(X)\\
&+\frac{\left(\acute{\beta}-\acute{\alpha}\right)^2-\acute{\beta}-\acute{\alpha}}{\left(1-X^2\right)^2}P^{(\kappa)}(X)\\
&+2\frac{\left(\acute{\alpha}-\acute{\beta}\right)\left(\acute{\beta}+\acute{\alpha}-1\right)X}{\left(1-X^2\right)^2}P^{(\kappa)}(X)\\
&+\frac{\left(\acute{\beta}+\acute{\alpha}\right)\left(\acute{\beta}+\acute{\alpha}-1\right)X^2}{\left(1-X^2\right)^2}P^{(\kappa)}(X),
\end{aligned} \tag{3.10}$$

then

$$\begin{aligned}
\frac{-2X\varphi^{(\kappa)\prime}(X)}{(1-X)^{\acute{\alpha}}(1+X)^{\acute{\beta}}} =& -2XP^{(\kappa)\prime}(X) + 2\frac{\left(\acute{\alpha}-\acute{\beta}\right)X+\left(\acute{\alpha}+\acute{\beta}\right)X^2}{1-X^2}P^{(\kappa)}(X),\\
\frac{-2X\varphi^{(\kappa)\prime}(X)}{(1-X)^{\acute{\alpha}}(1+X)^{\acute{\beta}}} =& -2XP^{(\kappa)\prime}(X) - 2\left(\acute{\alpha}+\acute{\beta}\right)P^{(\kappa)}(X)\\
&+2\frac{\acute{\alpha}+\acute{\beta}+\left(\acute{\alpha}-\acute{\beta}\right)X}{1-X^2}P^{(\kappa)}(X),
\end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
\frac{(1-X^2)\,\varphi^{(\kappa)\prime\prime}(X)}{(1-X)^{\acute{\alpha}}(1+X)^{\acute{\beta}}} &= \left(1-X^2\right)P^{(\kappa)\prime\prime}(X) + 2\left[\acute{\beta}-\acute{\alpha}-\left(\acute{\alpha}+\acute{\beta}\right)X\right]P^{(\kappa)\prime}(X) \\
&\quad \frac{\left(\acute{\beta}-\acute{\alpha}\right)^2-\acute{\beta}-\acute{\alpha}+\left(\acute{\beta}+\acute{\alpha}\right)\left(\acute{\beta}+\acute{\alpha}-1\right)X^2}{1-X^2}P^{(\kappa)}(X) \\
&\quad 2\frac{\left(\acute{\alpha}-\acute{\beta}\right)\left(\acute{\beta}+\acute{\alpha}-1\right)X}{1-X^2}P^{(\kappa)}(X), \\
\frac{(1-X^2)\,\varphi^{(\kappa)\prime\prime}(X)}{(1-X)^{\acute{\alpha}}(1+X)^{\acute{\beta}}} &= \left(1-X^2\right)P^{(\kappa)\prime\prime}(X) + 2\left[\acute{\beta}-\acute{\alpha}-\left(\acute{\alpha}+\acute{\beta}\right)X\right]P^{(\kappa)\prime}(X) \\
&\quad -\left(\acute{\beta}+\acute{\alpha}\right)\left(\acute{\beta}+\acute{\alpha}-1\right)P^{(\kappa)}(X) \\
&\quad +2\frac{\acute{\alpha}\left(\acute{\alpha}-1\right)+\acute{\beta}\left(\acute{\beta}-1\right)}{1-X^2}P^{(\kappa)}(X) \\
&\quad +2\frac{\left(\acute{\alpha}-\acute{\beta}\right)\left(\acute{\beta}+\acute{\alpha}-1\right)X}{1-X^2}P^{(\kappa)}(X).
\end{aligned}
\tag{3.12}
$$

The equation of motion (**??**) can be rewritten as

$$
\begin{aligned}
0 &= \left(1-X^2\right)P^{(\kappa)\prime\prime}(X) + 2\left[\acute{\beta}-\acute{\alpha}-\left(\acute{\beta}+\acute{\alpha}+1\right)X\right]P^{(\kappa)\prime}(X) \\
&\quad +\left[k^2r^2-\frac{1}{4}-\left(\acute{\alpha}+\acute{\beta}\right)\left(\acute{\alpha}+\acute{\beta}+1\right)\right]P^{(\kappa)}(X) \\
&\quad +\left[\frac{\acute{m}\kappa\frac{r}{d}X-\left(\acute{m}\frac{r}{d}\right)^2-\frac{1}{4}}{1-X^2}+2\frac{\acute{\alpha}^2+\acute{\beta}^2+\left(\acute{\alpha}^2-\acute{\beta}^2\right)X}{1-X^2}\right]P^{(\kappa)}(X),
\end{aligned}
\tag{3.13}
$$

we obtain

$$
\begin{aligned}
0 &= \left(1-X^2\right)P^{(\kappa)\prime\prime}(X) + 2\left[\acute{\beta}-\acute{\alpha}-\left(\acute{\beta}+\acute{\alpha}+1\right)X\right]P^{(\kappa)\prime}(X) \\
&\quad +\left[k^2r^2-\left(\acute{\beta}+\acute{\alpha}+\frac{1}{2}\right)^2\right]P^{(\kappa)}(X),
\end{aligned}
\tag{3.14}
$$

where we assume

$$
\begin{aligned}
\left(\acute{m}\frac{r}{d}\right)^2+\frac{1}{4} &= 2\acute{\alpha}^2+2\acute{\beta}^2, & \acute{m}\kappa\frac{r}{d} &= 2\acute{\beta}^2-2\acute{\alpha}^2, \\
\acute{\alpha} &= \Lambda_{\acute{\alpha}}\left(\frac{1}{4}-\acute{m}\frac{\kappa}{2}\frac{r}{d}\right), & \acute{\beta} &= \Lambda_{\acute{\beta}}\left(\frac{1}{4}+\acute{m}\frac{\kappa}{2}\frac{r}{d}\right),
\end{aligned}
\tag{3.15}
$$

where $\Lambda_{\acute{\alpha}}, \Lambda_{\acute{\beta}} = \pm 1$. Eq.(**??**) is Jacobi Differential Equation, the energy levels become

$$k^2 r^2 - \left(\acute{\beta} + \acute{\alpha} + \frac{1}{2}\right)^2 = n(n + 2\acute{\beta} + 2\acute{\alpha} + 1)$$
$$k^2 r^2 = \frac{E^2 - M^2 c^4}{\hbar^2 c^2} r^2 = \left(n + \acute{\beta} + \acute{\alpha} + \frac{1}{2}\right)^2,$$

then

$$\begin{aligned} E^2_{m,n,\kappa} &= M^2 c^4 + \hbar^2 c^2 k^2_{m,n,\kappa} \\ &= M^2 c^4 + \frac{\hbar^2 c^2}{r^2}\left(n + \frac{1}{2} + \acute{\beta} + \acute{\alpha}\right)^2. \end{aligned} \tag{3.16}$$

Depending on the sign choices of $\acute{\beta}, \acute{\alpha}$, the resulting equation of motion and the corresponding energy levels will be dependent or independent of the spin-orbit coupling term $\sim \kappa m r/d$. The exact solution can be expressed as

$$\Psi(t, \phi, \theta) = \mathrm{e}^{-\frac{i}{\hbar} E_{m,n,\kappa} t} \mathrm{e}^{im\theta} \begin{pmatrix} \dfrac{-i\hbar c \mathbf{D} \varphi(\phi)}{E_{m,n,\kappa} + Mc^2} \\ \varphi(\phi) \end{pmatrix}, \tag{3.17}$$

where $\varphi^{(\kappa)}(\phi) = \varphi_{m,n,\kappa}(1 + \sin\phi)^{\alpha}(1 - \sin\phi)^{\beta} P^{(\kappa)}(-\sin\phi)$. The function $P^{(\kappa)}(Y)$ is the Jacobi polynomials

$$\begin{aligned} P^{(\kappa)}(Y) &= P_n^{(2\acute{\alpha}, 2\acute{\beta})}(Y) \\ &= \frac{(-1)^n}{2^n n!}(1 - Y)^{-2\acute{\alpha}}(1 + Y)^{-2\acute{\beta}} \frac{d^n}{dY^n}\Big[(1 - Y)^{2\acute{\alpha}}(1 + Y)^{2\acute{\beta}}(1 - Y^2)^n\Big] \\ &= \sum_{j=0}^{n} \begin{pmatrix} n + 2\acute{\alpha} \\ n - j \end{pmatrix} \begin{pmatrix} n + 2\acute{\beta} \\ j \end{pmatrix} \Big(\frac{Y - 1}{2}\Big)^j \Big(\frac{Y + 1}{2}\Big)^{n-j}, \end{aligned} \tag{3.18}$$

for integer n and

$$\begin{pmatrix} z \\ n \end{pmatrix} = \begin{cases} \dfrac{\Gamma(z + 1)}{\Gamma(n + 1)\Gamma(z - n + 1)} & \text{for} \quad n > 0, \\ 0 & \text{for} \quad n < 0. \end{cases}$$

CHAPTER IV

FERMION FIELD IN NEGATIVE GAUSSIAN CURVATURE

In this chapter, we focus on a surface of negative constant Gaussian curvature when $R(u) = d\cosh_q(u/r)$. And $R(u)$ is based on a q-deformation of the usual hyperbolic functions [?, ?] which are defined by

$$\cosh_q(x) \equiv \frac{\mathrm{e}^x + q\mathrm{e}^{-x}}{2}, \quad \sinh_q(x) \equiv \frac{\mathrm{e}^x - q\mathrm{e}^{-x}}{2}, \quad \tanh_q(x) = \frac{\sinh_q(x)}{\cosh_q(x)}. \tag{4.1}$$

Hence the deformed hyperbolic functions satisfy trigonometric identity :

$$\begin{gathered} \cosh_q^2(x) - \sinh_q^2(x) = q, \\ \frac{d}{dx}\sinh_q(x) = \cosh_q(x), \qquad \frac{d}{dx}\tanh_q(x) = \frac{q}{\cosh_q^2(x)}. \end{gathered} \tag{4.2}$$

They reduce to hyperbolic functions when q equal 1. There are special cases, i.e. Wormhole, Beltrami wormhole, and Elliptic pseudosphere when the deformation parameter $q = 1, 0, -1$ respectively.

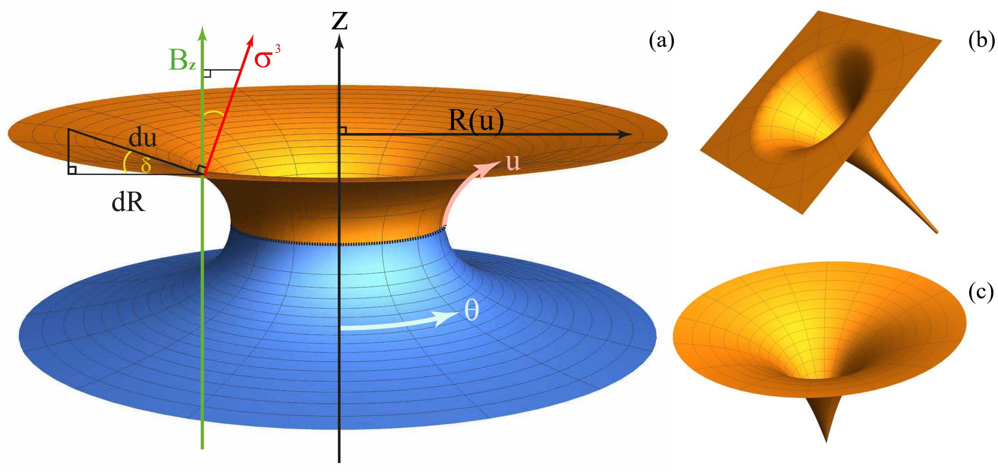

Figure 4.1: Geometric structure of the curved surface where d is a radius at minimum radius function $R(u = 0)$. And r is the scale parameter of the curved surfaces along u direction. **(a)** Hyperbolic pseudosphere(*wormhole*) surface where $R(u) = d\cosh_q(u/r)$. The Hilbert horizons of the *wormhole* are at $u_H = r\ln\left(r/d \pm \sqrt{r^2/d^2+1}\right)$ where $q = 1$. **(b)** Beltrami pseudosphere(*Beltrami wormhole*) surface. The Hilbert horizon of Beltrami surface is at $u_H = r\ln(2r/d)$ where $q = 0$. **(c)** elliptic pseudosphere surface, gives the Hilbert horizons at $u_H = r\ln\left(r/d \pm \sqrt{r^2/d^2-1}\right)$ where $q = -1$.

4.1 Wormhole Geometry

Now performing the similar transformations using

$$X(u) = rR'(u)/d = \sinh_q(u/r). \tag{4.3}$$

4.1.1 The constant magnetic flux scenario

The equation of motion (**??**) then takes the form

$$0 =(q+X^2)\varphi''_{(q)}(X) + 2X\varphi'_{(q)}(X) + \left[k^2r^2 + \frac{1}{4} + \frac{\acute{m}\sigma^3 X\frac{r}{d} - \left(\acute{m}\frac{r}{d}\right)^2 + \frac{q}{4}}{q+X^2}\right]\varphi_{(q)}(X), \tag{4.4}$$

and we now appeal to the solution from Section **??**, $\varphi_{(q,\kappa)}(X) = (\sqrt{q}+iX)^{\alpha}(\sqrt{q}-iX)^{\beta}\mathcal{P}_{(q,\kappa)}(X)$, where a next step follows from Eq.(**??**)$-$(**??**), the equation of motion (**??**) can be rewritten as

$$\begin{aligned}
0 = \left(q+X^2\right)\mathcal{P}''_{(q,\kappa)}(X) &+ 2\left[i\left(\beta-\alpha\right)\sqrt{q} + \left(\beta+\alpha+1\right)X\right]\mathcal{P}'_{(q,\kappa)}(X) \\
&+ \left[k^2r^2 + \frac{1}{4} + \left(\alpha+\beta\right)\left(\alpha+\beta+1\right)\right]\mathcal{P}_{(q,\kappa)}(X) \\
&+ \left[\frac{\acute{m}\kappa\frac{r}{d}X - \left(\acute{m}\frac{r}{d}\right)^2 + \frac{q}{4}}{q+X^2} + 2\frac{i\left(\alpha^2-\beta^2\right)\sqrt{q}X - \alpha^2 - \beta^2}{q+X^2}\right]\mathcal{P}_{(q,\kappa)}(X),
\end{aligned} \tag{4.5}$$

we obtain

$$\begin{aligned}
0 = (1-Y^2)\mathcal{P}''_{(q,\kappa)}(Y) &+ 2\left[(\alpha-\beta) - \left(\beta+\alpha+1\right)Y\right]\mathcal{P}'_{(q,\kappa)}(Y) \\
&- \left[k^2r^2 + \left(\beta+\alpha+\frac{1}{2}\right)^2\right]\mathcal{P}_{(q,\kappa)}(Y),
\end{aligned} \tag{4.6}$$

where $X \equiv -i\sqrt{q}Y$, and this is fulfilled if

$$\alpha = \Lambda_\alpha\left(\frac{1}{4} + \frac{i\kappa}{\sqrt{q}}\frac{\acute{m}r}{2d}\right), \qquad \beta = \Lambda_\beta\left(\frac{1}{4} - \frac{i\kappa}{\sqrt{q}}\frac{\acute{m}r}{2d}\right), \tag{4.7}$$

where $\Lambda_\alpha, \Lambda_\beta = \pm 1$. We then have the energy levels

$$E^2_{m,n,\kappa} = M^2c^4 + \hbar^2c^2k^2_{m,n,\kappa} = M^2c^4 - \frac{\hbar^2c^2}{r^2}\left(n + \frac{1}{2} + \alpha + \beta\right)^2. \tag{4.8}$$

Finally, the solutions of Eq.(**??**) is

$$\varphi_{(q,\kappa)}(X) = \varphi_0(\sqrt{q}+iX)^{\alpha}(\sqrt{q}-iX)^{\beta}P_n^{(2\alpha,2\beta)}(iX/\sqrt{q}). \tag{4.9}$$

Note that the solutions have the following properties,

$$P_n^{(2\alpha,2\beta)*}(Y) = (-1)^n P_n^{(\pm 2\alpha,\pm 2\beta)}(Y), \quad \text{for} \quad \alpha = \pm\beta^*. \tag{4.10}$$

For $\alpha = \beta^*$, the Jacobi polynomial is real for even n, or imaginary for odd n. Notably for this case, the spatial wave function $\varphi_{(q,\kappa)}(X)$ for even n is also real and the energy $E_{m,n,\kappa}(\Lambda_\alpha, \Lambda_\beta)$ given by

$$E_{m,n,\kappa}(+,+) = \pm\sqrt{M^2c^4 - \frac{\hbar^2c^2}{r^2}(n+1)^2}, \tag{4.11}$$

and

$$E_{m,n,\kappa}(-,-) = \pm\sqrt{M^2c^4 - \frac{\hbar^2c^2}{r^2}n^2}, \tag{4.12}$$

these depend only on the quantum number n. This energy is independent of the spin-orbit and magnetic field. In addition, They have negative momentum square p_u^2 along u-direction, that also have dependency on the coupling between orbital angular momentum and the magnetic field. This leads to a new p_u-diffusive modes that depend on the spin of the fermion in the wormhole. The p_u-diffusive modes can have either real or imaginary energy depending on the quantum number n and the magnetic field in comparison to the rest-mass energy.

On the other hand, for another sign choice $\alpha = -\beta^*$

$$E_{m,n,\kappa}(+,-) = \pm\sqrt{M^2c^4 - \frac{\hbar^2c^2}{r^2}\left(n + \frac{1}{2} + \frac{i\kappa}{\sqrt{q}}\frac{\acute{m}r}{d}\right)^2}, \tag{4.13}$$

and

$$E_{m,n,\kappa}(-,+) = \pm\sqrt{M^2c^4 - \frac{\hbar^2c^2}{r^2}\left(n + \frac{1}{2} - \frac{i\kappa}{\sqrt{q}}\frac{\acute{m}r}{d}\right)^2}. \tag{4.14}$$

The energy from Eqs.(**??**) and (**??**) contains the interaction between the spin-orbit coupling $\sim \kappa m$ (independent of the magnetic field), and the Landau coupling between the magnetic field and the spin (orbital) angular momentum $\sim \kappa BmB$. Notably, it also contains the term proportional to the spin-orbit-flux interaction $\sim \kappa m\Phi$, orbit-orbit couplings $\sim m^2$, and the flux-flux couplings $\sim \Phi^2$. The spin-orbit and orbit-orbit terms are purely gravitational and kinematical since they are independent of the magnetic field. Regardless of the magnetic field, the imaginary parts in the energy expression have the gravitational origin. They are originated from the curvature of the wormhole and we will see in Sections **??** and **??** where the curvature vanishes. A complex quantity which can be interpreted as the quasinormal modes (QNMs). For the QNMs with negative imaginary parts, the curvature effects leak the energy of the fermion away from the wormhole as long as the angle $\delta = \arccos R'$ between the σ^3-spin component and orbital angular momentum is not $\pi/2$ (where δ is the angle between the σ^3-spin component and the z-axis). For these states, the fermion will either slowly decay away or be spun off the wormhole due to the curvature effect. A special case occurs when $m - \Phi/\Phi_0 = 0$ where the imaginary spin-orbit coupling term vanishes.

4.1.2 The constant magnetic field scenario

The equation of motion (**??**) then takes the form

$$\begin{aligned} 0 = \varphi''_{(q)}(u) + \frac{R'}{R}\varphi'_{(q)}(u) + \left(k^2 + \frac{eB}{\hbar c}m + \frac{R''}{2R}\right)\varphi_{(q)}(u) \\ + \left[\frac{R'\sigma^3 m - m^2 - \left(\frac{R'}{2}\right)^2}{R^2} - \left(\frac{eB}{2\hbar c}\right)^2 R^2 + \frac{eB}{2\hbar c}\sigma^3 R'\right]\varphi_{(q)}(u). \end{aligned} \tag{4.15}$$

By using relationships from Eq.(**??**)−(**??**), and we assume the solution $\varphi_{(q,\kappa)}(X) = (\sqrt{q}+iX)^{\alpha_0}(\sqrt{q}-iX)^{\beta_0}\mathcal{P}_{0(q,\kappa)}(X)$, Eq.(**??**) now becomes

$$0 = (q+X^2)\mathcal{P}''_{0(q,\kappa)}(X) + [\mathcal{A}+\mathcal{B}X]\,\mathcal{P}'_{0(q,\kappa)}(X) + \left[\mathcal{C}+\mathcal{D}X+\mathcal{E}X^2+k^2r^2\right]\mathcal{P}_{0(q,\kappa)}(X). \tag{4.16}$$

The coefficient parameters are defined as the following

$$\begin{aligned} &\mathcal{A} = 2i(\alpha_0-\beta_0)\sqrt{q}, \qquad \mathcal{B} = 2(\alpha_0+\beta_0+1), \\ &\mathcal{C} = (\alpha_0+\beta_0)(\alpha_0+\beta_0+1) + \frac{1}{4} + \frac{eB}{\hbar c}mr^2 - \left(\frac{rd}{2}\frac{eB}{\hbar c}\right)^2 q, \\ &\mathcal{D} = \frac{rd}{2}\frac{eB}{\hbar c}\kappa, \qquad \mathcal{E} = -\left(\frac{rd}{2}\frac{eB}{\hbar c}\right)^2, \end{aligned} \tag{4.17}$$

where we again assume

$$\alpha_0 = \Lambda_{\alpha_0}\left(\frac{1}{4}+\frac{i\kappa}{\sqrt{q}}\frac{mr}{2d}\right), \qquad \beta_0 = \Lambda_{\beta_0}\left(\frac{1}{4}-\frac{i\kappa}{\sqrt{q}}\frac{mr}{2d}\right) \qquad ; \quad \Lambda_{\alpha_0},\Lambda_{\beta_0} = \pm 1. \tag{4.18}$$

So as to find the solution to Eq.(**??**), we first obtain the asymptotic solution for large X, having the solutions: $\mathcal{P}_{Large}(X) = \exp[\pm\sqrt{-\mathcal{E}}X] = \exp[\pm\frac{rd}{2}\frac{eB}{\hbar c}X]$. Rewriting the solution for all region as

$$\mathcal{P}_{0(q,\kappa)}(X) \Longrightarrow \mathcal{P}_{Large}(X)\mathcal{P}_{0(q,\kappa)}(X).$$

The equation of motion (**??**) then takes the form

$$0 = (q+X^2)\mathcal{P}''_{0(q,\kappa)}(X) + \left[\mathcal{F}+\mathcal{G}X+\mathcal{H}X^2\right]\mathcal{P}'_{0(q,\kappa)}(X) + \left[\mathcal{I}+k^2r^2+\mathcal{J}X\right]\mathcal{P}_{0(q,\kappa)}(X), \tag{4.19}$$

where the parameters are defined as

$$\begin{aligned}
\mathcal{F} =&\mathcal{A} \pm 2q\sqrt{-\mathcal{E}} = 2i(\alpha_0 - \beta_0)\sqrt{q} \pm qrd\frac{eB}{\hbar c},\\
\mathcal{G} =&\mathcal{B} = 2(\alpha_0 + \beta_0 + 1),\\
\mathcal{H} =& \pm 2\sqrt{-\mathcal{E}} = \pm rd\frac{eB}{\hbar c},\\
\mathcal{I} =&\mathcal{C} - q\mathcal{E} \pm \sqrt{-\mathcal{E}}\mathcal{A} = \left(\alpha_0 + \beta_0 + \frac{1}{2}\right)^2 + rd\frac{eB}{\hbar c}\left[m\frac{r}{d} \pm i(\alpha_0 - \beta_0)\sqrt{q}\right],\\
\mathcal{J} =&\mathcal{D} \pm \mathcal{B}\sqrt{-\mathcal{E}} = rd\frac{eB}{\hbar c}\left[\frac{\kappa}{2} \pm (\alpha_0 + \beta_0 + 1)\right].
\end{aligned} \tag{4.20}$$

Now to find the solution to Eq.(**??**). The wave function can be solved exactly for small $X = \sinh_q(u/r) < 1$ implies that $R'(u_H) = \frac{d}{r}\sinh_q(u_H/r) = 1 \longrightarrow r < d$ in terms of the Jacobi polynomials as we will show in the following. For small X, so we obtain

$$0 \approx (1 - Y^2)\mathcal{P}''_{0(q,\kappa)}(Y) + \left[\frac{-i}{\sqrt{q}}\mathcal{F} - \mathcal{G}Y\right]\mathcal{P}'_{0(q,\kappa)}(Y) - \left[\mathcal{I} + k^2r^2\right]\mathcal{P}_{0(q,\kappa)}(Y), \tag{4.21}$$

where $X \equiv -i\sqrt{Y}$, this has solution in the form of the Jacobi polynomials ($\mathcal{P}_{0(q,\kappa)}(Y) = P_n^{(\alpha_1,\beta_1)}(Y)$), given conditions

$$\begin{aligned}
2\alpha_1 = \mathcal{G} - 2 + \frac{i}{\sqrt{q}}\mathcal{F} =& 4\beta_0 \pm \sqrt{q}rd\frac{eB}{\hbar c},\\
2\beta_1 = \mathcal{G} - 2 - \frac{i}{\sqrt{q}}\mathcal{F} =& 4\alpha_0 \mp i\sqrt{q}rd\frac{eB}{\hbar c},\\
n\left(n + 1 + \alpha_1 + \beta_1\right) =& -\left[\mathcal{I} + k^2r^2\right],
\end{aligned} \tag{4.22}$$

given the energy levels

$$\begin{aligned}
E^2_{m,n,\kappa} =& M^2c^4 + \hbar^2c^2k^2_{m,n,\kappa}\\
=& M^2c^4 - \frac{\hbar^2c^2}{r^2}\left(n + \frac{1}{2} + \alpha_0 + \beta_0\right)^2 - \frac{d}{r}\hbar ceB\left[m\frac{r}{d} \pm i(\alpha_0 - \beta_0)\sqrt{q}\right].
\end{aligned} \tag{4.23}$$

For the choice $\mathcal{P}_{Large}(X) = \exp[-\frac{rd}{2}\frac{eB}{\hbar c}X]$. On the one hand,

$$E_{m,n,\kappa}(+,+) = \pm\sqrt{M^2c^4 - \frac{\hbar^2c^2}{r^2}(n+1)^2 - \hbar cm(1+\kappa)eB}, \tag{4.24}$$

and

$$E_{m,n,\kappa}(-,-) = \pm\sqrt{M^2c^4 - \frac{\hbar^2c^2}{r^2}n^2 - \hbar cm(1-\kappa)eB}, \tag{4.25}$$

the energy given by Eqs.(**??**),(**??**) have an energy splitting between the spin up ($\kappa = +1$) and down ($\kappa = -1$) proportional to $2\hbar cmeB$. This is the spin-orbit-magnetic coupling. Nevertheless, for enough large n, m, B, the energy becomes purely imaginary since the negative interaction energy is larger than the rest-mass energy. On the other hand, the energy given by Eqs.(**??**),(**??**) become complex number with the imaginary part depending on both the spin-orbit and the external magnetic field. The QNMs always exist for nonzero m and magnetic field in this case.

$$E_{m,n,\kappa}(+,-) = \pm\sqrt{M^2c^4 - \frac{\hbar^2c^2}{r^2}\left(n + \frac{1}{2} + i\frac{\kappa mr}{d\sqrt{q}}\right)^2 - \hbar c\left(m - i\frac{rd}{2}\sqrt{q}\right)eB}, \tag{4.26}$$

and

$$E_{m,n,\kappa}(-,+) = \pm\sqrt{M^2c^4 - \frac{\hbar^2c^2}{r^2}\left(n + \frac{1}{2} - i\frac{\kappa mr}{d\sqrt{q}}\right)^2 - \hbar c\left(m + i\frac{rd}{2}\sqrt{q}\right)eB}. \tag{4.27}$$

Consequently, The solutions of Eq.(**??**) is

$$\varphi_{(q,\kappa)}(X) = \varphi_0(\sqrt{q} + iX)^{\alpha_0}(\sqrt{q} - iX)^{\beta_0}\exp\left[-\frac{rd}{2}\frac{eB}{\hbar c}X\right]P_n^{(\alpha_1,\beta_1)}(Y), \tag{4.28}$$

where $X = -i\sqrt{q}Y = \sinh_q(u/r)$.

4.2 Beltrami Geometry

For $q = 0$ and $R(u) = \frac{d}{2}\mathrm{e}^{u/r}$, the Beltrami actually ends at the Hilbert horizon $u_H = r\ln(2r/d)$, $R(u_H) = r$ (where $R'(u_H = 1)$).

4.2.1 The constant magnetic flux scenario

The equation of motion (**??**) becomes

$$0 = \varphi''(u) + \frac{1}{r}\varphi'(u) + \left[k^2 + \frac{1}{4r^2} + 2\frac{\acute{m}\sigma^3}{dr}\mathrm{e}^{-u/r} - \left(2\frac{\acute{m}}{d}\right)^2 \mathrm{e}^{-2u/r}\right]\varphi(u). \quad (4.29)$$

The general solution can be expressed in the form

$$\varphi_{(q=0,m,\kappa)}(u) = Z^{\frac{\rho}{2}}\mathrm{e}^{-Z/2}\left[C_1\, {}_1F_1\left(\rho - \frac{\kappa}{2}, \rho, Z\right) + C_2\, U\left(\rho - \frac{\kappa}{2}, \rho, Z\right)\right], \quad (4.30)$$

where we define $\rho \equiv (1 - 2ikr)/2$ and $Z(u) = \frac{4\acute{m}r}{d}\mathrm{e}^{-u/r}$ that takes the value $Z \in [4\acute{m}r/d, 0]$ for $u \in [0, \infty)$ respectively. ${}_1F_1$ and U is the confluent hypergeometric function of the First kind and second kind.

Regularity at $Z = 4\acute{m}r/d > 1$ demands that the series of the hypergeometric function truncates at finite power of Z giving the energy quantization

$$E_n^2 = M^2c^4 + \hbar^2c^2k_n^2 = M^2c^4 - \frac{\hbar^2c^2}{r^2}\left(n + \frac{1-\kappa}{2}\right)^2. \quad (4.31)$$

Significantly, the energies do not depend on the magnetic field and the orbital angular momentum at all, only the wave functions have $\acute{m}$ dependence. All $\acute{m}$ states degenerate in each energy level E_n. They have still the negative momentum square along u direction.

A special solution for $m = \Phi/\Phi_0$ where the magnetic flux is quantized to integer values, this gives the solutions are decaying plane wave travelling along the u direction, in and out of the surface. The wave has zero effective angular momentum.

$$\varphi_{(q=0,m=\Phi/\Phi_0,\kappa)}(u) = C_1\,\mathrm{e}^{-u/2r}\mathrm{e}^{-iku} + C_2\,\mathrm{e}^{-u/2r}\mathrm{e}^{iku}. \tag{4.32}$$

4.2.2 The constant magnetic field scenario

Which in the constant magnetic field scenario the equation of motion (**??**) takes the form

$$\begin{aligned} 0 = \varphi''(u) + \frac{1}{r}\varphi'(u) + \left[k^2 + \frac{1}{4r^2} + 2\frac{m\sigma^3}{rd}\mathrm{e}^{-u/r} - \left(2\frac{m}{d}\right)^2 \mathrm{e}^{-2u/r}\right]\varphi(u) \\ + \left[\frac{d\sigma^3}{4rL^2}\mathrm{e}^{u/r} - \frac{d^2}{16L^4}\mathrm{e}^{2u/r} + \frac{m}{L^2}\right]\varphi(u), \end{aligned} \tag{4.33}$$

where $L \equiv \sqrt{\hbar c/eB}$ is the magnetic length. In order to find the solution to Eq.(**??**), we approximate by considering the situation when the magnetic length is larger than d, the terms containing d/L^2 is insignificant. For representative $B = 10\ T \rightarrow L = 8.1\ nm$, the radius parameter d a needs to be smaller for the approximation to be valid. The resulting equation of motion (**??**) takes the form

$$0 \approx \varphi''(u) + \frac{1}{r}\varphi'(u) + \left[k^2 + \frac{1}{4r^2} + \frac{m}{L^2} + 2\frac{m\sigma^3}{rd}\mathrm{e}^{-u/r} - \left(2\frac{\acute{m}}{d}\right)^2 \mathrm{e}^{-2u/r}\right]\varphi(u), \tag{4.34}$$

which is exactly the same as Eq.(**??**) with substitution $\acute{m} \rightarrow m$ and $k^2 \rightarrow k^2 + m/L^2$. The solutions are thus the same with the replacement above. That leads to the energy quantization

$$E_n^2 = M^2c^4 - \frac{\hbar^2c^2}{r^2}\left[\left(n + \frac{1-\kappa}{2}\right)^2 + m\frac{r^2}{L^2}\right]. \tag{4.35}$$

4.3 Elliptic Geometry

For elliptic pseudosphere all formulae of the wormhole cases can be used. Notably since $q = -1$, the parameters α, β and α_0, β_0 become purely real and we can simply make replacement $\sqrt{q} \to i$ in all the results of the wormhole cases. The spin-orbit and all magnetic induced coupling terms in equations (**??**),(**??**),(**??**), (**??**) become real. For choice the sign of parameters $(\Lambda_\alpha, \Lambda_\beta)$ and $(\Lambda_{\alpha_0}, \Lambda_{\beta_0})$ as $(+, -)$ and $(-, +)$, the QNMs only occur for highly excited states which require the coupling terms $\sim n$ and/or $\acute{m}$ are larger than the rest-mass energy $\sim Mc^2$. In this case, the energies are purely imaginary when QNMs appear. Highly excited states with large n could become normal modes if the coupling terms $\sim \kappa, m, B$ has opposite sign and cancel with the n-term somehow under the square root. This depends on the relative sign between the interaction energy. Interestingly for fixed n, larger the interaction energy results in QNMs with shorter lifetime, i.e., larger $\mathbb{I}\mathrm{m}\ E$. For $(+, +)$ and $(-, -)$ modes are not affected by the wormhole geometry, they are diffusive modes for highly excited states. Topologically, the elliptic surface is distinctively different from the hyperbolic(wormhole) and Beltrami ones. They QNMs for low n states in contrast to the hyperbolic and Beltrami cases because the space starts at $R(u = 0) = 0$, so the modes cannot leak out through the hole, resulting in the absence of QNMs for low n states in contrast to the wormhole and Beltrami cases.

CHAPTER V

IMPLICATION FOR GRAPHENE SYSTEM

Graphene's hexagonal lattice, stability is due to its tightly packed carbon atoms. A carbon atom in graphene has one $\sigma-$bond with each of its three neighbors and one $\pi-$bond that is oriented out of plane. The analyses and main results of our work are generic for any charged fermion spatially confined to a two-dimensional surface in the presence of the axial magnetic field. The curved surface can be made from any kind of conductor, semimetal or semiconductor as long as the excited quasiparticles can move freely along the curved surface. A special case worthwhile mentioning is the zero-band gap semiconductor graphene where the electrons in the conducting band from the $2p_z$ orbitals behave like a massless fermion above the Fermi energy around the Dirac points in the momentum space. Because of this, electrons in graphene can thus be described by a relativistc Dirac equation with replacement $c \rightarrow v_F = 1 \times 10^6\ m/s$ (see e.g. Ref.[**?**] for an excellent review). In the continuum limit where the radius of the wormhole a is much larger than the lattice size [?], we can make the following identification in the above analyses

$$\begin{aligned} \chi^{\kappa=+,-} &\Longrightarrow \varphi_K^{A,B} \\ \varphi^{\kappa=+,-} &\Longrightarrow \varphi_{K'}^{A,B} \end{aligned} \tag{5.1}$$

where $K(K')$ is one of the Dirac point $K(K')$ and A, B are the two inequivalent atomic sites in the unit cell of the graphene lattice. $\varphi_{K(K')}^{A,B}$ are the corresponding wave functions of the fermion at each site. The energy of quasiparticles around the Dirac point $K(K')$ in the graphene wormhole can be calculated with replacement $M \to 0,\ c \to v_F$ (without changing the definition of the magnetic flux quantum $\Phi_0 = hc/e$ and the magnetic length $L = \sqrt{\hbar c/eB}$. In SI unit where the electric charge e is measured in Coulombs, Φ_0 and L do not originally contain c). Now the energies from (??),(??),(??),(??) are purely imaginary implying that they are purely unstable (exponentially growing) or decaying modes analogous to over damped modes of oscillator. The other modes have both real and imaginary parts of the energy and thus are QNMs. The QNM in graphene wormhole can be interpreted about the quasiparticles with a quantum state has a finite lifetime τ.

The energy scale of the fermion in the graphene wormhole is of the order of $E \sim \hbar v_F/r = 0.658\ eVnm/r$. Naturally this is the same order as the electronic energy in the carbon nanotube with the similar radius and length. Curvature effects, however, play a crucial role in the wormhole case where the imaginary energy is induced via the spin-orbit coupling and interaction between angular momentum and the external magnetic field. Remarkably in the constant flux scenario at mass less, the spin-orbit and spin-magnetic interaction energy become real as shown in Eqs. (??) and (??). In this case, the diffusive part of energy characterized by the quantum number n.

The lifetime of the fermionic states of the graphene wormhole is characterized by $\tau = 1/\mathbb{I}\mathrm{m}E$ which is also of the order of the inverse of $\hbar v_F/r = 0.658\ eVnm/r$. For $r = 1 - 1000\ nm \longrightarrow \tau = 10^{-6} - 10^{-3}\ ns$, a considerably short period of time. Interestingly for a macroscopic graphene wormhole of radius 100 μm, the lifetime could be as long as 0.1 ns. For the constant flux scenario, the lifetime is purely determined by the quantity $\hbar v_F/r$ and independent of the angular momentum and the flux. For Eqs. (**??**) and (**??**) of the constant field scenario, $m < 0$ states in the presence of magnetic field can become stable with infinite lifetime when the quantity under the square root becomes negative and the energy takes the real values. These are the Landau states. Such long-lived states require sufficiently high magnetic field B and low n so that the Landau interaction is dominant. For e.g. $n = 0 \longrightarrow r = 10\ nm$, the magnetic field B has to be larger than 6.58 T so that $L > r$ for the dominant Landau interaction in Eq.(**??**).

CHAPTER VI

Summary

We consider charged fermion in a two-dimensional wormhole in the presence of the external magnetic field with axial symmetry. Assuming uniform field in the plane perpendicular to the direction of the field, we consider energy levels of fermion in two scenarios, constant flux through the wormhole throat and constant field. The curvature connection of wormhole generates effective gauge connection resulting in the induced spin-orbit coupling of the fermion on the wormhole. The coupling is genuinely "gravitational" since it exists even in the absence of the magnetic field and it is vanishing when the wormhole is flat, e.g. cylindrical wormhole. When the magnetic field is turned on along the wormhole axis, the spin-orbit-magnetic coupling is also generated in addition to the conventional Landau coupling between the angular momentum of the fermion and the magnetic field. This new interaction is the combined effect of gravity and gauge field on the charged fermion.

When a fermion is confined to the 2-dimensional curved space its σ^3-spin component is perpendicular to the surface (since the dreibein is locally defined in the tangent space of the surface) while the orbital angular momentum is pointing along the z-direction. The spin-orbit coupling $\sim \vec{\sigma}^3 \cdot m\hat{z}$ is thus generated for generic curved spaces with curvature. For cylindrical tube, the spin-orbit coupling disappears together with the Landau coupling between spin and the magnetic field. The only remaining interaction is the orbital-magnetic Landau coupling. Note that the σ^3-spin component is pointing along the direction of the normal vector of the curvature of surface since the dreibein e^a_μ is defined on the tangent space of the curved surface. Also, because $R'(u) = \cos\delta$ where δ is the angle between the σ^3-spin component and the z axis, the spin-orbit coupling term can thus be rewritten as $\sim \vec{\sigma}^3 \cdot m\hat{z} = \sigma m \cos\delta = \sigma m R'$. The spin-orbit coupling vanishes when $R' = \cos\delta = 0$ or $\delta = \pi/2$, i.e. when the normal vector of the surface is perpendicular to $\hat{z}$ (see Figure **??**).

For both constant flux and constant field scenarios in every choice of solution parameters, sufficiently highly excited states with large n will always give QNMs. The energy naturally leaks out of the wormhole when the fermion is sufficiently excited. This is consistent with the existence of Hilbert horizons [**?**] at finite

$$u_H = r \ln\left(\frac{r}{d} \pm \sqrt{\frac{r^2}{d^2} + q}\right),$$

where the wormhole geometry ends. Highly excited fermion lives at larger u and it will leak out of the wormhole through the Hilbert horizons.

On the other hand, the spin-orbit coupling always generate QNMs since the coupling (on the wormhole) itself is imaginary $\sim i\kappa m$. The origin of this term can be traced back to the pseudo gauge connection $A_{\acute{u}}$ which is purely imaginary. Remarkably, the curvature connection Γ_θ (i.e. "gravity") generates an effective (pseudo) gauge connection that is purely imaginary resulting in the complexity of the energy and the existence of the QNMs. Physically, imaginary energy should be interpreted as the energy dissipation and instability. Energy dissipation corresponds to the case with $\mathbb{I}\text{m}E < 0$. Instability stems from the enhancement in time of the wave function when $\mathbb{I}\text{m}E > 0$. A state with high orbital angular momentum m tends to leak energy faster due to larger imaginary part of the QNMs. Note that the spin-orbit coupling term in the equation of motion is zero when $R' = \cos\delta = 0$ (at midpoint of the throat or in the case of cylindrical tube) and maximum when $R' = \cos\delta = 1$ at the Hilbert horizon. At Hilbert horizon, the surface is merging to the plane and perpendicular to $\hat{z}$.

Emergence of QNMs in this $(2+1)-$dimensional wormhole should be compared with the situation in astrophysical or gravitational traversable wormhole in $(3+1)-$dimensions. In Refs.[**?**, **?**, **?**], ringing of astrophysical wormhole results in the QNMs due to the leaking-out waves into the asymptotically flat infinity and the throat (and subsequently to the other asymptotically flat region). The traversable $(3+1)-$dimensional wormhole has no event horizon and is the analog of our $(2+1)-$dimensional "wormhole" (minus the time dilatation in the latter). Choosing only the leaking-out bound ry condition leads to singling out only the decaying QNMs. In our case, we keep all possible boundary conditions in this work since they are useful in the generic scattering processes.

The curvature connection of curved spaces generates effective gauge connection resulting in the induced spin-orbit coupling of the fermion on the surface. The coupling is genuinely gravitational since it exists even in the absence of the magnetic field. Adding external magnetic fields in the tangent direction to the surface, the new interaction is the combined effect of gravity and gauge field on the charged fermions, appearing in the terms of the angular momentum of the fermion and the magnetic field. The interplay between the curvature connection of the wormhole and the induced (pseudo) gauge connection demonstrates an interesting kind of gauge-gravity duality. The real gravity connection can be interpreted as the imaginary (effective) gauge connection (in the locally perpendicular direction on the surface) that leads to the complexity of the energy and the emergence of the QNMs and unstable modes. Adding external magnetic field induces a new imaginary coupling term proportional to the field that only exists when there is curvature and they will vanish when the curvature is zero as we can see again in sections **??** and **??**. The new curvature-spin-magnetic field coupling similarly leads to the emergence of QNMs and unstable modes.

The gauge field in the wormhole can change the total angular momentum of the charged fermions, altering their statistics accordingly (see Ref.[**?**] for discussion in cylinder). The effective orbital quantum number is given by $\acute{m} = m - \Phi/\Phi_0$. Since the magnetic flux quantum is $\Phi_0 = hc/e = 4.13567 \times 10^{-15}\ Tm^2$; for $B = 10\ T$ and $d = \sqrt{\Phi_0/2\pi B} = 8.11\ nm$, we then have $\acute{m} = m - 1/2$ and the total angular momentum becomes integer. The fermion quasiparticle(e.g. electron or hole) would behave like a boson *with charge e, the electron charge.* It should be emphasized that these boson-like fermions are not pairs of fermions like the Cooper pairs, their statistics are simply altered by global boundary condition in the presence of the magnetic field. It is possible to store condensated boson-like (when the flux is half of odd-integer number of Φ_0) fermions in the wormhole connecting e.g. two graphene sheets and control their behaviour by changing either the magnetic field or the shape of the wormhole. This could potentially lead to a number of profound electronic properties and future applications.

Conversely, if the constant flux is trapped within a vortex for instance in the type II superconductors, the quasiparticle in this case is the Cooper pairs with charge $2e$ which is a boson satisfying the Klein-Gordon equation given also by Eq.(**??**). The Cooper pairs on the vortex surface will remain boson for integer values of Φ/Φ_0. In the interior of the vortex where the flux is smaller, the statistics become arbitrary (non-Bose-Einstein statistics) and the Cooper pairs cannot condensate. In this way, we can conclude that there is a minimum magnetic flux (above zero) when $\Phi/\Phi_0 = 1/2$, i.e., below which the Bose-Einstein condensation cannot occur and the region within this vortex is in the non-superconducting phase (superconducting phase can occur for zero flux as long as the angular momentum of the charged particles is also zero, i.e. no vortex). This is one way to argue the existence of the magnetic flux quantum in type II superconductor. Generically, we can calculate the radius a at which statistics of quasiparticles exchange between fermion and boson to be

$$d = L\sqrt{2n+1} \qquad ; n = 0, 1, 2, ... \tag{6.1}$$

At these radii on the graphene wormholes, boson-like fermions should condensate at the ground state within the hole at sufficiently low temperature $T < \hbar v_F / k_B d = 7639\ Knm/d$. To achieve condensation at room temperature $T = 300\ K$, we need $d \lesssim 25.5\ nm$ implying also from Eq.(**??**) that $1 \lesssim B$ for $n = 0$.

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Vitae

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Publication

1. T. Rojjanason, P. Burikham, and K. Pimsamarn, Charged fermion in (1+2)-dimensional wormhole with axial magnetic field, *Eur. Phys. J. C* **79** 660, 2019.