CHAPTER III

QUANTUM DYNAMICS AND PATH INTEGRALS

3.1 TIME EVOLUTION AND THE SCHRÖDINGER EQUATION

3.1.1 Time Evolution Operator

Our basic concern in this section is, how does a state ket change with time. Suppose we have a physical system whose state ket at t_0 is represented by $|\Psi\rangle$ which corresponds to wave function $\psi(x)$. At later times, we do not, im general, expect the system to remain in the same state $|\Psi\rangle$. Let us denote the ket corresponding to the state at some later by

$$|\Psi,t_0;t\rangle, \quad t>t_0, \tag{3.1.1}$$

where we have written Ψ, t_0 to remind ourselves that the system used to be in state $|\Psi\rangle$ at some earlier reference time t_0. Because time is assumed to be a continuous parameter, we expect

$$\lim_{t\to t_0}|\Psi,t_0;t\rangle=|\Psi\rangle \tag{3.1.2}$$

and we may as well use a shorthand notation,

$$|\Psi,t_0;t_0\rangle = |\Psi,t_0\rangle, \tag{3.1.3}$$

for this. Our basic task is to study the time evolution of a state ket.

$$|\Psi,t_0\rangle = |\Psi\rangle \xrightarrow{\text{timeevolution}} |\Psi,t_0;t\rangle = |\Psi(t)\rangle. \tag{3.1.4}$$

Put in another way, we are interested in asking how the state ket changes under a time displacement $t_0 \longrightarrow t$.

As in the case of translation, the two kets are related by an operator which we call the **time-evolution operator** $U(t,t_0)$:

$$|\Psi,t_0;t\rangle = U(t,t_0)|\Psi,t_0\rangle, \tag{3.1.5a}$$

or we can rewrite

$$|\Psi(t)\rangle = U(t,t_0)|\Psi,(t_0)\rangle. \tag{3.1.5b}$$

Stated another way, if the state ket is initially normalized to unity, it must remain normalized to unity at all later times:

$$\langle\Psi(t_0)|\Psi(t_0)\rangle = 1 \Rightarrow \langle\Psi(t)|\Psi(t)\rangle = 1. \tag{3.1.6}$$

As in the translation case, this property is guaranteed if the time-evolution operator is taken to be unitary. For this reason we take unitarity,

$$U^{+}(t,t_0)U(t,t_0) = 1, \tag{3.1.7}$$

to be one of the fundamental properties of the U operator

Another feature we require the U operator is the composition property;

$$U(t_2,t_0) = U(t_2,t_1)U(t_1,t_0),(t_2 > t_1 > t_0). \tag{3.1.8}$$

This equation says that if we are interested in obtaining time evolution from t_0 to t_2, then we can obtain the same result by first considering time evolution from t_0 to t_1, then from t_1 to t_2 - a reasonable requirement. Note that we read (3.1.8) from right to left

It also turns out to be advantageous to consider an infinitesimal time-evolution operator $U(t_0+\varepsilon,t_0)$:

$$|\Psi(t_0+\varepsilon)\rangle = U(t_0+\varepsilon,t_0)|\Psi(t_0)\rangle. \tag{3.1.9}$$

Because of continuity [see (3.1.2)], the infinitesimal time-evolution operator must reduce to the identity operator as dt goes to zero.

$$\lim_{\varepsilon\to 0} U\left(t_0+\varepsilon, t_0\right)=1, \tag{3.1.10}$$

and as in the translation case, we expect the difference between $U(t_0+dt, t_0)$: and 1 to be of first order in ε

With the infinitesimal time - displacement operator satisfies the composition property

$$U\left(t_0+\varepsilon_1+\varepsilon_2, t_0\right)=U\left(t_0+\varepsilon_1+\varepsilon_2, t_0+\varepsilon_1\right) U\left(t_0+\varepsilon_1, t_0\right); \tag{3.1.11a}$$

it differs from the identity operator by a term of order ε. The infinitesimal time - evolution operator is writen as

$$U(t_0+\varepsilon, t_0)=1-\frac{i\varepsilon H}{\hbar} \tag{3.1.11b}$$

3.1.2 The Schrödinger Equation

We are now in a position to derive the fundamental differential equation for the time-evolution operator $U(t, t_0)$. We exploit the composition property of the time-evolution operator by letting $t_1 \rightarrow t$, $t_2 \rightarrow t+\varepsilon$ in (3.1.8):

$$U\left(t+\varepsilon, t_0\right)=U(t+\varepsilon, t)\, U\left(t, t_0\right)=\left(1-\frac{i\varepsilon H}{\hbar}\right) U\left(t, t_0\right), \tag{3.1.12}$$

where the time difference $t - t_0$ need not be infinitesimal. We have

$$U(t+\varepsilon,t_0) - U(t,t_0) = -i\left(\frac{\varepsilon H}{\hbar}\right)U(t,t_0), \tag{3.1.13}$$

which can be written in differential equation form:

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = HU(t,t_0). \tag{3.1.14}$$

This is the **Schrödinger Equation for the time-evolution operator.** Everything that has to do with time development follows this fundamental equation.

Equation (3.1.14) immediately leads to the Schrödinger equation for a state ket. Multiplying both sides of (3.1.14) by $|\Psi(t_0)\rangle$ on the right, we obtain

$$i\hbar\frac{\partial}{\partial t}U(t,t_0)|\Psi(t_0)\rangle = HU(t,t_0)|\Psi(t_0)\rangle. \tag{3.1.15}$$

But $|\Psi(t_0)\rangle$ does not depend on t, so this is the same as

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle, \tag{3.1.16}$$

where (3.1.5) has been used.

If we are given $U(t,t_0)$ and, in addition, know how $U(t,t_0)$ acts on the initial state ket $|\Psi(t_0)\rangle$, it is not necessary to bother with the Schrödinger equation for the state ket (3.1.16). All we have to do is apply $U(t,t_0)$ to $|\Psi,(t_0)\rangle$; in this manner we can obtain a state ket at any t. Our first task is therefore

to derive formal solutions to the Schrödinger equation for the time evolution operator (3.1.14). There are three cases to be treated separately:

The Hamiltonian operator is independent of time. By this we mean that even when the parameter t is changed, the H operator remains unchanged. The Hamiltonian for a spin-magnetic moment interacting with a time-independent magnetic field is an example of this. The solution to (3.1.14) in such a case is given by

$$U(t,t_0)=\exp\left[\frac{-iH(t-t_0)}{\hbar}\right]. \tag{3.1.17}$$

To prove this let us expand the exponential as follows:

$$\exp\left[\frac{-iH(t-t_0)}{\hbar}\right]=1-\frac{iH(t-t_0)}{\hbar}+\left[\frac{(-i)^2}{2}\right]\left[\frac{H(t-t_0)}{\hbar}\right]^2+\ldots \tag{3.1.18}$$

Because the time derivative of this expansion is given by

$$\frac{\partial}{\partial t}\exp\left[\frac{-iH(t-t_0)}{\hbar}\right]=-\frac{iH}{\hbar}+\left[\frac{(-i)^2}{2}\right]2\left(\frac{H}{\hbar}\right)^2(t-t_0)+\ldots \tag{3.1.19}$$

expression (3.1.17) obviously satisfies differential equation (3.1.14). The boundary condition is also satisfied because as $t \longrightarrow t_0$, (3.1.17) reduces to the identity operator. An alternative way to obtain (3.1.17) is to compound successively infinitesimal time - evolution operators just as we did to obtain for finite translation:

$$\lim_{N\to\infty}\left[1-\frac{(iH/\hbar)(t-t_0)}{N}\right]^N = \exp\left[\frac{-iH(t-t_0)}{\hbar}\right]. \tag{3.1.20}$$

The Hamiltonian operator H is time - dependent but the H's at different time commute. As an example, let us consider the spin - magnetic moment subjected to a magnetic field whose strength varies with time but whose direction is always unchanged. The formal solution to (3.1.14) in this case is

$$U(t,t_0) = \exp\left[-\left(\frac{i}{\hbar}\right)\int_{t_0}^{t} dt'H(t')\right]. \tag{3.1.21}$$

This can be proved in a similar way. We simply replace $H(t-t_0)$ in (3.1.18) and (3.1.19) by $\int_{t_0}^{t} dt'H(t')$.

3.1.3 Probability Amplitudes and Events Occurring in Succession

In quantum mechanics, an experiment is set up to give result a of the measurement A followed by the measurement C giving the result c, but no effort is made to make measurement B, then eq. (2.0.4) is found to be false and has to be relpaced. This is done by assigning a quantity called probability amplitude ψ to every route or path, say ψ_{ac} to the path form A to C, that the probability for this path is eq. (2.0.3). The probability amplitude $\psi^*(x_a,t_a)$ and $\psi(x_c,t_c)$ at any space-time points are known as the wave function satisfying the Schrödinger wave equation.[5]

The probability P_{ac} to go from a point x_a at the time t_a to the point x_c at t_c is the absotute square

$$P_{ac} = |K(c,a)|^2. \qquad (3.1.22)$$

So that we can obtain

$$\psi_{ac} = K(c,a). \qquad (3.1.23)$$

In the rule for two events, we have an important law for the composition of amplitudes for events which occur successively in time. Suppose t_b is some time between t_a and t_c. Then the probabilily amplitude along any path between a and c can be written *as*

$$K(c,a) = \int_{x_b} K(c,b)\ K(b,a)\ dx_b$$

(3.1.24)

which we can compare with eq.(2.0.5). In the rule for several events, it is perfectly possible to make two divisions in all the path: these at $t_b, t_c, t_d, \ldots$ and the end at, say, t_j. Then the probabilily amplitude or the kernel for a particle going from a to k can writen as

$$K(k,a) = \int_{x_b} \int_{x_c} \int_{x_d} \ldots \int_{x_j} K(k,j)\ldots K(f,e)K(e,d)K(d,c)K(c,b)K(b,a)dx_b dx_c dx_d \ldots dx_j$$

(3.1.25)

This means that we look at a particle which goes from a to k as if it went first from a to b, then from b to c, and finally from j to k. The amplitude to follow such a path is the product of the kernels for each part of the path. The kernel taken over all such paths that go from a to k is obtained by integrating this product over all possible values of $x_b, x_c, x_d \ldots and\ x_j$

3.1.4 The Path Integral From The Operator Formalism

We start from the position to position amplitude for Heisenberg eigenstates:

$$\langle q_b, t_b | q_a, t_a \rangle$$

Recall that for a Heisenberg eigenstate:

$$|q,t\rangle = \exp\left(\frac{i}{\hbar}\hat{H}(\hat{q},\hat{p})t\right)|q\rangle \tag{3.1.26}$$

$\hat{H}$ being a Hermitean operator and $|q\rangle$ the state at time zero, and so:

$$\langle q_b, t_b | q_a, t_a \rangle = \langle q_b \left| \exp\left(-\frac{i}{\hbar}\hat{H}(\hat{q},\hat{p})(t_b - t_a)\right) \right| q_a \rangle \tag{3.1.27}$$

At this stage an assumption is made to disassemble the exponential: form equation (3.1.20) we obtain

$$\langle q_b, t_b | q_a, t_a \rangle = \lim_{N \to \infty} \langle q_b \left| \left(1 - \frac{i}{\hbar}\hat{H}(\hat{q},\hat{p})\varepsilon\right)^N \right| q_a \rangle \tag{3.1.28}$$

where $\varepsilon \equiv (t_b - t_a)/N$.

Inserting position and momentum resolutions of unity:

$$1=\int_{-\infty}^{\infty} dq|q\rangle\langle q| \qquad 1=\int_{-\infty}^{\infty} dp|p\rangle\langle p|$$

leads to:

$$\langle q_b,t_b|q_a,t_a\rangle =$$

$$\lim_{N\to\infty} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\cdots \prod_{j=1}^{N-1} dq_i \prod_{i=1}^{N} dp_i \prod_{k=1}^{N} \langle q_k|p_k\rangle\langle p_k\left|\left(1-\frac{i}{\hbar}\hat{H}(\hat{q},\hat{p})\varepsilon\right)^N\right|q_{k-1}\rangle$$

(3.1.29)

where $q_b = q_N, q_a = q_0$, and are not integrated over.

Define a function $\mathcal{H}$ as:

$$\mathcal{H}(q,p) \equiv \frac{\langle p|\hat{H}(\hat{q},\hat{p})|q\rangle}{\langle p|q\rangle} \tag{3.1.30}$$

noting that general $\mathcal{H}$ (q,p) $\neq$ H (q,p). To evaluate H one should commute factors in the Hamiltonian operator (using $[\hat{q},\hat{p}]=i\hbar\hat{1}$), such that $\hat{q}$ operators are shifted to the right and can then be applied to the position eigenstate, while $\hat{p}$ operators (now on the left) apply to their eigenstates. Then noting that even though it is intended to take the $\varepsilon\to 0$ limit, one must work to order ε, since there are N such terms, where $N\varepsilon = T$ (the 'time of flight').

We proceed by making a further assumption to reassemble the exponential:

form equation (3.1.20) we obtain

$$\langle q_b,t_b|q_a,t_a\rangle =$$
$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\dots\prod_{j=1}^{N-1}dq_i\prod_{i=1}^{N}dp_i\prod_{k=1}^{N}\langle q_k|p_k\rangle\langle p_k|q_{k-1}\rangle\exp\left(-\frac{i}{\hbar}\mathcal{H}\left(q_{k-1},p_k\right)\varepsilon\right)$$

(3.1.31)

Now recall that

$$\langle q|p\rangle = \frac{\exp\left(\frac{i}{\hbar}qp\right)}{\sqrt{2\pi\hbar}} \tag{3.1.32}$$

Substituting this into the above leads to:

$$\langle q_b,t_b|q_a,t_a\rangle =$$
$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\dots\prod_{j=1}^{N-1}dq_i\prod_{i=1}^{N}dp_i\prod_{k=1}^{N}\frac{dp_i}{2\pi\hbar}\exp\left(\frac{i}{\hbar}\sum_{k=1}^{N}\left((p_k(q_k-q_{k-1})-\mathcal{H}(q_{k-1},p_k)\right)\varepsilon\right)$$

(3.1.33)

which is the traditional phase space (Hamiltonian) Path Integral, and can be formally written as:

$$\langle q_b,t_b|q_a,t_a\rangle = \int_{-\infty}^{\infty}\frac{Dp}{2\pi\hbar}\exp\left(\frac{i}{\hbar}\int_{t_a}^{t_b}(p\dot{q}-H)dt\right) \tag{3.1.34}$$

3.2 THE HEISENBERG EQUATION AND THE BAKER-HAUSDORFF LEMMA

We shall ofer be interested in knowing how expectation values of operators, in the state $|\Psi(t)\rangle$, change in the course of time. If $\hat{A}$ is an operators, then its expectation value at time t is

$$\langle \hat{A} \rangle_t = \langle \Psi(t)|\hat{A}|\Psi(t)\rangle. \tag{3.2.1}$$

Assume $\hat{A}$ not to depend on time explicitly. [The operator pt depends on t explicitly.] Using $|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle and \langle\Psi(t)| = \langle\Psi(0)|e^{iHt/\hbar}$ we have

$$\langle \hat{A} \rangle_t = \langle \Psi(0)|e^{iHt/\hbar}\,\hat{A}\,e^{-iHt/\hbar}|\Psi(0)\rangle. \tag{3.2.2}$$

If we define a time-dependent operator $\hat{A}$(t) by

$$\hat{A}(t) = e^{iHt/\hbar}\,\hat{A}\,e^{-iHt/\hbar}, \tag{3.2.3}$$

then we can also write

$$\langle \hat{A} \rangle_t = \langle \Psi(0)|A(t)|\Psi(0)\rangle. \tag{3.2.4}$$

In this equation we can regard the time development of $<\hat{A}>_t$ as occurring because the operator changes in time, while the state remains the same at all times. This way of regarding the time development of the system is called the ***Heisenberg picture***. Our old way of looking at the time development, as in Eq. (3.2.1), where the operators remain constand in time, but the states change according to equation (3.1.16) is called the **Schrödinger picture.** At t = 0, the states and operators are the same in both pictures. Both pictures give the same results for time dependent expectation values; we can solve for either the time dependence of the states in the Schrödinger pictures, or the time dependence of the operators in the Heisenberg pictures

Differentiating (3.2.3) with respect to t and assuming that $\hat{A}$ does not depend explicitly on time, we find

$$i\hbar\frac{d\hat{A}(t)}{dt} = i\hbar\left[\frac{d}{dt}e^{iHt/\hbar}\right]\hat{A}e^{-iHt/\hbar} + i\hbar e^{iHt/\hbar}\hat{A}\frac{d}{dt}e^{-iHt/\hbar}$$

$$= e^{iHt/\hbar}\left[-\hat{H}\hat{A}+\hat{A}\hat{H}\right]e^{-iHt/\hbar} \tag{3.2.5}$$

Since $\hat{H}$ commutes with $e^{-iHt/\hbar}$, (3.2.5) is simply

$$i\hbar\frac{d\hat{A}(t)}{dt} = \left[\hat{A}(t),\hat{H}\right]. \tag{3.2.6}$$

This is the equation of motion obeyed by operators in the Heisenberg picture.

In particular, $\hat{H}(t)$ is independent of time in the Heisenberg picture, if it is independent of time in the Schrödinger picture since

$$\hat{H}(t) = e^{-iHt/\hbar}\hat{H}\,e^{iHt/\hbar} = e^{-iHt/\hbar}e^{iHt/\hbar}\hat{H} = \hat{H}. \tag{3.2.7}$$

This is the quantum mechanical statement that the energy is a constant of the motion.

The Heisenberg picture position operator for a particle in one dimension obeys

$$i\hbar\frac{d\hat{x}(t)}{dt}=\left[\hat{x}(t),\hat{H}\right]. \tag{3.2.8}$$

Let us assume that $\hat{H}$ is of the form

$$\hat{H}=\frac{\hat{p}^2}{2m}+V(\mathrm{x})$$

Because $\hat{H}$ is independent of time, we can write it as

$$\hat{H}=\hat{H}(t)=\frac{\hat{p}^2(t)}{2m}+V(\mathrm{x},t) \tag{3.2.9}$$

In the commutator in (3.2.8), $\hat{x}(t)$ commutes with *V(x(t)* since an operator always commutes with a function of ifself. $\hat{x}(t)$ fails to commute, however, with $\hat{p}^2(t)/2m$. To evaluate their commutator, let us notice that,

$$\left[\hat{x}(t),\hat{p}(t)\right]=\hat{x}(t)\hat{p}(t)-\hat{p}(t)\hat{x}(t)=e^{iHt/\hbar}(\hat{x}(t)\hat{p}(t)-\hat{p}(t)\hat{x}(t))e^{-iHt/\hbar}$$

$$=e^{iHt/\hbar}i\hbar\, e^{-iHt/\hbar} \tag{3.2.10}$$

$$=i\hbar.$$

Thus $\hat{x}(t)$ and $\hat{p}(t)$ obey the same commutation relation as $\hat{x}$ and $\hat{p}$ in the Schrödinger picture. By simple calculation then

$$\left[\hat{x}(t), \frac{\hat{p}^2(t)}{2m}\right] = \frac{i\hbar\hat{p}(t)}{m},$$

so that

$$\frac{d\hat{x}(t)}{dt} = \frac{\hat{p}(t)}{m} \qquad (3.2.11)$$

The position operator in the Heisenberg picture obeys the usual classical equation of motion.

To find the equation of motion of $\hat{p}(t)$ we must evaluate $\hat{p}(t)$, $V(x(t))]$, which equals

$$e^{iHt/\hbar}\left[\hat{p}, V(\mathrm{x})\right]e^{-iHt/\hbar}.$$

Now

$$\left[\hat{p}, V(\mathrm{x})\right] = -i\hbar\frac{\partial}{\partial x}V(\mathrm{x}), \qquad (3.2.12)$$

since for any $|\Psi\rangle$

$$\langle \mathrm{x}|[\hat{p}, V(\mathrm{x})]|\Psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\left[V(\mathrm{x})\langle \mathrm{x}|\Psi\rangle\right] - V(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\langle \mathrm{x}|\Psi\rangle$$

$$= -i\hbar\frac{\partial}{\partial x}V(\mathrm{x})\langle \mathrm{x}|\Psi\rangle = \left\langle \mathrm{x}\left|-i\hbar\frac{\partial}{\partial x}V(\mathrm{x})\right|\Psi\right\rangle.$$

Thus

$$[\hat{p}(t), V(x(t))] = -i\hbar e^{iHt/\hbar}\frac{\partial}{\partial x}V(x)e^{-iHt/\hbar} = -i\hbar\frac{\partial}{\partial x}V(x(t)),$$

(3.2.13)

and $\hat{p}(t)$ obeys the equation of motion

$$\frac{d\hat{p}(t)}{dt} = -\frac{\partial}{\partial x}V(x(t)).$$

(3.2.14)

Again, this is the usual classical equation of motion; we can interpret $-\frac{\partial}{\partial x}V(x(t))$ as the operator for the force on the particle.

The similarity between the Heisenberg equations of motion and the classical Hamiltonian equations of motion arises from the fact that $(1/i\hbar)$ [A, B], the commutator of the operators for the physical quantities A and B, divided by $i\hbar$, plays a similar role in quantum mechanics, to that of the Poisson bracket of the classical quantities A and B, in classical mechanics.

Generally, the Heisenberg equations of motion are more difficult to solve than the corresponding classical equations because of the lack of commutation of quantum mechanical operators. There are a few cases, however, that we can easily solve.

For a free particle, the Heisenberg equation of motion for $\hat{p}(t)$ is

$$\frac{d\hat{p}(t)}{dt} = 0.$$

(3.2.15)

Thus the momentum operator is a constant of the motion,

$$\hat{p}(t) = \hat{p}(0)$$

and the position operator obeys

$$\hat{x}(t) = \hat{x}(0) + t\,\frac{\hat{p}(0)}{m}. \qquad (3.2.16)$$

If $|\Psi(0)\rangle$ is the wave packet of the particle at time 0, then the center of the wave packet , $<\hat{x}>_t$, is given by

$$<\hat{x}>_t = <\hat{x}>_0 + t\frac{<\hat{p}>_0}{m}, \qquad (3.2.17)$$

a familiar result.

The Heisenberg equation of motion for a simple harmonic oscillator are

$$\frac{d\hat{x}(t)}{dt} = \frac{\hat{p}(t)}{m} \qquad (3.2.18)$$

$$\frac{d\hat{p}(t)}{dt} = -m\omega^2\hat{x}(t),$$

and they have the solutions

$$\hat{x}(t) = \hat{x}(0)\cos\omega t + \frac{\hat{p}(0)}{m\omega}\sin\omega t \qquad (3.2.19)$$

$$\hat{p}(t) = \hat{p}(0)\cos\omega t - m\omega\hat{x}(0)\sin\omega t.$$

Taking the expectation value of the first equation in an arbitrary state $|\Psi\rangle$ of the oscillator, we see that $<\hat{x}>_t = <\hat{x}>_0 \cos\omega t + (<\hat{p}>_0 / m\omega)\sin\omega t$; the expectation value of the position oscillates exactly as in a classical oscillator.

These look the as the classical equations of motion. We see that the x and p operators "oscillate" just like their classical analogues.

For pedagogical reasons we now present an alternative derivation of (3.1.19). Instead of solving the Heisenberg equation of motion, we attempt to evaluate

$$\hat{x}(t) = \exp\left(\frac{iHt}{\hbar}\right)\hat{x}(0)\exp\left(\frac{-iHt}{\hbar}\right) \tag{3.2.20}$$

To this end we record a very useful formula :[8]

$$\exp(iG\lambda)A\exp(-iG\lambda) = A + i\lambda[G,A] + \left(\frac{i^2\lambda^2}{2!}\right)[G[G.A]] + \ldots + \left(\frac{i^n\lambda^n}{n!}\right)[G,[G,[G\ldots[G,A]]]\ldots] + \ldots, \tag{3.2.21}$$

where G is a Hermitian operator and λ is a real paramter. We can prove this formula as follows :

Consider

$$f(\beta) = e^{\beta G} A e^{-\beta G}$$

Make a Taylor series expansion of $f(\beta)$, obscrving that

$$\frac{df}{d\beta} = Gf(\beta) - f(\beta)G = [G, f(\beta)]$$

$$\frac{d^2 f}{d\beta^2} = \left[G, \frac{df(\beta)}{d\beta}\right] = [G,[G, f(\beta)]]$$

etc. Since f(0) = A , and $\beta = i\lambda$, we get

$$f(\beta) = A + \frac{i\lambda}{1!}[G, A] + \frac{i^2\lambda^2}{2!}[G,[G, A]] + \ldots$$

known as the **Baker - Hausdorff Lemma.** Applying this formula to (3.2.20), we obtain

$$\exp\left(\frac{iHt}{\hbar}\right)\hat{x}(0)\exp\left(\frac{-iHt}{\hbar}\right)$$

$$= \hat{x}(0) + \left(\frac{it}{\hbar}\right)\left[\hat{H}, \hat{x}(0)\right] + \left(\frac{i^2 t^2}{2!\hbar^2}\right)\left[\hat{H}, \left[\hat{H}, \hat{x}(0)\right]\right] + \ldots$$

(3.2.22)

Each term on the right-hand side can be reduced to either x or p by repeatedly using

$$\left[\hat{H}, \hat{x}(0)\right] = \frac{i\hbar \hat{p}(0)}{m} \tag{3.2.23}$$

and

$$\left[\hat{H}, \hat{p}(0)\right] = i\hbar m\omega^2 \hat{x}(0). \tag{3.2.24}$$

Thus

$$\exp\left(\frac{iHt}{\hbar}\right)\hat{x}(0)\exp\left(\frac{iHt}{\hbar}\right) = \hat{x}(0) + \left[\frac{\hat{p}(0)}{m}\right]t - \left(\frac{1}{2!}\right)t^2\omega^2\hat{x}(0)$$

$$-\left(\frac{1}{2}\right)\frac{t^3\omega^2\hat{p}(0)}{m} + \ldots \tag{3.2.25}$$

$$= \hat{x}(0)\cos\omega t + \left[\frac{\hat{p}(0)}{m\omega}\right]\sin\omega t,$$

in agreement with (3.2.19).

From eq. (3.2.25), if we assume that frequency w = 0, it can be written that

$$\hat{x}(t) = \hat{x}(0) + \frac{\hat{P}(0)}{m}t$$

which has been equal to the solution of Heisenberg equation of motion for free - particle eq. (3.2.16)

In the chapter IV, we play obviously the important role of Baker - Hausdorff Lemma or Rule on solving the problem about non - quadratic potentials of path integration. It is also used for finding free - particle and simple harmonic oscillator propagator.