A NEURAL NETWORK BASED METHOD FOR SOLVING A DYNAMIC INVESTMENT AND CONSUMPTION PROBLEM WITH TRANSACTION COSTS AND STOCHASTIC VOLATILITY

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A Thesis Submitted in Partial Fulfillment of the Requirements

for Degree of Master of Science Program in Financial Engineering

Department of Banking and Finance

Faculty of Commerce and Accountancy

Chulalongkorn University

Academic Year 2019

วิธีทางโครงข่ายใยประสาทสำหรับการแก้ปัญหา
การลงทุนและการบริโภคแบบพลวัต
ที่มีต้นทุนธุรกรรมและความแปรปรวนแบบสุ่ม

นายกิตติพงษ์ หนูน้อย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาวิศวกรรมการเงิน ภาควิชาการธนาคารและการเงิน

คณะพาณิชยศาสตร์และการบัญชี จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2562

Thesis Title: A NEURAL NETWORK BASED METHOD FOR SOLVING A DYNAMIC INVESTMENT AND CONSUMPTION PROBLEM WITH TRANSACTION COSTS AND STOCHASTIC VOLATILITY

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กิตติพงษ์ หนูน้อย: วิธีทางโครงข่ายใยประสาทสำหรับการแก้ปัญหาการลงทุนและการบริโภคแบบพลวัตที่มีต้นทุนธุรกรรมและความแปรปรวนแบบสุ่ม (A NEURAL NETWORK-BASED METHOD FOR SOLVING A DYNAMIC INVESTMENT AND CONSUMPTION PROBLEM WITH TRANSACTION COSTS AND STOCHASTIC VOLATILITY) อ.ที่ปรึกษาวิทยนิพนธ์หลัก: รศ.ดร.ไทยศิริ เวทไว, 44หน้า.

งานวิจัยในอดีตพบว่าผลเฉลยของปัญหาการลงทุนและการบริโภคที่มีต้นทุนธุรกรรมแบบสัดส่วนหรือปัญหาของเดวิสและนอร์แมน มีความเกี่ยวเนื่องกับ 3 ขอบเขต ได้แก่ ขอบเขตซื้อ, ขอบเขตนิ่งเฉย, และขอบเขตขาย โดยมีหลักปฏิบัติสำหรับนักลงทุนที่แตกต่างกันในแต่ละขอบเขต การค้นหานโยบายการลงทุนที่เหมาะสมมีความเกี่ยวเนื่องกับการค้นหาเส้นแบ่งขอบเขตเหล่านี้ อย่างไรก็ตาม การประมาณเส้นแบ่งขอบเขตและผลเฉลยของปัญหาดังกล่าวมีความซับซ้อนและไม่สามารถหาผลเฉลยโดยตรงได้ ในงานวิจัยนี้ ผู้วิจัยได้ปรับปรุงตัวแบบที่ใช้ในปัญหาดังกล่าวโดยกำหนดให้ความแปรปรวนของผลตอบแทนจากหุ้นมีลักษณะสุ่มโดยอาศัยตัวแบบความไม่แน่นอนแบบสุ่มของเฮสตัน ซึ่งสามารถแปลงปัญหาไปสู่สมการเชิงอนุพันธ์ย่อยสองมิติแบบไม่ทราบขอบเขตซึ่งแตกต่างจากสมการในปัญหาของเดวิสและนอร์แมน ทำให้วิธีทางตัวเลขที่ใช้ในปัญหาดังกล่าวไม่สามารถนำมาปรับใช้ในการประมาณเส้นแบ่งขอบเขตและผลเฉลยสำหรับปัญหานี้ได้ ดังนั้นผู้วิจัยจึงปรับปรุงวิธีทางโครงข่ายใยประสาทชื่อวิธีดีพกาเลอร์คิน ซึ่งถูกพัฒนาโดยซิริงงาโน่และสปิลิโอโพลอส และถูกตีพิมพ์ครั้งแรกในปี ค.ศ. 2018 เพื่อใช้ในการประมาณผลเฉลยและเส้นแบ่งขอบเขต ผลเฉลยสังเขปที่ได้จากโครงข่ายใยประสาทชี้ให้เห็นว่าเมื่อความแปรปรวนมีลักษณะสุ่ม อิทธิพลของระดับความแปรปรวนตั้งต้นที่มีต่อนโยบายลงทุนที่เหมาะสมนั้นเพิ่มสูงขึ้นเมื่อเทียบกับในกรณีของปัญหาเดวิสและนอร์แมน อย่างไรก็ตาม ในกรณีที่การเปลี่ยนแปลงระดับความเสี่ยงมีความสัมพันธ์เชิงลบกับผลตอบแทนจากหุ้น ผู้วิจัยพบว่า การเพิ่มขึ้นของอิทธิพลของระดับความแปรปรวนตั้งต้นที่มีต่อนโยบายลงทุนที่เหมาะสมนั้นกลับลดต่ำลงเมื่อเทียบกับกรณีที่ปราศจากความสัมพันธ์ดังกล่าว โดยผลเฉลยที่ได้ในกรณีที่ศึกษาชี้ให้เห็นว่าความเสี่ยงสามารถมีอิทธิพลเหนือผลตอบแทนส่วนเกินได้ ในกรณีที่ปราศจากความสัมพันธ์ระหว่างการเปลี่ยนแปลงของความแปรปรวนและผลตอบแทนจากหุ้น นอกจากนี้ ในกรณีที่การเปลี่ยนแปลงของความแปรปรวนมีความสัมพันธ์เชิงลบกับผลตอบแทนจากหุ้น เส้นแบ่งขอบเขตที่ได้แทบจะไม่เปลี่ยนแปลงเมื่อมีการเปลี่ยนแปลงระดับความแปรปรวนตั้งต้น

ภาควิชา การธนาคารและการเงิน	ลายมือชื่อนิสิต ....
สาขาวิชา วิศวกรรมการเงิน	ลายมือชื่ออ.ที่ปรึกษาหลัก ....
ปีการศึกษา 2562	

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KEYWORDS : OPTIMAL INVESTMENT AND CONSUMPTION PROBLEM / TRANSACTION COSTS / STOCHASTIC VOLATILITY/ RISK-RETURN TRADEOFF/ HJB EQUATION / FREE BOUNDARY PROBLEM / NEURAL NETWORK / DEEP GALERKIN METHOD /

KITTIPONG NOONOI: A NEURAL NETWORK-BASED METHOD FOR SOLVING A DYNAMIC INVESTMENT AND CONSUMPTION PROBLEM WITH TRANSACTION COSTS AND STOCHASTIC VOLATILITY. ADVISOR: ASSOC. PROF. THAISIRI WATEWAI, Ph.D., 44 pp.

Past research highlights the dependency between the solution of the optimal investment and consumption problem with proportional transaction costs known as Davis & Norman problem and the three different regions: no-trade region, buy region, and sell region, in which different actions are prescribed. As a result, discovering the optimal investment and consumption policy is equivalent to discovering these regions. However, obtaining the two unknown boundaries separating these regions are not straightforward and requires indirect solution method. In this study, we extend the model setup of the Davis & Norman problem to include stochastic variance based on the Heston stochastic volatility model. The problem becomes a two dimensional free-boundary partial differential equation problem in which the numerical methods for approximating boundaries of the Davis & Norman problem are no longer applicable. As a result, we propose a neural network-based method inspired by the original Deep Galerkin Method (DGM) proposed by Sirignano and Spiliopoulos (2018). Unlike the solution of the Davis & Norman problem, the approximated solution implies that the stochastic variance can increase the effect of the variance in the optimal policy. However, in the presence of a negative correlation between the change in the variance and stock return, the increased effect of the variance is less. Our numerical example shows that the variance can dominate the excess return for the zero-correlation case, and the boundaries can be quite insensitive to the volatility level in the negative correlation case.

Department: Banking and Finance | Student's signature..........

Field of Study: Financial Engineering | Advisor's signature..........

Academic Year: 2019

ACKNOWLEDGEMENTS

I would like to express my gratitude to those without whose support I would not achieve this. The first is, my advisor, Assoc Prof. Thaisiri Watewai, who has contributed as much as I have to this thesis project. The second is, all members of the thesis committee who have provided useful guidance. Moreover, my parents who have been supportive to me in every way possible. Last but not least, my peer thesis-track students, who have been constantly sharing useful tips for neural network training.

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Chapter I

Introduction

Deriving intertemporal consumption from investment is challenging for an investor who is given with a sufficiently large amount of endowment. A risk-free asset could be a safe-haven for the fund, yet offers unsatisfactory level of return which is, most of the time, lower than inflation rate. Therefore, stock comes into the picture to fulfill this need. Stock, on average, yields a much higher return compared to a risk-free asset but that comes along with uncertainty. This trade-off highlights the importance of constructing a portfolio consisting of a stock i.e. a risky asset and a money market account i.e. a risk-free asset. Given that consumption can only be derived from cash i.e. dollar-amount in the risk-free asset, the question of how to invest and consume optimally becomes naturally intriguing.

Merton (1971) is the first to study optimal investment and consumption problem in continuous time setting. The investor in the problem aims to maximize his expected total utility of consumption throughout infinite investment horizon. By assuming that there are no transaction costs in the market, Merton succeeded to provide a closed form solution to the problem. This is subsequently referred to as Merton investment ratio i.e. proportion of dollar-amount invested in stock and Merton consumption rate both of which can be derived from model parameters. This suggests that by maintaining the portfolio position in the two assets at the ratio the entire time, one would obtain the optimal investment. Nevertheless, the presence of transaction costs in the real market makes it impossible to continuously trade in order to always keep the portfolio at Merton ratio because doing so would incur infinite cost. Therefore, it is intuitive that the investor would trade only when the ratio sufficiently deviates from Merton ratio and take no action otherwise. As a result, investment ratios are split into the three regions i.e. buy region, sell region, and inactive or no-trade region where Merton ratio resides in the no-trade region. For the case where the transaction costs are of proportional type, the problem is known as Davis & Norman problem. Even if the problem is analytically unsolvable, a bundle of research including Davis and Norman (1990) have proposed numerical methods for approximating the boundaries between these regions.

One of the common beliefs in stock markets is *high risk, high expected return*. This emphasizes the trade-off between volatility and mean return of stocks. For a special case where excess return is linear in variance i.e. squared standard deviation, Merton ratio becomes insensitive to volatility while both of the approximate boundaries increase in volatility, see Figure 1. As can be seen in Figure 1 that, for Stock A which offers low-risk/low-return, the optimal no-trade

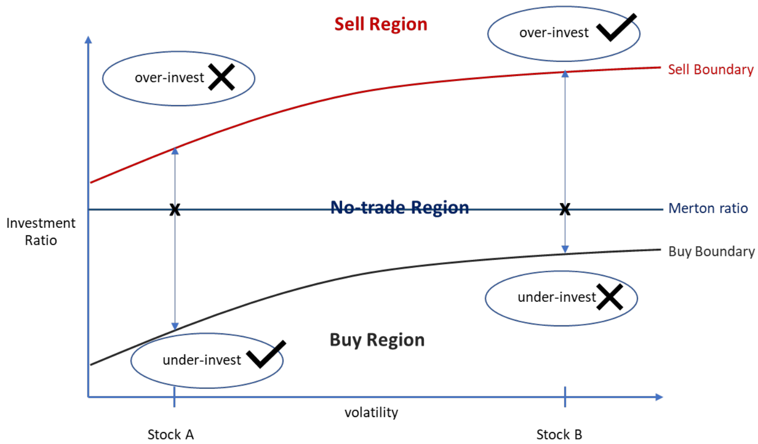


Figure 1: Trend of change in boundaries when volatility changed given that the excess return is linear in variance.

region expands upward modestly from the Merton point while it expands downward dramatically. Therefore, throughout the investment horizon, the investor is suggested to maintain the investment ratio to be in the relatively low range i.e. not allowed to over-invest but allowed to under-invest. On the contrary, for Stock B which offers high-risk/high-return, the optimal no-trade region expands upward dramatically from the Merton point while it expands downward modestly. Therefore, throughout the investment horizon, the investor is suggested to maintain the investment ratio to be in the relatively high range i.e. allowed to over-invest but not allowed to under-invest. The explanation for such a phenomenon is that, the benefit from the excess return outweighs the negative effect of volatility for both portfolios, in case of constant variance. This becomes contradictory to the belief after all and highlights the flaw of the model.

Another phenomenon found in the real market which receives much attention from researchers is the stochasticity of variance. This is also regarded as stochastic investment opportunities. A variety of research have documented the existence and effect of it in the real markets, see Jacquier et al. (1994, 2004), Mendoza (2011) for example. Stochastic volatility affects optimal policy because the investor has to take into account the prospect investment opportunity set when making a decision. Heston model is a class of stochastic volatility model which has been widely used in many different contexts e.g. option pricing, credit risk modeling. By letting the process governing variance be a mean-reverting square root process, Heston model is able to mimic real market variance. Kraft (2005) studied optimal investment problem without transaction costs under a special form of Heston model in which the excess return is linear in variance. In this research, we extend the model setup in Kraft (2005) to the case with proportional transaction costs to see whether in the presence of stochastic variance, the phenomenon, as in Figure 1, still persists or it will be redeemed by stochasticity of the variance. Lastly, the problem can be transformed into an HJB equation of the similar form to that of the case with constant investment opportunity set, see Muthuraman (2007), except for the fact that Merton point is now replaced by an optimal investment ratio derived from the result provided by Kraft (2005). This can be regarded as a kind of free-boundary problems of which the true boundaries are unknown and here is where the neural network-based method comes into the picture to approximate the solution simultaneously with its unknown boundaries.

The rest of this thesis is organized as follows. Chapter 2 presents the review of literature relevant to the optimal investment problems, conventional solution methods for the case without stochastic volatility, and neural network methods for approximating solution of boundary value problems as well. Chapter 3 provides the details of our problem while the proposed neural network-based method is described in Chapter 4. Chapter 5 provides details pertaining to neural network training, discussion of the obtained results from a particular set of model parameters, and the conjectured impact of each feature of the Heston model in general cases as well. Chapter 6 concludes this study.

Chapter II

Literature Review

2.1 Portfolio problems without transaction costs

2.1.1 Constant investment opportunity set

Merton (1971) is the pioneer of this field of research and the first to study optimal investment and consumption problem in continuous-time setting. The investor is assumed to have Constant Relative Risk Aversion (CRRA) utility function and aim to maximize his expected discounted total utility of intertemporal consumption while given with only one money market account and one stock of which the dynamics are as follows

$$dP_t = rP_t dt \tag{2.1}$$

and

$$dS_t = S_t(\alpha dt + \sigma_y dB_t^y). \tag{2.2}$$

where P_t is principle of the money market account, S_t is stock price, $r > 0$ is a constant continuous interest rate, α is mean return of stock, σ_y is volatility of stock, and B_t^y is a standard Brownian motion driving stock price. Thus, the dynamics of dollar-amounts in money market account X_t and in stock Y_t become

$$dX_t = (rX_t - c_t)dt \tag{2.3}$$

and

$$dY_t = Y_t(\alpha dt + \sigma_y dB_t^y) \tag{2.4}$$

where c_t is instantaneous consumption rate. Thus, the wealth process can be written as

$$d\omega_t = [((\alpha - r)\pi_t\omega_t + r\omega_t - c_t)dt + \pi_t\omega_t\sigma_y dB_t^y] \tag{2.5}$$

where $0 \leq \pi_t \leq 1$ is the proportion of wealth invested in stock at time t. Hence, the objective can be expressed as

$$V(\omega) = \max_{\pi_u, c_u} \mathrm{E}_\omega[\textstyle\int_s^\infty e^{-\theta(u-s)} U(c_u) du] \tag{2.6}$$

where $s < \infty$, V is a value function of the state variables in ω, $U(\omega) = \frac{\omega^{1-\gamma}}{1-\gamma}, \gamma > 0, \neq 1$ denotes CRRA utility function, and θ denotes the impatience factor.

The corresponding HJB equation is

$$\max_{\pi_u,c_u}[\omega V_\omega(\pi_u(\alpha-r)+r)-V_\omega c_u+\tfrac{1}{2}\omega^2\pi_u^2\sigma_y^2V_{\omega\omega}+\tfrac{c_u^{1-\gamma}}{1-\gamma}-\theta V]=0 \quad (2.7)$$

where V_ω and $V_{\omega\omega}$ denote first order and second order derivatives of $V(\omega)$ with respect to ω respectively. By solving (2.7), we obtain the closed-form solution $\pi_u^*=\frac{\alpha-r}{\gamma\sigma_y^2}$ and $c_u^*=C\omega_u$. According to this, the optimal policy is to always adjust the position in the two assets so that the proportion of wealth (π_u) stays at this ratio called Merton ratio and consume at a constant rate $(C=\frac{1}{\gamma}[\theta+(\gamma-1)r-\frac{(1-\gamma)(\alpha-r)^2}{2\sigma^2\gamma}])$ proportional to the total wealth.

2.1.2 Stochastic investment opportunity set

Kraft (2005) considered the case where the investor who has CRRA utility function aims to maximize his expected utility of terminal wealth. The investor is given with two assets as in Merton (1971) but stock price follows Heston's stochastic volatility model as follows

$$dS_t=S_t[(r+\beta q_t)dt+\sqrt{q_t}dB_t^S] \quad (2.8)$$

and

$$dq_t=\kappa(\delta-q_t)dt+\sigma_q\sqrt{q_t}dB_t^q \quad (2.9)$$

where q_t is a stochastic variance of stock return, B_t^q is a standard Brownian motion governing $q_t, \beta\in R\setminus\{0\}$ is a coefficient of market price of risk, $\kappa>0$ is a reversion rate of $q_t, \delta>0$ is long-run value of q_t, $\rho\in[-1,1]$ is a correlation coefficient of the two Brownian processes B_t^y and B_t^q. Therefore, the dynamics of dollar-amounts in both assets become

$$dX_t=rX_tdt \quad (2.10)$$

and

$$dY_t=Y_t[(r+\beta q_t)dt+\sqrt{q_t}dB_t^S]. \quad (2.11)$$

Thus, it is straightforward to derive the wealth process as

$$d\omega_t=\omega_t[(r+\beta q_t\pi_t)dt+\pi_t\sqrt{q_t}dB_t^y] \quad (2.12)$$

and the corresponding value function is

$$V(s,\omega_s,q_s)=\max_{\pi(\cdot)}\mathrm{E}_{s,\omega_s,q_s}[\tfrac{\omega_T^{(1-\gamma)}}{1-\gamma}] \quad (2.13)$$

where $s < t < T$, $\gamma > 0, \neq 1$ is the degree of relative risk aversion. By the Feynman-Kac representation theorem, Kraft succeeded to provide an explicit form of optimal investment policy as

$$\pi_t^* = \frac{\beta}{\gamma} + \frac{(1-\gamma)\rho\sigma\beta^2}{\gamma^2}\frac{e^{\widetilde{a}(T-t)}-1}{-\widetilde{\kappa}+\widetilde{a}+(\widetilde{\kappa}+\widetilde{a})e^{\widetilde{a}(T-t)}} \tag{2.14}$$

where $\widetilde{\kappa} = \kappa - (1-\gamma)\rho\beta\sigma, \widetilde{a} = \sqrt{\kappa^2 + 2\widetilde{\Phi}\sigma^2}$, $\widetilde{\Phi} = -\frac{(1-\gamma)\beta^2}{2c\gamma}, c = \frac{\gamma}{1-(1-\gamma)+\rho^2(1-\gamma)}$.

For cases where $\gamma \in (0,1)$, which are the focus of this study, $\widetilde{\kappa}$ and $\widetilde{a}$ must be positive. Besides, an extra condition $\beta\frac{(1-\gamma)}{\gamma}(\frac{\kappa\rho}{\sigma} + \frac{\beta}{2}) < \frac{\kappa^2}{2\sigma^2}$ must be satisfied so that the value function and optimal policy are well defined. On the other hand, for cases where $\gamma > 1$, the value function is always well defined.

2.2 Portfolio problems with transaction costs

According to Merton (1971) presented in the previous subsection, the investor has to trade continuously in order to always keep his portfolio at the Merton ratio. The presence of transaction costs of any types makes that impossible since doing so would incur infinite cost and apparently not optimal.

2.2.1 Proportional transaction costs

Proportional transaction costs are a type of transaction costs depending on the trading size. The larger the order size, the more cost incurred. The problem with proportional transaction costs can be modeled by introducing two independent non-decreasing processes representing cumulative dollar-amount paid (received) due to buying (selling) stock. Therefore, the problem is now in a two-dimensional domain of stock, and money market account and the dynamics of dollar-amounts in the two assets become

$$dX_t = (rX_t - c_t)dt - (1+\lambda)dL_t + (1-\mu)dD_t \tag{2.15}$$

and

$$dY_t = Y_t[\alpha dt + \sigma_y dB_t^y + dL_t - dD_t] \tag{2.16}$$

where L_t is a non-decreasing process representing cumulative dollar amount spent for buying stock, and D_t is a non-decreasing process representing cumulative dollar amount received from selling stock. $\lambda \in [0, \infty)$ denotes the buying proportional transaction cost, and $\mu \in [0,1]$ denotes the selling proportional transaction cost. The value function is

$$V(x,y) = \max_{(c_u,L_u,D_u)\in\mathcal{A}(x,y)} \mathrm{E}_{x,y}[\int_s^\infty e^{-\theta(u-s)}U(c_u)du] \tag{2.17}$$

where $\mathcal{A}(\cdot)$ denotes a set of all admissible policies. Thus, the corresponding HJB equation is as follows

$$\max_{(c_s,L_s,D_s)\in\mathcal{A}(x,y)}[JV(s)+MV(s), LV(s), BV(s)] = 0 \tag{2.18}$$

where $JV(s) = \frac{1}{2}\sigma_y^2 y^2 V_{yy} + \alpha y V_y + rxV_x - \theta V$,
$MV(s) = U(c_s) - c_s V_x$,
$LV(s) = (1-\mu)V_x - V_y$,
$BV(s) = -(1+\lambda)V_x + V_y.$,
V_x, V_y denote the first order partial derivative of $V(s)$ with respect to x and y respectively, and V_{yy} denotes the second order partial derivative of $V(s)$ with respect to y.

The term $JV(s) + MV(s)$ is the expected change of the value function when it is optimal not to trade, while $BV(s)$ and $LV(s)$ are the expected changes of the value function when it is optimal to buy and to sell respectively. The optimal solution is characterized by three regions on the $X-Y$ space. The first region is the no-trade region. When the values of (X,Y) fall into this region, it is optimal not to trade. The other two regions are the buy and sell regions. When the value of the dollar-amount invested in the stock is too high, the value of (X,Y) falls into the sell region, and makes it optimal to sell the stock just to bring the value (X,Y) back to the no-trade region. By the same token, when the value of the dollar-amount invested in the stock is too low, the value of (X,Y) falls into the buy region, and makes it optimal to buy the stock just to bring the value (X,Y) back to the no-trade region.

2.2.2 Fixed transaction costs

Fixed transaction costs are a type of transaction costs charged at a constant per trade. The more often the investor trades, the more cost incurred. In the presence of fixed transaction costs, the investor has to trade discretely, and consequently the problem requires a different technique called stochastic impulse control to solve. The solution is given as a set of intervention times containing both trading times and trade sizes i.e. impulses

$$\nu = (\tau_1, \tau_2, \ldots \tau_j, \ldots; \varsigma_1, \varsigma_2, \ldots, \varsigma_j, \ldots;)_{j\le N}, \quad N \le \infty \tag{2.19}$$

where ν denotes the solution, τ_j denotes the j^{th} intervention time, ς_j denotes the trade size of the j^{th} intervention, and N denotes the total number of interventions.

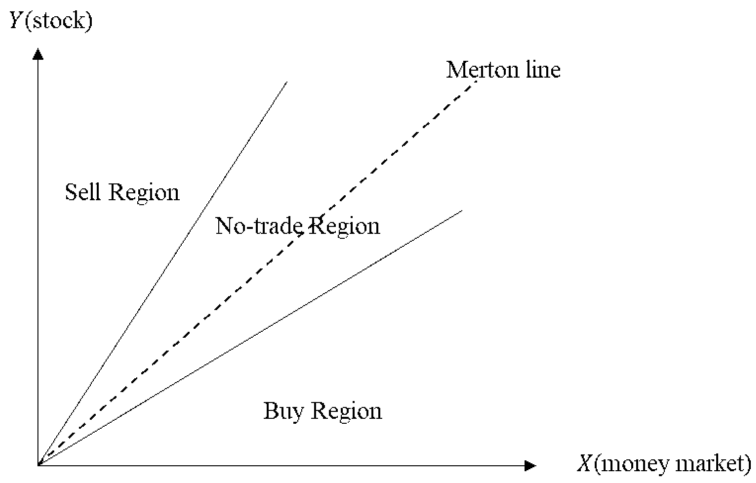


Figure 2: Optimal solution space. The Merton line represents the optimal ratio between X and Y in the Merton problem.

The solution of the problem is characterized by the Quasi-Variational Hamilton-Jacobi-Bellman inequality (QVI) of the value function, see Shaikhet (2003). The optimal investment policy can be described by a graph. The portfolio domain is separated into three regions as in the case of proportional transaction costs. The only difference is that when it is optimal to trade, the investor has to trade until the portfolio reaches the target boundary, which is independent of the magnitude of the fixed costs as shown in Figure 3. The investment horizon and his wealth composition are used to describe all the boundaries.

2.2.3 Proportional and fixed transaction costs

According to Chellathurai and Draviam (2007), portfolio problems with both proportional and fixed transaction costs can be solved using a non-singular stochastic control method. Instead of the trading rates or the cumulative shares, they use the number of the stock bought and sold as decision variables. The objective is to maximize the expected value of the discounted utility of terminal wealth. By deriving the HJB equation and separating the control set into three regions i.e. buy, sell, and no-trade, the optimal trading policy can be represented by three disjoint regions similar to that of the other two types previously discussed. The only difference is that there are four boundaries to describe the limit of the transactions, as shown in Figure 4. Two of them are the upper and the lower boundaries of the no-trade region while the other two are buy and sell targets. The no-trade region in this problem is wider than that of the other types. The investor immediately transacts when the portfolio hits buy (sell) boundary until it reaches its target boundary.

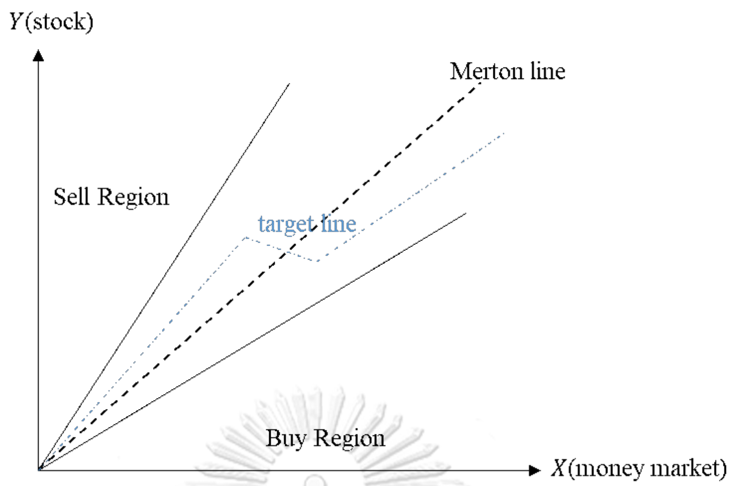


Figure 3: Solvency region for a problem with fixed transaction costs.

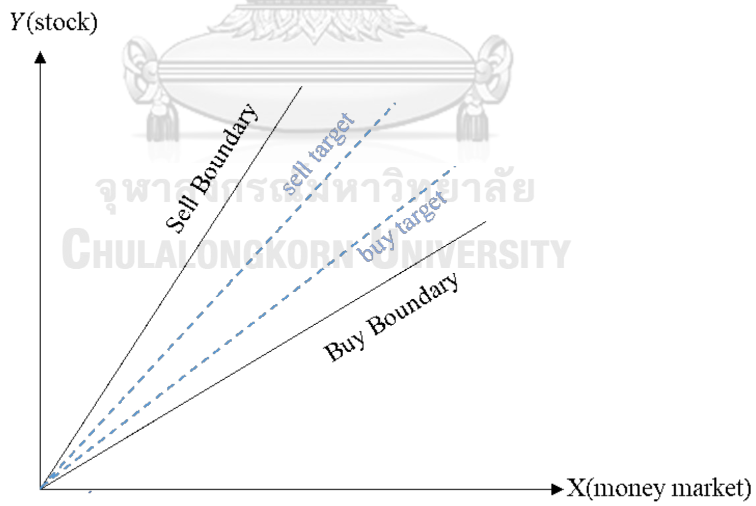


Figure 4: Solvency region for a problem with proportional and fixed transaction costs.

2.3 Variation of Portfolio Problems with Proportional Transaction Costs

Portfolio problems with proportional transaction costs have been widely studied. Apart from the setup shown in the previous subsection known as Davis & Norman problem which is a major building block of our problem, portfolio problems with proportional transaction costs are further extended in several directions by combining with other effects existing in the real market. Therefore, this section is devoted to reviewing some of them. Dumas and Luciano (1991) provided the exact solution to portfolio problem with proportional transaction costs for the case where consumption is postponed and paid at the end of investment horizon. Liu and Loewenstein (2002) showed that the *life-cycle investment advice* i.e. an older investor should hold less risky asset than a younger one, is advocated by the optimal allocation strategy when transaction costs are of proportional type and the investor has finite investment horizon. Liu et al. (2003) studied optimal investment problem with proportional transaction costs and jumps in both stock price and volatility of stock due to event risks. Framstad et al. (2006) obtained the optimal investment and consumption policy under the model with proportional transaction costs and jump diffusion. Jang et al. (2007) showed that under regime-switching model with infinite investment horizon, proportional transaction costs could have first order effect on liquidity premia. Liu and Loewenstein (2013) obtained the portfolio policy for the case where there exists a correlation between market crashes i.e. event risks and regime switching processes. Puopolo (2015) focused on the model with proportional transaction costs, intertemporal consumption, and default risks and discovered that in spite of relatively small magnitude compared to that of transaction costs, default risks may have first order effect on investment asset allocation.

2.4 Numerical methods for approximating solution of Davis & Norman problem

Magill and Constantinides (1976) is the first to consider optimal investment problem with consumption and proportional transaction costs. They formulated the problem on stock-money market plane as in Figure 2 and characterized the shape of the no-trade region as a wedge in the middle of the solvency region based on an economic approach. This work was much ahead of its time, in that, an essential ingredient namely the theory of local time and reflecting diffusion was unavailable. Therefore, this work gives no prescription on how to compute the location of the boundaries or what the controlled process should do when reaching them. Then Davis and Norman (1990) re-formulated the problem to make it more rigorous and declared conditions for which the HJB equation has a smooth solution. In addition, they clearly prescribed actions i.e. immediate sell (buy) when the controlled policy crosses sell (buy) boundary until it moves back to the sell (buy) boundary. Ever since, the problem with this setup is known as Davis & Norman problem and it is a challenging free-boundary problem to solve. However, Davis and Norman (1990) also provided the first numerical solution method. By solving the transformed ODE backward via numerical integration, one can eventually discover a path that crosses both boundaries. It has been proven that this searching procedure is always successful but computationally intensive. After that, Shreve and Soner (1994) considered a relaxation form of the problem and used viscosity solution techniques to ensure existence and uniqueness of the solution and also characterized the regularity of the value function. Moreover, Janacek and Shreve (2004) studied asymptotic expansion of the no-trade region and the value function for the case of power utility function in the power of $\lambda^{\frac{1}{3}}$ where λ denotes the transaction cost (identical for buying and selling). They provided the coefficient of the first term in the Taylor's series. Besides, Gerhold et al. (2010) extended the work of Janacek and Shreve (2004) to the case of log-utility function. They succeeded to show that for small bid-ask spread, there is no duality gap between the primal and the dual problems. Therefore, by considering the dual problem in which the market is frictionless i.e. no transaction costs, terms of arbitrary order in the power of $\lambda^{\frac{1}{3}}$ can be computed algorithmically where λ is a sufficiently small bid-ask spread. However, this condition is quite restrictive and not applicable to the case of power utility function.

Muthuraman (2007) considered the dimensionality-reduced form of the problem, i.e., one-dimensional problem of z which is defined as the proportion of dollar-amount in money market account with respect to that in stock and proposed a fast computational scheme to approximate the boundaries and the corresponding value function. By transforming the free-boundary problem into a series of fixed-boundary problems and solving them iteratively, starting by guessing boundaries with extreme values on both ends then stepping in toward Merton point with some optimal distance derived from the previously updated solution of each iteration, one would approach the true boundary eventually. For each iteration, the algorithm consists of two main steps. The first one is solving the fixed boundary problem of linear partial differential equation (approximate problem) and the second is computing optimal moving distance of the iteration from the solution obtained in the first step. The algorithm terminates when the difference between the boundary of the current iteration and that of the previous, for both ends, are less than some pre-defined tolerance level. This convergence determination method is also used in the first step of each iteration where the maximum consumption is approximated. The boundary update rules i.e. outer loop, are the main contribution of this work. Nonetheless, the convergence of it is sensitive and restricted to the approximate problem, therefore, this computational scheme is not applicable to other classes of models including Heston model.

2.5 Approximating solution of PDE problems using neural network

The idea of approximating solution of boundary value problems of Ordinary Differential Equation (ODE) or Partial Differential Equation (PDE) using neural network has long been established as early as 1997. Lagaris et al. (1997) showed that even a simple feed forward network can be used to approximate solution of some initial or boundary value problems. By expressing the solution as the product of the known part satisfying initial or boundary condition and neural network part satisfying ODE/PDE. This is applicable to several problems with boundaries of Dirichlet type or Neumann type. Over the latest two decades, plenty of neural networks have been invented and developed upon different underlying theories. Beck et al. (2017) exploited the connection between PDE and Backward Stochastic Differential Equation (BSDE) to develop a neural network-based method claimed to be available for high dimensional fully nonlinear problems. Some of them involve HJB equation stemming from many engineering and financial problems. Unfortunately, none of them contains unknown boundaries.

Sirignano and Spiliopoulos (2018) invented a so-called Long Short-Term Memory (LSTM)-inspired highway network. This type of network combines the advantage of the famous LSTM network which is the ability to model time-series data and that of highway network which is the ability for a node in each layer to simply pass on information without transformation. Figure 5 illustrates the configuration within a node of LSTM-inspired highway network.

As can be seen that the input consists of two parts i.e. non-transformed information (x), and the transformed one (S^{in}) which is the output from the previous layer. After entering the node, both of them are split into two independent streams. The main stream which is analogous to the cell state of the typical LSTM network contains two gate signals which are the non-linearly transformed products of the two inputs. The transformed input (S^{in}) will first be gated i.e. dot operation by the input gate signal (R) then non-linearly transformed to become the main stream signal (H) before being gated by forget gate signal ($1 - G$) to become a cell state. The other stream is called highway stream which is a product of a single gated S^{in} by highway gate signal (Z). The output of the node is simply the sum of the products of the two streams. This architecture allows for the possibility that S^{in} is passed on without transformation. Therefore, unlike the typical LSTM-network, this is free from gradient sparse/explosion problem and we are enabled to stack up

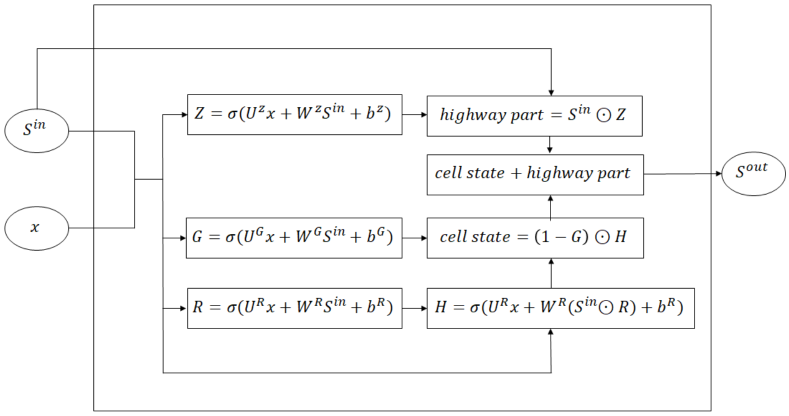


Figure 5: Operations in a DGM cell where $\sigma(\cdot)$ denotes activation function i.e. non-linear transformation function which is suggested by the author to be hyperbolic tangent function for this network, x denotes non-transformed input, S^{in} denotes transformed input, S^{out} denotes output, $U^{(\cdot)}, W^{(\cdot)}, b^{(\cdot)}$ are network weight/bias matrices, $\odot$ denotes gating operation or, mathematically, a dot operation.

multiple layers in a network. However, it is suggested by the authors that the network consisting of three layers with 50 nodes per layer suffices to handle high-dimensional PDE problems e.g. PDE problems with two hundred dimensions.

In addition, with this type of network, the author proposed a so called Deep Galerkin Method (DGM). The name DGM stems from the fact that the neural network of this type is a substitute of a basis function i.e. test functions used in conventional Galerkin method. DGM is a natural merger of Galerkin method and machine learning. Nevertheless, unlike conventional Galerkin method, DGM is totally independent of forming mesh-grid which could be problematic for high-dimensional problems. For any location i.e. an (multi-dimensional) array (x) in the (multi-dimensional) space of a problem, the network yields an approximate value of the solution function. As a result, one can calculate the approximate derivatives at a particular location and also use these derivatives to train the network to satisfy the target ODE/PDE objective terms. Over many iterations, the network will succeed to yield the highly accurate value for the solution function at the location. A major contribution of the DGM is an achievement in recovering the unknown boundary in an American option free-boundary problem.

In case of an option on k stocks, the problem becomes $k+1$ dimensional. The first k^{th} dimensions are stock prices while the last is time (t). Therefore, the input vector contains $k+2$ elements including bias. In order to recover the unknown boundary, the DGM algorithm requires three subsets of multidimensional-uniformly sampled points. The first subset represents locations on the terminal boundary on which we know exactly whether a point belongs to In-the-Money region or Out-of-the-Money region when the exercise function $g(x)$ is known. Apart from terminal time (T), one cannot tell if a particular location truly belongs to which region. As a result,after sampling, the sample points in the other two subsets need to be evaluated in order to select/reject points based on the results yielded from the current network to ensure that each of them satisfies the property of that region. The criterion for In-the-Money region is that the function value must be greater than the value from exercise function $g(x)$ while that for Out-of-the-Money region is that the function value must be less than the value from exercise function $g(x)$. The Objective function of each iteration is just the sum of the average value of the minimization target of each subset evaluated with the current network. For more details, see step 4 in the following list. Hence, for the n^{th} iteration, the DGM algorithm can be described in steps as follows

1. Uniformly sample x_h, $h = 1, 2, ..., M$, for subset 1 over the space of the first k^{th} dimensions while time (t) dimension is fixed at maturity time (T).

2. Uniformly sample (x_i, t_i), $i = 1, 2, ..., M$, for subset 2 (In-the-Money) over the space of $k + 1$ dimensions, then select only ones satisfying the property of the region which is $f(x_i, t_i, \epsilon_n) > g(x_i)$ where x_i denotes a k dimensional vector of stock prices, ϵ_n denotes the network weight at the n^{th} iteration .

3. Uniformly sample (x_j, t_j), $j = 1, 2, ..., M$, for subset 3 (Out-of-the-Money) over the space of $k+1$ dimensions, then select only ones satisfying the property of the region which is $f(x_j, t_j, \epsilon_n) < g(x_j)$ where x_j denotes a k dimensional vector of stock prices, ϵ_n denotes the network weight at the n^{th} iteration.

4. Compute gradient (with respect to the network weight) of the objective function of the following form

$$
\begin{aligned}
J(f; \epsilon_n, B_n) = {} & \tfrac{1}{|B^1|}\Sigma_{(x_h,T)\in B^1}(f - g(x_h))^2 \\
& + \tfrac{1}{|\widetilde{B}^2|}\Sigma_{(x_i,t_i)\in \widetilde{B}^2}[\tfrac{\partial}{\partial t}f + \mu(x_i)\cdot \tfrac{\partial}{\partial x}f + \tfrac{1}{2}\sigma(x_i)^2\tfrac{\partial^2}{\partial x^2}f - rf]^2 \\
& + \tfrac{1}{|\widetilde{B}^3}|\Sigma_{(x_j,t_j)\in \widetilde{B}^3}\max(g(x_j) - f, 0)^2
\end{aligned}
\tag{2.20}
$$

   where $B_n = \{B^1, \widetilde{B}^2, \widetilde{B}^3\}$ and B^1 is a subset of sample points representing the terminal boundary, $\widetilde{B}^2$ is a subset of selected sample points representing In-the-Money region, and $\widetilde{B}^3$ is a subset of selected sample points representing Out-of-the-Money region.

5. Take a descent step with learning rate (α_n) to update the network weight

$$\epsilon_{n+1} = \epsilon_n - \alpha_n \nabla_\epsilon J(f; \epsilon_n, B_n). \tag{2.21}$$

Theoretically, for Out-of-the Money region, the property is that the solution function is equal to the exercise function. However, this is a state-of-the-art by which the $L^2 - norm$ of the difference will gradually be shrunken towards zero.

Note that the appropriate leaning rate for each iteration α_n depends on the problem and network architecture. It needs to be tuned by trial and error, see Sirignano and Spiliopoulos (2018) for an example of a learning rate program suggested for this problem. Furthermore, in practice, Adaptive moment estimation optimizer (ADAM) is used in step 5 instead of Stochastic Gradient Descent (SGD).

It is not surprising that the algorithm succeeds to recover the boundary. On the terminal boundary, one can simply distinguish part of it belonging to in-the-money from the rest belonging to out-of-the-money. Since the objective function is just the sum of the three average values representing three different regions, within a few iterations after initialization, the first term representing terminal boundary will be the most consistent and powerful one to drive the optimizer: ADAM toward the minimum. Over several iterations, the first term will be settled and be able to induce sample points belonging to the other two subsets in the neighboring area to recover their gradient descent directions as well and consequently make the unknown boundary gradually recovered backward. The solution of the case of single stock option is presented in Aradi et al. (2018), in that the farther away from maturity time, the more error the approximate solution yields. Nevertheless, one can reach his satisfactory level of accuracy by increasing the number of iterations.

Chapter III

Optimal Investment and Consumption Problem

The model in this study is further developed from the Heston model presented in Kraft (2005) to include proportional transaction costs. The followings are the required basic assumptions. The market consists of one risky asset (stock) and one risk-free asset (money market account). The investor can only derive intertemporal consumption from money market account. The mean return of stock depends on the stochastic variance process. Transaction costs for buying/selling stock are constant and of proportional type. The investor has Constant Relative Risk Aversion (CRRA) utility function with degree of relative risk aversion $\gamma > 1$. Thus, the CRRA utility of consumption takes the following form $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$ where $\gamma > 1$, and c denotes the consumption rate. The investor must always be solvent, i.e. the realization of his portfolio value must always be non-negative. Lastly, the investment horizon is infinite. According to these assumptions and the original version of the model, see Kraft (2005), the dynamics of the money market account P_t, stock price S_t, and an underlying variance process q_t can be written as

$$dP_t = rP_t dt \tag{3.1}$$

$$dS_t = S_t[(r + \beta q_t)dt + \sqrt{q_t} dB_t^y] \tag{3.2}$$

$$dq_t = \kappa(\delta - q_t)dt + \sigma_q \sqrt{q_t} dB_t^q. \tag{3.3}$$

where $r > 0$ denotes interest rate of the money market account,
$\beta > 0$ denotes the coefficient of market price of risk,
$\kappa > 0$ denotes mean reversion rate of the volatility process,
$\delta > 0$ denotes long run value of the volatility,
$\sigma_q > 0$ denotes constant volatility coefficient of the volatility process,
B_t^y denotes a standard Brownian motion governing stock price process,
B_t^q denotes a standard Brownian motion governing volatility process.

Thus, the dynamics of dollar-amount in the money market account X_t, and stock Y_t can be written as

$$dX_t = (r - c_t)X_t dt - (1 + \lambda)dL_t + (1 - \mu)dD_t \tag{3.4}$$

and

$$dY_t = Y_t[(r + \beta q_t)dt + \sqrt{q_t} dB_t^y] + dL_t - dD_t \tag{3.5}$$

where L_t denotes a non-decreasing process representing cumulative dollar-amount spent when buying stock,

D_t denotes a non-decreasing process representing cumulative dollar-amount earned from selling stock,

$c_t > 0$ denotes intertemporal consumption rate at time t,

λ and μ are buying and selling proportional transaction costs respectively. We assume that the two Brownian motions B_t^y and B_t^q are correlated with correlation coefficient ρ. As mentioned earlier, this setup is based on that in Kraft (2005) which lets the volatility process be a mean-reverting square root process, therefore, the volatility can only take some non-negative values. The mean return of stock consists of two parts which are constant interest rate and the excess return depending on variance.

At initial time s, the investor is given with x dollars in the money market account and y dollars in stock and volatility q. He must choose a consumption rate and trading policy to maximize his objective. Consumption can only be derived from money market account and cannot be negative. We also require that $c(\cdot)$ be integrable for any finite time $u > s$. The solvency region is defined as

$$\mathcal{S} = \{(x, y, q) \in (0, \infty)^3\} \tag{3.6}$$

meaning that short-selling and borrowing are not allowed. A consumption-transaction policy (c_u, L_u, D_u) where $u > s$ is called admissible if X_u, Y_u, and q_u given by (3.3)- (3.5) respectively lie within $\mathcal{S}$ for all time $u > s$ and this will ensure that bankruptcy does not occur in finite time. Also, let $\mathcal{A}(x, y, q)$ denote the set of all admissible policies. The objective which is to maximize the expected value of total discounted utility of lifetime consumption can be expressed in the form of value function of the three state variables as

$$V(x, y, q) = \max_{(c_s, L_s, D_s) \in \mathcal{A}(x,y,q)} \mathrm{E}_{x,y,q}[\int_s^\infty e^{-\theta(u-s)} \tfrac{(c_u X_u)^{1-\gamma}}{1-\gamma} du] \tag{3.7}$$

where $s < \infty$. Then, the corresponding HJB equation is

$$\max_{(c_s, L_s, D_s) \in \mathcal{A}(x,y,q)} [\mathcal{J}V(s) + \mathcal{M}V(s), \mathcal{L}V(s), \mathcal{B}V(s)] = 0 \tag{3.8}$$

where $\mathcal{M}V(s) = \frac{(c_s x)^{1-\gamma}}{1-\gamma} - c_s x V_x$,

$\mathcal{J}V(s) = -\theta V + rxV_x + y(r + \beta q)V_y + \kappa(\delta - q)V_q + \frac{1}{2}y^2 q V_{yy} + \frac{1}{2}\sigma_q^2 q V_{qq} + \sigma_q y q \rho V_{yq}$,

$\mathcal{L}V(s) = V_x(1 - \mu) - V_y$,

$$\mathcal{B}V(s) = -V_x(1+\lambda) + V_y.$$

The term $\mathcal{J}V(s) + \mathcal{M}V(s)$ is the expected change of the value function when it is optimal not to trade, while $\mathcal{B}V(s)$ and $\mathcal{L}V(s)$ are the expected change of the value function when it is optimal to buy and to sell respectively.

It can be seen that the optimal consumption rate is given by $c_s = V_x^{-\frac{1}{\gamma}}$. Due to the homothetic property of CRRA utility function and linearity in X_t and Y_t, it can be shown that the value function is of the following form

$$V(x,y,q) = x^{1-\gamma}W(z,q) \tag{3.9}$$

for some function $W(z,q)$ where $z = \frac{y}{x}$. As a result, we also obtain

$$V_x = (1-\gamma)x^{-\gamma}W - x^{-\gamma-1}W_z y,$$

$$V_y = x^{-\gamma}W_z, V_{yy} = x^{-\gamma-1}W_{zz}, V_{yq} = x^{-\gamma}W_{zq}$$

$$V_q = x^{1-\gamma}W_q, V_{qq} = x^{1-\gamma}W_{qq}.$$

Hence, the HJB equation can be reduced to a two-dimensional problem as follows

$$\max_{(c_s,L_s,D_s)\in\mathcal{A}(z,q)}[\mathcal{J}W(s) + \mathcal{M}W(s), \mathcal{L}W(s), \mathcal{B}W(s)] = 0 \tag{3.10}$$

where

$$\begin{aligned}
\mathcal{M}W(s) &= x^{1-\gamma}\tfrac{\gamma}{(1-\gamma)}[(1-\gamma)W - zW_z]^{\frac{(\gamma-1)}{\gamma}},\\
\mathcal{J}W(s) &= x^{1-\gamma}[-\theta W + rW - \gamma rW - rzW_z + zW_z(r+\beta q)\\
&\quad + W_q\kappa(\delta - q) + \tfrac{1}{2}z^2 qW_{zz} + \tfrac{1}{2}\sigma_q^2 qW_{qq} + \sigma_q\rho zqW_{zq}],\\
\mathcal{L}W(s) &= x^{-\gamma}([(1-\gamma)W - zW_z](1-\mu) - W_z),\\
\mathcal{B}W(s) &= x^{-\gamma}(-[(1-\gamma)W - zW_z](1+\lambda) + W_z).
\end{aligned}$$

Heuristically, x is non-negative and in each region, only one of the three terms is exactly zero while the rest are negative, we then can simply eliminate x and re-define these terms as

$$\begin{aligned}
\mathcal{M}W(s) &= \tfrac{\gamma}{(1-\gamma)}[(1-\gamma)W - zW_z]^{\frac{(\gamma-1)}{\gamma}},\\
\mathcal{J}W(s) &= -\theta W + rW - \gamma rW - rzW_z + zW_z(r+\beta q)\\
&\quad + W_q\kappa(\delta - q) + \tfrac{1}{2}z^2 qW_{zz} + \tfrac{1}{2}\sigma_q^2 qW_{qq} + \sigma_q\rho zqW_{zq},\\
\mathcal{L}W(s) &= ([(1-\gamma)W - zW_z](1-\mu) - W_z),\\
\mathcal{B}W(s) &= (-[(1-\gamma)W - zW_z](1+\lambda) + W_z).
\end{aligned}$$

Chapter IV

Modified DGM Algorithm

4.1 Intuition behind the algorithm

According to the HJB equation derived in the previous chapter, the domain of the problem is two-dimensional. Without the stochastic volatility, the problem is simply the Davis & Norman problem. Thus, we conjecture that in our case where there exists an underlying stochastic volatility, at a fixed $q = q_m$, the domain splits into three regions in dimension z and each of them is consistent throughout i.e. no alternation, as in that of the Davis & Norman problem. Figure 6 exemplifies the domain corresponding to our conjecture.

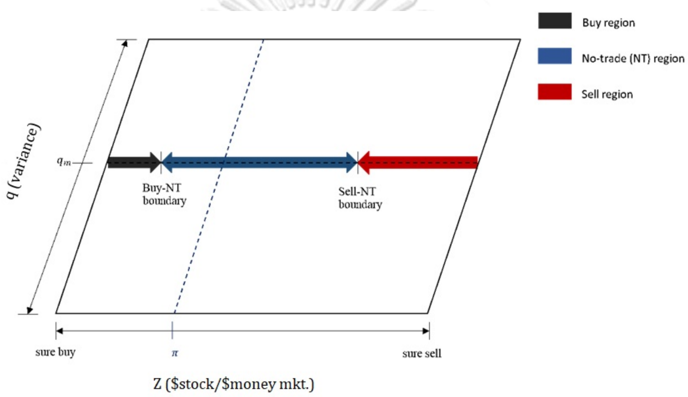


Figure 6: The two-dimensional domain of the problem where π denotes the optimal proportion of dollar-amount in stock with respect to that in money market account in the no transaction cost case and a designated sure member of our no-trade region.

As can be seen, there are three regions with two unknown boundaries. The problem can be regarded as a free-boundary problem and our goal is to find the approximate solution and these two boundaries. We are inspired by the success of the neural-network based method called DGM algorithm in recovering the boundary and the solution of American option free-boundary problem, therefore, we modify this method to suit our problem. However, the idea behind our algorithm is quite different from that of the original one.

In accordance with the fast computational scheme for approximating boundaries of the Davis & Norman problem proposed by Muthuraman (2007), we assume that the optimal proportion of dollar-amount in stock with respect to that in risk-free asset obtained in the no transaction cost case certainly is a member of our no-trade (NT) region. Apart from a slight difference in the objective, our problem is just an extension of Kraft (2005) to the case with proportional transaction costs. Thus, in order to derive the sure member of the no-trade region, it is compelling to rely on the optimal investment policy in Kraft (2005)

$$\pi^*(t) = \frac{\beta}{\gamma} + \rho\sigma_q\beta^2\frac{(1-\gamma)}{\gamma^2}\frac{e^{\widetilde{a}(T-t)}-1}{e^{\widetilde{a}(T-t)}(\widetilde{\kappa}+\widetilde{a})-\widetilde{\kappa}+\widetilde{a}} \tag{4.1}$$

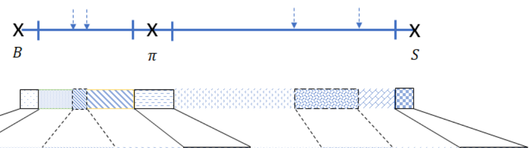
where the notations are as defined earlier, see subsection 2.1.2. By taking limit $T \to \infty$, we then obtain the sure member of no-trade region

$$\pi = \frac{\pi_y}{1-\pi_y} \tag{4.2}$$

where $\pi_y = \frac{\beta}{\gamma} + \frac{(1-\gamma)\rho\sigma_q\beta^2}{\gamma^2(\widetilde{\kappa}+\widetilde{a})}$ is the optimal portfolio weight in the Kraft (2005)'s problem when the investment horizon is infinite. Moreover, moving away from the point with sufficiently large distance, we can rest assured that we are no longer in the no-trade region. Therefore, we select a very small positive number in the z-space as a sure member of buy region and a sufficiently large number as a sure member of sell region. It is worth noting that the sure members of the three regions are the same and independent of the value of q_m.

From this point on, we refer to a sure member of each region as the center of it. Knowing these centers simplifies the problem dramatically. For any q_m, the locations, in dimension z, between the center of buy region and no-trade region can only have two possibilities of being in the buy region or in the no-trade one. By the same token, the locations, in dimension z, between the center of no-trade region and sell region can only have two possibilities of being in the no-trade region or in the sell one. As a result, we can partition the domain of the problem at the sure no-trade member into two independent zones one of which contains Buy-NT boundary while the other contains NT-Sell boundary.

In order to discover the unknown boundaries lying in the middle of these two zones, we define three classes of intervals named as confident interval, likely interval, and inconclusive interval. A confident interval is a preserved range with a very small pre-determined length expanding from the center of the region. The importance of having confident intervals is due to that neural network aims to minimize HJB quantities which are composed of function value and/or its derivatives, therefore, having only the center for each region is inadequate for the neural network to properly handle derivatives of the function. Since the locations within a confident interval is very close to the center, we have high confidence that these locations truly belong to the anticipated region. A likely interval reflects a lower degree of confidence that locations within it truly belong to the anticipated region. It is defined as an interval containing locations beyond the pre-determined distance-away



Quantity	C_B	L_B	$I_{B,N}$	L_N	C_N	L_N	$I_{N,S}$	L_S	C_S
$\mathcal{B}W^2$	✓	✓							
$(\mathcal{J}W+\mathcal{M}W)^2$				✓	✓	✓			
$\mathcal{L}W^2$								✓	✓
$\max(\mathcal{B}W,0)^2$	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\max(\mathcal{J}W+\mathcal{M}W,0)^2$	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\max(\mathcal{L}W,0)^2$	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\max(W,0)^2$	✓	✓	✓	✓	✓	✓	✓	✓	✓

Figure 7: Required objective function components for executing a point belonging to each interval where B denotes the center of the buy region, π denotes the center of the no-trade region, S denotes the center of the sell region, $C_{(\cdot)}$ denotes the confident interval of $(\cdot)$ region, $L_{(\cdot)}$ denotes the likely interval of $(\cdot)$ region, and $I_{(\cdot,\odot)}$ denotes the inconclusive interval between $(\cdot)$ region and $(\odot)$ region, a solid partition separates a confident interval from the adjacent likely interval, a dashed arrow separates a likely interval from the adjacent inconclusive interval.

from the center. Besides, moving away from the center through these locations, one must always obtain the result of maximum HJB quantity (compared between the two candidate regions) in favor of the anticipated region i.e. none of the locations in the middle yields the maximum HJB quantity that fails the anticipated region. The last class which reflects the lowest degree of confidence is inconclusive intervals. An inconclusive interval is defined as the locations beyond the likely intervals of the two candidate regions, therefore, one can expect to see alternation of the maximum HBJ quantity over this region.

According to the HJB equation derived in the previous chapter, the three HJB quantities must take zero value over its region and take some negative value otherwise. This can be achieved using the combination of two different tasks which are regularization and $L^2 - norm$ minimization. As a result, we come up with seven objective function components by which the two tasks will be fulfilled. The first three components are straightforward because they are simply the $L^2 - norm$ of the three HJB quantities which serve $L^2 - norm$ minimization task:

$$(\mathcal{J}W + \mathcal{M}W)^2 = 0, \mathcal{B}W^2 = 0, \mathcal{L}W^2 = 0.$$

The other four terms serve regularization task for which the three HJB quantities and the value function must be non-positive:

$$\max(\mathcal{J}W + \mathcal{M}W, 0)^2 = 0, \max(\mathcal{B}W, 0)^2 = 0, \max(\mathcal{L}W, 0)^2 = 0, \max(W, 0)^2.$$

The last due to the assumption on CRRA utility function with $\gamma > 1$ of the investor by which the yielded value of utility is always negative.

Hence, when q is fixed at q_m, we can split the locations, in dimension z, into nine intervals. Figure 7 lists the required components of the neural network objective function for a point belonging to each interval. As can be seen, unlike an inconclusive interval, if a point belongs to a confident or a likely interval, the objective function at the point will not only contain four penalty terms but also $L^2 - norm$ of HJB quantity representing the region as well. The benefit of doing this is that the neural network will not be misguided during the training session because, basically, we update the neural network model so that it satisfies regularization conditions first and only minimize the $L^2 - norm$ of HJB terms toward zero for those locations that we have high confidence. As a result, over many iterations, the likely intervals will expand toward each other resulting the inconclusive interval in the middle to vanish and the approximate solution will eventually converge to the true solution.

4.2 Implementation details

Like a typical deep neural network model, the major concern of this method is the high computational cost. We aim to minimize the computational intensity with the following choices of implementation.

4.2.1 Training order

Since the 2-dimensional domain of the problem is relatively compact, we have no trouble using a grid with some sufficiently small grid-size instead of sampling; consequently, the set of input is always the same for every iteration. For the sake of computational efficiency, we split the grid into blocks along dimension q by letting $\widetilde{q}$ denote the vector of grid values in this dimension and also exploit the assumption of a known sure no-trade member to split the grid along dimension z into two zones as well, see Figure 8. As mentioned in the previous section, the buy (sell) boundary at a fixed $q = q_m$ is definitely contained in $zone = 1 (zone = 2)$. Therefore, each block can be trained independently of one another. The challenge remains inside each block in which we have to evaluate inputs along with discovering the interval they belong to.

According to the definition of the three classes of intervals, the partition between a confident interval and the adjacent likely one is set by the predetermined small distance away from the center while the partition between a likely interval and the adjacent inconclusive one is unsettling and depends on the first location that yields an inferior HJB quantity compared to that of the candidate region when

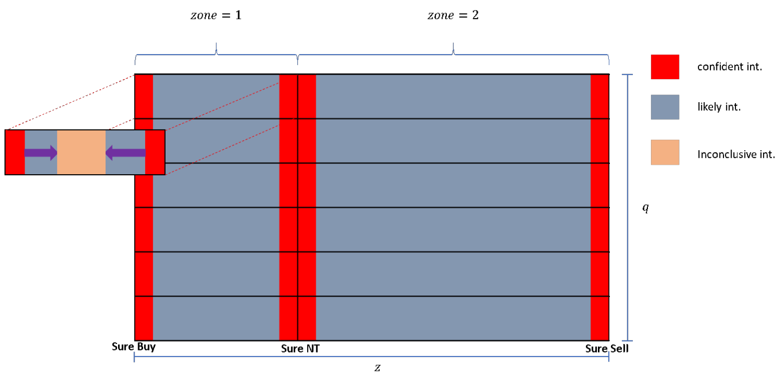


Figure 8: Training order for each block.

propagating away from the center. Given that, for the k^{th} iteration, there still exists an inconclusive interval in the middle of a block, there are two partitions need to be discovered, so that every input in the block can properly be assigned to its interval. However, such a task can be achieved by location-based sequentially training. For each block, regardless of the $zone$, it is a competition of the two candidate regions. By letting vector $\widetilde{z}(zone)$ store the grid values, in dimension z, in ascending order, we first propagate from left to right. After reaching the left partition, we jump to the last grid i.e. the farthest right one, then move in the opposite direction i.e. from right to left toward the left partition previously determined.

4.2.2 Evaluation

In the previous section, we introduce two main ideas on how to tackle the problem. The first one is the separated two tasks consisting of regularization and $L^2 - norm$ minimization. We prioritize regularization task ahead of $L^2 - norm$ minimization task. As a result, when an input is evaluated, those objective function components serving this task will always be present regardless of what class of intervals the input belongs to, revisit Figure 7 for components to be included when evaluating an input belonging to each interval. On the contrary, those components serving $L^2 - norm$ minimization task will be included only when an input belongs to a confident or a likely class. Moreover, in order to emphasize the importance of serving regularization task for every input, we magnify these regularization terms using a multiplier (Λ) where $\Lambda >> 1$. This results in a well-behaved approximate function throughout the training session. The other idea as important as the first one is the difference in the level of confidence. Since the result from inputs in a confident interval should be taken more seriously than that from a likely interval while that from a likely interval should be taken more seriously than that from an inconclusive interval, we then introduce the multipliers $\tau^C > \tau^L > 1$ to empower the evaluation result of an input from a confident and a likely interval respectively. Hence, in mathematics, the objective of the k^{th} iteration can be expressed as

$$J(W_k^f;\epsilon_k,B) = \Sigma_{h\in\{Buy,NT,Sell\}}[\Sigma_{B_h^C} obj^C(q_m,z_n) + \Sigma_{B_h^L} obj^L(q_m,z_n) + \Sigma_{B_h^I} obj^I(q_m,z_n)] \quad (4.3)$$

where $obj^C(q_m,z_n) = \tau^C \cdot L^2 - min + \Lambda \cdot pen(q_m,z_n)$,

$obj^L(q_m,z_n) = \tau^L \cdot L^2 - min + \Lambda \cdot pen(q_m,z_n)$,

$obj^I(q_m,z_n) = \Lambda \cdot pen(q_m,z_n)$,

$$L^2 - min = (\mathcal{B}W_k^f)^2 \quad \text{if } h = Buy$$
$$= (\mathcal{J}W_k^f + \mathcal{M}W_k^f)^2 \quad \text{if } h = NT$$
$$= (\mathcal{L}W_k^f)^2 \quad \text{if } h = Sell,$$

$$pen(q_m,z_n) = \max(W_k^f,0)^2 + \max(\mathcal{J}W_k^f + \mathcal{M}W_k^f,0)^2 + \max(\mathcal{L}W_k^f,0)^2 + \max(\mathcal{B}W_k^f,0)^2,$$

W_k^f is a neural-network approximate solution function at the iteration,

$B = \{B_{Buy} \cup B_{NT} \cup B_{Sell}\}$ is the set of grid points in the training space,

$B_h = B_h^C \cup B_h^L \cup B_h^I$ where $h \in \{Buy, NT, Sell\}$,

ϵ_k is the neural network weight at the iteration,

Λ is a coefficient of regularization terms, τ^C, τ^L denote significance coefficients for confident intervals and likely intervals respectively.

4.2.3 Algorithm

For the k^{th} iteration, the modified DGM algorithm consists of steps as follows

1. Split the domain into blocks which can be trained independently.
2. Sequentially evaluate inputs contained in a block using the rules specified in subsection 4.2.1 and the evaluation criterion expressed in (4.3).
3. Compute objective function and its gradient using the cumulative results of the entire grid.
4. Input the gradient of the objective function and the learning rate* into ADAM optimizer.
5. Use the result from ADAM optimizer to update network weight.

Note that the appropriate learning rate for each iteration needs to be tuned via trial and error. The steps listed above are the main steps, for the detailed version of the algorithm, see Figure 9.

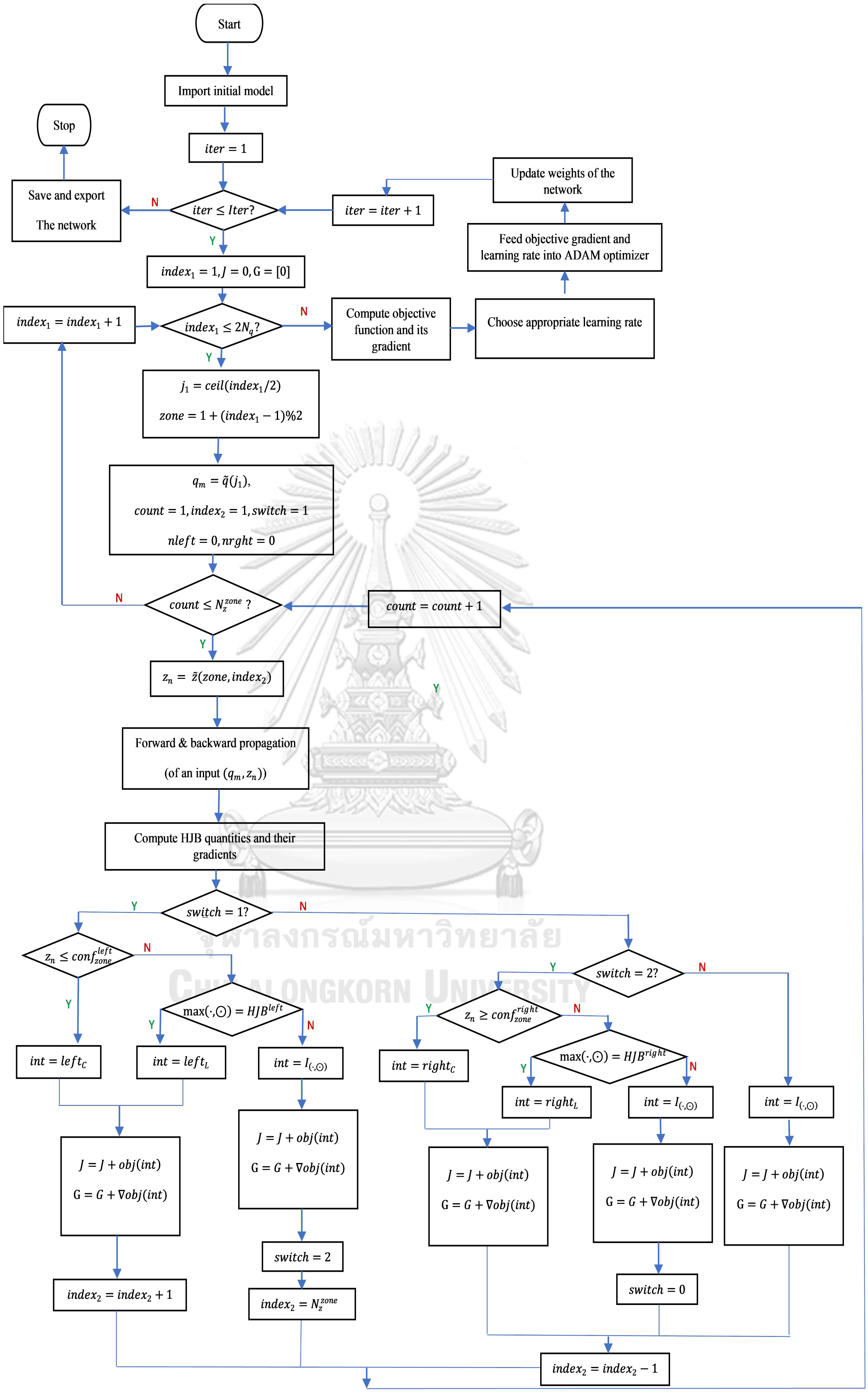


Figure 9: Detailed steps of the modified DGM algorithm.

Chapter V

Training and Numerical Results

5.1 Neural Network Training

It is crucial to select model parameters that is consistent with the real market and the proper training options for it. In this research, we follow Wang et al (2017) who obtain Heston model parameters from S&P 500 using a Kalman filter technique. Moreover, the investor is still assumed to possess impatience factor (θ) of 0.1 and degree of relative risk aversion (γ) of 2 as in Muthuraman (2007). According to this set of model parameters, we determine the two-dimensional training space in variance (q) and proportion between dollar-amount in stock with respect to that in money-market account (z) using 95% confidence interval by which the range of variance between 0.04 and 0.15 and the range of z between 0.02 and 2 are qualified. However, apart from stochastic variance, a phenomenon typically observed in the real market is the negative correlation between changes in variance and stock returns. Therefore, in this research, we compare two different cases with zero and negative correlation to see the explicit effect on the boundaries of each feature. The transaction costs i.e. buying (λ) and selling (μ) are set at 5% which is quite high compared to what we normally encountered in the real market. This is just for the sake of study to make the effect obvious and to ensure that the three regions i.e. Buy, NT, Sell sufficiently span. Since the training space is relatively large, we use a grid size of 0.01 for both dimensions in order to maintain an appropriate runtime. Hence, the model parameters are listed in Table 1 while Table 2 contains parameters related to the optimization problem. Moreover, the neural network of depth 1 is proven to be able to handle our two-dimensional problem with which the penalty coefficients and the effective learning rate schedule can be tuned and discovered subsequently. Besides, we found that the learning rate schedule as shown in Table 3 is effective.

5.2 Fitting Results

According to equation 4.3 and by substituting the penalty coefficient (Λ) with 10, significance coefficient for confident intervals (τ^C) with 5 and that for likely intervals (τ^L) with 2, we obtain the objective function as follows

$$J(W_k^f;\epsilon_k,B)=\Sigma_{h\in\{Buy,NT,Sell\}}[\Sigma_{B_h^C}obj^C(q_m,z_n)+\Sigma_{B_h^L}obj^L(q_m,z_n)$$
$$+\Sigma_{B_h^I}obj^I(q_m,z_n)] \qquad (5.1)$$

where $obj^C(q_m,z_n)=5\cdot L^2-min+10\cdot pen(q_m,z_n)$,

$$
\begin{aligned}
obj^L(q_m, z_n) &= 2 \cdot L^2 - min + 10 \cdot pen(q_m, z_n), \\
obj^I(q_m, z_n) &= 10 \cdot pen(q_m, z_n), \\
L^2 - min &= (\mathcal{B}W_k^f)^2 && \text{if } h = Buy \\
&= (\mathcal{J}W_k^f + \mathcal{M}W_k^f)^2 && \text{if } h = NT \\
&= (\mathcal{L}W_k^f)^2 && \text{if } h = Sell, \\
pen(q_m, z_n) &= \max(W_k^f, 0)^2 + \max(\mathcal{J}W_k^f + \mathcal{M}W_k^f, 0)^2 \\
&\quad + \max(\mathcal{L}W_k^f, 0)^2 + \max(\mathcal{B}W_k^f, 0)^2,
\end{aligned}
$$

W_k^f is a neural-network approximate solution function at the iteration,
$B = \{B_{Buy} \cup B_{NT} \cup B_{Sell}\}$ is the set of grid points in the training space,
$B_h = B_h^C \cup B_h^L \cup B_h^I$ where $h \in \{Buy, NT, Sell\}$,
ϵ_k is the neural network weight at the iteration.

The fitted neural networks which were both trained over 1 million iterations yield the minimum objective values of 1.23 and 2.99 for the case with zero and negative correlation (between variance and stock return) respectively as shown in Table 4. These are equivalent to the root mean square objective value per grid point of 0.0247 and 0.0383 respectively. However, the magnitude of objective function depends on the magnitude of the penalty coefficients, see Subsection 4.2.3. Therefore, an alternative and reliable measure of the model error is the statistics of the realized error at each grid point. Since, over each region, the ideal solution function must satisfy the HJB equation, it is reasonable to measure the absolute error of the maximum HJB quantity i.e. $|max(JW_k^f + MW_k^f, LW_k^f, BW_k^f)|$ at a particular grid point. The simple statistics of the absolute error over the entire training space ($q \in [0.05 : 0.01 : 0.14], z \in [0.02 : 0.01 : 2]$) are as shown in Table 4. As can be seen that the maximum absolute error is 0.1209 and 0.1204 for the case with zero and negative correlation respectively. This is acceptable when compared to the error reported in Aradi et al (2018), who demonstrate implementation of the original DGM algorithm on a one-dimensional American-put-option free-boundary problem of which the maximum error is above 10% over some region within its training space.

5.3 Optimal Strategies

This section discusses the optimal investment strategies based on the parameter set given in Table 1. Figure 10 shows the optimal buy and sell boundaries of the stochastic variance model with zero correlation (solid lines) compared to those of the Davis & Norman problem (dashed lines). According to Figure 10, it can be seen that when the mean and variance are increased while the Merton ratio i.e. the zero-cost optimality is fixed, the two boundaries are upward sloping in q in case of the Davis & Norman problem. This is equivalent to suggesting the investor to maintain relatively low proportion of wealth in stock for a portfolio consisting of

Table 1: Model parameters.

Model parameter	Value
Impatience factor (θ)	0.1
Degree of risk aversion (γ)	2
Risk-free rate (r)	0.03
Market price of risk (β)	1
Reversion rate of variance (κ)	4.46
Long-run variance (δ)	0.09
Volatility coeff. of variance (σ_q)	0.2
Transaction costs: buy (λ), sell (μ)	0.05

Table 2: Optimization problem parameters.

Optimization parameter	Value
Min variance	0.05
Max variance	0.14
Min z	0.02
Max z	2
Grid size	0.01

Table 3: An effective learning rate schedule.

Number of iteration	Learning rate ($\times 10^{-5}$)
$iter \leq 2{,}000$	50
$2{,}000 < iter \leq 6{,}000$	25
$6{,}000 < iter \leq 10{,}000$	5
$10{,}000 < iter \leq 20{,}000$	2.5
$20{,}000 < iter \leq 40{,}000$	1.25
$40{,}000 < iter \leq 80{,}000$	0.625
$iter > 80{,}000$	0.313

Table 4: The summary statistics of neural network model error.

Measure	w/o correlation	w/ correlation
Number of iteration	1,103,000	1,089,000
Min objective	1.2297	2.9906
Sqrt(min obj per grid point)	0.0247	0.0383
Min error	0.0000	0.0000
Mean error	0.0104	0.0164
Max error	0.1209	0.1204

a low-risk/low-return stock and maintain relatively high proportion in stock for a portfolio consisting of a high-risk/high-return stock. As a result, the principle of risk-return tradeoff is challenged in this class of models since the benefit from the increased return outweighs the risk due to the increased variance.

However, in our model where stochastic variance is present and with zero correlation between the variance and stock returns, the two boundaries behave differently as shown in Figure 10. The buy boundary is still upward sloping in q but less steep. This indicates that even in the presence of stochastic variance, the benefit from the excess return still outweighs the risk due to variance but with smaller effect for this set of parameters. From another perspective, this implies that the presence of stochastic variance does not affect the balance between (buying) trading cost and the cost of being away from the zero-cost optimality that much since the buy boundary does not shift evidently. On the other hand, the sell boundary is evidently downward sloping in q for the lower range of variance (smaller than long-run value $\delta = 0.09$) and levels off in the higher range. It is no surprise that the effect on the sell boundary is more obvious than that on the buy boundary because the Heston model assumes positive mean return of stock by which the sell boundary is more likely to be crossed and makes it more sensitive to changes in the nature of stock movement compared to the buy boundary. The downward slope implies that in the presence of stochastic variance, variance dominates the excess return on the sell boundary. Besides, with its mean-reversion property, it can be conjectured that when the variance is extremely high, it is more likely that the prospect variance will be lower, therefore, the sell boundary for the high range of variance levels off instead of continues decreasing. From another perspective, the fact that the sell boundary on the lower end is wider while narrower on the upper end compared to that of the case with constant variance highlights the effect of stochastic variance on the balance between (selling) trading cost and the cost of being away from the zero-cost optimality. On the lower end, stochastic variance is likely to complement the trading cost against the cost of being

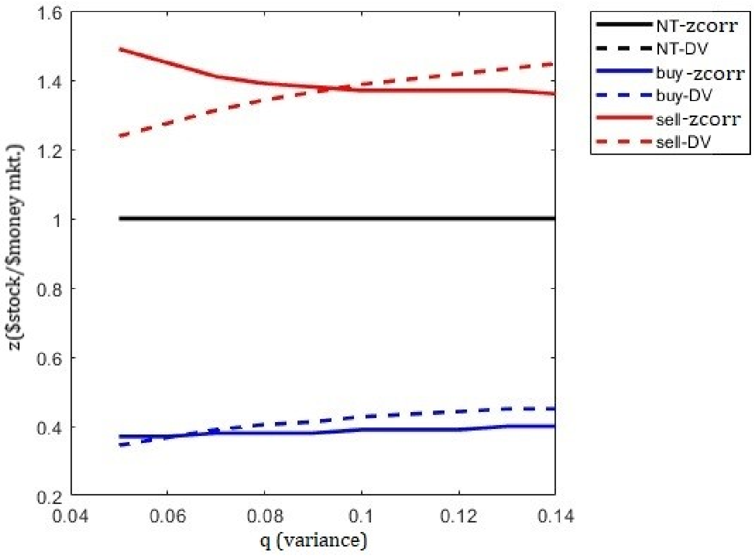


Figure 10: Comparison plot of buy and sell boundaries for the case with stochastic variance and zero correlation compared with that of the Davis & Norman problem where the dashed line in red (upper) denotes the sell boundary in case of the Davis & Norman problem, the dashed line in blue (lower) denotes the buy boundary in case of the Davis & Norman problem, the solid line in red (upper) denotes the sell boundary in case with stochastic variance and zero correlation, the solid line in blue (lower) denotes the buy boundary in case with stochastic variance and zero correlation, and the horizontal solid line in black (middle) denotes our sure NT member derived from Kraft ratio which is ,by chance, equal to Merton ratio in this setup.

away from the zero-cost optimality resulting in a wider NT region and the upward shift of it while on the upper end, stochastic variance is likely to complement the cost of being away from the zero-cost optimality against the trading cost resulting in a narrower NT region and the downward shift of the boundary.

Note that in order to make it reasonable to compare the boundaries in our model with those of the Davis & Norman problem, we select the same set of parameters as shown in Table 1 except for reversion rate of variance (κ), long-run variance (δ) and volatility coefficient of variance (σ_q) which are not present in the model of the Davis & Norman problem. Besides, when correlation coefficient is zero, our sure NT member approximated from Kraft ratio becomes the same as Merton ratio.The other case of our interest is when the stochastic variance is negatively correlated with stock return. Figure 11 compares the boundaries of the two cases of our interest i.e. with zero and negative correlation. As can be seen that the buy boundaries look almost exactly the same. The explanation for this is that the

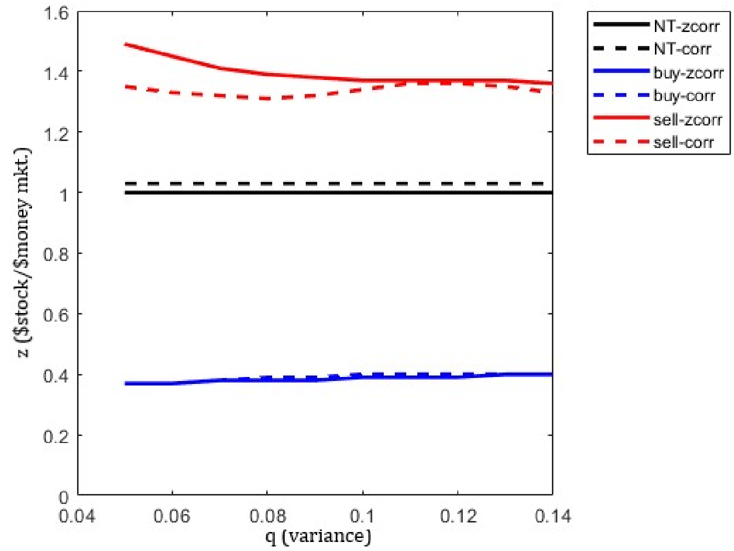


Figure 11: Comparison plot of buy and sell boundaries for the case with stochastic variance and zero correlation compared with that of the case with stochastic variance and negative ($\rho = -0.9$) correlation where the dashed line in red (upper) denotes the sell boundary in the case with negative correlation, the dashed line in blue (lower) denotes the buy boundary in the case with negative correlation, the solid line in red (upper) denotes the sell boundary in the case with zero correlation, the solid line in blue (lower) denotes the buy boundary in the case with zero correlation, the horizontal dashed line in black (middle) denotes our sure NT member derived from Kraft ratio for the case with negative correlation, and the horizontal solid line in black (middle) denotes our sure NT member derived from Kraft ratio for the case with zero correlation.

negative correlation ($\rho = -0.9$) in conjunction with stochastic variance is not strong enough to leverage the stochastic part to overcome the positive mean return part of stock return to make a difference to the boundary. On the other hand, the sell boundary for the case with negative correlation becomes less steep particularly on the lower end. This indicates that the presence of negative correlation in conjunction with stochastic variance results in the less volatile stock process. This is supported by the evidence from the simulation of variation in the final wealth in stock over the period of 10, 20, and 30 trading days at the 95^{th} percentile and the 5^{th} percentile as shown in Figure 12 and 13 respectively.

The result in Figure 12 highlights the relatively high upward volatility of the stock when there is zero correlation between stock returns and the variance. This supports our results that the sell boundary in case with negative correlation is

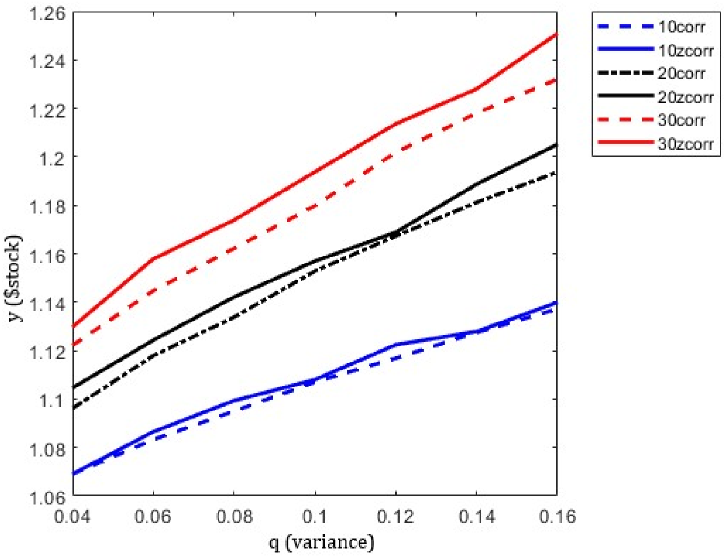


Figure 12: Comparison plot of the 95^{th} percentile variation in the final wealth in stock over the period of 10,20, and 30-trading-days when the initial endowment is $1 and variance is varied from 0.04 to 0.16 where the dashed blue line (lower pair) denotes the 95^{th} percentile over 10-day-period in case with negative correlation, the solid blue line (lower pair) denotes the 95^{th} percentile over 10-day-period in case with zero correlation, the dashed-dotted black line (middle pair) denotes the 95^{th} percentile over 20-day-period in case with negative correlation, the solid black line (middle pair) denotes the 95^{th} percentile over 20-day-period in case with zero correlation, the dashed red line (upper pair) denotes the 95^{th} percentile over 30-day-period in case with negative correlation, and the solid red line (upper pair) denotes the 95^{th} percentile over 30-day-period in case with zero correlation.

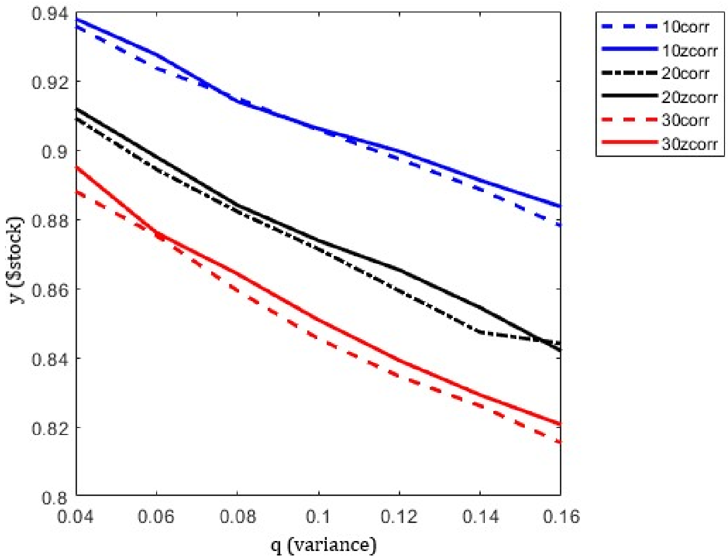


Figure 13: Comparison plot of the 5^{th} percentile variation in the final wealth in stock over the period of 10, 20, and 30-trading-days when the initial endowment is \$1 and variance is varied from 0.04 to 0.16 where the dashed blue line (lower pair) denotes the 5^{th} percentile over 10-day-period in case with negative correlation, the solid blue line (lower pair) denotes the 5^{th} percentile over 10-day-period in case with zero correlation, the dashed-dotted black line (middle pair) denotes the 5^{th} percentile over 20-day-period in case with negative correlation, the solid black line (middle pair) denotes the 5^{th} percentile over 20-day-period in case with zero correlation, the dashed red line (upper pair) denotes the 5^{th} percentile over 30-day-period in case with negative correlation, and the solid red line (upper pair) denotes the 5^{th} percentile over 30-day-period in case with zero correlation.

less distant from the sure NT compared to that of the case with zero correlation. On the other hand, the result in Figure 13 highlights the relatively high downward volatility of the stock when there is the negative correlation between stock returns and changes in variance. However, the difference is not as obvious as in Figure 12 and we can also see some crossovers at every period, this eventually might explain why the buy boundaries of the two cases are indistinguishable. Nevertheless, the approximate solution functions from the neural network still possess relatively high absolute errors around the buy boundary for both cases i.e. zero and negative correlations, see Figure a3 and a4 in the appendix for the error plots over the entire training space for both cases. This can be improved with dedication of computer resources and time but for this research, this is sufficient for us to make conclusion on the boundaries. Note that by using Amazon Web Service's Elastic Compute Cloud (AWS-EC2) of the specified type *m5.2xlarge* with 8 virtual CPU cores (Intel Xeon Platinum 3.1 GHz max) and 30 GiB memory, it took more than 3 weeks to reach 1 million iterations and achieve this level of accuracy.

In summary, the results from our modified DGM method suggest that, with this set of model parameters, an investor who assumes that the market possesses stochastic variance with zero correlation between the variance and stock returns should be more vigilant on when to liquidate stock than when to buy it because the buy boundary turns out to be less steep throughout the entire range of variance while the sell boundary still shows a big difference between both ends as shown in Figure 10. When the variance is low, the investor is allowed to over-invest in stock i.e. let profit run due to the elevated sell boundary while inhibited to over-invest in stock when the variance is high due to the restrictive sell boundary on that end. Moreover, for an investor who assumes that the market possesses stochastic variance with negative correlation ($\rho = -0.9$) between the changes in variance and stock returns, he could be carefree with the level of variance and rather stick to the same optimal policy throughout the invest horizon because both the buy and the sell boundaries stay almost exactly the same throughout the entire range of variance as shown in Figure 11.

Furthermore, based on the results that we obtain, we conjecture the impact of stochastic variance on the boundaries as follows. According to the boundaries of the Davis Norman problem, when the model assumes constant variance, the total volatility is solely from the randomness of stock return. The presence of transaction costs reduces the expected profit, and to compensate for this lower profit, the expected excess return becomes relatively more significant than the variance compared to the case of zero transaction cost. Thus, when the variance and expected excess return are relatively low, the loss due to small excess return has greater impact than the benefit from the small variance. As a result, the optimal policy is to maintain the relatively low position in stock and that is why we see the sell boundary closer to the zero-cost optimality (Merton ratio) compared to the buy boundary. Likewise, when the variance and expected excess return are relatively high, the benefit from excess return has greater impact than the risk due to the increase in variance. As a result, the optimal policy is to maintain the relatively high position in stock and that is why we see the buy boundary closer to the zero-cost optimality (Merton ratio) compared to the sell boundary.

However, when the model includes stochastic variance, the randomness in investment opportunity set would increase the total volatility, then the significance of the excess return over the variance would eventually be reduced. As a result, the boundaries would become less sensitive to variance compared to those in constant variance case. Moreover, the mean-reversion assumption as in the Heston model also plays a role in the transformation of the boundaries. When the variance is extremely low, it is highly likely that the prospect variance will be higher and the investor would eventually obtain the benefit from the bigger excess return. That is why the optimal policy suggests the investor to maintain relatively high portfolio weight in stock as can be seen that both of the boundaries are higher than those in constant variance case on the lower end. Likewise, when the variance is extremely high, it is highly likely that the prospect variance will be lower and by the same token, the investor would eventually lose the benefit from the big excess return. That is why the optimal policy suggests the investor to maintain relatively low portfolio weight in stock as can be seen that both of the boundaries are lower than those in constant variance case on the upper end. On the contrary, in the presence of negative correlation between the change in the variance and stock return, the total volatility is reduced. Therefore, the empowered impact of the variance against the excess return due to stochastic variance is smoothed out. As a result, the two boundaries become less sensitive to the variance compared to those in case with zero correlation.

Chapter VI

Conclusion

The objective of this study is to find the optimal investment and consumption policy for an investor who possesses CRRA utility function and aims to maximize the expected value of total discounted utility of intertemporal consumption throughout infinite investment horizon, given that there exist only one risk-free asset and one risky asset in the market where proportional transaction costs and stochastic variance are present. The obtained HJB equation is originally three-dimensional. However, by homothetic property of CRRA utility function, we reduce dimensionality of the problem to two i.e. the proportion between initial dollar-amount in stock with respect to that in money market account and the current level of the variance of the stock return. As a result, the problem becomes two-dimensional with two unknown boundaries. This can be regarded as a free-boundary problem which is analytically intractable. Therefore, a neural-network based method is proposed for approximating solutions together with the unknown boundaries. This is inspired by the success of the original DGM algorithm proposed by Sirignano and Spiliopoulos (2018) in recovering the unknown boundary of American option free-boundary problems.

The approximated solutions highlight the importance of taking into account stochastic variance because even in the presence of stochastic variance alone i.e. no correlation between the variance and stock returns, the optimal investment policy is clearly different from the case with constant variance i.e. the Davis & Norman problem. Unlike the optimal policy in the problem of Davis & Norman in which the benefit of the excess return dominates the risk from the variance, the stochastic variance can balance the effect of the excess return and variance on the buy boundary, resulting in a less steep boundary compared to that of the Davis & Norman problem. Besides, based on the set of model parameters that we used, stochastic variance completely gains the dominance over the excess mean return on the sell boundary making it downward-sloping. That is, the sell boundary decreases when the variance level is increased. The other case of our interest reflecting a more realistic situation in the market is when there is the negative correlation between the stochastic variance and stock returns. Surprisingly, with this set of model parameters, the presence of negative correlation can barely affect the buy boundary while reduces the relative effect of the variance, causing the sell boundary to be less steep.

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APPENDIX

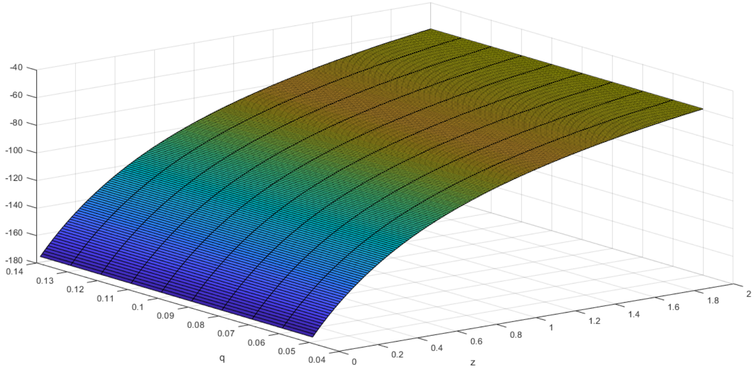

Figure a1: Solution function derived from the fitted neural network for the case with zero correlation.

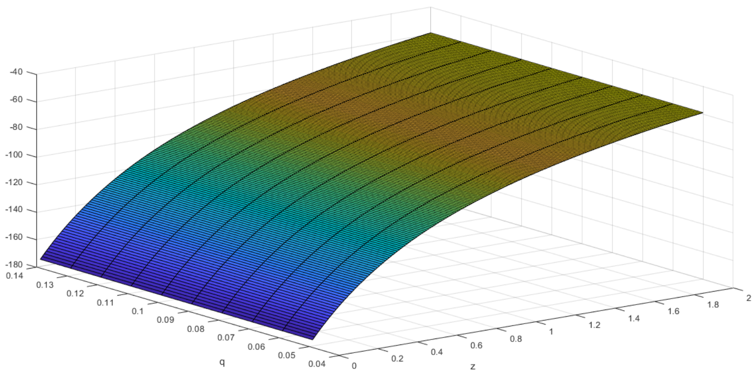

Figure a2: Solution function derived from the fitted neural network for the case with negative correlation ($\rho = -0.9$).

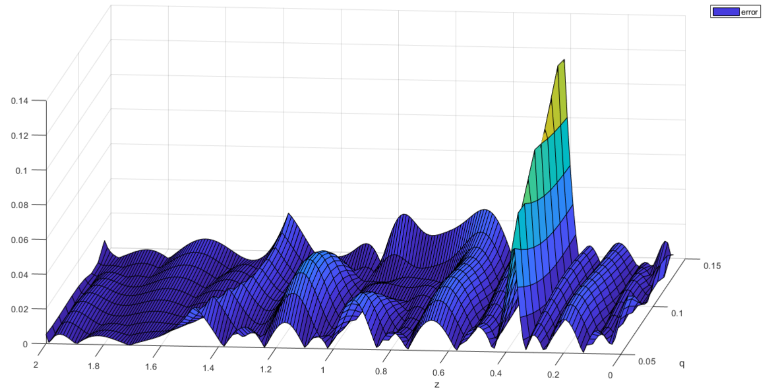

Figure a3: Absolute error over the training space for the case with zero correlation.

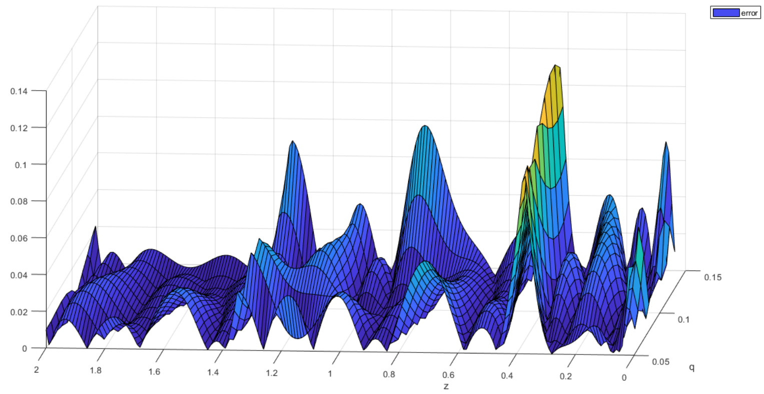

Figure a4: Absolute error over the training space for the case with negative correlation ($\rho = -0.9$).

Table a1: Simulation results of 10-trading-day-variation of the final wealth in stock with the total of 10,000 repetitions when the initial wealth in stock is 1 and variance is varied from 0.04 to 0.16.

		Initial variance						
		0.04	0.06	0.08	0.1	0.12	0.14	0.16
95%	w/o corr	1.0690	1.0857	1.0985	1.1090	1.1200	1.1296	1.1404
	w/ corr	1.0691	1.0833	1.0950	1.1070	1.1168	1.1274	1.1372
90%	w/o corr	1.0536	1.0665	1.0779	1.0846	1.0950	1.1016	1.1084
	w/ corr	1.0544	1.0663	1.0760	1.0825	1.0918	1.0998	1.1092
75%	w/o corr	1.0287	1.0362	1.0412	1.0460	1.0515	1.0546	1.0583
	w/ corr	1.0311	1.0378	1.0416	1.0476	1.0506	1.0566	1.0586
50%	w/o corr	1.0015	1.0034	1.0029	1.0030	1.0050	1.0053	1.0050
	w/ corr	1.0046	1.0043	1.0055	1.0058	1.0051	1.0073	1.0050
25%	w/o corr	0.9749	0.9709	0.9669	0.9617	0.9599	0.9552	0.9530
	w/ corr	0.9769	0.9721	0.9682	0.9640	0.9604	0.9573	0.9529
10%	w/o corr	0.9521	0.9423	0.9339	0.9266	0.9215	0.9151	0.9082
	w/ corr	0.9502	0.9418	0.9347	0.9273	0.9204	0.9142	0.9060
5%	w/o corr	0.9385	0.9251	0.9142	0.9052	0.8994	0.8915	0.8817
	w/ corr	0.9358	0.9236	0.9149	0.9057	0.8973	0.8886	0.8780

Table a2: Simulation results of 20-trading-day-variation of the final wealth in stock with the total of 10,000 repetitions when the initial wealth in stock is 1 and variance is varied from 0.04 to 0.16.

		Initial variance						
		0.04	0.06	0.08	0.1	0.12	0.14	0.16
95%	w/o corr	1.1046	1.1243	1.1421	1.1569	1.1688	1.1886	1.2051
	w/ corr	1.0961	1.1179	1.1338	1.1528	1.1673	1.1811	1.1937
90%	w/o corr	1.0812	1.0963	1.1101	1.1224	1.1283	1.1458	1.1578
	w/ corr	1.0781	1.0942	1.1062	1.1209	1.1299	1.1436	1.1505
75%	w/o corr	1.0415	1.0517	1.0591	1.0665	1.0697	1.0783	1.0840
	w/ corr	1.0453	1.0538	1.0615	1.0667	1.0741	1.0786	1.0836
50%	w/o corr	1.0035	1.0050	1.0059	1.0074	1.0072	1.0081	1.0064
	w/ corr	1.0070	1.0075	1.0090	1.0096	1.0106	1.0117	1.0122
25%	w/o corr	0.9653	0.9591	0.9555	0.9504	0.9469	0.9423	0.9362
	w/ corr	0.9671	0.9609	0.9575	0.9515	0.9471	0.9435	0.9393
10%	w/o corr	0.9323	0.9196	0.9108	0.9033	0.8937	0.8858	0.8798
	w/ corr	0.9323	0.9202	0.9100	0.9012	0.8909	0.8822	0.8799
5%	w/o corr	0.9120	0.8980	0.8840	0.8738	0.8653	0.8544	0.8419
	w/ corr	0.9092	0.8944	0.8821	0.8713	0.8593	0.8473	0.8441

Table a3: Simulation results of 30-trading-day-variation of the final wealth in stock with the total of 10,000 repetitions when the initial wealth in stock is 1 and variance is varied from 0.04 to 0.16.

		Initial variance						
		0.04	0.06	0.08	0.1	0.12	0.14	0.16
95%	w/o corr	1.1297	1.1578	1.1739	1.1937	1.2136	1.2280	1.2508
	w/ corr	1.1221	1.1447	1.1622	1.1797	1.2018	1.2180	1.2320
90%	w/o corr	1.1022	1.1226	1.1354	1.1493	1.1654	1.1734	1.1916
	w/ corr	1.0975	1.1157	1.1322	1.1456	1.1606	1.1727	1.1831
75%	w/o corr	1.0565	1.0653	1.0732	1.0798	1.0887	1.0932	1.1006
	w/ corr	1.0573	1.0656	1.0751	1.0821	1.0907	1.0976	1.1039
50%	w/o corr	1.0059	1.0094	1.0093	1.0074	1.0104	1.0098	1.0112
	w/ corr	1.0104	1.0111	1.0130	1.0136	1.0140	1.0145	1.0166
25%	w/o corr	0.9585	0.9543	0.9481	0.9407	0.9349	0.9323	0.9270
	w/ corr	0.9604	0.9554	0.9509	0.9443	0.9392	0.9355	0.9308
10%	w/o corr	0.9174	0.9064	0.8966	0.8840	0.8741	0.8668	0.8575
	w/ corr	0.9166	0.9046	0.8943	0.8809	0.8721	0.8662	0.8557
5%	w/o corr	0.8952	0.8761	0.8642	0.8509	0.8391	0.8292	0.8206
	w/ corr	0.8880	0.8752	0.8593	0.8454	0.8345	0.8260	0.8153

BIOGRAPHY

Mr. Kittipong Noonoi was born on August 13, 1990 in Prachuap Khiri Khan province, Thailand. He graduated with a Bachelor of Science (Chemistry) degree (First Class Honors, Distinction Program) from the Faculty of Science, Mahidol University, Thailand in 2012. He joined Master Program in Financial Engineering, Faculty of Commerce and Accountancy, Chulalongkorn University, Thailand in 2017.