# การวิเคราะห์ความเป็นปรกติแบบโฮลเดอร์โดยการแปลงเคิร์ฟเล็ตและการแปลงที่คล้ายกัน

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# ANALYSIS OF HÖLDER REGULARITY BY CURVELET AND SIMILAR TRANSFORMS

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วิทยานิพนธ์นี้เราใช้การแปลงริดจ์เล็ตและการแปลงของสมิทเป็นเงื่อนไขจำเป็นและเงื่อนไขเพียงพอที่ทำให้ฟังก์ชันมีเลขชี้กำลังโฮล์เดอร์ชนิดเอกรูปและเฉพาะจุดเป็น $\alpha \in (0,1)$ นอกจากนี้เรายังได้เงื่อนไขจำเป็นที่ทำให้ฟังก์ชันมีความเป็นปรกติแบบโฮลเดอร์อยู่ในรูปการแปลงเคิร์ฟเล็ตของฟังก์ชันนั้น ซึ่งเงื่อนไขเหล่านี้จะอยู่ในรูปขอบเขตของการแปลงที่ขึ้นอยู่กับตัวแปรเสริมส่วนมาตราเช่นเดียวกับที่เคยศึกษาจากการวิเคราะห์ความเป็นปรกติแบบโฮลเดอร์โดยการแปลงเวฟเล็ตมาแล้ว จากการศึกษาการแปลงริดจ์เล็ตใน 2 มิติเราพบว่า อันดับของขอบเขตในเงื่อนไขเพียงพอกับเงื่อนไขจำเป็นแตกต่างกัน 1 ทั้งกรณีเอกรูปและเฉพาะจุด นอกจากนี้เนื่องจากการใช้มาตราเชิงพาราโบลาในการแปลงของสมิททำให้อับดับของขอบเขตในเงื่อนไขเพียงพอกับเงื่อนไขจำเป็นแตกต่างกัน 3/2 ทั้งกรณีเอกรูปและเฉพาะจุด อย่างไรก็ตามการลดลงของขอบเขตการแปลงริดจ์เล็ตของฟังก์ชันที่มีความเป็นปรกติแบบโฮลเดอร์เฉพาะจุดไม่ขึ้นอยู่กับเลขชี้กำลัง $\alpha$ และเนื่องจากมีการใช้ตัวแปรเสริมระบุทิศทางในการแปลงเหล่านี้เราจึงสนใจที่จะวิเคราะห์ฟังก์ชันที่มีความเป็นปรกติระบุทิศทางด้วยการแปลงของฟังก์ชันนั้น เราได้เงื่อนไขจำเป็นที่ทำให้ฟังก์ชันมีความเป็นปรกติระบุทิศทางอยู่ในรูปการแปลงริดจ์เล็ตของฟังก์ชันนั้น

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KEY WORDS : CONTINUOUS WAVELET TRANSFORM/ CONTINUOUS RIDGELET TRANSFORM/ HÖLDER REGULARITY/ DIRECTIONAL REGULARITY

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Using ridgelet transform and Smith wavelet-like transform, new necessary conditions and sufficient conditions for a function to have uniform and pointwise Hölder exponent $\alpha \in (0,1)$ are given. We also obtain necessary conditions for a function to have some Hölder regularity in terms of its continuous curvelet transform. Similar to the characterization of Hölder regularity by continuous wavelet transform, the conditions here are in terms of bounds of the transforms across fine scales. In 2-dimensional ridgelet transform, order of bounds in the sufficient condition and necessary condition differ by 1 in both uniform and pointwise regularity cases. Moreover, due to the parabolic scaling of the Smith transform, orders of bounds in the sufficient condition and necessary condition differ by 3/2 in both uniform and pointwise cases. However, the decay of bound of the ridgelet transform of a function with pointwise Hölder regularity does not depend upon exponent $\alpha$. Because of the directional nature of these transforms, we are also interested in characterizing functionns with directional regularity via its transform. We obtain a necessary condition for a function to have directional regularity in terms of its continuous ridgelet transform across fine scales.


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I would like to thank all of the teachers who have taught me, for their valuable knowledge. Thanks to all of my friends for giving me happiness. Finally, I would like to express my greatly gratitude to my family for their love and encouragement during my study.

# CONTENTS

# CHAPTER I

# INTRODUCTION AND PRELIMINARIES

## 1.1 Introduction

This thesis is concerned with the continuous ridgelet transform, the Smith transform and the continuous curvelet transform which are wavelet-like transforms with directional dilation. We study these transforms of function with Hölder regularity and then analyze the Hölder regularity of function by these transforms. The problem of this kind has been studied in the wavelet transform. In 1989 which found in [1,2,3,4,5], M. Holschneider and P.Tchamitchian gave a result and a proof that the wavelet transform is its ability to characterize the Hölder regularity over interval of functions, gave a necessary and sufficient condition, in condition that the wavelet is continuously differentiable, with real values and compact support. Jaffard shows that one can also estimate the Hölder regularity of function, precisely at a point, which gives a necessary condition and sufficient condition, but not a necessary and sufficient condition, and he supposed that the wavelet type function has $n$ vanishing moments, has continuously differentiable and compact support. In 1998 [6,7], Emmanuel J. Candés has been defined the ridgelet transform with three parameters: scale, location and orientation parameter and then in 2002 [8,9,10,11], Emmanuel J. Candés and David L. Donoho have been constructed the continuous curvelet transform, developed from the continuous ridgelet transform and closely related to a continuous transform used by Hart Smith in his study of Fourier Integral operators. The Smith's transform [8,9,12,13] is based on strict affine parabolic scaling of a single mother wavelet, while for the continuous curvelet transform they discuss gernerating wavelet changes(slightly) scale by scale and affine based on polar parabolic scaling. These transforms are motivated by the need for finding better representations for natural images with edges where several geometric wavelets have been proposed.

These seem like the natural tools for analyzing the directional regularity of function.

In our work, we confine on the continuous ridgelet transform, the Smith's transform and the continuous curvelet transfrom, analyze the Hölder regularity by these transform. Generally speaking the amount of Hölder regularity of a function reflected in its ridgelet, Smith's or curvelet transform by the decrease of their cofficient at small scale. And then we turn our attention to the reciprocal problem. The next section consists of basic definitions, theorems and some interesting properties that will be used in our investigations. The next chapter we represent the continuous wavelet transform and demonstrate the characterization of pointwise and uniform Hölder regularity with the wavelet transform. In chapter III we definitively indicate the continuous ridgelet transform, the Smith transform and the continuous curvelet transfrom and discuss their properties. Final chapter, we analyze the Hölder regularity with these transforms and the directional regularity with the ridgelet transform.

First, we indicate state fundamental definitions, examples, theorems and some interesting properties that will be used in the proceeding chapters.

## 1.2 The $L^p$-spaces

**Definition** If $0 < p < \infty$ and if $f$ is a complex measurable function on $X$, define

$$\|f\| = \left\{ \int_x |f|^p \, d\mu \right\}^{\frac{1}{p}}$$

and let $L^p(\mu)$ consist of all $f$ for which

$$\|f\| < \infty.$$

We call $\|f\|$ the $L^p - norm$ of $f$.

**Theorem 1.1.** *(Hölder inequality) If $p$ and $q$ are conjugate expands, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ $1 \leq p, q \leq \infty$, and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

In this thesis, we shall deal exclusively with Lebesgue measure on $\mathbb{R}^d$ and hence denote the integral of function by the usual integral notation.

**Theorem 1.2.** *(Fubini's theorem). If $\int\int |f(x,y)|\,dydx < \infty$, then*

$$\int\int f(x,y)\,dy\,dx \;=\; \int\left[\int f(x,y)\,dy\right]dx \;=\; \int\left[\int f(x,y)\,dx\right]dy,$$

*i.e., the order of the integrations can be permuted.*

On a given Hilbert space $\mathcal{H}$, we will follow the mathematician's convention and use inner product which is linear in the first argument, i.e.,

$$\langle \lambda_1 u_1 + \lambda_2 u_2, v\rangle = \lambda_1 \langle u_1, v\rangle + \lambda_2 \langle u_2, v\rangle \quad \text{for} \;\; \lambda_1, \lambda_2 \in \mathbb{C} \;\; \text{and} \;\; u_1, u_2, v \in \mathcal{H}.$$

As usual, we have $\langle u, v\rangle = \overline{\langle v, u\rangle}$ where $\overline{\alpha}$ denotes the complex conjugate of $\alpha$, and $\langle u, u\rangle \geq 0$ for all $u \in \mathcal{H}$. We define the norm $\|u\|$ of $u$ by $\|u\|^2 = \langle u, u\rangle$. A standard inequality in a Hilbert space is the Cauchy-Schwarz inequality,

$$|\langle v, w\rangle| \leq \|v\|\,\|w\|$$

for all $v, w \in \mathcal{H}$. A standard example of such a Hilbert space is $L^2(\mathbb{R}^2)$, with

$$\langle f, g\rangle = \int_{\mathbb{R}^2} f(x)\overline{g(x)}\,dx.$$

We will often drop the integration bound when the integral runs over the whole $\mathbb{R}^2$.

## 1.3 Rotation Matrices

Let $v_1 = (x_1, y_1),\;\; v_2 = (x_2, y_2) \in \mathbb{R}^2$. We say that vector $v_1$ is rotated to $v_2$ by an angle $\theta$, with $\theta \in [0, 2\pi)$, if we have the following relationships;

$$\begin{aligned} x_2 &= x_1 \cos\theta - y_1 \sin\theta \\ y_2 &= y_1 \cos\theta + x_1 \sin\theta, \text{ i.e.,} \end{aligned}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

or we can also write $v_2 = R_\theta v_1$ where $R_\theta$ is the $2 \times 2$ matrix on the right of the equation, called the rotation matrix by the angle $\theta$.

Indeed, since $x_1 = r\cos\alpha$ and $y_1 = r\sin\alpha$ for some $\alpha \in [0, 2\pi)$ and $r \in \mathbb{R}$ we have

$$x_2 = r\cos(\alpha+\theta) = r(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = x_1\cos\theta - x_2\sin\theta$$

$$y_2 = r\sin(\alpha+\theta) = r(\sin\alpha\cos\theta + \cos\alpha\sin\theta) = y_1\cos\theta + x_1\sin\theta.$$

## 1.4 Fourier Transform

Let $f \in L^1(\mathbb{R})$. We let

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx, \qquad \xi \in \mathbb{R}$$

be the definition of the Fourier transform of $f$.

The inverse of the Fourier transform of $f$ is defined by

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i \xi x}\, d\xi \quad \text{for all } \ x \in \mathbb{R}.$$

In 2-dimensional space, the Fourier transform becomes

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}^2} f(x)e^{-2\pi i x\cdot\xi}\, dx \\ &= \int_{\mathbb{R}^2} f(x_1, x_2)e^{-2\pi i(x_1\xi_1 + x_2\xi_2)}\, dx_1\, dx_2 \qquad \text{for all } \ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.\end{aligned}$$

And, the inversion formula is

$$\begin{aligned}f(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)e^{2\pi i\xi\cdot x}\, d\xi \\ &= \int_{\mathbb{R}^2} \hat{f}(\xi_1, \xi_2)e^{2\pi i(x_1\xi_1 + x_2\xi_2)}\, d\xi_1\, d\xi_2 \qquad \text{for all } \ x = (x_1, x_2) \in \mathbb{R}^2.\end{aligned}$$

We assume that functions $f$ and $g$ are pointwise continuous and absolutely integrable on the plane. The convolution of $f$ and $g$ is defined and denoted by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy.$$

We then have

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

and

$$\widehat{(fg)}(\xi) = \hat{f}(\xi) * \hat{g}(\xi).$$

## 1.5 The Hölder Regularity

Let $f$ be a real-valued function defined on $\mathbb{R}^d$and $x_0 \in \mathbb{R}^d$. We recall the following definitions of uniform and pointwise Hölder regularity with exponent $\alpha$ where $\alpha > 0$.

**Definition : Uniform Hölder Regularity**

For $\alpha \notin \mathbb{N}$. A locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ is uniform Hölder regular with exponent $\alpha$, denoted by $f \in C^\alpha(\mathbb{R}^d)$, if there exists a constant $C > 0$ for which at any $x_0 \in \mathbb{R}^d$, there is a polynomial $P_{x_0}$ of degree less than $\alpha$ such that, for all $x$ in a neighborhood of $x_0$, $|f(x) - P_{x_0}(x - x_0)| \leq C \|x - x_0\|^\alpha$. (1.3)

**Definition : Pointwise Hölder Regularity**

i. $f : \mathbb{R}^d \to \mathbb{R}$ is pointwise Hölder regular at $x_0$ with exponent $\alpha$, denoted by $f \in C^\alpha(x_0)$, if there exist a constant $C > 0$ and a polynomial $P_{x_0}$ of degree less than $\alpha$ such that, for all $x$ in a neighborhood of $x_0$, $|f(x) - P_{x_0}(x - x_0)| \leq C \|x - x_0\|^\alpha$.

It can be shown that if $f \in C^\alpha(x_0)$, then $f$ belongs to $C^\beta(x_0)$ for any $\beta < \alpha$.

Remark: For $0 < \alpha < 1$, $P_{x_0}(x - x_0) = f(x_0)$. Indeed if $\alpha < 1$ then $P \equiv$ *constant*. If $x = x_0$ we have $|f(x_0) - P_{x_0}(x_0 - x_0)| \leq C \|x_0 - x_0\|^\alpha = 0$, then $P_{x_0}(x - x_0) = f(x_0)$. Thus, for $0 < \alpha < 1$, the inequality (1.3) becomes $|f(x) - f(x_0)| \leq C \|x - x_0\|^\alpha$.

ii. The pointwise Hölder exponent, $h_f(x_0)$, of $f$ at $x_0$ is defined by

$$h_f(x_0) = \sup \{\alpha : f \in C^\alpha(x_0)\},$$

where we say that $h_f(x_0) = 0$ if $f$ is not in $C^\alpha(x_0)$ for any $\alpha > 0$. That is, we are taking the supremum of the set of $\alpha > 0$ for which $f \in C^\alpha(x_0)$ as a subset of $[0, \infty)$. Note that the function $h_f(x_0)$ takes value in $[0, \infty]$. This exponent measures the pointwise regularity of the function $f$ at $x_0$. The larger the exponent $h_f(x_0)$ is, the 'smoother' the function $f$ is at the point $x_0$.

Note that (1) If $f$ is Hölder continuous with exponent $\alpha$ then for each $\beta < \alpha$, $f$ is Hölder continuous with exponent $\beta$.

(2) The uniform Hölder exponent of a function need not be the infimum of the pointwise Hölder exponents. For example the function $f(x) = x\sin\left(\frac{1}{x}\right)$ is $C^1$ at the origin and $C^\infty$ elsewhere, while its uniform Hölder exponent is only $1/2$. The following example demonstrates that an irregular function, in this case a continuous nowhere-differentiable function, might not be a multifractal function.

**Example** It is well known, see also [2,3,4], that the Weiertrass function

$$\mathcal{W}_\alpha(x) = \sum_{j=0}^{\infty} \frac{\sin(2^j x)}{2^{j\alpha}}$$

is continuous but nowhere differentiable for $\alpha \in (0,1)$. We shall prove that the Hölder exponent of the Weierstrass function is $\alpha$ everywhere. That is, we show that for all $\beta < \alpha < \gamma$, $\mathcal{W}_\alpha \in C^\beta(x_0)$ but $\mathcal{W}_\alpha \notin C^\gamma(x_0)$ at a point $x_0 \in \mathbb{R}$.

*Proof.* We shall only show that $\mathcal{W}_\alpha \in C^\beta(x_0)$ for all $\beta < \alpha$ and for all $x_0 \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ and $\beta < \alpha$. By the mean value theorem, $|\sin(2^j x) - \sin(2^j x_0)| \leq 2^j|x - x_0|$. But from the boundedness of the sine function, we also have 2 as another upper bound of the left hand side. So we will have to choose better bounds for each $j$. Suppose that $|x - x_0| < 1$. Then $\frac{1}{2^{n+1}} \leq |x - x_0| < \frac{1}{2^n}$ for some $n \geq 0$.

$$|\mathcal{W}_\alpha(x) - \mathcal{W}_\alpha(x_0)| \leq \sum_{2^j 2^{-n} \leq 2} \frac{|\sin(2^j x) - \sin(2^j x_0)|}{2^{j\alpha}} + \sum_{2^j 2^{-n} > 2} \frac{|\sin(2^j x) - \sin(2^j x_0)|}{2^{j\alpha}}.$$

By the mean value theorem and boundedness of sine function, we have

$$
\begin{aligned}
|\mathcal{W}_\alpha(x) - \mathcal{W}_\alpha(x_0)| &\le \sum_{2^j 2^{-n} \le 2} \frac{2^j |x - x_0|}{2^{j\alpha}} + \sum_{2^j 2^{-n} > 2} \frac{|\sin(2^j x) - \sin(2^j x_0)|}{2^{j\alpha}} \\
&\le |x - x_0| \sum_{j \le n+1} 2^{j(1-\alpha)} + \sum_{j > n+1} \frac{2}{2^{j\alpha}} \\
&\le |x - x_0| \left( \frac{2^{(n+2)(1-\alpha)} - 2^{1-\alpha}}{2^{1-\alpha} - 1} \right) + 2 \left( \frac{2^{-\alpha(n+2)}}{1 - 2^{-\alpha}} \right) \\
&\le \frac{|x - x_0|^\beta}{2^{n(1-\beta)}} \left( \frac{2^{(n+1)(1-\alpha)} - 2^{1-\alpha}}{2^{1-\alpha} - 1} \right) + \frac{2}{1 - 2^{-\alpha}} \left( \frac{1}{2} |x - x_0| \right)^\alpha \\
&\le |x - x_0|^\beta \left[ \frac{2^{1-\alpha}}{2^{1-\alpha} - 1} \left( \frac{2^{1-\alpha}}{2^{n(\alpha-\beta)}} - \frac{1}{2^{n(\alpha-\beta)}} \right) + \frac{2}{2^\alpha - 1} \frac{1}{2^{n(\alpha-\beta)}} \right] \\
&\le C_{\beta,\alpha} |x - x_0|^\beta.
\end{aligned}
$$

□

# CHAPTER II

# CONTINUOUS WAVELET TRANSFORM AND CHARACTERIZATION OF HÖLDER REGULARITY

A concise exposition of the theory of characterization of the uniform and pointwise Hölder regularity (irregularity) of functions via the continuous wavelet transforms is given, see also [1,2,3,4,5]. Section 2.1 introduces the definitions and theorems of the continuous wavelet transforms. Some characterizations of the uniform(global) Hölder regularity by the wavelet transform are then listed and proved in section 2.2. The last section, gives and proves some characterization of the pointwise(local) Hölder regularity.

## 2.1 Continuous Wavelet Transform

We will not give an exhaustive definition of what we will call a wavelet, since in the literature the term wavelet is used for various kinds of functions depending on the application. However, to fix the ideas, we shall consider a locally integrable complex-valued function $\psi \in L^2$, which is in general well localized and regular in the sense that

$$|\psi(t)| + |\psi'(t)| \leq \frac{C}{1+|t|^{2+\epsilon}} \quad \text{for some fixed constant } C > 0, \quad \epsilon > 0.$$

In addition, suppose that the first two moments of $\psi$ vanish:

$$\int_{-\infty}^{\infty} \psi(t)\, dt = \int_{-\infty}^{\infty} \psi(t) t\, dt = 0.$$

These two conditions are maximal in the sense that all theorems stated in this chapter hold for those functions taken as wavelets. Depending on the problem we shall be able to relax these conditions considerably. Since some moments of $\psi$ vanish it necessarily has some oscillations, which justify the term wavelet. Dilating and translating the wavelet $\psi$

we obtain a parameter family of functions

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right).$$

The parameter $b \in \mathbb{R}$ is a position(translation) parameter, where as $a > 0$ may be interpreted as a scale parameter. We can define the wavelet transform of an arbitrary function $f \in L^2(\mathbb{R})$ with respect to a wavelet $\psi$ as follows.

**Definition** (Continuous Wavelet Transform). The continuous wavelet transform of an $L^2(\mathbb{R})$ function $f$ is defined by

$$\begin{aligned} W_f(a,b) &= \int f(x)\overline{\psi_{a,b}(x)}\,dx \\ &= \frac{1}{\sqrt{a}}\int f(x)\overline{\psi\left(\frac{x-b}{a}\right)}\,dx \end{aligned}$$

where this Lebesgue integral is well-defined for $a \in (0,\infty)$ and $b \in \mathbb{R}$.

Note that $|W_f(a,b)| \leq \|f\|_2\,\|\psi\|_2$, and it also should be noted that the continous wavelet transform maps $L^2(\mathbb{R})$ function into $L^2(\mathbb{R}^+ \times \mathbb{R}, a^{-2}da\,db)$.

**Definition** (Admissibility Condition). A function $\psi$ is said to be admissible if

$$0 < C_\psi = \int_0^\infty |\hat{\psi}(a)|^2 \frac{da}{a} < \infty.$$

Observe that for any $\xi \in \mathbb{R}$,

$$\int_0^\infty |\hat{\psi}(a\xi)|^2 \frac{da}{a} = \int_0^\infty |\hat{\psi}(a)|^2 \frac{da}{a}.$$

In general, $W_f$ is a smooth function over the position-scale half-plane. Analyzing a function with the help of the wavelet transform amounts to analyzing it on different length scales around arbitrary positions. This transform is a sort of mathematical microscope, where $\frac{1}{a}$ is the enlargement and $b$ is its position over the function to be analyzed.

The wavelet transform is invertible. An explicit inversion formula is given by the following theorem.

**Theorem 2.1.** *(Inversion Formula)*

*Let $\psi \in L^2(\mathbb{R})$ be admissible. If $f \in L^1(\mathbb{R} \cap L^2(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then for each $x \in \mathbb{R}$, we have the inversion formula*

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{\mathbb{R}} W_f(a,b)\psi_{ab}(x)\, db \frac{da}{a^2}$$

*and the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2\, dx = \frac{1}{C_\psi} \int_0^\infty \int_{\mathbb{R}} |W_f(a,b)|^2\, db \frac{da}{a^2}.$$

Let $\alpha > 0$ and $k = [\alpha]$. Here and below we choose a wavelet-type function $\psi$ satisfying the following smoothness, decaying and oscillating properties

$$|\psi^{(i)}(x)| \leq C(1+|x|)^{-k-2} \quad \text{for} \;\; i = 0, ..., k+1,$$

$$\int_{\mathbb{R}} x^j \psi(x)\, dx = 0 \quad \text{for} \;\; j = 0, ..., k,$$

and

$$\int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi}\, d\xi = 1 \quad \text{with} \;\; \hat{\psi}(\xi) = 0 \;\; \text{if} \;\; \xi < 0.$$

We use this wavelet-type function $\psi$ to analyze uniform and pointwise Hölder regularity by the wavelet transform in the next two sections.

## 2.2 Wavelet Transform Analysis of Uniform(Global) Hölder Regularity

In this section we recall how to analyze uniform Hölder regularity by the wavelet transform. Generally speaking the amount of uniform regularity of function is reflected in the decrease of its wavelet transform at small scale as shown by the following well known theorem which gives a necessary and sufficient condition.

**Theorem 2.2.** *A bounded function $f \in L^2(\mathbb{R})$ is Hölder continuous with exponent $\alpha$, $0 < \alpha < 1$ if and only if its wavelet transform with respect to a compactly supported wavelet-type function $\psi$ satisfies $|W_f(a,b)| \leq Ca^{\alpha+\frac{1}{2}}$ for some constant $C > 0$, for all $(a,b) \in (0,\infty) \times \mathbb{R}$.*

*Proof.* Because of the oscillating property, $\int_{\mathbb{R}} \psi(y)\,dy = 0$, we have $\int_{\mathbb{R}} \overline{\psi\left(\frac{x-b}{a}\right)} f(b)\,dx = 0$. We apply the uniform Hölder regularity of $f$ and obtain

$$\begin{aligned}
|W_f(a,b)| &= \left| \frac{1}{\sqrt{a}} \int_{\mathbb{R}} f(x) \overline{\psi\left(\frac{x-b}{a}\right)}\,dx \right| \\
&= \left| \frac{1}{\sqrt{a}} \int_{\mathbb{R}} (f(x)-f(b)) \overline{\psi\left(\frac{x-b}{a}\right)}\,dx \right| \\
&\leq \frac{1}{\sqrt{a}} \int_{\mathbb{R}} |f(x)-f(b)| \left| \overline{\psi\left(\frac{x-b}{a}\right)} \right| dx \\
&\leq \frac{1}{\sqrt{a}} \int_{\mathbb{R}} C\,|x-b|^{\alpha} \left| \psi\left(\frac{x-b}{a}\right) \right| dx, \quad \text{since} \;\; f \in C^{\alpha}(\mathbb{R}), \\
&= C\frac{1}{\sqrt{a}} \int_{\mathbb{R}} a^{\alpha+1}\,|y|^{\alpha}\,|\psi\,(y)|\;dy \\
&\leq C a^{\alpha+\frac{1}{2}} \int_{\mathbb{R}} C' \frac{|y|^{\alpha}}{(1+|y|)^2}\,dy, \quad \text{by the decaying property,} \\
&\leq C'' a^{\alpha+\frac{1}{2}}, \quad \text{since} \;\; 0<\alpha<1.
\end{aligned}$$

Conversely, by the inversion formula we have

$$f(x) = \frac{1}{C_{\psi}} \int_0^{\infty} \int_{-\infty}^{\infty} W_f(a,b)\psi_{a,b}(x)\,db\,\frac{da}{a^2} \quad \text{for all} \;\; x \in \mathbb{R}.$$

Now, $C_{\psi} = \int \frac{|\hat{\psi}(\xi)|^2}{\xi}\,d\xi = 1$. Let $0<\alpha<1$. We will break the interval of integration over $a$ into parts, $a \leq 1$ and $a \geq 1$, and call the respective integrals $f_{SS}$ (small scales) and $f_{LS}$ (large scales).

Let $x \in \mathbb{R}$. First of all, note that $f_{LS}$ is bounded unifromly in $x$ as a simple change of variable and integrability of $\psi$ yield

$$\begin{aligned}
|f_{LS}(x)| &\leq \int_{a\geq 1} \int_{-\infty}^{\infty} |W_f(a,b)|\,|\psi_{a,b}(x)|\;db\,\frac{da}{a^2} \\
&\leq \int_{a\geq 1} \int_{-\infty}^{\infty} \|f\|_2\,\|\psi\|_2\,|\psi_{a,b}(x)|\;db\,\frac{da}{a^2} \\
&= C \int_{a\geq 1} a^{-\frac{1}{2}+1-2}\,\|\psi\|_1\;da \\
&\leq C' < \infty.
\end{aligned}$$

Next, let $h \in \mathbb{R}$ be such that $|h| \le 1$. Then

$$\begin{aligned}
|f_{LS}(x+h) - f_{LS}(x)| &\le \int_{a\ge 1}\int_{-\infty}^{\infty} |W_f(a,b)|\,|\psi_{a,b}(x+h) - \psi_{a,b}(x)|\;db\,\frac{da}{a^2} \\
&\le \int_{a\ge 1} a^{-3}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} f(y)\psi\left(\frac{y-b}{a}\right)\,dy\right| \\
&\quad \left|\psi\left(\frac{x+h-b}{a}\right) - \psi\left(\frac{x-b}{a}\right)\right|\,db\,da \qquad (1)
\end{aligned}$$

Since $\psi$ is differentiable everywhere so there is a constant $C > 0$ such that $|\psi(z+t) - \psi(z)| \le C|t|$ and, also, since $\operatorname{supp}(\psi) \subseteq [-R, R]$ for some $R < \infty$, we can bound (1) by

$$\begin{aligned}
(1) &\le C\int_{a\ge 1} a^{-3}\int_{|x-b|\le aR+1}\frac{|h|}{a}\left(\int_{|y-b|\le aR}\left|\psi\left(\frac{y-b}{a}\right)\right||f(y)|\,dy\right)db\,da \\
&\le C|h|\int_{a\ge 1} a^{-4}\int_{|y-b|\le aR}|f(y)|\left(\int_{|x-b|\le aR+1}\left|\psi\left(\frac{y-b}{a}\right)\right|\,db\right)dy\,da \\
&\le C'|h|\int_{a\ge 1} a^{-3}\left(\int_{|y-x|\le 2aR+1}|f(y)|\,dy\right)da \\
&\le C'|h|\,\|f\|_2\int_{a\ge 1} a^{-3}\,(4aR+2)^{\frac{1}{2}}\;da \\
&\le C''|h|.
\end{aligned}$$

This holds for all $|h| < 1$. This, together with the uniform boundedness of $f_{LS}$, implies that $|f_{LS}(x+h) - f_{LS}(x)| \le C|h|^{\alpha}$ for all $h \in \mathbb{R}$, uniformly in $x \in \mathbb{R}$. The small scale part $f_{SS}$ is also uniformly bounded as a simple change of variable and integrability of $\psi$ yield

$$\begin{aligned}
|f_{SS}(x)| &= \left|\int_{a\le 1}\int_{-\infty}^{\infty} W_f(a,b)\psi_{a,b}(x)\,db\,\frac{da}{a^2}\right| \\
&\le \int_{a\le 1}\int_{-\infty}^{\infty}|W_f(a,b)|\,|\psi_{a,b}(x)|\;db\,\frac{da}{a^2} \\
&\le C\int_{a\le 1}\int_{-\infty}^{\infty} a^{\alpha+\frac{1}{2}}a^{-\frac{1}{2}}\left|\psi\left(\frac{x-b}{a}\right)\right|\,db\,\frac{da}{a^2} \\
&\le C\int_{a\le 1} a^{\alpha-2+1}\int_{-\infty}^{\infty}|\psi(y)|\;dy\,da \\
&= C'\int_{a\le 1} a^{\alpha-1}\,da \\
&= C'' \quad < \quad \infty
\end{aligned}$$

We therefore again only have to bound $|f_{SS}(x+h) - f_{SS}(x)|$ for small $h$, say $|h| \leq 1$.By the assumtion, we have

$$\begin{aligned}
|f_{SS}(x+h) - f_{SS}(x)| &= \left| \int_{a\leq 1} \int_{-\infty}^{\infty} W_f(a,b) \left(\psi_{a,b}(x+h) - \psi_{a,b}(x)\right) \, db \, \frac{da}{a^2} \right| \\
&\leq \int_{a\leq 1} \int_{-\infty}^{\infty} |W_f(a,b)| \, |\psi_{a,b}(x+h) - \psi_{a,b}(x)| \, db \, \frac{da}{a^2} \\
&\leq C \int_{a\leq 1} \int_{-\infty}^{\infty} a^{\alpha+\frac{1}{2}} \, |\psi_{a,b}(x+h) - \psi_{a,b}(x)| \, db \, \frac{da}{a^2}. \qquad (2)
\end{aligned}$$

We split the integral into fine and coarse scale ranges and using again $|\psi(z+t) - \psi(z)| \leq C|t|$

$$\begin{aligned}
(2) &\leq C \int_{a\leq |h|} \int_{-\infty}^{\infty} a^{\alpha+\frac{1}{2}-2-\frac{1}{2}} \left( \left| \psi\left(\frac{x+h-b}{a}\right) \right| + \left| \psi\left(\frac{x-b}{a}\right) \right| \right) db \, da \\
&\quad + C \int_{|h|\leq a\leq 1} \int_{-\infty}^{\infty} a^{\alpha+\frac{1}{2}-\frac{1}{2}-2} \left| \psi\left(\frac{x+h-b}{a}\right) - \psi\left(\frac{x-b}{a}\right) \right| db \, da \\
&\leq C \int_{a\leq |h|} a^{\alpha-2+1} \left( \int_{-\infty}^{\infty} \psi(y) \, dy \right) da \\
&\quad + C' \int_{|h|\leq a\leq 1} a^{\alpha-2} \left( \int_{|x-b|\leq aR+|h|} \left| \frac{h}{a} \right| db \right) da \quad \text{since } \operatorname{supp}(\psi) \subseteq [-R, R] \\
&= C'' \int_{a\leq |h|} a^{\alpha-1} \, da + C'|h| \int_{|h|\leq a\leq 1} a^{\alpha-3}(aR + |h|) \, da \leq C'''|h|^{\alpha}.
\end{aligned}$$

This holds for all $|h| \leq 1$, which, together with the proven fact that $f_{SS}$ is bounded uniformly, implies that $|f_{SS}(x+h) - f_{SS}(x)| \leq C|h|^{\alpha}$ for all $h$ and uniformly in $x$. It follows that $f$ is Hölder continuous with exponent $\alpha$. □

It already contains, in a very simple form, main ingredients of the proofs in the subsequent theorem.

## 2.3 Wavelet Transform Analysis of Pointwise(Local) Hölder Regularity

Above theorems gives a characterization of the Hölder regularity over $\mathbb{R}$ but not at a point. The next theorem proved by Jaffard [1,2,3,4,5] show that one can also estimate the Hölder regularity of a function precisely at a point. The theorems give a necessary condition and a sufficient condition, but not a necessary and sufficient condition. We still assume that $\psi$ satisfies the smoothness, decaying and oscillating properties in page 9.

**Theorem 2.3.** *If a bounded function $f$ is Hölder continuous at $x_0$ with exponent $\alpha \in (0,1)$, then $|W_f(a, b+x_0)| \leq Ca^{\frac{1}{2}}\left(a^\alpha + |b|^\alpha\right)$.*

*Proof.* By translating everything of translation parameter $b$ of the continuous wavelet transform, we can assume that $x_0 = 0$.

Because $\int \psi(x)\, dx = 0$, we have $\int \psi_{a,b}(x) f(0)\, dx = 0$. We obtain

$$|W_f(a,b)| \leq \int |\psi_{a,b}(x)||f(x) - f(0)|\, dx.$$

Since $f$ is Hölder continuous at 0 with exponent $\alpha$, it follows that

$$\begin{aligned}
|W_f(a,b)| &\leq C \int |x - 0|^\alpha a^{-\frac{1}{2}} \left|\psi\left(\frac{x-b}{a}\right)\right| dx \\
&\leq C a^{\alpha+\frac{1}{2}} \int \left|y + \frac{b}{a}\right|^\alpha |\psi(y)|\, dy \\
&\leq C a^{\alpha+\frac{1}{2}} \left(\int |y|^\alpha |\psi(y)|\, dy + \int \left|\frac{b}{a}\right|^\alpha |\psi(y)|\, dy\right).
\end{aligned}$$

As a result of the decay condition on $\psi$ and its integrability, the last two integrals are finite, and hence

$$\begin{aligned}
|W_f(a,b)| &\leq C' a^{\alpha+\frac{1}{2}} \left(1 + \left|\frac{b}{a}\right|^\alpha\right) \\
&= C' a^{\frac{1}{2}} \left(a^\alpha + |b|^\alpha\right).
\end{aligned}$$

□

We now turn our attention to the reciprocal proplem. The following theorem is similar to a theorem proved before by S.Jaffard.

**Theorem 2.4.** *Suppose that $\psi$ is compactly supported, and $f \in L^2(\mathbb{R})$ is bounded and continuous. If, for some $\beta > 0$ and $\alpha \in (0,1)$, there exists a constant $C$ such that*

$$|W_f(a,b)| \leq C a^{\beta+\frac{1}{2}}$$

*and*

$$|W_f(a, b+x_0)| \leq C a^{\frac{1}{2}} \left(a^\alpha + \frac{|b|^\alpha}{|\log|b||}\right) \quad \text{for all } a, b.$$

*then $f$ is Hölder continuous at $x_0$ with exponent $\alpha$.*

*Proof.* We will split the integral over $a$ into two parts, $a \le 1$ and $a \ge 1$, and call the two terms $f_{SS}$ (small scale) and $f_{LS}$ (large scale). Clearly, the large scale part $f_{LS}$ is always regular by the same arguments as in the proof of Theorem 2.2.

Let $x \in \mathbb{R}$. We start by proving that $f_{SS}$ is bounded at $x_0$ as a simple change of variable and integrability of $\psi$ yield

$$\begin{aligned}
|f_{SS}(x_0)| &\le \int_{a\le 1}\int_{-\infty}^{\infty} |W_f(a,b)|\,|\psi_{a,b}(x_0)|\; db\,\frac{da}{a^2} \\
&\le C\int_{a\le 1}\int_{-\infty}^{\infty} a^{\beta+\frac{1}{2}}a^{-\frac{1}{2}}\left|\psi\left(\frac{x_0-b}{a}\right)\right|\; db\,\frac{da}{a^2} \\
&\le C\int_{a\le 1} a^{\beta-2+1}\int_{-\infty}^{\infty}|\psi(y)|\; dy\,da \\
&= C'\int_{a\le 1} a^{\beta-1}\,da \\
&= C'' \quad < \quad \infty.
\end{aligned}$$

We therefore only have bound $|f_{SS}(x_0+h)-f_{SS}(x_0)|$ for small $h$, i.e. $|h| \le 1$. By translating everything, we can assume $x_0 = 0$. We split the integral into three ranges of the scale $a$ and get

$$\begin{aligned}
|f_{SS}(h)-f_{SS}(0)| &\le \int_0^1\int_{-\infty}^{\infty}|W_f(a,b)||\psi_{a,b}(h)-\psi_{a,b}(0)|\,db\,\frac{da}{a^2} \\
&\le \int_{0\le a\le |h|^{\frac{\alpha}{\beta}}}\int_{-\infty}^{\infty}|W_f(a,b)|\,(|\psi_{a,b}(h)|+|\psi_{a,b}(0)|)\; db\,\frac{da}{a^2} \\
&\quad + \int_{|h|^{\frac{\alpha}{\beta}}\le a\le |h|}\int_{-\infty}^{\infty}|W_f(a,b)|\,(|\psi_{a,b}(h)|+|\psi_{a,b}(0)|)\; db\,\frac{da}{a^2} \\
&\quad + \int_{|h|\le a\le 1}\int_{-\infty}^{\infty}|W_f(a,b)||\psi_{a,b}(h)-\psi_{a,b}(0)|\,db\,\frac{da}{a^2}
\end{aligned}$$

Using the asumption,

$$\begin{aligned}
|f_{SS}(h)-f_{SS}(0)| \le\; & C\int_{0\le a\le |h|^{\frac{\alpha}{\beta}}}\int_{-\infty}^{\infty} a^{\beta}\left|\psi\left(\frac{h-b}{a}\right)\right|\; db\,\frac{da}{a^2} \\
&+ C\int_{|h|^{\frac{\alpha}{\beta}}\le a\le |h|}\int_{-\infty}^{\infty}\left(a^{\alpha}+\frac{|b|^{\alpha}}{|\log|b||}\right)\left|\psi\left(\frac{h-b}{a}\right)\right|\; db\,\frac{da}{a^2} \\
&+ C\int_{a\le |h|}\int_{-\infty}^{\infty}\left(a^{\alpha}+\frac{|b|^{\alpha}}{|\log|b||}\right)\left|\psi\left(\frac{-b}{a}\right)\right|\; db\,\frac{da}{a^2} \\
&+ C\int_{|h|\le a\le 1}\int_{-\infty}^{\infty}\left(a^{\alpha}+\frac{|b|^{\alpha}}{|\log|b||}\right)\left|\psi\left(\frac{h-b}{a}\right)-\psi\left(\frac{-b}{a}\right)\right|\; db\,\frac{da}{a^2},
\end{aligned}$$

where we assume without loss of generality that $\alpha > \beta$. Let us denote the four terms on the right-hand side of inequality by $T_1$, $T_2$, $T_3$ and $T_4$, respectively. After a change of variable, integrability of $\psi$ yields

$$\begin{aligned}
T_1 &= C\int_{0\le a\le |h|^{\frac{\alpha}{\beta}}}\int_{-\infty}^{\infty} a^{\beta}\left|\psi\left(\frac{h-b}{a}\right)\right| db\,\frac{da}{a^2}\\
&\le C\int_{0\le a\le |h|^{\frac{\alpha}{\beta}}} a^{\beta-2}\left(\int_{-\infty}^{\infty}|\psi(y)|a\,dy\right) da\\
&\le \int_0^{|h|^{\frac{\alpha}{\beta}}} a^{\beta-1}||\psi||_1\,da\\
&\le C|h|^{\alpha}.
\end{aligned}$$

By assumption,

$$\begin{aligned}
T_2 &= C\int_{|h|^{\frac{\alpha}{\beta}}\le a\le |h|}\int_{-\infty}^{\infty}\left(a^{\alpha}+\frac{|b|^{\alpha}}{|\log|b||}\right)\left|\psi\left(\frac{h-b}{a}\right)\right| db\,\frac{da}{a^2}\\
&\le C\int_{0\le a\le |h|} a^{\alpha-2}\left(\int_{-\infty}^{\infty}|\psi(y)|a\,dy\right) da\\
&\quad + C\int_{|h|^{\frac{\alpha}{\beta}}\le a\le |h|} a^{-2}\int_{-\infty}^{\infty}\frac{|b|^{\alpha}}{|\log|b||}\left|\psi\left(\frac{h-b}{a}\right)\right| db\,da
\end{aligned}$$

For the second term, since $\text{supp}(\psi)\subseteq[-R,R]$ for some $0<R<\infty$ we can bound the two integrals in $T_2$ as follows

$$\begin{aligned}
T_2 &\le C\int_{0\le a\le |h|} a^{\alpha-1}||\psi||_1\,da\\
&\quad + C\int_{|h|^{\frac{\alpha}{\beta}}\le a\le |h|} a^{-1}||\psi||_1\frac{(aR+|h|)^{\alpha}}{|\log(aR+|h|)|}\,da
\end{aligned}$$

Since we integrate over $a\le |h|$, for sufficiently small $|h|\le 1$,

$$\begin{aligned}
T_2 &\le C'|h|^{\alpha}\left[1+\frac{1}{|\log|h||}\int_{|h|^{\frac{\alpha}{\beta}}\le a\le |h|} a^{-1}\,da\right]\\
&\le C'|h|^{\alpha}.
\end{aligned}$$

Similarly, for sufficiently small $|h|\le 1$,

$$\begin{aligned}
T_3 &= C\int_{a\le |h|}\int_{-\infty}^{\infty}\left(a^{\alpha}+\frac{|b|^{\alpha}}{|\log|b||}\right)\left|\psi\left(\frac{-b}{a}\right)\right| db\,\frac{da}{a^2}\\
&\le \int_{a\le |h|} a^{\alpha-1}||\psi||_1\,da + \int_{a\le |h|} a^{-1}||\psi||_1\frac{(aR)^{\alpha}}{|\log(aR)|}\,da\\
&\le C|h|^{\alpha}.
\end{aligned}$$

Finally, we use the properties of $\psi$ that it has compact support and bounded derivative to obtain the following bound of $T_4$:

$$\begin{aligned}
T_4 &= C\int_{|h|\le a\le 1}\int_{-\infty}^{\infty}\left(a^\alpha+\frac{|b|^\alpha}{|\log|b||}\right)\left|\psi\left(\frac{h-b}{a}\right)-\psi\left(\frac{-b}{a}\right)\right| db\,\frac{da}{a^2}\\
&\le C|h|\int_{|h|\le a\le 1} a^{-3}\left[a^\alpha+\frac{(aR+|h|)^\alpha}{|\log(aR+|h|)|}\right](aR+|h|)\,da\\
&\le C'|h|\left[1+|h|^{-1+\alpha}+|h|(1+|h|^{\alpha-2})\right]\\
&\le C''|h|^\alpha.
\end{aligned}$$

Thus $|f_{SS}(h)-f_{SS}(0)|\le C|h|^\alpha$ for all $|h|\le 1$, which, together with the bound of $f_{SS}$, implies that $|f_{SS}(h)-f_{SS}(0)|\le C|h|^\alpha$ for all $h$. Hence $f$ is Hölder continuous at $x_0=0$ with exponent $\alpha$ as desired. Therefore by translating everything we can conclude that $f$ is Hölder regularity at $x_0$ with exponent $\alpha$. □

It is well known that the Weierstrass function $\mathcal{W}_\alpha(x)=\displaystyle\sum_{j=1}^{\infty}\frac{\sin(2^j x)}{2^{j\alpha}}$, $\alpha\in(0,1)$ has Hölder continuous exponent $\alpha$ everywhere, this can be proved by the continuous wavelet transform.

# CHAPTER III

# CONTINUOUS CURVELET AND SIMILAR TRANSFORMS

Energized by the success of wavelets, the last two decades saw the rapid develop new field, computational harmonic analysis, which aims to develop new systems for effecitively representing phenomena of scientific interest. The curvelet transform is a recent addition to the family of mathematical tools this community enthusiastically builds up. In short, this is a new multiscale transform with strong directional character in which elements are highly anisotropic at fine scale, with effective support shaped according to the parabolic scaling principle length$^2 \sim$ width. In this chapter we construct a continuous curvelet transform, projects $f \in \mathbb{R}^2$ onto a curvelet $\gamma_{a,\bar{b},\theta}$, yielding coefficient $\Gamma_f(a, \bar{b}, \theta) = \langle f, \gamma_{a,\bar{b},\theta} \rangle$; with parameter space indexed by scale $a > 0$, location $\bar{b} \in \mathbb{R}^2$ and orientation $\theta \in [0, 2\pi)$. The corresponding curvelet $\gamma_{a,\bar{b},\theta}$ is defined by parabolic dilation in polar frequency(Fourier) domain coordinates. The continuous cuvelet transform is developed from the ridgelet transform and it is also closely related to a continuous transform introduced by Hart Smith in his study of Fourier integral operators. So we also study the continuous ridgelet transform and the Smith transform. The ridgelet transform is a wavelet-like transform with directional dilation, while the curvelet transform and the Smith transform is a wavelet-like transform with directional parabolic dilation. The Smith transform is based on true affine palabolic scaling of a single mother wavelet, while the continuous cuvelet transform can only be viewed as affine palabolic scaling in Euclidean coordinate by taking a slightly different mother wavelets at each scale. The geometry of a curvelet is now apparent: if the function $\gamma$ is supported near the unit square, we see that the envelope of $\gamma_{a,\bar{b},\theta}$ is supported near an $a$ by $\sqrt{a}$ rectangle with minor axis pointing in the direction $\theta$. An important property is possibility to analyze and reconstruct an arbitrary function $f$ as a superposition of such templates.

**contents:** Section 3.1 we recall that the continuous ridgelet transform provides a reproducing formula and Parseval relation and Section 3.2 then construct a continuous curvelet transform based on a polar parabolic scaling and provides a Calderón reproducing formula and the Parseval relation. Section 3.3 discusses our reformulation of Smith transform based on true palabolic scaling. The last section discusses some properties of continuous curvelet transfrom.

## 3.1 The Continuous Ridgelet Transform

The success of wavelets is mainly due to the good performance for piecewise smooth functions in 1-dimension. Unfortunately, such is not the case in 2-dimension. To overcome the weakness of wavelets in higher dimensions, Candés and Donoho recently pioneered a new system of representations named ridgelets which deals effectively with line singularities in 2-dimensional space. The idea is to map a line singularity into a point singularity using the Radon transform. Then, the wavelet transform can be used to effectively handle the point singularity in the Radon domain. Their initial proposal was intended for functions defined in the continuous $\mathbb{R}^d, d > 1$ space. We start by briefly reviewing the ridgelet transform, see [6,7], showing its connections with other transform in the domain and then present a reproducing formula and Parseval relation. Now, we have introduced the parameter space

$$\Gamma = \left\{(a, \bar{u}, b); a, b \in \mathbb{R}, a > 0, \bar{u} \in S^{d-1}\right\}$$

with $S^{d-1}$ denoting the unit sphere in $\mathbb{R}^d$.

For each $a > 0$, $b \in \mathbb{R}$ and $\bar{u} \in S^{d-1}$, we define the bivariate ridgelet $\psi_{a,b,\bar{u}}$ by

$$\psi_{a,b,\bar{u}}(\bar{x}) = \frac{1}{\sqrt{a}}\psi\left(\frac{\bar{x}\cdot\bar{u} - b}{a}\right) \quad \text{for all } \bar{x} \in \mathbb{R}^d$$

where the ridgelet parameter $(a, \bar{u}, b)$ has a natural interpretation; $a$ indexes the scale of the ridgelet, $\bar{u}$ represents its orientation and $b$ is its location. The measure on space $\Gamma$ is defined by $\frac{da}{a^{d+1}}\sigma_d d\bar{u}db$, where $\sigma_d$ is the surface area of the unit sphere $S^{d-1}$ in dimension $d$ and $d\bar{u}$ the uniform probability measure on $S^{d-1}$. Finally, we will always asumme that the wavelet-type 1-dimension function $\psi : \mathbb{R} \to \mathbb{R}$ belongs to the Schwartz space $S(\mathbb{R})$. A ridgelet is constant along the line $\bar{u} \cdot \bar{x} = constant$. Transverse to these ridges is a wavelet. The results permented here hold under weaker conditions on $\psi$.

**Definition** Let $\psi : \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$K_\psi = \int \frac{|\hat{\psi}(\xi)|^2}{|\xi|^d}\, d\xi < \infty.$$

Then $\psi$ is called an Admissible Neural Activation Function.

We will call the ridge function $\psi_{a,b,\bar{u}}$ generated by an admissible function $\psi$ a ridgelet.

In 2-dimension, we have $\bar{u} = (\cos\theta, \sin\theta)$. Given an integrable bivariate function $f$, its continuous ridgelet transform in $\mathbb{R}^2$ is defined by

$$\mathcal{R}(f)(a, b, \theta) = \int_{\mathbb{R}^2} \psi_{a,b,\theta}(\bar{x}) f(\bar{x})\, d\bar{x}$$

where the ridgelets $\psi_{ab\theta}(\bar{x})$ are defined from a wavelet-type function in 1-dimension $\psi$ as

$$\psi_{a,b,\theta}(\bar{x}) = \frac{1}{\sqrt{a}}\psi\left(\frac{x_1\cos\theta + x_2\sin\theta - b}{a}\right) \quad \text{for all } \ \bar{x} = (x_1, x_2) \in \mathbb{R}^2.$$

As can be seen, the continuous ridgelet transform is similar to the 2-dimension continuous wavelet transform except that the point parameters $\bar{b} = (b_1, b_2)$ are replaced by the line parameters $(b, \theta)$. In other words, these 2-D multiscale transforms are related by:

Wavelets : $\longrightarrow \psi_{scale,\ point-position}$

Ridgelets : $\longrightarrow \psi_{scale,\ line-position}$

As a consequence, wavelets are very effective in representing objects with isolated point singularities, while ridgelets are very effective in representing objects with singularities along lines. In fact, one can think of ridgelets as a way of concatenating 1-dimension wavelets along lines. Hence the motivation for using ridgelets in image processing tasks is apparent since singularities are often joined together along edges or contours in images. In 2-dimension, points and lines are related via the Radon transform, thus the wavelet and ridgelet transforms are lined via the Radon transform. More precisely, let us denote the Radon transform by

$$R_\theta f(t) = \int_{\mathbb{R}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds$$

then the ridgelet transform is the application of 1-dimension wavelet transform to the slices (also refered to as projections) of the Radon transform,

$$\mathcal{R}(f)(a,b,\theta) = \int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right) R_\theta f(t)\, dt.$$

More specifically, for Fourier transform we have

$$\hat{f}(\xi\bar{u}) = \hat{f}(\xi\cos\theta, \xi\sin\theta) = \int_{\mathbb{R}} e^{-2\pi i\xi t} R_\theta f(t)\, dt = \widehat{R_\theta f}(\xi).$$

**Theorem 3.1.** *(Reproducing formula) Suppose that $f$ and $\hat{f} \in L^1(\mathbb{R}^d)$. If $\psi$ is admissible, then*

$$f(x) = \frac{1}{K_\psi} \int \langle f, \psi_{a,b,\theta}\rangle\, \psi_{a,b,\theta}(x)\, \sigma_d\, db\, d\bar{u}\, \frac{da}{a^{d+1}} \quad \text{for all } x \in \mathbb{R}^d.$$

**Remark** In fact, the admissibility condition on $\psi$ is essentially equivalent to the requirement of vanishing moments $\int t^k \psi(t)\, dt = 0,\ k \in \left\{0, 1, ..., \left[\frac{d+1}{2}\right] - 1\right\}$. This clearly shows the similarity of admissibility condition to the 1-dimensional wavelet admissibility condition, however, unlike wavelet theory, the number of necessary vanishing moments grows linearly in the $d$-dimensional space.

**Theorem 3.2.** *(Parseval relation) Assume $f \in L^1 \cap L^2(\mathbb{R}^d)$ and $\psi$ is admissible, then*

$$\|f\|_2^2 = \frac{1}{K_\psi} \int |\langle f, \psi_{ab\theta}\rangle|^2\, \sigma_d\, db\, d\bar{u}\, \frac{da}{a^{d+1}}.$$

## 3.2 The Continuous Curvelet Transform (A Transform Based on Polar Parabolic Scaling)

Candés and Donoho [8,9,10,11] introduced continuous curvelet transform which has much simpler inversion formula and still enjoys properties reminiscent to parabolic scaling. We work throughout in $\mathbb{R}^2$ with variable $\bar{x}$ and frequency domain variable $\bar{\xi}$ with polar coordinates (r,$\omega$). We shall define a continuous curvelet with a continuous scale/location/direction parameter space.

We begin by introducing the notation pair of windows.

i) Radial Window: $W(r)$ is a positive real valued function on $[0,\infty)$ with support in $\left(\frac{1}{2},2\right)$. This window will always obey the admissibility condition:

$$\int_0^\infty W(r)^2 \frac{dr}{r} = 1$$

or we can say that $\displaystyle\int_0^\infty W(ar)^2 \frac{da}{a} = 1 \quad \text{for all} \quad r > 0.$

$$\text{Indeed, since} \int_0^\infty W(r)^2 \frac{dr}{r} = 1, \text{ by letting } r = a\acute{r} \text{ we get that}$$

$$1 = \int_0^\infty W(r)^2 \frac{dr}{r} = \int_0^\infty W(a\acute{r})^2 \acute{r} \frac{da}{a\acute{r}} = \int_0^\infty W(a\acute{r})^2 \frac{da}{a}.$$

ii) Angular Window: $V(t)$ is a real-valued function for which $\mathrm{supp}(V) \subseteq [-1,1]$. This window will always obey the admissibility condition:

$$\int_{-1}^1 V(t)^2\, dt = 1.$$

We use these windows in the frequency domain to construct a family of analyzing elements with three parameters:

1. the scale parameter $a \in \mathbb{R}$ with $0 < a < a_0$. Here and below, $a_0$ is a fixed number, the coarsest scale for our proplem. It is fixed once and for all, and must obey $a_0 < \pi^2$ for the construction of continuous curvelet trasform to work property. $a_0 = 1$ seems a natural choice.

2. The location(translation) parameter $\bar{b} \in \mathbb{R}^2$.

3. The oreintation (direction) parameter $\theta \in [0, 2\pi)$ (or $[-\pi, \pi)$ according to convenience below).

At scale $a$, the family of curvelets elements is generated by translation and rotation of a basic element $\gamma_{a,\bar{0},0}$, For each $\bar{x} \in \mathbb{R}^2$, define

$$\gamma_{a,\bar{b},\theta}(\bar{x}) = \gamma_{a,\bar{0},0}\left(R_\theta\left(\bar{x} - \bar{b}\right)\right)$$

where $R_\theta$ is the 2-by-2 rotation matrix effecting planar rotation by $\theta$ radians. The generating element at scale $a$ ,$\gamma_{a,\bar{0},0}$, is defined by going to polar Fourier coordinates

$$\hat{\gamma}_{a,\bar{0},0}\left(r,\omega\right) = a^{\frac{3}{4}} W\left(ar\right) V\left(\omega/\sqrt{a}\right),$$

with radial variable $r > 0$ and angular variable $\omega \in [0, 2\pi)$ being polar coordinates in the frequency domain. Now we note the definition of $\hat{\gamma}_{a,\bar{0},\theta}$, if we put $\overline{e_\omega} = (\cos\omega, \sin\omega)$;

$$\hat{\gamma}_{a,\bar{0},\theta}\left(r,\omega\right) = a^{\frac{3}{4}} W\left(ar\right) V\left(\left(\omega - \theta\right)/\sqrt{a}\right).$$

The support of each $\hat{\gamma}_{a,\bar{b},\theta}$ is a polar wedge defined by the supports of $W$ and $V$, the radial and angular windows, with scale dependent window widths in each direction. Since $W(r)$ is supported in $\left(\frac{1}{2}, 2\right)$, we get that $W(ar)$ is supported $\left(\frac{1}{2a}, \frac{2}{a}\right)$. Since $V(t)$ is supported on [-1,1], we get that $V\left(\frac{\omega}{\sqrt{a}}\right)$ is supported in $[-\sqrt{a}, \sqrt{a}]$. Thus $\hat{\gamma}_{a00}\left(\xi\right)$ is supported on

$$\left\{(r,\omega) \left| \frac{1}{2a} \le r \le \frac{2}{a}, -\sqrt{a} \le \omega \le \sqrt{a} \right.\right\},$$

and now from the definition of $\hat{\gamma}_{a,\bar{0},\theta}$, we see that its support lies in

$$\left\{(r,\omega) \left| \frac{1}{2a} \le r \le \frac{2}{a}, \theta - \sqrt{a} \le \omega \le \theta + \sqrt{a} \right.\right\}.$$

In effect, the scaling is parabolic in the polar variables $r$ and $\omega$, with $\omega$ being the thin variable. In accordance with the use of the term curvelet to denote families exhibiting such parabolic scaling, we call this system of analyzing elements curvelets. However, note that the curvelet $\gamma_{a,\bar{0},0}$ is not a simple affine change-of-variable acting on $\gamma_{\acute{a},\bar{0},0}$ for $\acute{a} \neq a$. We initially omit description of the transform at coarse scales. Note that these curvelets are highly oriented and they become very needle-like at fine scales.

Equipped with this family of high-frequency elements $\gamma_{a,\bar{b},\theta}$, we can define a continuous curvelet transfrom

$$\Gamma_f\left(a,\bar{b},\theta\right)=\left\langle \gamma_{a,\bar{b},\theta},f\right\rangle \quad \text{for} \quad 0<a<a_0, \quad \bar{b}\in\mathbb{R}^2 \quad \text{and} \quad \theta\in[0,2\pi)\,.$$

This transform has an exact reconstruction formula and parseval relation.

**Theorem 3.3.** *Let $f\in L^2$ have a Fourier transform vanishing for $|\xi|<\frac{2}{a_0}$. Let $V$ and $W$ obey the admissibility conditions. We have a Calderon-like reproducing formula, valid for such high-frequency functions;*

$$f(\bar{x})=\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\Gamma_f\left(a,\bar{b},\theta\right)\gamma_{a,\bar{b},\theta}\left(\bar{x}\right)\frac{da}{a^3}\,d\bar{b}\,d\theta, \quad \text{for all} \;\; \bar{x}\in\mathbb{R}^2$$

*and a Parseval formula for high-frequency funtions;*

$$\|f\|_{L^2}^2=\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left|\Gamma_f\left(a,\bar{b},\theta\right)\right|^2\frac{da}{a^3}\,d\bar{b}\,d\theta.$$

## 3.3 The Smith Transform (A Transform Based on Affine Parabolic Scaling)

Candés and Donoho [8,9,12,13] defined a wavelet-like transform in $\mathbb{R}^2$ with parabolic directional dilation as follows. The three parameters are scale $a>0$, location $\bar{b}\in\mathbb{R}^2$, and orientation $\theta\in[0,2\pi)$. Let $P_{a,\theta}$ be the parabolic directional dilation of $\mathbb{R}^2$ given in matrix by $P_{a,\theta}=D_{\frac{1}{a}}R_{-\theta}$ where $D_{\frac{1}{a}}=\operatorname{diag}\left(\frac{1}{a},\frac{1}{\sqrt{a}}\right)$ and $R_{-\theta}$ is planar rotation by $-\theta$ radians. This matrix has ellipsoidal contours with minor axis pointing in direction $\theta$.

Assume that $\varphi\in L^2(\mathbb{R}^2)$ is a single mother wavelet, then we define the family elements generated by parabolic dialation, translation and rotation of a single mother wavelet $\varphi$;

$$\varphi_{a,\bar{b},\theta}=\varphi\left(P_{a,\theta}\left(\bar{x}-\bar{b}\right)\right)Det\left(P_{a,\theta}\right)^{\frac{1}{2}}=\varphi\left(P_{a,\theta}\left(\bar{x}-\bar{b}\right)\right)a^{-\frac{3}{4}}.$$

Classically, the term wavelet transfrom has been understood to mean that a single waveform is operated on by a family of affine transformations producing a family of analysing waveforms. So this transfrom fits in with the classical notation of wavelet family, except that the family of parabolic affine transform is nonstandard.

Hart F. Smith (1998)[10] studied essentially this construction, with two inessential differences. First, instead of working with scale $a$ and direction $\theta$, he worked with the frequency variable $\xi \equiv a^{-1}e_\theta$ and second, instead of using the $L^2$ normalizing factor $Det\,(P_{a,\theta})^2$, he used the $L^1$ normalizing factor $Det\,(P_{a,\theta})$. In any event, we pretend that Smith has used the scale/location/direction parametrization and the $L^2$ normalization and we can define a Hart Smith directional wavelet transform based on affine parabolic scaling

$$\overline{\Gamma}_f\left(a, \bar{b}, \theta\right) = \langle f, \varphi_{a,\bar{b},\theta}\rangle \quad \text{where} \quad 0 < a < a_0, \bar{b} \in \mathbb{R}^2 \ \text{ and } \ \theta \in [0, 2\pi)\,,$$

where $a_0$ is a fixed coarsest scale. Smith gave a reconstruction formula and a Parsevel relation.

**Theorem 3.4.** *There is a Fourier multiplier $M$ of order 0 so that whenever $f$ is a high-frequency function supported in frequency space $\|\bar{\xi}\| > \frac{2}{a_0}$,*

$$f(\bar{x}) = \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{\infty} \langle \varphi_{a,\bar{b},\theta}, Mf\rangle\, \varphi_{a,\bar{b},\theta}(\bar{x})\,\frac{da}{a^3}\,d\bar{b}\,d\theta \qquad \textit{for all} \ \ \bar{x} \in \mathbb{R}^2$$

*and*

$$\|f\|_2^2 = \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{\infty} \left|\left\langle \varphi_{a,\bar{b},\theta}, M^{\frac{1}{2}} f\right\rangle\right|^2 \frac{da}{a^3}\,d\bar{b}\,d\theta.$$

*The function $Mf$ is defined in the frequency domain by a multiplier formula $\widehat{Mf}(\xi) = m(\|\xi\|)\hat{f}(\xi)$, where the multiplier such that $\log m(\exp(u)) \to 0$ as $u \to \infty$, together with all its derivaties.*

Observe that the reconstruction formula for the Smith's transform is not as simple as those of many other variants of wavelet transform.

Now, one has to work not with the coefficients of $f$ but with those of $Mf$. An alternate approach is to define dual elements $\varphi^*_{a,\bar{b},\theta} \equiv M\varphi_{a,\bar{b},\theta}$ and change the transform definition to either

$$f = \int \left\langle \varphi^*_{a,\bar{b},\theta}, f\right\rangle \varphi_{a,\bar{b},\theta}\,\frac{da}{a^3}\,d\bar{b}\,d\theta$$

or

$$f = \int \langle \varphi_{a,\bar{b},\theta}, f \rangle \, \varphi^*_{a,\bar{b},\theta} \, \frac{da}{a^3} \, d\bar{b} \, d\theta.$$

This more complicated set of formulas gaives a few annoyances which are avoided using the continuous curvelet transform defined in the previous section. However, for many purposes, the two transfroms have similar behavior. For an elementary example; see also [2], we have the following lemma.

**Lemma 3.5.** *Suppose that the windows $V$ and $W$ underlying the continuous curvelet transform are $C^\infty$ and that the mother wavelet generating the Smith transform $\bar{\Gamma}$ has the frequency-domain representation*

$$\hat{\varphi}_{a,\bar{0},0}(\xi) = Ca^{\frac{3}{4}} W(a\xi_1) V\left(\frac{\xi_2}{\sqrt{a}\xi_1}\right), \quad a < \bar{a}_0$$

*for the same windows $V$ and $W$, where $C$ is some normalizing constant, and $\bar{a}_0$ is the transform's coarsest scale. Then at fine scales we have $\sup_{b,\theta} \left\|\gamma_{a,\bar{b},\theta} - \varphi_{a,\bar{b},\theta}\right\|_2 \to 0$ as $a \to 0$.*

## 3.4 Some Properties of Continuous Curvelet Transform

### 3.4.1 Directional Transform

In the standard wavelet transform there is a way to create a directional wavelet transform. Suppose we have a classical admissible wavelet $\varphi$ which is centered at the origin. We stretchs it preferentially in one direction, say according to $\widetilde{\varphi}(x_1, x_2) = \varphi(10x_1, x_2/10)$, so it has an elongated support ( in this case, one hundred times longer than its width ), and consider each rotation $\varphi_\theta(\bar{x}) = \widetilde{\varphi}(R_\theta \bar{x})$ of that wavelet, where $R_\theta$ is rotation by $\theta$ radians. Next, we take the generated scale-location family

$$\varphi_{a,\bar{b},\theta}(\bar{x}) \;=\; \frac{1}{a}\varphi_\theta\left(\frac{\bar{x}-\bar{b}}{a}\right) \;=\; \frac{1}{a}\widetilde{\varphi}\left(\frac{1}{a}R_\theta(\bar{x}-\bar{b})\right).$$

This would provide a wavelet transform with strongly oriented wavelets and a direction parameter.

### 3.4.2 Parabolic Scaling

In harmonic analysis there have been a number of important applications of decompositions based on parabolic dilations

$$f_a(x_1, x_2) = f\left(\frac{1}{a}x_1, \frac{1}{\sqrt{a}}x_2\right)$$

so called because they leave invariant the parabola $x_1 = x_2^2$. In the above equation the dilation is always twice as powerful in one fixed direction as in the orthogonal one. Decompositions also can be based on directional parabolic dilation of the form

$$f_{a,\theta}(x_1, x_2) = f_a(R_\theta(x_1, x_2)) = f\left(D_{\frac{1}{a}} R_\theta(x_1, x_2)\right)$$

where $R_\theta$ is a rotation matrix by $\theta$ radians, and $D_{\frac{1}{a}} = \mathrm{diag}\left(\frac{1}{a}, \frac{1}{\sqrt{a}}\right)$. The directional transform we defined uses curvelets $\gamma_{a,\bar{b},\theta}$ which are essentially the result of such directional parabolic dilations. This means that at fine scales they are increasingly long compared to their width : $width \approx length^2$.

The motivation for decomposition into parabolic dilations comes from several sources. Starting in the 1970's they were used in harmonic analysis, for example by Fefferman and later Seeger, Sogge, and Stein to study the boundedness of certain operators. More recently, Hart Smith proposed parabolic scaling in defining molecular decompositions of Fourier integeral operators, while Candés and Donoho proposed its use in decompositions of image-like objects which are smooth apart from edges. So parabolic dilations are useful in representing operators and singularities along curves.

### 3.4.3 Localization

From definition of the continuous curvelet transform, in this thesis we always suppose that $V$ and $W$ are $C^\infty$ ; this will imply that $\gamma_{a,\bar{b},\theta}(\bar{x})$ and it derivatives are each of rapid decay as $\|\bar{x}\| \to \infty$:

$$\gamma_{a,\bar{b},\theta}(\bar{x}) = O(\|\bar{x}\|^{-N}), \qquad \forall N > 0.$$

We can describe the decay properties of $\gamma_{a,\bar{b},\theta}$ much more precisely; roughly the right norm to measure distance from $b$ is associated with an anisotropic ellipse with sides $a$ and $\sqrt{a}$ and minor axis in direction $\theta$ radians, and $\gamma_{a,\bar{b},\theta}$ decays as a function of distance in that norm. So, suppose we let $P_{a,\theta}$ be the parabolic directional dilation of $\mathbb{R}^2$ given in matrix form by

$$P_{a,\theta} = D_{\frac{1}{a}} R_{-\theta}$$

where $D_{\frac{1}{a}} = \text{diag}(1/a, 1/\sqrt{a})$ and $R_{-\theta}$ is planar rotation by $-\theta$ radians. For a vector $\bar{v} \in \mathbb{R}^2$, define the norm

$$\|\bar{v}\|_{a,\theta} = \|P_{a,\theta}(\bar{v})\| = \left\| D_{\frac{1}{a}} R_{-\theta}(\bar{v}) \right\|$$

this norm has ellipsoidal unit ball with minor axis pointing in direction $\theta$. It follows immediately that $\frac{\|\bar{v}\|}{\sqrt{a}} \leq \|\bar{v}\|_{a,\theta} \leq \frac{\|\bar{v}\|}{a}$. The following pointwise bound of their curvelets transform will be needed in the next chapter, see [8,9]. Also, here and below, we use the notation $\langle y \rangle = (1 + y^2)^{1/2}$ for all $y \in \mathbb{R}$.

**Lemma 3.6.** *Suppose that the windows $V$ and $W$ are $C^\infty$ and have compact supports. Then, for each $N$ = 1,2,... and corresponding constant $C_N$,*

$$|\gamma_{a,\bar{b},\theta}(\bar{x})| \leq C_N \cdot a^{-3/4} \cdot \left\langle \left\|\bar{x} - \bar{b}\right\|_{a,\theta} \right\rangle^{-N} \qquad \textit{for all } x \in \mathbb{R}^2.$$

These estimates are compatible with the view that curvelets are affine transforms of a single mother wavelet, where the analyzing elements are of the form $\varphi(P_{a,\theta}(\bar{x} - \bar{b}))\text{Det}(P_{a,\theta})^{1/2}$. However, it is important to emphasize that $\gamma_{ab\theta}$ does not obey true parabolic scaling, i.e., there is not a single mother curvelet $\gamma_{1,\bar{0},0}$ so that

$$\gamma_{a,\bar{b},\theta} = \gamma_{1,\bar{0},0}(P_{a,\theta}(\bar{x} - \bar{b}))\text{Det}(P_{a,\theta})^{1/2}.$$

A transform based on such true parabolic scaling can of course be defined. In fact, essentially this has been done by Hart F. Smith.

# CHAPTER IV

# CHARACTERIZATION OF HÖLDER REGULARITY WITH THE CONTINUOUS CURVELET AND SIMILAR TRANSFORM

A classical tool for measuring the Hölder regularity of a function $f$ is to look at the asymptotic decay of its Fourier transform $\hat{f}$. One can prove that a bounded function $f$ is uniformly Hölder exponent $\alpha$ over $\mathbb{R}$ if $\int_{-\infty}^{\infty} |\hat{f}(\omega)| (1 + |\omega|^{\alpha})\ d\omega < \infty$. This condition is sufficient but not necessary. It gives a global regularity condition over the whole real line but from this condition, one can not determine whether the function is locally more regular at a particular point $x_0$. This is because the Fourier transform unlocalizes the information along the spatial variable $x$. The Fourier transform is therefore not well adapted to measure the local Hölder regularity of functions. As an efficient mathematical microscope, wavelets has been one of the better tools for analyzing regularity of functions. Holschneider and Tchamitchian [4] have given chacterizations of uniform and pointwise Hölder regularity of functions. See also [3,5,6,7,8]. It says roughly that a function has Hölder regularity with exponent $\alpha$ if and only if its wavelet transform satisfies a corresponding bound condition across scales.

In Section 1, we show how the uniform and pointwise Hölder regularity of a function can be characterized by its ridgelet transform. Similar characterizations by the Smith transform are given in Section 2. We then give bounds of the continuous curvelet transform of uniform and pointwise Hölder continuous functions in Section 3. Since the system of functions used in the ridgelet transform depends on the direction parameter, it is belived that this transform can be adapted to studying the directional regularity. So in the last section we give some idea of the directional regularity and use it to analyze the bounds of the ridgelet transform.

## 4.1 Characterizations of Hölder Regularity by The Ridgelet Transform

Let $\alpha > 0$ and $k = [\alpha]$. We pick a smooth univariate wavelet-type function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying the following smoothness, decaying and oscillating properties

$$|\psi^{(i)}(x)| \leq C(1+|x|)^{-k-2} \quad \text{for} \quad i = 0, 1, ..., k+1 \quad \text{for some constant} \quad C > 0,$$

$$\int_{\mathbb{R}^2} x^j \psi(x)\, dx = 0 \quad \text{for} \quad j = 0, ..., k,$$

and

$$\int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{|\xi|^2}\, d\xi = 1 \quad \text{with} \quad \hat{\psi}(\xi) = 0 \quad \text{if} \quad \xi < 0.$$

This wavelet-type function $\psi$ gives rise to the ridgelet transform used to analyze uniform and pointwise Hölder regularity of functions. In this section, we show how to analyze uniform and poinwise Hölder regularity by the ridgelet transform. Generally speaking, the amount of uniform and pointwise regularity of a function is reflected in its ridgelet transform by the decrease of the ridgelet coefficients at small scales as shown by the following theorem.

**Theorem 4.1.** *If $f : \mathbb{R}^2 \to \mathbb{R}$ has compact support and uniform Hölder exponent $\alpha \in (0, 1]$ on $\mathbb{R}^2$, then there is a constant $C > 0$ such that $|\mathcal{R}(f)(a, b, \theta)| \leq Ca^{\alpha+\frac{1}{2}}$ for all $a > 0, b \in \mathbb{R}$, and $\theta \in [0, 2\pi)$.*

*Proof.* We write the ridgelet transform in terms of the Radon transform,

$$\begin{aligned}|\mathcal{R}(f)(a,b,\theta)| &= \left|\frac{1}{\sqrt{a}}\int_{\mathbb{R}^2} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\psi\left(\frac{t-b}{a}\right) ds\, dt\right| \\ &= \left|\frac{1}{\sqrt{a}}\int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right)\left(\int_{\mathbb{R}} f(R_\theta(t,s))\, ds\right) dt\right|.\end{aligned}$$

Since $f$ is compactly supported and $\int_{\mathbb{R}} \psi(x)\, dx = 0$ we have

$$\begin{aligned}|\mathcal{R}(f)(a,b,\theta)| &= \left|\frac{1}{\sqrt{a}}\int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right)\int_{\mathbb{R}} f(R_\theta(t,s))\, ds\, dt - \frac{1}{\sqrt{a}}\int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right)\int_{\mathbb{R}} f(R_\theta(b,s))\, ds\, dt\right| \\ &\leq \frac{1}{\sqrt{a}}\int_{\mathbb{R}}\left|\psi\left(\frac{t-b}{a}\right)\right| |R_\theta f(t) - R_\theta f(b)|\, dt.\end{aligned}$$

Since $f$ is Hölder continuous with exponent $\alpha$ and $\text{supp}(f) \subseteq [-R, R] \times [-R, R]$ for some $R < \infty$, we can bound $|R_\theta f(t) - R_\theta f(b)|$ by

$$\begin{aligned}
|R_\theta f(t) - R_\theta f(b)| &\leq \int_{\mathbb{R}} |f(R_\theta(t,s)) - f(R_\theta(b,s))|\, ds \\
&\leq \int_{-2R}^{2R} C \left\|((t-b)\cos\theta, (t-b)\sin\theta)\right\|^\alpha \, ds \\
&= C \int_{-2R}^{2R} \left[((t-b)\cos\theta)^2 + ((t-b)\sin\theta)^2\right]^{\frac{\alpha}{2}} \, ds \\
&= C \int_{-2R}^{2R} |t-b|^\alpha \, ds \\
&= C\,|t-b|^\alpha \int_{-2R}^{2R} 1^\alpha \, ds \\
&= C'\,|t-b|^\alpha .
\end{aligned}$$

Thus we have

$$\begin{aligned}
|\mathcal{R}(f)(a,b,\theta)| &\leq \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \left|\psi\left(\frac{t-b}{a}\right)\right| C'|t-b|^\alpha \, dt \\
&= C' a^{\alpha - \frac{1}{2} + 1} \int_{\mathbb{R}} |\psi(y)|\, |y|^\alpha \, dy \\
&= C'' a^{\alpha + \frac{1}{2}}
\end{aligned}$$

since the last integral is finite as a result of the decay condition on $\psi$. □

The following is a converse theorem.

**Theorem 4.2.** *Suppose that $\psi$ is compactly supported. Suppose also that $f \in L^1(\mathbb{R})$ is bounded, continuous, and compactly supported. If, for some $\alpha \in (0,1)$, there is a constant $C > 0$ such that $|\mathcal{R}(f)(a,b,\theta)| \leq C|a|^{\alpha+\frac{3}{2}}$ for all $a > 0, b \in \mathbb{R}$, and $\theta \in [0, 2\pi)$, then $f$ is Hölder continuous with exponent $\alpha$.*

*Proof.* Let $\bar{x}, \bar{h} \in \mathbb{R}^2$. By the reconstruction formula of ridgelet transform we have

$$f(\bar{x}) = \frac{1}{K_\psi} \int_{-\infty}^{\infty} \int_{S^1} \int_{-\infty}^{\infty} \mathcal{R}(f)(a,b,\theta) \frac{1}{\sqrt{a}} \psi\left(\frac{\bar{x}\cdot\bar{u} - b}{a}\right) \sigma_2 \, db\, d\bar{u}\, \frac{da}{a^3}$$

where $\bar{u} = (\cos\theta, \sin\theta)$ belongs to $S^1$. Note that $K_\psi = 1$ and $\sigma_2$ is the surface area of $S^1$. We will split integral over $a$ into two parts, $|a| \leq 1$ and $|a| \geq 1$, and call the two terms $f_{SS}$

(small scale) and $f_{LS}$ (large scales). First of all, note that $f_{LS}$ is bounded uniformly in $\bar{x}$ as a simple change of variable and integrability of $\psi$ yield:

$$\begin{aligned}
|f_{LS}(\bar{x})| &\leq C\int_1^\infty \int_{S^1}\int_{-\infty}^\infty |\mathcal{R}(f)(a,b,\theta)|\frac{1}{\sqrt{a}}\left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\,\frac{da}{a^3} \\
&= C\int_1^\infty \int_{S^1}\int_{-\infty}^\infty \left|\int_{-\infty}^\infty \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)R_\theta f(t)\,dt\right|\frac{1}{\sqrt{a}}\left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\,\frac{da}{a^3} \\
&\leq C\int_1^\infty \frac{1}{a}\int_{S^1}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty \left|\psi\left(\frac{t-b}{a}\right)\right| |R_\theta f(t)|\,dt\right]\left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\,\frac{da}{a^3} \\
&\leq C\int_1^\infty \int_{S^1}\int_{-\infty}^\infty a^{-3-1+\frac{1}{2}}\,\|\psi\|_2\,\|R_\theta f\|_2\left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\,da.
\end{aligned}$$

Since $f$ has compact support, we have $\|R_\theta f\|_2 \leq \|f\|_2$. So the integrability of $\psi$ and $f$ yield

$$\begin{aligned}
|f_{LS}(\bar{x})| &\leq C\int_1^\infty \int_{S^1}\int_{-\infty}^\infty a^{-\frac{7}{2}}\,\|\psi\|_2\,\|f\|_2\left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\,da \\
&= C'\int_1^\infty a^{-\frac{7}{2}+1}\int_{S^1}\int_{-\infty}^\infty |\psi(y)|\,dy\,d\bar{u}\,da \\
&= C''\int_1^\infty a^{-\frac{5}{2}}\,da \\
&= C''\left[a^{-\frac{3}{2}}\right]_1^\infty \\
&= C''' < \infty.
\end{aligned}$$

Next, we show $|f_{LS}(\bar{x}+\bar{h}) - f_{LS}(\bar{x})| \leq C\left\|\bar{h}\right\|^\alpha$ for $\left\|\bar{h}\right\| \leq 1$;

$$\begin{aligned}
|f_{LS}(\bar{x}+\bar{h}) - f_{LS}(\bar{x})| &\leq C\int_1^\infty \int_{S^1}\int_{-\infty}^\infty |\mathcal{R}(f)(a,b,\theta)|\frac{1}{\sqrt{a}} \\
&\qquad \left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right) - \psi\left(\frac{(\bar{x}+\bar{h})\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\,\frac{da}{a^3} \\
&\leq C\int_1^\infty \int_{S^1}\int_{-\infty}^\infty a^{-\frac{1}{2}-\frac{1}{2}-3}\int_{-\infty}^\infty |\psi(\frac{t-b}{a})|\,|R_\theta f(t)|\,dt \\
&\qquad \left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right) - \psi\left(\frac{(\bar{x}+\bar{h})\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\,da \qquad (1)
\end{aligned}$$

Since $\psi$ is differentiable everywhere with uniformly bounded derivative, there is a constant $C$ such that $|\psi(x) - \psi(y)| \leq C\,|x-y|$ for all $x, y \in \mathbb{R}$. Consequently,

$$\begin{aligned}
\left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right) - \psi\left(\frac{(\bar{x}+\bar{h})\cdot\bar{u}-b}{a}\right)\right| &\leq C\left|\frac{\bar{x}\cdot\bar{u}-b}{a} - \frac{(\bar{x}+\bar{h})\cdot\bar{u}-b}{a}\right| \\
&= C\left|\frac{\bar{h}\cdot\bar{u}}{a}\right| \\
&\leq C\frac{\left\|\bar{h}\right\|}{a}
\end{aligned}$$

and, since $\operatorname{supp}(\psi) \subseteq [-M, M]$ for some $M < \infty$, if $\psi\left(\frac{(\bar{x}+\bar{h})\cdot\bar{u}-b}{a}\right) \neq 0$ then

$$\left|\frac{(\bar{x}+\bar{h})\cdot\bar{u}-b}{a}\right| \leq M$$

which implies that $|\bar{x}\cdot\bar{u}-b| - |\bar{h}\cdot\bar{u}| \leq aM$ and hence $|\bar{x}\cdot\bar{u}-b| \leq aM + |\bar{h}\cdot\bar{u}| \leq aM + \left\|\bar{h}\right\| \leq aM+1$. $\|R_\theta f(t)\|_2 \leq \|f\|_2$ since $f$ is compact support, we can bound (1) by

$$\begin{aligned}
(1) &\leq C' \int_1^\infty \int_{S^1} a^{-4} \int_{|\bar{x}\cdot\bar{u}-b|\leq aM+1} \left(\int_{|t-b|\leq aM} |\psi(\frac{t-b}{a})|\,|R_\theta f(t)|\,dt\right) \frac{\left\|\bar{h}\right\|}{a}\,db\,d\bar{u}\,da \\
&= C' \left\|\bar{h}\right\| \int_1^\infty \int_{S^1} a^{-5} \int_{|\bar{x}\cdot\bar{u}-t|\leq 2aM+1} \left(\int_{|\bar{x}\cdot\bar{u}-b|\leq aM+1} |\psi(\frac{t-b}{a})|\,db\right) |R_\theta f(t)|\,dt\,d\bar{u}\,da \\
&\leq C'' \left\|\bar{h}\right\| \int_1^\infty \int_{S^1} a^{-5+1} \int_{|\bar{x}\cdot\bar{u}-t|\leq 2aM+1} |R_\theta f(t)|\,dt\,d\bar{u}\,da \\
&\leq C''' \left\|\bar{h}\right\| \|R_\theta f(t)\|_2 \int_1^\infty a^{-4} \left(\int_{|\bar{x}\cdot\bar{u}-t|\leq 2aM+1} 1\,dt\right)^{\frac{1}{2}} da \\
&\leq C''' \left\|\bar{h}\right\| \|f\|_2 \int_1^\infty a^{-4}\sqrt{2aM+1}\,da \\
&\leq C'''' \left\|\bar{h}\right\|^\alpha.
\end{aligned}$$

This estimate holds for all $\left\|\bar{h}\right\| \leq 1$. Hence, together with uniformly boundedness of $f_{LS}$, we conclude that $|f_{LS}(\bar{x}+\bar{h}) - f_{LS}(\bar{h})| \leq C\left\|\bar{h}\right\|^\alpha$ for all $\bar{h}$, uniformly in $\bar{x}$.

Next, by the assumption on the decay of the ridgelet transform of $f$, the small scale part $f_{SS}$ is also uniformly bounded as a simple change of variable, and integrability of $\psi$ yield

$$\begin{aligned}
|f_{SS}(x)| &\leq \int_0^1 \int_{S^1} \int_{-\infty}^\infty \frac{1}{\sqrt{a}} |\mathcal{R}(f)(a,b,\theta)| \left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right)\right| db\,d\bar{u}\,\frac{da}{a^3} \\
&\leq C \int_0^1 \int_{S^1} \int_{-\infty}^\infty a^{\alpha+\frac{3}{2}-\frac{1}{2}-3} \left|\psi\left(\frac{\bar{x}\cdot\bar{u}-b}{a}\right)\right| db\,d\bar{u}\,da \\
&= C \int_0^1 a^{\alpha-2+1} \left(\int_{-\infty}^\infty |\psi(y)|\,dy\right) da \\
&= C \int_0^1 a^{\alpha-1}\,da \\
&= C\left[\frac{a^\alpha}{\alpha}\right]_0^1 \\
&= C < \infty.
\end{aligned}$$

Finally, we again only have to check $|f_{SS}(\bar{x}+\bar{h}) - f_{SS}(\bar{x})| \leq C\left\|\bar{h}\right\|^\alpha$ for small $\bar{h}$, say $\left\|\bar{h}\right\| \leq 1$. We apply the asummed inequality on the ridgelet transform and split the integral into fine and coarse scale ranges,

$$
\begin{aligned}
&|f_{SS}(\bar{x}+\bar{h}) - f_{SS}(\bar{x})| \\
&\le \int_0^1 \int_{S^1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} |\mathcal{R}(f)(a,b,\theta)| \left| \psi\left(\frac{\bar{u}\cdot(\bar{x}+\bar{h})-b}{a}\right) - \psi\left(\frac{\bar{u}\cdot\bar{x}-b}{a}\right)\right| db\, d\bar{u}\, \frac{da}{a^3} \\
&\le C \int_0^1 \int_{S^1} \int_{-\infty}^{\infty} a^{\alpha+\frac{3}{2}-\frac{1}{2}-3} \left| \psi\left(\frac{\bar{u}\cdot(\bar{x}+\bar{h})-b}{a}\right) - \psi\left(\frac{\bar{u}\cdot\bar{x}-b}{a}\right)\right| db\, d\bar{u}\, da \\
&\le C \int_0^{\|\bar{h}\|} \int_{S^1} \int_{-\infty}^{\infty} a^{\alpha-2} \left| \psi\left(\frac{\bar{u}\cdot(\bar{x}+\bar{h})-b}{a}\right)\right| + \left| \psi\left(\frac{\bar{u}\cdot\bar{x}-b}{a}\right)\right| db\, d\bar{u}\, da \\
&\quad + C \int_{\|\bar{h}\|}^1 \int_{S^1} \int_{-\infty}^{\infty} a^{\alpha-2} \left| \psi\left(\frac{\bar{u}\cdot(\bar{x}+\bar{h})-b}{a}\right) - \psi\left(\frac{\bar{u}\cdot\bar{x}-b}{a}\right)\right| db\, d\bar{u}\, da. \qquad (2)
\end{aligned}
$$

By the decay condition of $\psi$ and its is differentiability everywhere with uniformly bounded derivative, there is a constant $C > 0$ such that $|\psi(x) - \psi(y)| \le C|x-y|$ for all $x, y \in \mathbb{R}$. Also, since $\operatorname{supp}(\psi) \subseteq [-M, M]$, we can bound (2) by

$$
\begin{aligned}
(2) &\le C \int_0^{\|\bar{h}\|} \int_{S^1} a^{\alpha-2+1} \|\psi\|_1 \, d\bar{u}\, da \\
&\quad + C \int_{\|\bar{h}\|}^1 \int_{S^1} \int_{|\bar{u}\cdot\bar{x}-b|\le aM+\|\bar{h}\|} a^{\alpha-2} \frac{\|\bar{h}\|}{a} \, db\, d\bar{u}\, da \\
&\le C' \int_0^{\|\bar{h}\|} a^{\alpha-1}\, da + C'' \|\bar{h}\| \int_{\|\bar{h}\|}^1 a^{\alpha-3} \int_{|\bar{u}\cdot\bar{x}-b|\le aM+\|\bar{h}\|} 1 \, db\, da \\
&\le C' \int_0^{\|\bar{h}\|} a^{\alpha-1}\, da + C'' \|\bar{h}\| \int_{\|\bar{h}\|}^1 a^{\alpha-3} \left(aM + \|\bar{h}\|\right) da \\
&= C' \left[\frac{a^\alpha}{\alpha}\right]_0^{\|\bar{h}\|} + C'' \|\bar{h}\| \left[\frac{M a^{\alpha-1}}{\alpha-1} + \frac{\|\bar{h}\| a^{\alpha-2}}{\alpha-2}\right]_{\|\bar{h}\|}^1 \\
&= C''' \|\bar{h}\|^\alpha + C'' \|\bar{h}\| \left[\frac{M}{\alpha-1} + \frac{\|\bar{h}\|}{\alpha-2} - \frac{\|\bar{h}\|^{\alpha-1} M}{\alpha-1} + \frac{\|\bar{h}\| \|\bar{h}\|^{\alpha-2}}{\alpha-2}\right] \\
&= C''' \|\bar{h}\|^\alpha + C'' \|\bar{h}\|^\alpha \left[\frac{M}{1-\alpha}\left(1 - \|\bar{h}\|^{1-\alpha}\right) + \frac{1}{2-\alpha}(1 - \|\bar{h}\|^{2-\alpha})\right] \\
&\le C'''' \|\bar{h}\|^\alpha .
\end{aligned}
$$

Since this holds for all $\|\bar{h}\| \le 1$, which, together with the uniformly boundedness of $f_{SS}$, we conclude that $|f_{SS}(\bar{x}+\bar{h}) - f_{SS}(\bar{h})| \le C\|\bar{h}\|^\alpha$ for all $\bar{h}$, uniformly in $\bar{x}$. It follows that $f$ is Hölder continuous with exponent $\alpha$. □

Theorems 4.1 and 4.2 give a necessary condition and a sufficient condition for a function to be in $C^\alpha(\mathbb{R}^2)$ in terms of its ridgelet transform, respectively. The last two theorems in this section provide a characterization of pointwise regularity by the same transform.

**Theorem 4.3.** *If $f$ is compactly supported and Hölder continuous at $\bar{x}_0$ with exponent $\alpha \in (0,1)$, then there is a constant $C > 0$ for which*

$$|\mathcal{R}(f)(a,b,\theta)| \le Ca^{\frac{1}{2}}\left(a^\alpha + |b - \bar{u}\cdot\bar{x}_0|^\alpha + 1\right) \quad \textit{for all} \ \ a > 0, b \in \mathbb{R}, \ \textit{and}\ \theta \in [0,2\pi)$$

*where $\bar{u}$ is the unit vecter forming the angle $\theta$ radians counter clockwise with the positive $x$-axis, i.e. $\bar{u} = (\cos\theta, \sin\theta)$.*

*Proof.* By translating everything we assume that $\bar{x}_0 = \bar{0}$. Using the assumptions $\int \psi(y)\,dy = 0$ that is $\int \psi_{a,b,\theta}(x)\,dx = 0$ for all $a > 0, b \in \mathbb{R}, \theta \in [0,2\pi)$, and that $f$ has Hölder continuous at $x_0$ with exponent $\alpha$, we have

$$\begin{aligned}
|\mathcal{R}(f)(a,b,\theta)| &\le \int_{\mathbb{R}^2} \frac{1}{\sqrt{a}}\left|\psi\left(\frac{\bar{x}\cdot\bar{u} - b}{a}\right)\right| |f(\bar{x}) - f(\bar{0})|\ d\bar{x} \\
&\le \int_{\mathbb{R}^2} \frac{1}{\sqrt{a}}\left|\psi\left(\frac{\bar{x}\cdot\bar{u} - b}{a}\right)\right| C\,\|\bar{x}\|^\alpha\ d\bar{x} \\
&= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}}\left|\psi\left(\frac{x_1\cos\theta + x_2\sin\theta - b}{a}\right)\right| \left(x_1^2 + x_2^2\right)^{\frac{\alpha}{2}}\ dx_1\,dx_2.
\end{aligned}$$

Put $x_1 = t\cos\theta - s\sin\theta$ and $x_2 = t\sin\theta + s\cos\theta$ and, using the assumption that $f$ is compactly supported, we have

$$\begin{aligned}
|\mathcal{R}(f)(a,b,\theta)| &\le C\int_{-\infty}^{\infty}\int_{-M}^{M} \frac{1}{\sqrt{a}}\left|\psi\left(\frac{t-b}{a}\right)\right| \left(t^2 + s^2\right)^{\frac{\alpha}{2}}\ ds\,dt \\
&\le C\int_{-\infty}^{\infty}\int_{-M}^{M} \frac{1}{\sqrt{a}}\left|\psi\left(\frac{t-b}{a}\right)\right| |t|^\alpha\,ds\,dt + C\int_{-\infty}^{\infty}\int_{-M}^{M} \frac{1}{\sqrt{a}}\left|\psi\left(\frac{t-b}{a}\right)\right| |s|^\alpha\,ds\,dt \\
&= C\int_{-M}^{M}\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} a\,|\psi\,(y)|\,|ay + b|^\alpha\,dy\,ds + C\int_{-M}^{M}\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} a\,|\psi(y)|\,|s|^\alpha\,dy\,ds \\
&\le C\int_{-M}^{M}\int_{-\infty}^{\infty} a^{\frac{1}{2}}\,|\psi\,(y)|\,(|ay|^\alpha + |b|^\alpha)\ dy\,ds + C\int_{-M}^{M}\int_{-\infty}^{\infty} a^{\frac{1}{2}}\,|\psi\,(y)|\,|s|^\alpha\,dy\,ds \\
&\le C'a^{\frac{1}{2}}\left((a)^\alpha + |b|^\alpha\right)\int_{-M}^{M} 1\,ds + C'a^{\frac{1}{2}}\int_{-M}^{M} |s|^\alpha\,ds \\
&\le C''a^{\frac{1}{2}}(a^\alpha + |b|^\alpha + 1),
\end{aligned}$$

where we have used the integrability of $\int |\psi(y)||y|^\alpha\,dy$ and $\int |\psi(y)|\,dy$ . □

We can see that the amount of pointwise Hölder regularity of function is reflected in its ridgelet transform by the decrease of the ridgelet transform at small scales. However, the decay dose not depend on the exponent $\alpha$ of $a$. In Section 5.4, we shall obtain a better bound by means of directional regularity.

However, we can characterize the pointwise Hölder regularity of function by the ridgelet transform in the following theorem.

**Theorem 4.4.** *Suppose that $\psi$ is compactly supported and suppose also that $f \in L^1(\mathbb{R}^2)$ is bounded and continuous. If, for some $\beta \in (0,1)$, there is a constant $C > 0$ such that*

$$|\mathcal{R}(f)(a,b,\theta)| \leq Ca^{\beta+\frac{3}{2}} \quad \text{for all } a > 0, b \in \mathbb{R} \text{ and } \theta \in [0,2\pi),$$

*and for some $\alpha \in (0,1)$, there is a constant $C > 0$ for which*

$$|\mathcal{R}(f)(a,b,\theta)| \leq Ca^{\frac{3}{2}} \left( a^{\alpha} + \frac{|b - \bar{x}_0 \cdot \bar{u}|^{\alpha}}{|\log|b - \bar{x}_0 \cdot \bar{u}||} \right) \quad \text{for all } a > 0, b \in \mathbb{R} \text{ and } \theta \in [0,2\pi)$$

*where $\bar{u}$ is the unit vecter forming the angle $\theta$ radians counter clockwise with the positive $x$-axis, i.e., $\bar{u} = (\cos\theta, \sin\theta)$, then $f$ is Hölder continuous with exponent $\alpha$ at $\bar{x}_0$.*

*Proof.* We will split the integral over $a$ into two parts, $a \leq 1$ and $a > 1$, and call the two terms $f_{SS}$ (small scale) and $f_{LS}$ (large scale). Clearly the large part $f_{LS}$ is always regular. See the proof of the large scale part of Theorem 4.2. Thus we start the proof by showing that $f_{SS}$ is bound for each $\bar{x} \in \mathbb{R}^2$.

$$\begin{aligned}
|f_{SS}(\bar{x})| &\leq \int_{a\leq 1}\int_{S^1}\int_{\mathbb{R}} |\mathcal{R}(f)(a,b,\theta)| \frac{1}{\sqrt{a}} \left|\psi\left(\frac{\bar{x}\cdot\bar{u} - b}{a}\right)\right| db\, d\bar{u}\, \frac{da}{a^3} \\
&\leq C \int_{a\leq 1}\int_{S^1}\int_{\mathbb{R}} a^{\beta+\frac{3}{2}-\frac{1}{2}-3} \left|\psi\left(\frac{\bar{x}\cdot\bar{u} - b}{a}\right)\right| db\, d\bar{u}\, da \\
&= C \int_{a\leq 1}\int_{S^1}\int_{\mathbb{R}} a^{\beta-2+1} \left|\psi\left(y\right)\right| dy\, d\bar{u}\, da \\
&= C' \int_{a\leq 1} a^{\beta-1}\, da \\
&= C' \left[\frac{a^{\beta}}{\beta}\right]_0^1 \\
&= C'' < \infty.
\end{aligned}$$

We therefore only have to check $|f_{SS}(\bar{x}_0+\bar{h})-f_{SS}(\bar{x}_0)| \leq C\left\|\bar{h}\right\|^{\alpha}$ for small $h$, say $\left\|\bar{h}\right\| \leq 1$. By an overall translation and dilation which does not change the local regularity of $f$, we may suppose that the support of $\psi$ is contained in $[-R, R]$ for some $0 < R \leq \frac{1}{2}$ and assume $\bar{x}_0 = \bar{0}$. We then obtain

$$
\begin{aligned}
&|f_{SS}(\bar{h}) - f_{SS}(\bar{0})| \\
&\leq \int_{a\leq 1}\int_{S^1}\int_{\mathbb{R}} |\mathcal{R}(f)(a,b,\theta)| \frac{1}{\sqrt{a}} \left|\psi\left(\frac{\bar{h}\cdot\bar{u}-b}{a}\right) - \psi\left(\frac{-b}{a}\right)\right| db\, d\bar{u}\, \frac{da}{a^3} \\
&\leq \int_0^{\|\bar{h}\|^{\frac{\alpha}{\beta}}}\int_{S^1}\int_{\mathbb{R}} |\mathcal{R}(f)(a,b,\theta)| \frac{1}{\sqrt{a}} \left|\psi\left(\frac{\bar{h}\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\, \frac{da}{a^3} \\
&\quad + \int_{\|\bar{h}\|^{\frac{\alpha}{\beta}}}^{\|\bar{h}\|}\int_{S^1}\int_{\mathbb{R}} |\mathcal{R}(f)(a,b,\theta)| \frac{1}{\sqrt{a}} \left|\psi\left(\frac{\bar{h}\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\, \frac{da}{a^3} \\
&\quad + \int_0^{\|\bar{h}\|}\int_{S^1}\int_{\mathbb{R}} |\mathcal{R}(f)(a,b,\theta)| \frac{1}{\sqrt{a}} \left|\psi\left(\frac{-b}{a}\right)\right| db\, d\bar{u}\, \frac{da}{a^3} \\
&\quad + \int_{\|\bar{h}\|}^{1}\int_{S^1}\int_{\mathbb{R}} |\mathcal{R}(f)(a,b,\theta)| \frac{1}{\sqrt{a}} \left|\psi\left(\frac{\bar{h}\cdot\bar{u}-b}{a}\right) - \psi\left(\frac{-b}{a}\right)\right| db\, d\bar{u}\, \frac{da}{a^3}
\end{aligned}
$$

where we have assumed $\alpha > \beta$. (If $\alpha \leq \beta$, since $f$ is Hölder continuous with exponent $\beta$, it is Hölder continuous with exponent $\alpha$). Let us denote the four terms on the right-hand side of inequality by $T_1, T_2, T_3$, and $T_4$, respectively.

Using the uniform Hölder continuous of $f$ we have $|\mathcal{R}(f)(a,b,\theta)| \leq Ca^{\beta+\frac{3}{2}}$ which leads to

$$
\begin{aligned}
T_1 &\leq C\int_0^{\|\bar{h}\|^{\frac{\alpha}{\beta}}}\int_{S^1}\int_{\mathbb{R}} a^{\beta+\frac{3}{2}-\frac{1}{2}-3}\left|\psi\left(\frac{\bar{h}\cdot\bar{u}-b}{a}\right)\right| db\, d\bar{u}\, da \\
&= C\int_0^{\|\bar{h}\|^{\frac{\alpha}{\beta}}}\int_{S^1}\int_{\mathbb{R}} a^{\beta-2+1}\left|\psi\left(y\right)\right| dy\, d\bar{u}\, da \\
&= C'\int_0^{\|\bar{h}\|^{\frac{\alpha}{\beta}}} a^{\beta-1}\left(\int_{\mathbb{R}}\left|\psi\left(y\right)\right| dy\right) da \\
&= C''\int_0^{\|\bar{h}\|^{\frac{\alpha}{\beta}}} a^{\beta-1}\, da \\
&= C''\left[\frac{a^{\beta}}{\beta}\right]_0^{\|\bar{h}\|^{\frac{\alpha}{\beta}}} \\
&= C'''\left\|\bar{h}\right\|^{\alpha}.
\end{aligned}
$$

By the assumption, we get

$$T_2 \le C \int_{\|\bar{h}\|^{\frac{\alpha}{\beta}}}^{\|\bar{h}\|} \int_{S^1} \int_{\mathbb{R}} a^{\frac{3}{2}-\frac{1}{2}-3} \left( a^{\alpha} + \frac{|b|^{\alpha}}{|\log |b||} \right) \left| \psi \left( \frac{\bar{h} \cdot \bar{u} - b}{a} \right) \right| db\, d\bar{u}\, da$$
$$= C \int_{\|\bar{h}\|^{\frac{\alpha}{\beta}}}^{\|\bar{h}\|} \int_{S^1} \int_{\mathbb{R}} a^{\alpha-2} \left| \psi \left( \frac{\bar{h} \cdot \bar{u} - b}{a} \right) \right| db\, d\bar{u}\, da$$
$$+ C \int_{\|\bar{h}\|^{\frac{\alpha}{\beta}}}^{\|\bar{h}\|} \int_{S^1} \int_{\mathbb{R}} a^{-2} \frac{|b|^{\alpha}}{|\log |b||} \left| \psi \left( \frac{\bar{h} \cdot \bar{u} - b}{a} \right) \right| db\, d\bar{u}\, da.$$

After a simple change of variable $y = \dfrac{\bar{h} \cdot \bar{u} - b}{a}$ in both terms, and for the second term, since $\operatorname{supp}(\psi) \subseteq [-R, R]$, we have $|b| \le aR + \|\bar{h}\|$ and then $|b| \le (R+1)\|\bar{h}\|$ because we have under the integral $a \le \|\bar{h}\|$. For sufficiently small $\|\bar{h}\|$,

$$T_2 \le C \int_{0}^{\|\bar{h}\|} \int_{S^1} \int_{\mathbb{R}} a^{\alpha-2+1} |\psi(y)|\, d\, d\bar{u}\, da$$
$$+ C \int_{\|\bar{h}\|^{\frac{\alpha}{\beta}}}^{\|\bar{h}\|} \int_{S^1} \int_{\mathbb{R}} a^{-2+1} \frac{(R+1)^{\alpha} \|\bar{h}\|^{\alpha}}{|\log(R+1)\|\bar{h}\||} |\psi(y)|\, dy\, d\bar{u}\, da.$$

Since $\psi \in L^1(\mathbb{R})$, we obtain that

$$T_2 \le C' \|\bar{h}\|^{\alpha} + C'' \int_{\|\bar{h}\|^{\frac{\alpha}{\beta}}}^{\|\bar{h}\|} a^{-1} \frac{(R+1)^{\alpha} \|\bar{h}\|^{\alpha}}{|\log((R+1)\|\bar{h}\|)|}\, da.$$

For sufficiently small $\|\bar{h}\|$,

$$T_2 \le C' \|\bar{h}\|^{\alpha} + C''' \int_{\|\bar{h}\|^{\frac{\alpha}{\beta}}}^{\|\bar{h}\|} a^{-1} \frac{\|\bar{h}\|^{\alpha}}{|\log \|\bar{h}\||}\, da$$
$$= C' \|\bar{h}\|^{\alpha} + C''' \frac{\|\bar{h}\|^{\alpha}}{|\log \|\bar{h}\||} \left[ \log \|\bar{h}\| - \log \|\bar{h}\|^{\frac{\alpha}{\beta}} \right]$$
$$= C' \|\bar{h}\|^{\alpha} + C''' \|\bar{h}\|^{\alpha} \left[ 1 - \frac{\log \|\bar{h}\|^{\frac{\alpha}{\beta}}}{|\log \|\bar{h}\||} \right].$$

For the second term, we can see that costant $> 0$. Thus $T_2 \le C \|\bar{h}\|^{\alpha}$.

By the assumption,

$$T_3 \le C \int_{0}^{\|\bar{h}\|} \int_{S^1} \int_{\mathbb{R}} a^{\frac{3}{2}-\frac{1}{2}-3} \left( a^{\alpha} + \frac{|b|^{\alpha}}{|\log |b||} \right) \left| \psi \left( \frac{-b}{a} \right) \right| db\, d\bar{u}\, da$$
$$= C \int_{0}^{\|\bar{h}\|} \int_{S^1} \int_{\mathbb{R}} a^{\alpha-2} \left| \psi \left( \frac{-b}{a} \right) \right| db\, d\bar{u}\, da$$
$$+ C \int_{0}^{\|\bar{h}\|} \int_{S^1} \int_{\mathbb{R}} a^{-2} \frac{|b|^{\alpha}}{|\log |b||} \left| \psi \left( \frac{-b}{a} \right) \right| db\, d\bar{u}\, da.$$

After a simple change of variable $y = \dfrac{-b}{a}$ in both terms, and for the second term, since $\operatorname{supp}(\psi) \subseteq [-R, R]$, we have $|b| \leq aR$ and then $|b| \leq R\left\|\bar{h}\right\|$ because we have under the integral $a \leq \left\|\bar{h}\right\|$. For sufficiently small $\left\|\bar{h}\right\|$,

$$T_3 \leq C' \int_0^{\|\bar{h}\|} a^{\alpha-1}\, da + C'' \int_0^{\|\bar{h}\|} \int_{\mathbb{R}} a^{-2+1} \frac{(aR)^\alpha}{|\log(aR)|} \left|\psi\left(y\right)\right|\, dy\, da.$$

Since $\psi \in L^1(\mathbb{R})$ and for sufficiently small $\left\|\bar{h}\right\|$, we obtain that

$$\begin{aligned} T_3 &\leq C' \left\|\bar{h}\right\|^\alpha + C''' \int_0^{\|\bar{h}\|} \frac{a^{\alpha-1}}{|\log(\left\|\bar{h}\right\|)|}\, da \\ &\leq C' \left\|\bar{h}\right\|^\alpha + C''' \int_0^{\|\bar{h}\|} a^{\alpha-1}\, da \\ &\leq C' \left\|\bar{h}\right\|^\alpha + C''' \left[\frac{\left\|\bar{h}\right\|^\alpha}{\alpha}\right] \\ &\leq C \left\|\bar{h}\right\|^\alpha. \end{aligned}$$

Finally, by assumption, we derive

$$T_4 \leq C \int_{\|\bar{h}\|}^1 \int_{S^1} \int_{\mathbb{R}} a^{\frac{3}{2}-\frac{1}{2}-3} \left(a^\alpha + \frac{|b|^\alpha}{|\log|b||}\right) \left|\psi\left(\frac{\bar{h}\cdot\bar{u}-b}{a}\right) - \psi\left(\frac{-b}{a}\right)\right| db\, d\bar{u}\, da.$$

Since $\psi$ is differentiable everywhere so there is a constant $C > 0$ such that

$$\left|\psi\left(\frac{\bar{h}\cdot\bar{u}-b}{a}\right) - \psi\left(\frac{-b}{a}\right)\right| \leq C \left|\frac{\bar{h}\cdot\bar{u}}{a}\right| \leq C \frac{\left\|\bar{h}\right\|}{a}$$

and, also, since $\operatorname{supp}(\psi) \subseteq [-R, R]$ we have $|b| \leq aR + \left\|\bar{h}\right\|$, we can bound $T_4$ by

$$\begin{aligned} T_4 &\leq C \int_{\|\bar{h}\|}^1 \int_{S^1} \int_{|b| \leq aR + \|\bar{h}\|} a^{-2} \left(a^\alpha + \frac{(aR + \left\|\bar{h}\right\|)^\alpha}{|\log(aR + \left\|\bar{h}\right\|)|}\right) \frac{\left\|\bar{h}\right\|}{a}\, db\, d\bar{u}\, da \\ &= C' \left\|\bar{h}\right\| \int_{\|\bar{h}\|}^1 a^{-3} \left(a^\alpha + \frac{(aR + \left\|\bar{h}\right\|)^\alpha}{|\log(aR + \left\|\bar{h}\right\|)|}\right) (aR + \left\|\bar{h}\right\|)\, da. \end{aligned}$$

This integral runs over $\left\|\bar{h}\right\| \leq a \leq 1$, we get

$$\begin{aligned} T_4 &\leq C' \left\|\bar{h}\right\| \int_{\|\bar{h}\|}^1 a^{-3} \left(a^\alpha + \frac{a^\alpha (R+1)^\alpha}{|\log(\left\|\bar{h}\right\|)|}\right) (aR + \left\|\bar{h}\right\|)\, da. \\ &\leq C'' \left\|\bar{h}\right\| \int_{\|\bar{h}\|}^1 a^{-3} \left(a^\alpha + \frac{a^\alpha}{|\log(\left\|\bar{h}\right\|)|}\right) (aR + \left\|\bar{h}\right\|)\, da. \end{aligned}$$

For sufficiently small $\|\bar{h}\|$,

$$\begin{aligned} T_4 &\leq C''' \|\bar{h}\| \int_{\|\bar{h}\|}^{1} a^{\alpha-3}(aR + \|\bar{h}\|)\, da \\ &\leq C'''' \|\bar{h}\| \left[ \int_{\|\bar{h}\|}^{1} a^{\alpha-2}\, da + \int_{\|\bar{h}\|}^{1} a^{\alpha-3} \|\bar{h}\|\, da \right] \\ &= C'''' \|\bar{h}\| \left( \frac{1}{\alpha-1} - \frac{\|\bar{h}\|^{\alpha-1}}{\alpha-1} + \frac{\|\bar{h}\|}{\alpha-2} - \frac{\|\bar{h}\|^{\alpha-1}}{\alpha-2} \right) \\ &= C'''' \|\bar{h}\|^{\alpha} \left( \frac{1-\|\bar{h}\|^{1-\alpha}}{1-\alpha} + \frac{1-\|\bar{h}\|^{2-\alpha}}{2-\alpha} \right) \\ &= C''''' \|\bar{h}\|^{\alpha}. \end{aligned}$$

Thus $|f_{SS}(\bar{h}) - f_{SS}(\bar{0})| \leq C \|\bar{h}\|^{\alpha}$ for $\|\bar{h}\| \leq 1$. Together with the bound of $f_{SS}$, we conclude that $|f_{SS}(\bar{h}) - f_{SS}(\bar{0})| \leq C \|\bar{h}\|^{\alpha}$ for all $\bar{h} \in \mathbb{R}^2$. It follows that $f$ is Hölder continuous at $\bar{0}$ and hence at any $\bar{x}_0$, with exponent $\alpha$. □

## 4.2 Characterization of Hölder Regularity by The Smith Transform

Partial derivative of function $f : \mathbb{R}^d \to \mathbb{R}$ is denoted by $\partial^{\nu} f = \partial_1^{\nu_1} \partial_2^{\nu_2} ... \partial_d^{\nu_d} f$ where $\partial_i$ means the partial derivative with respect to the $i^{th}$-coordinate and the index $\nu = (\nu_1, \nu_2, ..., \nu_d) \in \mathbb{N}_0^d$ with $|\nu| = \nu_1 + \nu_2 + ... + \nu_d$. We pick a wavelet-type function $\varphi \in L^2(\mathbb{R}^2)$ is compactly supported function obeying

$$|\partial^{\nu} \varphi(\bar{x})| \leq C(1 + \|\bar{x}\|)^{-2} \quad \text{for} \quad |\nu| \leq 2, \bar{x} \in \mathbb{R}^2$$

and $\int \varphi_{a\bar{b}\theta}(\bar{x})\, d\bar{x} = 0$ for all $0 < a \leq a_0, b \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$. This wavelet-type function $\psi$ gives rise to the smith transform used to analyze Uniform and poinwise Hölder regularity of functions.

In this section we show how to analyze uniform and pointwise Hölder regularity by Smith transform. Generally speaking the amount of uniform and pointwise regularity of a function is reflected in its Smith transform by the decrease of the Smith transform at small scales as shown by the following theorem.

**Theorem 4.5.** *If a bounded function $f \in L^2(\mathbb{R}^2)$ has Hölder continuous exponent $\alpha$ for some $0 < \alpha < 1$, then there is a constant $C > 0$ such that*

$$|\overline{\Gamma}_f(a, \bar{b}, \theta)| \leq C a^{\frac{\alpha}{2} + \frac{3}{4}} \quad \text{for all} \;\; 0 < a < 1, \bar{b} \in \mathbb{R}^2, \;\; \text{and } \theta \in [0, 2\pi).$$

*Proof.* We have

$$\begin{aligned}
|\overline{\Gamma}_f(a, \bar{b}, \theta)| &= \left| \int_{\mathbb{R}^2} \varphi\left( D_{\frac{1}{a}} R_{-\theta}(\bar{x} - \bar{b}) \right) f(\bar{x})\, d\bar{x} \right| \\
&= \left| \int_{\mathbb{R}^2} a^{-\frac{3}{4}} \varphi\left( y_1, y_2 \right) f(x_1, x_2)\, dx \right|
\end{aligned}$$

where $y_1 = \frac{1}{a}\left( (x_1 - b_1)\cos\theta + (x_2 - b_2)\sin\theta \right)$ and $y_2 = \frac{1}{\sqrt{a}}\left( -(x_1 - b_1)\sin\theta + (x_2 - b_2)\cos\theta \right)$. A simple change of variable yields

$$\begin{aligned}
|\overline{\Gamma}_f(a, \bar{b}, \theta)| \leq \int_{\mathbb{R}^2} & a^{-\frac{3}{4} + \frac{3}{2}} |\varphi(y_1, y_2)| \\
& |f(ay_1\cos\theta - \sqrt{a} y_2 \sin\theta + b_1, ay_1\sin\theta + \sqrt{a} y_2\cos\theta + b_2)|\, dy_1\, dy_2.
\end{aligned}$$

Since $\int \varphi(\bar{x})\, d\bar{x} = 0$, we apply the uniform Hölder regularity of $f$ and get

$$\begin{aligned}
|\overline{\Gamma}_f(a, \bar{b}, \theta)| &\leq \int_{\mathbb{R}^2} a^{-\frac{3}{4} + \frac{3}{2}} |\varphi(y_1, y_2)| \\
& \quad |f(ay_1\cos\theta - \sqrt{a} y_2\sin\theta + b_1, ay_1\sin\theta + \sqrt{a} y_2\cos\theta + b_2) - f(b_1, b_2)|\, dy_1\, dy_2 \\
&\leq C \int_{\mathbb{R}^2} a^{-\frac{3}{4} + \frac{3}{2}} |\varphi(y_1, y_2)| \\
& \quad \left\| ay_1\cos\theta - \sqrt{a} y_2\sin\theta, ay_1\sin\theta + \sqrt{a} y_2\cos\theta \right\|^{\alpha}\, dy_1\, dy_2 \\
&= \int_{\mathbb{R}^2} a^{-\frac{3}{4} + \frac{3}{2}} |\varphi(y_1, y_2)| \left( a^2 y_1^2 + a y_2^2 \right)^{\frac{\alpha}{2}}\, dy_1\, dy_2 \\
&= C a^{\frac{\alpha}{2} + \frac{3}{4}} \int_{\mathbb{R}^2} |\varphi(\bar{y})| \left( a y_1^2 + y_2^2 \right)^{\frac{\alpha}{2}}\, d\bar{y} \\
&\leq C a^{\frac{\alpha}{2} + \frac{3}{4}} \int_{\mathbb{R}^2} |\varphi(\bar{y})|\, \|\bar{y}\|^{\alpha}\, d\bar{y}.
\end{aligned}$$

Since the last integral is finite as a result of the decay condition on $\varphi$, we get

$$|\overline{\Gamma}_f(a, b, \theta)| \leq C' a^{\frac{\alpha}{2} + \frac{3}{4}}.$$

□

The following is a converse theorem.

**Theorem 4.6.** *If, for some $\alpha \in (0,1)$, there is a Fourier multiplier $M$ of order 0 so that whenever $f \in L^2(\mathbb{R}^2)$ is a high-frequency function supported in frequency space $\|\bar{\xi}\| > \frac{2}{a_0}$ and a constant $C > 0$ for which*

$$|\langle \varphi_{a\bar{b}\theta}, Mf\rangle| \leq C' a^{\frac{\alpha}{2}+\frac{9}{4}} \text{ for all } 0 < a < a_0, \bar{b} \in \mathbb{R}^2 \text{ and } \theta \in [0, 2\pi),$$

*then $f$ is Hölder continuous with exponent $\alpha$.*

*Proof.* Without loss of generality, we suppose that $a_0 = 1$ (otherwise is always see the proof of the large scale part of Theorem 4.2). First of all, note that $f$ is bounded uniformly in $x$ as the decay of Smith transform, a simple change of variable and integrability of $\varphi$ yield

$$\begin{aligned}
|f(\bar{x})| &\leq \int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} |\langle \varphi_{a\bar{b}\theta}, Mf\rangle| \left|\varphi_{a,\bar{b},\theta}(\bar{x})\right| d\bar{b}\, d\theta \frac{da}{a^3} \\
&\leq C \int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}} a^{-\frac{3}{4}} \left|\varphi\left(D_{\frac{1}{a}} R_{-\theta}(\bar{x} - \bar{b})\right)\right| d\bar{b}\, d\theta \frac{da}{a^3} \\
C &\int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{3}{2}+\frac{3}{2}-3} |\varphi(\bar{y})|\, d\bar{y}\, d\theta\, da \\
&= C' \int_0^1 a^{\frac{\alpha}{2}}\, da \\
&= C' \left[\frac{2}{2+\alpha}\right] \\
&= C'' \quad < \quad \infty.
\end{aligned}$$

Let $\bar{x}, \bar{x}_0 \in \mathbb{R}^2$. Next, we look at $|f(\bar{x}) - f(\bar{x}_0)| \leq C \|\bar{x} - \bar{x}_0\|^\alpha$ for $|\bar{x} - \bar{x}_0| \leq 1$. We apply the inequality of Smith transform bounds and split the integral into fine and coarse scale ranges.

$$\begin{aligned}
|f(\bar{x}) - f(\bar{x}_0)| &\leq \int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} \left|\langle \varphi_{a,\bar{b},\theta}, Mf\rangle\right| \left|\varphi_{a,\bar{b},\theta}(\bar{x}) - \varphi_{a,\bar{b},\theta}(\bar{x}_0)\right| d\bar{b}\, d\theta \frac{da}{a^3} \\
&\leq C \int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}} \left|\varphi_{a,\bar{b},\theta}(\bar{x}) - \varphi_{a,\bar{b},\theta}(\bar{x}_0)\right| d\bar{b}\, d\theta \frac{da}{a^3} \\
&\leq C \int_0^{\|\bar{x}-\bar{x}_0\|^2} \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}} \left|\varphi_{a,\bar{b},\theta}(\bar{x})\right| + \left|\varphi_{a,\bar{b},\theta}(\bar{x}_0)\right| d\bar{b}\, d\theta \frac{da}{a^3} \\
&\quad + C \int_{\|\bar{x}-\bar{x}_0\|^2}^1 \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}} \left|\varphi_{a,\bar{b},\theta}(\bar{x}) - \varphi_{a,\bar{b},\theta}(\bar{x}_0)\right| d\bar{b}\, d\theta \frac{da}{a^3}. \qquad (3)
\end{aligned}$$

By the decay condition of $\varphi$ and all its first and second derivatives, $\varphi$ is differentiable everywhere with uniformly bounded gradients and so there is a constant $C > 0$ such that

$|\varphi(\bar{x})-\varphi(\bar{x}_0)| \le C\,\|\bar{x}-\bar{x}_0\|$. Also, since $\mathrm{supp}(\varphi)$ is contained in a ball of radius $R$ for some $R>0$, we can bound (3) by

$$\begin{aligned}
(3) &\le C\int_0^{\|\bar{x}-\bar{x}_0\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}-\frac{3}{4}+\frac{3}{2}}\,|\varphi(\bar{y})|\,d\bar{y}\,d\theta\,\frac{da}{a^3}\\
&\quad + C\int_{\|\bar{x}-\bar{x}_0\|^2}^{1}\int_0^{2\pi}\int_{\|\bar{x}-\bar{b}\|\le\sqrt{a}R+\|\bar{x}-\bar{x}_0\|} a^{\frac{\alpha}{2}+\frac{9}{4}-\frac{3}{4}-1}\,\|\bar{x}-\bar{x}_0\|\,d\bar{b}\,d\theta\,\frac{da}{a^3}\\
&= C'\int_0^{\|\bar{x}-\bar{x}_0\|} a^{\frac{\alpha}{2}}\,da + C''\,\|\bar{x}-\bar{x}_0\|\int_{\|\bar{x}-\bar{x}_0\|^2}^{1} a^{\frac{\alpha}{2}-\frac{5}{2}}\int_{\|\bar{x}-\bar{b}\|\le\sqrt{a}R+\|\bar{x}-\bar{x}_0\|} d\bar{b}\,da\\
&= C'''\,\|\bar{x}-\bar{x}_0\|^{\alpha+2}\\
&\quad + C''\,\|\bar{x}-\bar{x}_0\|\int_{\|\bar{x}-\bar{x}_0\|^2}^{1} a^{\frac{\alpha}{2}-\frac{5}{2}}\pi(\sqrt{a}R+\|\bar{x}-\bar{x}_0\|)^2\,da\\
&= C'''\,\|\bar{x}-\bar{x}_0\|^{\alpha+2} + C''\,\|\bar{x}-\bar{x}_0\|\left[\frac{1}{\alpha-1}+\frac{\|\bar{x}-\bar{x}_0\|}{\alpha-2}+\frac{\|\bar{x}-\bar{x}_0\|^2}{\alpha-3}\right]\\
&\quad - C''\,\|\bar{x}-\bar{x}_0\|\left[\frac{\|\bar{x}-\bar{x}_0\|^{\alpha-1}}{\alpha-1}+\frac{\|\bar{x}-\bar{x}_0\|^{\alpha-1}}{\alpha-2}+\frac{\|\bar{x}-\bar{x}_0\|^{\alpha-1}}{\alpha-3}\right]\\
&= C'''\,\|\bar{x}-\bar{x}_0\|^{\alpha+2}\\
&\quad + C''\,\|\bar{x}-\bar{x}_0\|^{\alpha}\left[\frac{1-\|\bar{x}-\bar{x}_0\|^{1-\alpha}}{1-\alpha}+\frac{1-\|\bar{x}-\bar{x}_0\|^{2-\alpha}}{2-\alpha}+\frac{1-\|\bar{x}-\bar{x}_0\|^{3-\alpha}}{3-\alpha}\right]\\
&\le C\,\|\bar{x}-\bar{x}_0\|^{\alpha}.
\end{aligned}$$

This holds for all $\|\bar{x}-\bar{x}_0\|\le 1$. Together with the uniform bounded of $f$ in $\bar{x}$, we conclude that $|f(\bar{x})-f(\bar{x}_0)|\le C\,\|\bar{x}-\bar{x}_0\|^{\alpha}$ for all $\bar{x},\bar{x}_0\in\mathbb{R}^2$. Therefore $f$ is Hölder continuous with exponent $\alpha$. □

The pointwise (local) regularity of a function implies an equivalent local decrease of its Smith transform at small scale as shown by the following theorem.

**Theorem 4.7.** *If a bounded function $f\in L^2(\mathbb{R}^2)$ is Hölder continuous at $\bar{x}_0$ with exponent $\alpha\in(0,1)$, then there is a constant $C>0$ such that*

$$|\bar{\Gamma}_f(a,\bar{b},\theta)| \le C a^{\frac{\alpha}{2}+\frac{3}{4}}\left(1+\frac{\|\bar{b}-\bar{x}_0\|^{\alpha}}{a^{\frac{\alpha}{2}}}\right)\quad \text{for all}\ \ 0<a\le 1,\bar{b}\in\mathbb{R}^2\ \text{and}\ \theta\in[0,2\pi),$$

*Proof.* By translating everything we can assume that $\bar{x}_0=\bar{0}$. Using the assumption that $\int\varphi(\bar{x})\,d\bar{x}=0$, i.e., $\int\varphi_{a,\bar{b},\theta}(\bar{x})\,d\bar{x}=0$ for all $0<a<1,\bar{b}\in\mathbb{R}^2$ and $\theta\in[0,2\pi)$ and that $f$ is

Hölder regular at 0 with exponent $\alpha$, we again have

$$\begin{aligned}
|\Gamma_f(a,\bar{b}+\bar{0},\theta)| &\leq \int_{\mathbb{R}^2} \left|\varphi_{a,\bar{b},\theta}(\bar{x})\right| |f(\bar{x})-f(\bar{0})| \, d\bar{x} \\
&\leq Ca^{-\frac{3}{4}} \int_{\mathbb{R}^2} \left|\varphi(D_{\frac{1}{a}}R_{-\theta}(\bar{x}-\bar{b}))\right| \|\bar{x}-\bar{0}\|^{\alpha} \, d\bar{x} \\
&\leq Ca^{-\frac{3}{4}} \int_{\mathbb{R}^2} |\varphi(\bar{y})| \left(a^{\frac{1}{2}} \|\bar{y}\| + \left\|\bar{b}\right\|\right)^{\alpha} a^{\frac{3}{2}} \, d\bar{y} \\
&= Ca^{\frac{\alpha}{2}+\frac{3}{4}} \int_{\mathbb{R}^2} |\varphi(\bar{y})| \left(\|\bar{y}\| + \frac{\left\|\bar{b}\right\|}{a^{\frac{1}{2}}}\right)^{\alpha} d\bar{y} \\
&\leq C'a^{\frac{\alpha}{2}+\frac{3}{4}} \int_{\mathbb{R}^2} |\varphi(\bar{y})| \left(\|\bar{y}\|^{\alpha} + \frac{\left\|\bar{b}\right\|^{\alpha}}{a^{\frac{\alpha}{2}}}\right) d\bar{y} \\
&\leq C'a^{\frac{\alpha}{2}+\frac{3}{4}} \left(\int_{\mathbb{R}^2} |\varphi(\bar{y})| \, \|\bar{y}\|^{\alpha} \, d\bar{y} + \int_{\mathbb{R}^2} |\varphi(\bar{y})| \frac{\left\|\bar{b}\right\|^{\alpha}}{a^{\frac{\alpha}{2}}} \, d\bar{y}\right) \\
&= C''a^{\frac{\alpha}{2}+\frac{3}{4}} \left(1 + \frac{\left\|\bar{b}\right\|^{\alpha}}{a^{\frac{\alpha}{2}}}\right)
\end{aligned}$$

where we have used the integrability of $|\varphi(\bar{y})| \, \|\bar{y}\|^{\alpha}$ and $|\varphi(\bar{y})|$ in the last inequality. Thus

$$|\Gamma_f(a,\bar{b}+\bar{x}_0,\theta)| \leq Ca^{\frac{\alpha}{2}+\frac{3}{4}} \left(1 + \frac{\left\|\bar{b}\right\|^{\alpha}}{a^{\frac{\alpha}{2}}}\right).$$

Therefore

$$|\Gamma_f(a,\bar{b},\theta)| \leq Ca^{\frac{\alpha}{2}+\frac{3}{4}} \left(1 + \frac{\left\|\bar{b}-\bar{x}_0\right\|^{\alpha}}{a^{\frac{\alpha}{2}}}\right).$$

□

Above theorem shows that the Smith wavelet transform can also be used to characterize local regularity. The following is a converse theorem.

**Theorem 4.8.** *If, for some $\beta > 0$ and $\alpha \in (0,1)$, there is a Fourier multiplier $M$ of order 0 so that whenever $f$ is a high-frequency function supported in frequency space $\left\|\xi\right\| > \frac{2}{a_0}$ and a constant $C > 0$, such that*

$$|\langle \varphi_{a\bar{b}\theta}, Mf \rangle| \leq Ca^{\frac{\beta}{2}+\frac{9}{4}} \quad \text{for } 0 < a < a_0 \quad \text{uniformly in } \bar{b} \in \mathbb{R}^2$$

*and there is a constant $C > 0$ for which*

$$\left|\langle \varphi_{a,\bar{b},\theta}, Mf \rangle\right| \leq Ca^{\frac{\alpha}{2}+\frac{9}{4}} \left(1 + \frac{\left\|\bar{b}-\bar{x}_0\right\|^{\alpha}}{a^{\frac{\alpha}{2}}}\right) \quad \text{for all } \ 0 < a \leq 1, \bar{b} \in \mathbb{R}^2 \text{ and } \theta \in [0,2\pi),$$

*then $f$ is Hölder continuous at $\bar{x}_0$ with exponent $\alpha$.*

*Proof.* First of all, note that $f$ is bounded uniformly in $\bar{x}$ as a simple change of variable and integrability of $\varphi$ yield

$$\begin{aligned}
|f(\bar{x})| &\le \int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} \left|\langle \varphi_{a,\bar{b},\theta}, Mf\rangle\right| \left|\varphi_{a,\bar{b},\theta}(\bar{x})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&\le C \int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}} a^{-\frac{3}{4}} \left|\varphi\left(D_{\frac{1}{a}} R_\theta(\bar{x}-\bar{b})\right)\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&= C \int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{3}{2}+\frac{3}{2}-3} \left|\varphi(\bar{y})\right| \, d\bar{y} \, d\theta \, da \\
&= C' \int_0^1 a^{\frac{\beta}{2}} \, da \\
&= C' \left[\frac{2}{2+\beta}\right] \le C'' \quad < \quad \infty.
\end{aligned}$$

We therefore only have to check $|f(\bar{x}_0+\bar{h}) - f(\bar{x}_0)| \le C \left\|\bar{h}\right\|^\alpha$ for small $\bar{h}$, i.e. $\left\|\bar{h}\right\| \le 1$. By translating everything, we can assume $\bar{x}_0 = 0$, and we obtain

$$\begin{aligned}
|f(\bar{h}) - f(\bar{0})| &\le \int_0^1 \int_0^{2\pi} \int_{\mathbb{R}} \left|\langle \varphi_{a,\bar{b},\theta}, Mf\rangle\right| \left|\varphi_{a,\bar{b},\theta}(\bar{h}) - \varphi_{a,\bar{b},\theta}(\bar{0})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&\le \int_0^{\left\|\bar{h}\right\|^{\frac{2\alpha}{\beta}}} \int_0^{2\pi} \int_{\mathbb{R}} \left|\langle \varphi_{a,\bar{b},\theta}, Mf\rangle\right| \left|\varphi_{a,\bar{b},\theta}(\bar{h})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&\quad + \int_{\left\|\bar{h}\right\|^{\frac{2\alpha}{\beta}}}^{\left\|\bar{h}\right\|^2} \int_0^{2\pi} \int_{\mathbb{R}} \left|\langle \varphi_{a,\bar{b},\theta}, Mf\rangle\right| \left|\varphi_{a,\bar{b},\theta}(\bar{h})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&\quad + \int_0^{\left\|\bar{h}\right\|^2} \int_0^{2\pi} \int_{\mathbb{R}} \left|\langle \varphi_{a,\bar{b},\theta}, Mf\rangle\right| \left|\varphi_{a,\bar{b},\theta}(\bar{0})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&\quad + \int_{\left\|\bar{h}\right\|^2}^1 \int_0^{2\pi} \int_{\mathbb{R}} \left|\langle \varphi_{a,\bar{b},\theta}, Mf\rangle\right| \left|\varphi_{a,\bar{b},\theta}(\bar{h}) - \varphi_{a,\bar{b},\theta}(\bar{0})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3}.
\end{aligned}$$

By the assumption, we get

$$\begin{aligned}
|f(\bar{h}) - f(\bar{0})| &\le C \int_0^{\left\|\bar{h}\right\|^{\frac{2\alpha}{\beta}}} \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\beta}{2}+\frac{9}{4}} \left|\varphi_{a,\bar{b},\theta}(\bar{h})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&\quad + C \int_{\left\|\bar{h}\right\|^{\frac{2\alpha}{\beta}}}^{\left\|\bar{h}\right\|^2} \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}} \left(1 + \frac{\left\|\bar{b}\right\|^\alpha}{a^{\frac{\alpha}{2}}}\right) \left|\varphi_{a,\bar{b},\theta}(\bar{h})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&\quad + C \int_0^{\left\|\bar{h}\right\|^2} \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}} \left(1 + \frac{\left\|\bar{b}\right\|^\alpha}{a^{\frac{\alpha}{2}}}\right) \left|\varphi_{a,\bar{b},\theta}(\bar{0})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3} \\
&\quad + C \int_{\left\|\bar{h}\right\|^2}^1 \int_0^{2\pi} \int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}} \left(1 + \frac{\left\|\bar{b}\right\|^\alpha}{a^{\frac{\alpha}{2}}}\right) \left|\varphi_{a,\bar{b},\theta}(\bar{h}) - \varphi_{a,\bar{b},\theta}(\bar{0})\right| \, d\bar{b} \, d\theta \, \frac{da}{a^3}.
\end{aligned}$$

Let us denote the four terms on the right-hand side by $T_1$, $T_2$, $T_3$, and $T_4$.

After a change of variable, integrability of $\varphi$ yields

$$
\begin{aligned}
T_1 &= C\int_0^{\|\bar{h}\|^{\frac{2\alpha}{\beta}}}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\beta}{2}+\frac{9}{4}}a^{-\frac{3}{4}}\left|\varphi\left(D_{\frac{1}{a}}R_{-\theta}(\bar{x}-\bar{b})\right)\right| d\bar{b}\,d\theta\,\frac{da}{a^3}\\
&= C\int_0^{\|\bar{h}\|^{\frac{2\alpha}{\beta}}}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\beta}{2}+\frac{3}{2}+\frac{3}{2}-3}\left|\varphi\left(\bar{y}\right)\right| d\bar{y}\,d\theta\,da\\
&= C'\int_0^{\|\bar{h}\|^{\frac{2\alpha}{\beta}}} a^{\frac{\beta}{2}}\,da\\
&= C'\left[\frac{2}{2+\beta}\right] \le C'' \quad < \quad \infty.
\end{aligned}
$$

For $T_2$, we have

$$
\begin{aligned}
T_2 = C&\int_{\|\bar{h}\|^{\frac{2\alpha}{\beta}}}^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\alpha}{2}-\frac{3}{4}-\frac{3}{4}}\left|\varphi\left(D_{\frac{1}{a}}R_{-\theta}(\bar{h}-\bar{b})\right)\right| d\bar{b}\,d\theta\,da\\
&+C\int_{\|\bar{h}\|^{\frac{2\alpha}{\beta}}}^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\alpha}{2}-\frac{3}{4}-\frac{3}{4}}\frac{\|\bar{b}\|^{\alpha}}{a^{\frac{\alpha}{2}}}\left|\varphi\left(D_{\frac{1}{a}}R_{-\theta}(\bar{h}-\bar{b})\right)\right| d\bar{b}\,d\theta\,da.
\end{aligned}
$$

After a change of variable in both terms by $\bar{y} = D_{\frac{1}{a}}R_{-\theta}(\bar{h}-\bar{b})$, since $\operatorname{supp}(\varphi) \subseteq B(0,R)$, the support of $\varphi_{a\bar{b}\theta}(\bar{h}) = \varphi\left(D_{\frac{1}{a}}R_{-\theta}(\bar{h}-\cdot)\right)$ lies in the ball $B(\bar{h},\sqrt{a}R)$ for each $\bar{h}\in\mathbb{R}^2, 0 < a < 1$ and $\theta\in[0,2\pi)$. Using this fact, we obtain that $\|\bar{b}\| \le \|\bar{h}\| + \sqrt{a}\,\|\bar{y}\| \le \|\bar{h}\| + \sqrt{a}R$, we can bound the two integrals in $T_2$ as follows

$$
\begin{aligned}
T_2 \le C&\int_0^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\alpha}{2}-\frac{3}{2}+\frac{3}{2}}\left|\varphi\left(\bar{y}\right)\right| d\bar{y}\,d\theta\,da\\
&+C\int_{\|\bar{h}\|^{\frac{2\alpha}{\beta}}}^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{3}{2}-\frac{3}{2}}\left(\sqrt{a}R+\|\bar{h}\|\right)^{\alpha}\left|\varphi\left(\bar{y}\right)\right| d\bar{y}\,d\theta\,da\\
&= C'\int_0^{\|\bar{h}\|^2} a^{\frac{\alpha}{2}}\,da + C''\int_{\|\bar{h}\|^{\frac{2\alpha}{\beta}}}^{\|\bar{h}\|^2}\left(\sqrt{a}R+\|\bar{h}\|\right)^{\alpha}\,da\\
&\le C'\frac{2}{2+\alpha}\|\bar{h}\|^{\alpha+2} + C'''\|\bar{h}\|^{\alpha}\int_{\|\bar{h}\|^{\frac{2\alpha}{\beta}}}^{\|\bar{h}\|^2} 1\,da\\
&= C''''\|\bar{h}\|^{\alpha} + C''\|\bar{h}\|^{\alpha}\left(\|\bar{h}\|^2 - \|\bar{h}\|^{\frac{2\alpha}{\beta}}\right)\\
&= C\|\bar{h}\|^{\alpha}
\end{aligned}
$$

where we have used $a \le \|\bar{h}\|^2$ in the last inequality. Estimate of $T_3$ can be derived in the same fashion as that of $T_2$.

$$
\begin{aligned}
T_3 &= C\int_0^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} Ca^{\frac{\alpha}{2}+\frac{9}{4}-3}\left(1+\frac{\|\bar{b}\|^{\alpha}}{a^{\frac{\alpha}{2}}}\right)\left|\varphi_{a,\bar{b},\theta}(\bar{0})\right|\, d\bar{b}\, d\theta\, da \\
&= C\int_0^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\alpha}{2}-\frac{3}{4}-\frac{3}{4}}\left|\varphi\left(D_{\frac{1}{a}}R_{-\theta}(-\bar{b})\right)\right|\, d\bar{b}\, d\theta\, da \\
&\quad + C\int_0^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\alpha}{2}-\frac{3}{4}-\frac{3}{4}}\frac{\|\bar{b}\|^{\alpha}}{a^{\frac{\alpha}{2}}}\left|\varphi\left(D_{\frac{1}{a}}R_{-\theta}(-\bar{b})\right)\right|\, d\bar{b}\, d\theta\, da \\
&= C\int_0^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\alpha}{2}-\frac{3}{2}+\frac{3}{2}}\left|\varphi\left(\bar{y}\right)\right|\, d\bar{y}\, d\theta\, da \\
&\quad + C\int_0^{\|\bar{h}\|^2}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{3}{2}-\frac{3}{2}}|\sqrt{a}R|^{\alpha}\left|\varphi\left(y\right)\right|\, d\bar{y}\, d\theta\, da \\
&= C'\int_0^{\|\bar{h}\|^2} a^{\frac{\alpha}{2}}\, da + C''\int_0^{\|\bar{h}\|^2}|\sqrt{a}R|^{\alpha}\, da \\
&\le C'\frac{2}{2+\alpha}\|\bar{h}\|^{\alpha+2} + C'''\|\bar{h}\|^{\alpha}\int_0^{\|\bar{h}\|^2} 1\, da \\
&= C''''\|\bar{h}\|^{\alpha} + C''\|\bar{h}\|^{\alpha}\left(\|\bar{h}\|^2\right) \\
&= C\|\bar{h}\|^{\alpha}
\end{aligned}
$$

where we have again used $a \le \|\bar{h}\|^2$ in the last inequality.

Finally for $T_4$,

$$
T_4 = C\int_{\|\bar{h}\|^2}^{1}\int_0^{2\pi}\int_{\mathbb{R}} a^{\frac{\alpha}{2}+\frac{9}{4}-\frac{3}{4}}\left(1+\frac{\|\bar{b}\|^{\alpha}}{a^{\frac{\alpha}{2}}}\right)\left|\varphi\left(D_{\frac{1}{a}}R_{-\theta}(\bar{h}-\bar{b})\right)-\varphi\left(D_{\frac{1}{a}}R_{-\theta}(-\bar{b})\right)\right|\, d\bar{b}\, d\theta\, \frac{da}{a^3}.
$$

We then use the properties of $\varphi$ that it has bounded derivatives so there is a constant $C > 0$ such that

$$
\begin{aligned}
\left|\varphi\left(D_{\frac{1}{a}}R_{-\theta}(\bar{h}-\bar{b})\right)-\varphi\left(D_{\frac{1}{a}}R_{-\theta}(-\bar{b})\right)\right| &\le C\left\|D_{\frac{1}{a}}R_{-\theta}(\bar{h}-\bar{b})-D_{\frac{1}{a}}R_{-\theta}(-\bar{b})\right\| \\
&\le C\left\|D_{\frac{1}{a}}R_{-\theta}(\bar{h})\right\| \\
&\le \frac{C}{a}\left\|R_{-\theta}(\bar{h})\right\| \\
&= \frac{C}{a}\left\|\bar{h}\right\|,
\end{aligned}
$$

and, also, since $\psi$ has compact support, we have $\|\bar{b}\| \leq \sqrt{a}R + \|\bar{h}\|$ we can bound $T_4$ by

$$\begin{aligned}
T_4 &\leq C \int_{\|\bar{h}\|^2}^{1} \int_0^{2\pi} \int_{\|\bar{b}\| \leq \sqrt{a}R + \|\bar{h}\|} a^{\frac{\alpha}{2}+\frac{3}{2}-3} \left(1 + \frac{\left(\sqrt{a}R + \|\bar{h}\|\right)^{\alpha}}{a^{\frac{\alpha}{2}}}\right) \frac{1}{a} \|\bar{h}\| \, d\bar{b} \, d\theta \, da \\
&= C' \|\bar{h}\| \int_{\|\bar{h}\|^2}^{1} a^{\frac{\alpha}{2}-\frac{3}{2}-1} \left(1 + \frac{\left(\sqrt{a}R + \|\bar{h}\|\right)^{\alpha}}{a^{\frac{\alpha}{2}}}\right) \pi \left(\sqrt{a}R + \|\bar{h}\|\right)^2 \, da.
\end{aligned}$$

Since we have integrate over $\|\bar{h}\| \leq a \leq 1$, we get

$$\begin{aligned}
T_4 &\leq C' \|\bar{h}\| \int_{\|\bar{h}\|^2}^{1} a^{\frac{\alpha}{2}-\frac{5}{2}} \left(1 + \frac{(\sqrt{a}R + \sqrt{a})^{\alpha}}{a^{\frac{\alpha}{2}}}\right) \pi \left(\sqrt{a}R + \|\bar{h}\|\right)^2 \, da \\
&= C'' \|\bar{h}\| \int_{\|\bar{h}\|^2}^{1} a^{\frac{\alpha}{2}-\frac{5}{2}} \left(aR^2 + 2\sqrt{a}R + \|\bar{h}\|^2\right) da \\
&= C'' \|\bar{h}\| \left(\frac{2R^2}{\alpha-1} + \frac{4R\|\bar{h}\|}{\alpha-2} + \frac{2\|\bar{h}\|^2}{\alpha-3} - \frac{2R^2\|\bar{h}\|^{\alpha-1}}{\alpha-1} + \frac{4R\|\bar{h}\|^{\alpha-1}}{\alpha-2} + \frac{2\|\bar{h}\|^{\alpha-1}}{\alpha-3}\right) \\
&= C'' \|\bar{h}\|^{\alpha} \left(\frac{2R^2}{1-\alpha}\left(1 - \|\bar{h}\|\right) + \frac{4R}{2-\alpha}\left(1 - \|\bar{h}\|^2\right) + \frac{2}{3-\alpha}\left(1 - \|\bar{h}\|^3\right)\right) \\
&= C''' \|\bar{h}\|^{\alpha}.
\end{aligned}$$

Thus $|f_{SS}(\bar{h}) - f_{SS}(\bar{0})| \leq C \|\bar{h}\|^{\alpha}$ for $\|\bar{h}\| \leq 1$. Together with the bound of $f_{SS}$, we conclude that $|f_{SS}(\bar{h}) - f_{SS}(\bar{0})| \leq C \|\bar{h}\|^{\alpha}$ for all $\bar{h} \in \mathbb{R}^2$. Therefore, $f$ is Hölder continuous at $\bar{0}$ and hence at $\bar{x}_0$, with exponent $\alpha$ as desired. □

Theorem 4.5 gives a characterization of the Hölder regularity over an interval but not at a point. Theorem 4.7 shows that one can also estimate the Hölder regularity of function, precisely at a point $x_0$. Both uniform and local Hölder regularity give a necessary condition and a sufficient condition, but not a necessary and sufficient condition.



## 4.3 The Curvelet Transform of Functions with Hölder Regularity

In this section we show how to analyze uniform and pointwise Hölder regularity by curvelet transforms. Generally speaking the amount of uniform regularity of a function is reflected in its curvelet transform at small scales as shown in the following theorem.

**Theorem 4.9.** *Suppose the windows $V$ and $W$ are infinitely differentiable and of compact support. If a bounded function $f$ is Hölder continuous with exponent $\alpha$ , $0 < \alpha < 1$, then there is a constant $C > 0$ such that $|\Gamma_f(a, \bar{b}, \theta)| \leq Ca^{\frac{\alpha}{2}+\frac{1}{4}}$ for all $0 < a < 1, \bar{b} \in \mathbb{R}^2$ and, $\theta \in [0, 2\pi)$.*

*Proof.* Since $W(0) = 0$ we have $\hat{\gamma}_{a,\bar{b},\theta}(0) = 0$ we get that $\int \gamma_{a\bar{b}\theta}(\bar{x})\, d\bar{x} = 0$. By Lemma 3.6, $|\gamma_{a,\bar{b},\theta}(\bar{x})| \leq C_N a^{-\frac{3}{4}} \left\langle \left\|\bar{x} - \bar{b}\right\|_{a,\theta} \right\rangle^{-N}$ for all $N \geq 4$, then

$$\begin{aligned}
|\Gamma_f(a, \bar{b}, \theta)| &\leq \int_{\mathbb{R}^2} \left|\gamma_{a,\bar{b},\theta}(\bar{x})\right| \left|f(\bar{x}) - f(\bar{b})\right|\, d\bar{x} \\
&\leq C \int_{\mathbb{R}^2} \left|\gamma_{a,\bar{b},\theta}(\bar{x})\right| \left\|\bar{x} - \bar{b}\right\|^{\alpha}\, d\bar{x} \\
&\leq C \int_{\mathbb{R}^2} C_N a^{-\frac{3}{4}} \left\langle \left\|\bar{x} - \bar{b}\right\|_{a,\theta} \right\rangle^{-N} \left\|\bar{x} - \bar{b}\right\|^{\alpha}\, d\bar{x} \\
&= C' a^{-\frac{3}{4}} \int_{\mathbb{R}^2} \frac{\left\|\bar{x} - \bar{b}\right\|^{\alpha}}{\left(1 + \left\|\bar{x} - \bar{b}\right\|_{a,\theta}^2\right)^{\frac{N}{2}}}\, d\bar{x} \\
&\leq C' a^{-\frac{3}{4}} \int_{\mathbb{R}^2} \frac{\left\|\bar{x} - \bar{b}\right\|^{\alpha}}{\left(1 + \frac{\left\|\bar{x} - \bar{b}\right\|^2}{a}\right)^{\frac{N}{2}}}\, d\bar{x} \qquad (4)
\end{aligned}$$

where we have used the fact that $\|v\|_{a,\theta} \geq \frac{\|v\|}{\sqrt{a}}$ in the last inequality.

In polar coorinates, (4) becomes

$$\begin{aligned}
(4) &= C' a^{-\frac{3}{4}} \int_0^{\infty} \int_0^{2\pi} \frac{r^{\alpha+1}}{\left(1 + \frac{r^2}{a}\right)^{\frac{N}{2}}}\, d\omega dr \\
&= C'' a^{-\frac{3}{4}} \int_0^{\infty} \frac{r^{\alpha+1}}{\left(1 + \frac{r^2}{a}\right)^{\frac{N}{2}}}\, dr \\
&= C'' a^{-\frac{3}{4}} \int_0^{\infty} \frac{y^{\alpha+1} a^{\frac{\alpha}{2}+\frac{1}{2}}}{(1 + y^2)^{\frac{N}{2}}} a^{\frac{1}{2}}\, dy \\
&= C'' a^{\frac{\alpha}{2}+\frac{1}{4}} \int_0^{\infty} \frac{y^{\alpha+1}}{(1 + y^2)^{\frac{N}{2}}}\, dy \quad = \quad C''' a^{\frac{\alpha}{2}+\frac{1}{4}}.
\end{aligned}$$

Note that we let $y = \frac{r}{\sqrt{a}}$ and the last integral is integrable for all $N \geq 4$. □

The pointwise (local) regularity of a function implies an equivalent local decrease of its curvelet coefficients at small scale as shown in the following theorem.

**Theorem 4.10.** *Suppose the windows $V$ and $W$ are infinitely differentiable and of compact support. If a bounded function $f$ is Hölder continuous at $\bar{x}_0$ with exponent $\alpha$ , $0 < \alpha < 1$, then there is a constant $C > 0$ such that*

$$|\Gamma_f(a, \bar{b}, \theta)| \leq C a^{\frac{\alpha}{2}+\frac{1}{4}} \left(1 + \frac{\left\|\bar{b} - \bar{x}_0\right\|^\alpha}{a^{\frac{\alpha}{2}}}\right)$$

*for all $0 < a < 1, \bar{b} \in \mathbb{R}^2$ and, $\theta \in [0, 2\pi)$.*

*Proof.* Since $W(0) = 0$ we have $\hat{\gamma}_{a\bar{b}\theta}(0) = 0$, i.e., $\int \gamma_{a\bar{b}\theta}(\bar{x})\, d\bar{x} = 0$. By translating everything we can assume that $\bar{x}_0 = \bar{0}$. Then we consider $|\Gamma_f(a, \bar{b} + \bar{x}_0, \theta)|$ at $\bar{x}_0 = \bar{0}$.

$$\begin{aligned}
|\Gamma_f(a, \bar{b} + \bar{0}, \theta)| &\leq \int_{\mathbb{R}^2} \left|\gamma_{a,\bar{b},\theta}(\bar{x})\right| |f(\bar{x}) - f(\bar{0})|\, d\bar{x} \\
&\leq C \int_{\mathbb{R}^2} \left|\gamma_{a,\bar{b},\theta}(\bar{x})\right| \|\bar{x} - \bar{0}\|^\alpha\, d\bar{x} \\
&\leq C \int_{\mathbb{R}^2} C_N a^{-\frac{3}{4}} \left\langle \left\|\bar{x} - \bar{b}\right\|_{a,\theta} \right\rangle^{-N} \|\bar{x} - \bar{0}\|^\alpha\, d\bar{x} \\
&= C' a^{-\frac{3}{4}} \int_{\mathbb{R}^2} \frac{\|\bar{x}\|^\alpha}{\left(1 + \left\|\bar{x} - \bar{b}\right\|_{a,\theta}^2\right)^N}\, d\bar{x}
\end{aligned}$$

where we have applied Lemma 3.6 with $N \geq 4$. Using the fact that $\left\|\bar{x} - \bar{b}\right\|_{a,\theta}^2 \geq \frac{\left\|\bar{x} - \bar{b}\right\|^2}{a}$, we get

$$|\Gamma_f(a, \bar{b}, \theta)| \leq C' a^{-\frac{3}{4}} \int_{\mathbb{R}^2} \frac{\|\bar{x}\|^\alpha}{\left(1 + \frac{\left\|\bar{x} - \bar{b}\right\|^2}{a}\right)^{\frac{N}{2}}}\, d\bar{x}.$$

After a change of variable,

$$\begin{aligned}
|\Gamma_f(a, \bar{b}, \theta)| &\leq C' a^{-\frac{3}{4}} \int_{\mathbb{R}^2} \frac{\left\|\bar{x} + \bar{b}\right\|^\alpha}{\left(1 + \frac{\|\bar{x}\|^2}{a}\right)^{\frac{N}{2}}}\, d\bar{x} \\
&\leq C' a^{-\frac{3}{4}} \int_{\mathbb{R}^2} \frac{\left(\|\bar{x}\| + \|\bar{b}\|\right)^\alpha}{\left(1 + \frac{\|\bar{x}\|^2}{a}\right)^{\frac{N}{2}}}\, d\bar{x}.
\end{aligned}$$

In polar coodinates,

$$\begin{aligned}
|\Gamma_f(a, \bar{b}, \theta)| &\leq C' a^{-\frac{3}{4}} \int_0^\infty \int_0^{2\pi} \frac{\left(r + \|\bar{b}\|\right)^\alpha}{\left(1 + \frac{r^2}{a}\right)^{\frac{N}{2}}} r\, d\phi\, dr \\
&\leq C'' a^{-\frac{3}{4}} \int_0^\infty \frac{\left(r + \|\bar{b}\|\right)^\alpha}{\left(1 + \frac{r^2}{a}\right)^{\frac{N}{2}}} r\, dr.
\end{aligned}$$

We again have, after a change of variable $y = \frac{r}{\sqrt{a}}$,

$$
\begin{aligned}
|\Gamma_f(a,\bar{b},\theta)| &\leq C'' a^{-\frac{3}{4}} \int_0^\infty \frac{\left(\sqrt{a}y + \|\bar{b}\|\right)^\alpha}{(1+y^2)^{\frac{N}{2}}} \sqrt{a}y\sqrt{a}\,dy \\
&\leq C'' a^{\frac{1}{4}} \int_0^\infty \frac{\left(a^{\frac{\alpha}{2}} y^\alpha + \|\bar{b}\|^\alpha\right)}{(1+y^2)^{\frac{N}{2}}} y\,dy \\
&\leq C'' a^{\frac{1}{4}} \left[ a^{\frac{\alpha}{2}} \int_0^\infty \frac{y^{\alpha+1}}{(1+y^2)^{\frac{N}{2}}}\,dy + \|\bar{b}\|^\alpha \int_0^\infty \frac{y}{(1+y^2)^{\frac{N}{2}}}\,dy \right] \\
&= C''' a^{\frac{1}{4}} \left( a^{\frac{\alpha}{2}} + \|\bar{b}\|^\alpha \right) \\
&= C''' a^{\frac{\alpha}{2}+\frac{1}{4}} \left( 1 + \frac{\|\bar{b}\|^\alpha}{a^{\frac{\alpha}{2}}} \right).
\end{aligned}
$$

Note that the last integral is integrable for all $N \geq 4$. Hence

$$
|\Gamma_f(a,\bar{b}+\bar{x}_0,\theta)| \leq C a^{\frac{\alpha}{2}+\frac{1}{4}} \left( 1 + \frac{\|\bar{b}\|^\alpha}{a^{\frac{\alpha}{2}}} \right)
$$

and so

$$
|\Gamma_f(a,\bar{b},\theta)| \leq C a^{\frac{\alpha}{2}+\frac{1}{4}} \left( 1 + \frac{\|\bar{b}-\bar{x}_0\|^\alpha}{a^{\frac{\alpha}{2}}} \right).
$$

□

## 4.4 Directional Regularity

See also [14,15] for the following definition.

**Definition** Let $\bar{x} \in \mathbb{R}^d$, $d > 1$ and let $\alpha > 0$, $\bar{\Phi}$ be a vector in $\mathbb{R}^d$ of modulus 1. A function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $C^\alpha_{\bar{\Phi}}(\bar{x})$ if the one dimensional function $g : t \longmapsto f(\bar{x} + \bar{\Phi}t)$ belongs to $C^\alpha(0)$, i.e., there exists a constant $C > 0$ s.t. $|g(t) - g(0)| \leq C|t|^\alpha$ for all $t$ in a neighborhood of 0. We can say that $|f(\bar{x} + \bar{\Phi}t) - f(\bar{x})| \leq C|t|^\alpha$.

For $d = 2$ a given vector $\bar{\Phi}$ of modulus 1 is of the form $\bar{\Phi} = (\cos\phi, \sin\phi)$ for some $\phi \in [0, 2\pi)$.

**Example** Let $\alpha : S^1 \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^2$, then $\bar{x} = r\bar{\Phi} = r(\cos\phi, \sin\phi)$ where $r \in [0, \infty)$ and $\phi \in (0, 2\pi]$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(\bar{x}) = r^{\alpha(\bar{\Phi})}$. Then $f$ has the directional Hölder exponent $\alpha(\bar{\Phi})$ at $\bar{0}$, i.e. $f \in C^{\alpha(\bar{\Phi})}_{\bar{\Phi}}(\bar{0})$.

*Proof.* Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(t) = f(t\bar{\Phi})$ for all $t \in \mathbb{R}$. Let $t$ be in a neighborhood of 0. Then

$$|g(t) - g(0)| = |f(t\bar{\Phi}) - f(\bar{0})| = ||t|^{\alpha(\bar{\Phi})} - 0| = |t|^{\alpha(\bar{\Phi})}.$$

Thus $g \in C^{\alpha(\bar{\Phi})}(0)$ and then by definition we have $f \in C^{\alpha(\bar{\Phi})}_{\bar{\Phi}}(\bar{0})$. □

**Theorem 4.11.** *Suppose that $f$ has compact support. Let $\phi \in [0, 2\pi)$ and $\bar{\Phi} = (\cos\phi, \sin\phi)$. If $f$ belongs to $C^{\alpha(\bar{\Phi})}_{\bar{\Phi}}(\bar{x})$ for all $\bar{x} \in \mathbb{R}^2$ such that $\bar{x} \perp \bar{\Phi}$ then $|\mathcal{R}(f)(a, b, \phi)| \leq Ca^{\frac{1}{2}} \left(a^{\alpha(\bar{\Phi})} + |b|^{\alpha(\bar{\Phi})}\right)$ for $a > 0$, and $b \in \mathbb{R}$.*

*Proof.* Let $\phi \in [0, 2\pi)$ and $\bar{\Phi} = (\cos\phi, \sin\phi)$. Suppose that $f \in C^{\alpha(\bar{\Phi})}_{\bar{\Phi}}(\bar{x})$ for all $\bar{x} \in \mathbb{R}^2$ such that $\bar{x} \perp \bar{\Phi}$, i.e., there is a one-dimensional function $g_{\bar{x}} : \mathbb{R} \to \mathbb{R}$ defined by $g_{\bar{x}}(k) = f(\bar{x} + (k\cos\phi, k\sin\phi))$, $k \in \mathbb{R}$, satisfying the inequality $|g_{\bar{x}}(t) - g_{\bar{x}}(0)| \leq C|t|^{\alpha(\bar{\Phi})}$ for some constant $C > 0$, for all $t$ in a neighborhood of 0, i.e. $|f(\bar{x}+(t\cos\phi, t\sin\phi))-f(\bar{x})| \leq C|t|^{\alpha(\bar{\Phi})}$ for all $t$ in a neighborhood of 0. Letting $x_1 = t\cos\phi - s\sin\phi$ and $x_2 = t\sin\phi + s\cos\phi$, we have

$$\begin{aligned}\mathcal{R}(f)(a, b, \phi) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\sqrt{a}} f(x_1, x_2)\psi\left(\frac{x_1\cos\phi + x_2\sin\phi - b}{a}\right) dx_1\, dx_2 \\ &= \frac{1}{\sqrt{a}}\int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right)\left(\int_{\mathbb{R}} f(t\cos\phi - s\sin\phi, t\sin\phi + s\cos\phi)\, ds\right) dt.\end{aligned}$$

Since $\int_{\mathbb{R}} \psi(\bar{x})\, d\bar{x} = 0$, we have

$$\begin{aligned}\mathcal{R}(f)(a, b, \phi) = &\frac{1}{\sqrt{a}}\int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right)\int_{\mathbb{R}} f(t\cos\phi - s\sin\phi, t\sin\phi + s\cos\phi)\, ds\, dt \\ &- \frac{1}{\sqrt{a}}\int_{\mathbb{R}} \psi\left(\frac{t-b}{a}\right)\int_{\mathbb{R}} f(-s\sin\phi, s\cos\phi)\, ds\, dt.\end{aligned}$$

Thus

$$\begin{aligned}|\mathcal{R}(f)(a, b, \phi)| \leq &\frac{1}{\sqrt{a}}\int_{\mathbb{R}^2} \left|\psi\left(\frac{t-b}{a}\right)\right| \\ &|f(t\cos\phi - s\sin\phi, t\sin\phi + s\cos\phi) - f(-s\sin\phi, s\cos\phi)|\, ds\, dt.\end{aligned}$$

Since $f$ is compactly supported and $f \in C_{\bar{\Phi}}^{\alpha(\bar{\Phi})}(\bar{x})$ for all $\bar{x} \in \mathbb{R}^2$ such that $\bar{x} \perp \bar{\Phi}$, we have

$$
\begin{aligned}
|\mathcal{R}(f)(a,b,\phi)| &\leq \frac{1}{\sqrt{a}} \int_{-M}^{M} \int_{-M}^{M} \left| \psi\left(\frac{t-b}{a}\right) \right| C |t|^{\alpha(\bar{\Phi})} \, ds \, dt \\
&= \frac{C}{\sqrt{a}} \int_{-M}^{M} 1 \, ds \int_{-M}^{M} \left| \psi\left(\frac{t-b}{a}\right) \right| |t|^{\alpha(\bar{\Phi})} \, dt \\
&= \frac{C'}{\sqrt{a}} \int_{-M}^{M} \left| \psi\left(\frac{t-b}{a}\right) \right| |t|^{\alpha(\bar{\Phi})} \, dt \\
&= \frac{C'}{\sqrt{a}} \int_{\frac{-M-b}{a}}^{\frac{M-b}{a}} |\psi(y)| \, |ay+b|^{\alpha(\bar{\Phi})} a \, dy \\
&\leq C' a^{\frac{1}{2}} \int_{-\infty}^{\infty} |\psi(y)| \left( |ay|^{\alpha(\bar{\Phi})} + |b|^{\alpha(\bar{\Phi})} \right) dy \\
&\leq C'' a^{\frac{1}{2}} \left( a^{\alpha(\bar{\Phi})} + |b|^{\alpha(\bar{\Phi})} \right).
\end{aligned}
$$

□

**Example:** Let $\theta \in [0, 2\pi)$, we apply the Weierstrass function in 2-dimensional;

$$
\mathcal{W}_{\alpha,\beta,\theta}(x_1,x_2) = e^{-|-x_1 \sin\theta + x_2 \cos\theta|} \sum_n \alpha^n \sin\left(\beta^n (x_1 \cos\theta + x_2 \sin\theta)\right)
$$

where $\beta$ is assumed to be larger than 1, so that the series is lacunary, and $\alpha$ is assumed to be smaller than 1, so that the series converges normally. This function is continuous but nowhere differentiable if $\alpha\beta > 1$.

*Proof.* Let $a > 0$ and $b \in \mathbb{R}$, we have

$$
\begin{aligned}
\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a,b,\theta) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}^2} & e^{-|-x_1 \sin\theta + x_2 \cos\theta|} \sum_n \alpha^n \sin\left(\beta^n (x_1 \cos\theta + x_2 \sin\theta)\right) \\
& \psi\left( \frac{x_1 \cos\theta + x_2 \sin\theta - b}{a} \right) dx_1 dx_2.
\end{aligned}
$$

Putting $x_1 = t\cos\theta - s\sin\theta$ and $x_2 = t\sin\theta + s\cos\theta$, we obtain that

$$
\begin{aligned}
\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a,b,\theta) &= \frac{1}{\sqrt{a}} \int_{\mathbb{R}^2} e^{-|s|} \sum_n \alpha^n \sin(\beta^n t) \, \psi\left(\frac{t-b}{a}\right) dsdt \\
&= \frac{1}{\sqrt{a}} \left( \int_{-\infty}^{\infty} e^{-|s|} \, ds \right) \sum_n \alpha^n \int_{-\infty}^{\infty} \sin(\beta^n t) \, \psi\left(\frac{t-b}{a}\right) dt.
\end{aligned}
$$

Because of the integrable of $e^{-|s|}$ and $\sin z = \frac{e^{2\pi i z} - e^{-2\pi i z}}{2i}$, we have

$$\begin{aligned}
\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a,b,\theta) &= \frac{C}{\sqrt{a}} \sum_n \alpha^n \int_{-\infty}^{\infty} \left[ \frac{e^{2\pi i \beta^n t} - e^{-2\pi i \beta^n t}}{2i} \right] \psi\left(\frac{t-b}{a}\right) dt \\
&= \frac{C}{2i\sqrt{a}} \sum_n \alpha^n \int_{-\infty}^{\infty} e^{2\pi i \beta^n t} \psi\left(\frac{t-b}{a}\right) dt \\
&\quad - \frac{C}{2i\sqrt{a}} \sum_n \alpha^n \int_{-\infty}^{\infty} e^{-2\pi i \beta^n t} \psi\left(\frac{t-b}{a}\right) dt \\
&= \frac{C}{2i}\sqrt{a} \sum_n \alpha^n \int_{-\infty}^{\infty} e^{2\pi i \beta^n (ay+b)} \psi(y)\, dy \\
&\quad - \frac{C}{2i}\sqrt{a} \sum_n \alpha^n \int_{-\infty}^{\infty} e^{-2\pi i \beta^n (ay+b)} \psi(y)\, dy \\
&= \frac{C}{2i}\sqrt{a} \sum_n \alpha^n e^{2\pi i \beta^n b} \int_{-\infty}^{\infty} e^{2\pi i \beta^n a y} \psi(y)\, dy \\
&\quad - \frac{C}{2i}\sqrt{a} \sum_n \alpha^n e^{-2\pi i \beta^n b} \int_{-\infty}^{\infty} e^{-2\pi i \beta^n a y} \psi(y)\, dy \\
&= \frac{C}{2i}\sqrt{a} \sum_n \alpha^n e^{2\pi i \beta^n b} \hat{\psi}(\beta^n a) - \frac{C}{2i}\sqrt{a} \sum_n \alpha^n e^{-2\pi i \beta^n b} \hat{\psi}(-\beta^n a).
\end{aligned}$$

Since $\hat{\psi}(z) = 0$ for all $z < 0$, we get

$$\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a,b,\theta) = \frac{C}{2i}\sqrt{a} \sum_n \alpha^n e^{2\pi i \beta^n b} \hat{\psi}(\beta^n a).$$

Thus $|\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a,b,\theta)| = \frac{C}{2}\sqrt{a}\sum_n \alpha^n |\hat{\psi}(\beta^n a)|$.

Choosing $a_m = \beta^{-m} \to 0$ as $m \to \infty$, we have

$$\begin{aligned}
|\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a_m,b,\theta)| &= \frac{C}{2}\sqrt{a_m}\alpha^m |\hat{\psi}(1)| + \frac{C}{2}\sqrt{a_m}\alpha^{m+1}|\hat{\psi}(\beta)| \\
&\geq C\frac{|\hat{\psi}(1)|}{2}\beta^{-m}\sqrt{a_m} + C\frac{|\hat{\psi}(\beta)|}{2}\beta^{-m-1}\sqrt{a_m} \\
&\geq C\beta^{-m}\sqrt{a_m} \\
&= C a_m^{1+\frac{1}{2}}
\end{aligned}$$

provided $\psi$ is chosen in such a way that $\mathrm{supp}(\hat{\psi}) \subseteq [1,\beta]$. Thus $\mathcal{W}_{\alpha,\beta,\theta} \notin C^1(\mathbb{R}^2)$. □

Let us check that $\mathcal{W}_{\alpha,\beta,\theta}$ is $C^{-\frac{\log\alpha}{\log\beta}}(\bar{x}_0)$ for any $\bar{x}_0 \in \mathbb{R}^2$. In the difference

$$|\mathcal{W}_{\alpha,\beta,\theta}(\bar{x}) - \mathcal{W}_{\alpha,\beta,\theta}(\bar{x}_0)| \leq \alpha^n \sum_n \left|\sin\left(\beta^n(x_1\cos\theta + x_2\sin\theta)\right) - \sin\left(\beta^n(x_{01}\cos\theta + x_{02}\sin\theta)\right)\right|.$$

We can either bound the difference of sines simply by 2 or, using the mean value theorem, by $\beta^n|(\bar{x}-\bar{x}_0)\cdot(\cos\theta,\sin\theta)| \leq \beta^n \|\bar{x}-\bar{x}_0\|$.

Let $N = \frac{-\log\|\bar{x}-\bar{x}_0\|}{\log\beta}$. Using the first bound for $n \geq N$ and the second one for $n < N$, we get

$$|\mathcal{W}_{\alpha,\beta,\theta}(\bar{x}) - \mathcal{W}_{\alpha,\beta,\theta}(\bar{x}_0)| \leq \sum_{n\leq N} \alpha^n\beta^n \|\bar{x}-\bar{x}_0\| + 2\sum_{n>N}^{\infty} \alpha^n$$
$$\leq \|\bar{x}-\bar{x}_0\| \left(\frac{(\alpha\beta)^{N+1}-\alpha\beta}{\alpha\beta-1}\right) + 2\left(\frac{\alpha^{N+1}}{1-\alpha}\right).$$

We have to sum up two geometric series. Because of the value taken for $N$, the first sum is bounded by $C(\alpha\beta)^N \leq C\|\bar{x}-\bar{x}_0\|^{-\frac{\log\alpha}{\log\beta}}$, and the second one is bounded by $C\alpha^N \leq C\|\bar{x}-\bar{x}_0\|^{-\frac{\log\alpha}{\log\beta}}$.

Indeed, $\log\alpha^N = N\log\alpha = -\frac{\log\alpha}{\log\beta}\log\|\bar{x}-\bar{x}_0\| = \log\|\bar{x}-\bar{x}_0\|^{-\frac{\log\alpha}{\log\beta}}$, then $\alpha^N = \|\bar{x}-\bar{x}_0\|^{-\frac{\log\alpha}{\log\beta}}$ and $\log\beta^N = N\log\beta = -\frac{\log\|\bar{x}-\bar{x}_0\|}{\log\beta}\log\beta = -\log\|\bar{x}-\bar{x}_0\|$, thus $\beta^N = \frac{1}{\|\bar{x}-\bar{x}_0\|}$.

Therefore $|\mathcal{W}_{\alpha,\beta,\theta}(\bar{x}) - \mathcal{W}_{\alpha,\beta,\theta}(\bar{x}_0)| \leq C\|\bar{x}-\bar{x}_0\|^{-\frac{\log\alpha}{\log\beta}}$.

Next we can show that $|\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a,b,\theta)| \leq Ca^{\frac{1}{2}-\frac{\log\alpha}{\log\beta}}$ if $\alpha\beta > 1$.

*Proof.* Let $a > 0$. We can see that $\beta^{-m} \leq a \leq \beta^{-m+1}$ for some integer $m > 0$ and then for each integer $n$, $\beta^{n-m} \leq a\beta^n \leq \beta^{n-m+1}$. Suppose that $\psi$ is chosen in such a way that *supp*$(\hat{\psi}) \subseteq (1,\beta]$, the only nonvanishing integral corresponds to $n = m$. Then

$$|\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a,b,\theta)| = \frac{C}{2}\sqrt{a}\sum_n \alpha^n|\hat{\psi}(\beta^n a)|$$
$$= C'\sqrt{a}\alpha^{m+1}|\hat{\psi}(\beta)|$$
$$= C''\sqrt{a}\alpha^m.$$

We have to show that $\alpha^m \leq a^{-\frac{\log\alpha}{\log\beta}}$. Since $\alpha\beta > 1$ we have $\frac{1}{\beta} < \alpha$ and then $\frac{1}{\beta^m} < \alpha^m$.

Thus $\log a \geq \log(\frac{1}{\beta^m}) = -m\log\beta$, then $m \geq -\frac{\log a}{\log\beta}$. Since $0 < \alpha < 1 < \beta$ we obtain that $\log\alpha^m = m\log\alpha \leq -\frac{\log a}{\log\beta}\log\alpha = \log a^{-\frac{\log\alpha}{\log\beta}}$, thus $\alpha^m \leq a^{-\frac{\log\alpha}{\log\beta}}$.

Hence $|\mathcal{R}(\mathcal{W}_{\alpha,\beta,\theta})(a,b,\theta)| \leq Ca^{\frac{1}{2}-\frac{\log\alpha}{\log\beta}}$. $\square$

Example: Let $\theta \in [0, 2\pi)$, we again apply the Weierstrass function in 2-dimensional;

$$\mathcal{W}_{\alpha,\beta,\theta}(x_1, x_2) = \sum_n \alpha^n \sin\left(\beta^n (x_1 \cos\theta + x_2 \sin\theta)\right)$$

where $\beta$ is assumed to be large than 1, so that the series is lacunary, and $\alpha$ is assume to be smaller than 1, so that the series converges normally. This function is continuous but nowhere differetiable if $\alpha\beta > 1$, we can prove by the Smith transform.

*Proof.* We can see that, for any $a > 0$ and $\bar{b} \in \mathbb{R}^2$,

$$\bar{\Gamma}_{\mathcal{W}_{\alpha,\beta,\theta}}(a, \bar{b}, \theta) = a^{-\frac{3}{4}} \int_{\mathbb{R}^2} \sum_n \alpha^n \sin\left(\beta^n (x_1 \cos\theta + x_2 \sin\theta)\right) \varphi\left(D_{\frac{1}{a}} R_{-\theta}(\bar{x} - \bar{b})\right)\, dx.$$

Putting $y_1 = \frac{1}{a}\left[(x_1 - b_1)\cos\theta + (x_2 - b_2)\sin\theta\right]$ and $y_2 = \frac{1}{\sqrt{a}}\left[-(x_1 - b_1)\sin\theta + (x_2 - b_2)\cos\theta\right]$. So, we have $x_1 = ay_1 \cos\theta - \sqrt{a} y_2 \sin\theta + b_1$ and $x_2 = ay_1 \sin\theta + \sqrt{a} y_2 \cos\theta + b_2$ and obtain that

$$\begin{aligned}
\bar{\Gamma}_{\mathcal{W}_{\alpha,\beta,\theta}}(a, \bar{b}, \theta) &= a^{-\frac{3}{4}+\frac{3}{2}} \int_{\mathbb{R}^2} \sum_n \alpha^n \sin\left(\beta^n (ay_1 + b_1 \cos\theta + b_2 \sin\theta)\right) \varphi(y_1, y_2)\, dy_1 dy_2 \\
&= a^{\frac{3}{4}} \int_{\mathbb{R}^2} \sum_n \alpha^n \frac{e^{2\pi i \beta^n a y_1} e^{2\pi i \beta^n (b_1 \cos\theta + b_2 \sin\theta)} - e^{-2\pi i \beta^n a y_1} e^{-2\pi i \beta^n (b_1 \cos\theta + b_2 \sin\theta)}}{2i} \\
&\quad \varphi(y_1, y_2)\, dy_1 dy_2 \\
&= \frac{C a^{\frac{3}{4}}}{2i} \left[\sum_n \alpha^n e^{2\pi \beta^n b \cdot (\cos\theta, \sin\theta)} \hat{\varphi}(\beta^n a, 0) - \sum_n \alpha^n e^{-2\pi \beta^n b \cdot (\cos\theta, \sin\theta)} \hat{\varphi}(-\beta^n a, 0)\right].
\end{aligned}$$

We also to suppose that $\hat{\varphi}(z_1, z_2) = 0$ if $z_1 < 0$ or $z_2 < 0$, then

$$\bar{\Gamma}_{\mathcal{W}_{\alpha,\beta,\theta}}(a, \bar{b}, \theta) = \frac{C a^{\frac{3}{4}}}{2} \sum_n \alpha^n |\hat{\varphi}(\beta^n a, 0)|.$$

Choosing $a_m = \beta^{-m} \to 0$ as $m \to \infty$, we have

$$\begin{aligned}
|\bar{\Gamma}_{\mathcal{W}_{\alpha,\beta,\theta}}(a_m, \bar{b}, \theta)| &= \frac{C}{2} a_m^{\frac{3}{4}} \alpha^m |\hat{\varphi}(1, 0)| + \frac{C}{2} a_m^{\frac{3}{4}} \alpha^{m+1} |\hat{\varphi}(\beta, 0)| \\
&\geq C' a_m^{\frac{3}{4}} \beta^{-m} + C' a_m^{\frac{3}{4}} \beta^{-m-1} \\
&\geq C' a_m^{\frac{3}{4}} \beta^{-m} \\
&= C'' a_m^{\frac{3}{4}+1}
\end{aligned}$$

provided $\varphi$ is choosen in such a way that $supp(\hat{\varphi}) \subseteq [1, \beta] \times \mathbb{R}$. □

Let us check that $\mathcal{W}_{\alpha,\beta,\theta}$ is $C^{-\frac{\log\alpha}{\log\beta}}(\bar{x}_0)$ for any $\bar{x}_0 \in \mathbb{R}^2$.

In the difference

$$|\mathcal{W}_{\alpha,\beta,\theta}(\bar{x}) - \mathcal{W}_{\alpha,\beta,\theta}(\bar{x}_0)| \leq \alpha^n \sum_n |\sin(\beta^n(x_1\cos\theta + x_2\sin\theta)) - \sin(\beta^n(x_{01}\cos\theta + x_{02}\sin\theta))| .$$

Similar the previous example we show that $|\mathcal{W}_{\alpha,\beta,\theta}(\bar{x}) - \mathcal{W}_{\alpha,\beta,\theta}(\bar{x}_0)| \leq C\,\|\bar{x} - \bar{x}_0\|^{-\frac{\log\alpha}{\log\beta}}$.

Finally, we can show that $|\bar{\Gamma}_{\mathcal{W}_{\alpha,\beta,\theta}}(a, \bar{b}, \theta)| \leq Ca^{\frac{3}{4}-\frac{\log\alpha}{\log\beta}}$.

*Proof.* Let $a > 0$. We can see that $\beta^{-m} \leq a \leq \beta^{-m+1}$ for some integer $m > 0$ and then for each integer $n$, $\beta^{n-m} \leq a\beta^n \leq \beta^{n-m+1}$. Suppose that $\psi$ is chosen in such a way that $supp(\hat{\psi}) \subseteq (1, \beta] \times \mathbb{R}$, the only nonvanishing integral corresponds to $n = m$. Then

$$\begin{aligned} |\bar{\Gamma}_{\mathcal{W}_{\alpha,\beta,\theta}}(a, \bar{b}, \theta)| &= \frac{C}{2}a^{\frac{3}{4}} \sum_n \alpha^n |\hat{\psi}(\beta^n a, 0)| \\ &= C'\sqrt{a}\alpha^{m+1}|\hat{\psi}(\beta, 0)| \\ &= C''a^{\frac{3}{4}}\alpha^m. \end{aligned}$$

We have to show that $\alpha^m \leq a^{-\frac{\log\alpha}{\log\beta}}$.Since $\alpha\beta > 1$ we have $\frac{1}{\beta} < \alpha$ and then $\frac{1}{\beta^m} < \alpha^m$.

Thus $\log a \geq \log(\frac{1}{\beta^m}) = -m\log\beta$, then $m \geq -\frac{\log a}{\log\beta}$. Since $0 < \alpha < 1 < \beta$ we obtain that $\log\alpha^m = m\log\alpha \leq -\frac{\log a}{\log\beta}\log\alpha = \log a^{-\frac{\log\alpha}{\log\beta}}$, thus $\alpha^m \leq a^{-\frac{\log\alpha}{\log\beta}}$.

Hence $|\bar{\Gamma}_{\mathcal{W}_{\alpha,\beta,\theta}}(a, \bar{b}, \theta)| \leq Ca^{\frac{3}{4}-\frac{\log\alpha}{\log\beta}}$. □

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