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LONG-TIME BEHAVIOR OF A NONLOCAL DISPERSAL EQUATION

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$$u_t(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)\,dy - u(x,t) \quad \text{ใน } \mathbb{R}_+,$$

ที่กำหนดเงื่อนไขค่าขอบไม่เอกพันธ์ ผลลัพธ์ของวิทยานิพนธ์นี้ขยายผลลัพธ์ของซุนในปี 2558 โดย J ในงานนี้สามารถมีเซตค้ำจุนไม่กระชับ วิทยานิพนธ์นี้พิสูจน์การมีผลเฉลยในวงกว้างขวางของสมการการกระจาย และพิสูจน์หลักการเปรียบเทียบ (สำหรับสมการที่แปรเปลี่ยนไปตามเวลาและไม่แปรเปลี่ยนไปตามเวลา) ผลลัพธ์ที่สำคัญในงานนี้คือพฤติกรรมของผลเฉลยของสมการการกระจายเมื่อเวลาผ่านไปนาน ซึ่งได้แสดงว่าขึ้นกับเงื่อนไขค่าขอบ

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In this thesis, study the nonlocal dispersal equation

$$u_t(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)\,dy - u(x,t) \quad \text{in } \mathbb{R}_+,$$

with a non-homogeneous boundary condition. We generalize the result of J.W. Sun (2015), where in this work the kernel J has a non-compact support. The global well-posedness and comparison principles (both time-dependent and stationary problems) are established. The main result is the long-time behavior of solutions, which is proved to depend on the boundary data.

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CONTENTS

CHAPTER I
INTRODUCTION

The purpose of this thesis is to study the long-time behavior of solutions to the following boundary-initial value problem for a nonlocal dispersal equation:

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}} J(x-y)u(y,t)dy - u(x,t) & \text{in } \mathbb{R}_+ \times (0,\infty), \\ u(x,t) = g(x,t) & \text{in } \mathbb{R}_- \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}, \end{cases} \tag{1.1}$$

where g, u_0 are given bounded functions and the kernel J satisfies $\int_{\mathbb{R}} J(x)dx = 1$. Here, we denote $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+$.

Let us give a brief physical motivation for our model of study; see [1] for a more complete description. If $u(x,t)$ is thought of as the density of moving particles at a point x and time t and $J(x-y)$ is the probability distribution of jumping from a location y to x, then $\int_{\mathbb{R}} J(y-x)u(y,t)\,dy = J * u(x,t)$ is the rate at which individuals are arriving at x from all other places. On the other hand, $-u(x,t) = -\int_{\mathbb{R}} J(x-y)u(x,t)\,dy$ is the rate at which they are leaving x to travel to all other sites. Thus, (1.1) is a conservation equation.

In recent years, there have been many studies on a nonlocal dispersal equation of the form

$$u_t(x,t) = \int_{\Omega} J(x,y)\left(u(y,t) - u(x,t)\right)\,dy + f(x,t) \tag{1.2}$$

where Ω is an open set in some Euclidean space.

In 2007, Cortázar, Elgueta, Rossi and Wolanski [5] studied (1.2) in a bounded domain $\Omega \subseteq \mathbb{R}^n$ with non-homogeneous boundary conditions

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)\left(u(y,t)-u(x,t)\right)\,dy \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^n \setminus \Omega} J(x-y)g(y,t)\,dy & \text{in } \Omega \times (0,\infty), \\ u(x,t) = g(x,t) & \text{in } (\mathbb{R}^n \setminus \Omega) \times (0,\infty), \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

with $J : \mathbb{R}^n \to \mathbb{R}$ is nonegative and symmetric $J(x) = J(-x)$ with unit integral $\int_{\mathbb{R}^n} J(x)\,dx = 1$. Furthermore, J is strictly positive in $B(0,d) := \{x \in \mathbb{R}^n \mid \|x\| < d\}$ and vanishes in $\mathbb{R}^n \setminus B(0,d)$. They also investigated the stationary problem, namely,

$$\int_\Omega J(x-y)\left(\varphi(y)-\varphi(x)\right)\,dy + \int_{\mathbb{R}^n \setminus \Omega} J(x-y)h(y)\,dy = 0, \tag{1.4}$$

where $g(x,t) = h(x)$ in (1.3). One of the main results of [5] is the following Theorem.

Theorem 1.1 (see [5]). *If u is a continuous solution of (1.3) with $g(x,t) = h(x)$, where*

$$\int_\Omega \int_{\mathbb{R}^n \setminus \Omega} J(x-y)h(y)\,dy\,dx = 0,$$

and if φ be the unique solution of (1.4) such that

$$\int_\Omega \varphi(x)\,dx = \int_\Omega u_0(x)\,dx,$$

then

$$\lim_{t\to\infty} u(x,t) = \varphi(x) \quad \textit{uniformly in } \overline{\Omega}.$$

In 2012, Cortázar, Elgueta, Quirós and Wolanski [3] studied (1.2) in a bounded open set $\mathcal{H}$ with homogeneous boundary condition,

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^n} J(x-y)\,(u(y,t)-u(x,t))\;dy & \text{in } \mathcal{H}\times(0,\infty), \\ u(x,t) = 0 & \text{in } \Omega\times(0,\infty), \\ u(x,0) = u_0(x) & x\in\mathbb{R}^n, \end{cases} \tag{1.5}$$

where $\mathcal{H} = \mathbb{R}^n \setminus \Omega$ $(n \geq 3)$. Under the same conditions on J in (1.3) and $J(0) > 0$. For this nonlocal problem, a conservation law holds:

$$\int_{\mathbb{R}^n} u(x,t)\phi(x)\,dx = \int_{\mathbb{R}^n} u_0(x)\phi(x)\,dx =: M^*,$$

where ϕ is the unique solution to

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} J(x-y)\phi(y)\,dy & \text{in } \mathcal{H}, \\ \phi(x) = 0 & \text{in } \Omega, \\ \displaystyle\lim_{|x|\to\infty} \phi(x) = 1. \end{cases} \tag{1.6}$$

The asymptotic behavior for (1.5) can be expressed in terms of the fundamental solution to the heat equation with diffusivity $\alpha := \frac{1}{2n}\displaystyle\int_{\mathbb{R}^n} |z|^2 J(z)\,dz$ and $\Gamma_\alpha(x,t) = t^{-n/2}U_\alpha(\frac{x}{t^{1/2}})$ where $U_\alpha = (4\pi\alpha)^{-n/2}e^{-\frac{|y|^2}{4\alpha}}$ as follows.

Theorem 1.2 (see [3])**.** *Let u be the solution of (1.5). Then for every $\delta > 0$,*

$$\lim_{t\to\infty} t^{n/2}\left\|u(x,t) - M^*\Gamma_\alpha(x,t)\right\|_{L^\infty(\{|x|^2\geq\delta t\})} = 0.$$

Theorem 1.3 (see [3])**.** *Let u be the solution of (1.5) and ϕ the solution of (1.6). Then there exists $\delta > 0$,*

$$\lim_{t\to\infty} t^{n/2}\left\|u(x,t) - M^*\phi(x)\Gamma_\alpha(x,t)\right\|_{L^\infty(\{|x|^2<\delta t\})} = 0.$$

After that, in 2015, Cortázar, Elgueta, Quirós and Wolanski [4] studied (1.2) on $\Omega = \mathbb{R}_+$ with homogeneous boundary condition,

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}} J(x-y)\left(u(y,t)-u(x,t)\right)\,dy & \text{in } \mathbb{R}_+ \times (0,\infty), \\ u(x,t) = 0 & \text{in } \mathbb{R}_- \times (0,\infty), \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.7}$$

under the same assumption on J in (1.3) with the stationary problem:

$$\begin{cases} \displaystyle\int_{\mathbb{R}} J(x-y)\phi(y)\,dy - \phi(x) = 0 & \text{for } x \in \mathbb{R}_+, \\ \phi(x) = 0 & \text{for } x \in (-d,0), d > 0. \end{cases} \tag{1.8}$$

They preserved as follow

$$M_1(t) := \int_0^\infty u(x,t)x\,dx = \int_0^\infty u_0(x)x\,dx =: M_1^*.$$

In terms of the fundamental solution $\Gamma(x,t) = \frac{e^{x^2/4t}}{2\sqrt{\pi}t^{1/2}}$, the long-time behavior of the solution is as follows.

Theorem 1.4 (see [4])**.** *Let $u_0 \in L^\infty(\mathbb{R}_+)$ be such that $\int_{\mathbb{R}_+} u_0(x)(1+x+x^2)\,dx < \infty$ and let $q = \frac{1}{2}\int_{\mathbb{R}} J(x)|x|^2\,dx$. If u is the solution of (1.7) and ϕ is the solution of (1.8), then*

$$\lim_{t\to\infty} \frac{t^{3/2}}{x+1}\left|u(x,t) + 2M_1^* \frac{\phi(x)}{x}\Gamma_x(x,qt)\right| = 0 \quad \textit{uniformly in } \overline{\mathbb{R}}_+.$$

Now, we play attention to (1.1). Consider the BVP

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ v(0,t) = g(t) & t > 0 \\ v(x,0) = v_0(x) & x \in \mathbb{R}_+ \end{cases} \tag{1.9}$$

The long-time behavior of the solution to (1.9) is uniquely determined by the boundary value $v(0,t)$. In fact, $v(x,t)$ is given by

$$\begin{aligned} v(x,t) = v(0,t) &+ \int_0^\infty \left(\Gamma(x-y,t) - \Gamma(x+y,t)\right) v_0(y)\, dy \\ &+ \int_0^t \int_0^\infty \left(\Gamma(x-y,t-s) - \Gamma(x+y,t-s)\right) h(s)\, dy\, ds. \end{aligned}$$

where $\Gamma(x,t) = \frac{e^{x^2/4t}}{2\sqrt{\pi}t^{1/2}}$ and $h(t) = g'(t)$. Assume that g is a bounded continuous function and

$$\lim_{t\to\infty} g(t) = \theta_*.$$

Then, the long-time behavior of solution is given by (see [6, 8]),

$$\lim_{t\to\infty} v(x,t) = \theta_* \quad \text{uniformly on } \mathbb{R}_+.$$

Hypothesis 1.5. We assume that the kernel satisfies $J : \mathbb{R} \to \mathbb{R}$ is continuous, $J \geq 0$, $J(x) = J(-x)$ and $\int_{\mathbb{R}} J(x)\, dx = 1$. We also assume that $g : \mathbb{R}_- \times [0,\infty) \to \mathbb{R}$ and $u_0 : \mathbb{R} \to \mathbb{R}$ satisfying:

(A1) g is continuous and there are constants $C_0, \theta_* > 0$ such that $-C_0 \leq g \leq C_0$ and

$$\lim_{t\to\infty} g(x,t) = \theta_* \quad \text{locally uniformly in } \mathbb{R}_-.$$

(A2) $u_0 \in C(\mathbb{R})$ with $u_0(x) = g(x,0)$ in $\mathbb{R}_-$, $-C_1 \leq u_0 \leq C_1$ in $\mathbb{R}$ for some $C_1 > 0$, and $\lim_{x\to\infty} u_0(x) = \theta_*$.

Sun [10] proved that the same property also holds for the nonlocal problem (1.1), that is

Theorem 1.6 (see [10])**.** *Assume Hypothesis 1.5 holds and that J has compact support. Let $u(x,t)$ be the unique solution of (1.1). Then,*

$$\lim_{t\to\infty} u(x,t) = \theta_* \quad \textit{locally uniformly on } \mathbb{R}_+.$$

The proof of this theorem [10] strongly depends on the assumption that J has compact support. In this work, we generalize the concepts to the case that J can have non-compact support.

The outline of this thesis is as follows. In Chapter II, we state some background results used in this work. The existence and uniqueness of a solution to (1.1) and the comparison principle are established in Chapter III. In Chapter IV, we investigate the stationary problem. We prove the long-time behavior of the solution to (1.1) in Chapter V. Finally, conclusion and discussion about our work in Chapter VI.

CHAPTER II
PRELIMINARIES

2.1 Notation and function spaces

- $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$
- $C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is the set of bounded continuous functions on $\mathbb{R}_+$.
- For a function $f(x,t)$, $f(t)$ denotes a function of x for each t.

For $T > 0$, we denote

$$\mathcal{X}_T = C([0,T]; C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+))$$

the space of functions $f : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ such that $f(t) \in C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ for each $t \in [0,T]$ and for $t_0 \in [0,T]$,

$$\lim_{t \to t_0} \|f(t) - f(t_0)\|_{L^\infty(\mathbb{R}_+)} = 0.$$

$\mathcal{X}_T$ is equipped with the norm

$$\|f\|_{\mathcal{X}_T} = \sup_{0 \le t \le T} \|f(t)\|_{L^\infty(\mathbb{R}_+)}.$$

We denote

$$\mathcal{X} = C([0,\infty); C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)) \quad \text{and} \quad \|f\|_{\mathcal{X}} = \sup_{0 \le t < \infty} \|f(t)\|_{L^\infty(\mathbb{R}_+)}.$$

2.2 Basic Tools

We recall some lemmas which will be used in this work. These results can be found in standard texts such as [2, 7, 9, 11, 12].

Definition 2.1. Let (X, d) be a metric space. A map $T : X \to X$ is called a **contraction** on X if there exists $\lambda \in [0, 1)$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y) \quad \text{for all } x, y \in X.$$

Lemma 2.2 (Banach's Fixed Point Theorem)**.** *If $T : X \to X$ is a contraction on a complete metric space X, then there exists a unique x^* in X such that $T(x^*) = x^*$.*

Lemma 2.3 (Gronwall's inequality)**.** *Let α, β, r be non-negative continuous real-valued functions defined on $[0, T]$. Suppose that*

$$r(t) \leq \alpha(t) + \int_0^t \beta(s) r(s)\, ds \quad \textit{for all } t \in [0, T].$$

Then,

$$r(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \left(\exp \int_s^t \beta(\tau)\, d\tau \right) ds$$

for all $t \in [0, T]$. In particular, if $\alpha = 0$, then $r \equiv 0$.

Lemma 2.4. *Assume that $1 \leq p < \infty$. If $f \in L^p(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f(x + z) - f(z)|^p\, dz = 0.$$

as $x \to 0$.

Lemma 2.5 (Dominated Convergence Theorem)**.** *Let (X, μ) be a measure space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that*

1. $\lim_{n\to\infty} f_n(x) = f(x)$ *for all $x \in X$ and*

2. *there exists $g \in L^1(\mu)$ such that $|f_n(x)| \leq g(x)$ for all $x \in X$.*

Then, $f \in L^1(\mu)$ *and*

$$\lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu.$$

Definition 2.6. A sequence $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ of functions defined on a metric space (X, d) is called **uniformly bounded** if there exists a constant M such that $|f_n(x)| \leq M$ for all $x \in X$ and all $n \in \mathbb{N}$. $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ is called **uniformly equicontinuous** if given $\epsilon > 0$, there is a $\delta > 0$ that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$ and all $x, y \in X$ such that $d(x, y) < \delta$.

Lemma 2.7 (Arzéla-Ascoli theorem)**.** *Let* (X, d) *be a metric space. Let* $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ *be a uniformly bounded and equicontinuous sequence of functions. Then, there is a subsequence* $\{f_{n_k} : X \to \mathbb{R}\}_{k=1}^{\infty}$ *and a bounded continuous function* $f : X \to \mathbb{R}$ *such that*

$$\lim_{k\to\infty} f_{n_k} = f \quad \textit{uniformly on any compact subsets of } X.$$

2.3 Definitions of solutions for (1.1)

To explore (1.1), it suffices to investigate the following nonlocal dispersal equation on $\mathbb{R}_+$:

$$u_t(x,t) = \int_{\mathbb{R}_+} J(x-y)u(y,t)\,dy - u(x,t) + \int_{\mathbb{R}_-} J(x-y)g(y,t)\,dy \quad \text{in } \mathbb{R}_+ \times (0,\infty) \tag{2.1}$$

with the initial condition $u(x, 0) = u_0$ on $\mathbb{R}_+$.

Proposition 2.8. *u solves (1.1) if and only if u solves (2.1)*

Proof. Assume that u solves (1.1). Since $\int_{\mathbb{R}} J(x)dx = 1$ and $u = g$ on $\mathbb{R}_- \times (0, \infty)$, we have

$$\begin{aligned} u_t(x,t) &= \int_{\mathbb{R}} J(x-y)u(y,t)\,dy - u(x,t) \\ &= \int_{\mathbb{R}_+} J(x-y)u(y,t)dy - u(x,t) + \int_{\mathbb{R}_-} J(x-y)g(y,t)dy, \end{aligned}$$

Thus, u satisfies (2.1). Conversely, if u solves (2.1), setting $u(x,t) = g(x,t)$ on $\mathbb{R}_- \times (0,\infty)$, it follows that u satisfies (1.1). $\square$

In view of the preceding proposition, we will mainly investigate (2.1). We give the notion of solutions for (2.1). To provide a motivation, assume that u solves (2.1), we get

$$u_t(x,t) = \int_{\mathbb{R}_+} J(x-y)u(y,t)\,dy - u(x,t) + \int_{\mathbb{R}_-} J(x-y)g(y,t)\,dy.$$

Multiplying e^t on both sides, we get

$$e^t u_t(x,t) = e^t \int_{\mathbb{R}_+} J(x-y)u(y,t)\,dy - e^t u(x,t) + e^t \int_{\mathbb{R}_-} J(x-y)g(y,t)\,dy$$

Integrating from 0 to t on both sides, we get

$$\begin{aligned}\int_0^t e^s u_s(x,s)\,ds = &\int_0^t e^s \int_{\mathbb{R}_+} J(x-y)u(y,s)\,dy\,ds - \int_0^t e^s u(x,s)\,ds \\ &+ \int_0^t e^s \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds\end{aligned}$$

Integrating by parts, we have

$$\int_0^t e^s u_s(x,s)\,ds = e^t u(x,t) - u_0(x) - \int_0^t e^s u(x,s)\,ds.$$

By substitution, we get

$$\begin{aligned}e^t u(x,t) - u_0(x) - \int_0^t e^s u(x,s) = &\int_0^t e^s \int_{\mathbb{R}_+} J(x-y)u(y,s)\,dy\,ds - \int_0^t e^s u(x,s)\,ds \\ &+ \int_0^t e^s \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds\end{aligned}$$

Thus, we obtain

$$u(x,t) = e^{-t}u_0(x) + \int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)u(y,s)dyds$$
$$+ \int_0^t e^{-(t-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)dyds. \tag{2.2}$$

Definition 2.9. By a solution to (2.1), we mean a function $u \in \mathcal{X}$ that satisfies (2.2) at each point $(x,t) \in \mathbb{R}_+ \times (0,\infty)$.

Proposition 2.10. *The following semigroup property holds:*

$$u(x,t+t_0) = e^{-t}u(x,t_0) + \int_{t_0}^{t+t_0} e^{-(t+t_0-s)} \int_{\mathbb{R}_+} J(x-y)u(y,s)\,dy\,ds$$
$$+ \int_{t_0}^{t+t_0} e^{-(t+t_0-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds. \tag{2.3}$$

Proof. Fix $t_0 > 0$. Assume that u is a solution of (2.1). By (2.2), we get

$$u(x,t_0) = e^{-t_0}u_0(x) + \int_0^{t_0} e^{-(t_0-s)} \int_{\mathbb{R}_+} J(x-y)u(y,s)\,dy\,ds$$
$$+ \int_0^{t_0} e^{-(t_0-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds.$$

Multiplying e^{-t} on the both sides, we get

$$e^{-(t+t_0)}u_0(x) = e^{-t}u(x,t_0) - \int_0^{t_0} e^{-(t+t_0-s)} \int_{\mathbb{R}_+} J(x-y)u(y,s)\,dy\,ds$$
$$- \int_0^{t_0} e^{-(t+t_0-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds,$$

By substitution in (2.2) by $t \to t + t_0$ and above result, we get

$$
\begin{aligned}
u(x, t + t_0) = e^{-(t+t_0)}u_0(x) &+ \int_0^{t+t_0} e^{-(t+t_0-s)} \int_{\mathbb{R}_+} J(x-y)u(y,s)dyds \\
&+ \int_0^{t+t_0} e^{-(t+t_0-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)dyds. \\
&= e^{-t}u(x,t_0) + \int_{t_0}^{t+t_0} e^{-(t+t_0-s)} \int_{\mathbb{R}_+} J(x-y)u(y,s)\,dy\,ds \\
&+ \int_{t_0}^{t+t_0} e^{-(t+t_0-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds.
\end{aligned}
$$

□

To derive the comparison principles, we need the following variant of solutions.

Definition 2.11. A function $u \in \mathcal{X}$ is called a **subsolution** for (2.1) if it satisfies

$$
\begin{aligned}
u(x,t) \le e^{-t}u_0(x) &+ \int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)u(y,s)dyds \\
&+ \int_0^t e^{-(t-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)dyds,
\end{aligned} \tag{2.4}
$$

at each (x,t) in the domain. It is called a **supersolution** for (2.1) if it satisfies

$$
\begin{aligned}
u(x,t) \ge e^{-t}u_0(x) &+ \int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)u(y,s)dyds \\
&+ \int_0^t e^{-(t-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)dyds.
\end{aligned} \tag{2.5}
$$

CHAPTER III
EXISTENCE, UNIQUENESS AND COMPARISON

In this chapter, we establish our first part of the main results: the existence and uniqueness of a solution to (2.2), which implies the result for (2.1). We also derive the comparison principle.

3.1 Existence and uniqueness

The existence and uniqueness of solutions of (2.1) is proved by a fixed point argument.

Theorem 3.1. *Assume that the Hypothesis 1.5 holds. Then, (2.1) admits a unique solution $u \in \mathcal{X}$. In particular, (1.1) has a unique solution as well.*

Proof. Fix $T > 0$ and define $\mathcal{M} : \mathcal{X}_T \to \mathcal{X}_T$ by

$$\begin{aligned}(\mathcal{M}w)(x,t) = e^{-t}u_0(x) + \int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds \\ + \int_0^t e^{-(t-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds,\end{aligned}$$

for $w \in \mathcal{X}_T$. Of course, u is a fixed point for $\mathcal{M}$ if and only if it is a solution to (2.1).

Let us check that $\mathcal{M}w \in \mathcal{X}_T$ for all $w \in \mathcal{X}_T$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}_+$ which converges to $x \in \mathbb{R}_+$. Since u_0 is continuous, we get $\lim_{n\to\infty} u_0(x_n) = u_0(x)$. Let $y \in \mathbb{R}_+$ and let $J_{x_n}(y) = J(x_n - y)$ for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, $J_{x_n} \in L^1(\mathbb{R}_+)$ because $J \in L^1(\mathbb{R})$. Since $J \geq 0$, we get

$$|J(x_n - y)w(y,t)| \leq \sup_{\mathbb{R}_+} |w(y,t)| J_{x_n} \in L^1(\mathbb{R}_+)$$

for each $n \in \mathbb{N}$ and $t \in [0,t]$. By Lemma 2.5 and the continuity of J, we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^t e^{-(t-s)}\int_{\mathbb{R}_+} J(x_n-y)w(y,s)\,dy\,ds &= \int_0^t e^{-(t-s)}\int_{\mathbb{R}_+}\lim_{n\to\infty} J(x_n-y)w(y,s)\,dy\,ds\\ &= \int_0^t e^{-(t-s)}\int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds.\end{aligned}$$

By a similarly argument, we have

$$\lim_{n\to\infty}\int_0^t e^{-(t-s)}\int_{\mathbb{R}_-} J(x_n-y)g(y,s)\,dy\,ds = \int_0^t e^{-(t-s)}\int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds.$$

Thus, $\mathcal{M}w(x_n,t) \to \mathcal{M}w(x,t)$ as $n \to \infty$. Therefore, $\mathcal{M}w(t) \in C(\mathbb{R}_+)$ for each $t \in [0,T]$. By the triangle inequality, we have

$$\begin{aligned}\|\mathcal{M}w(t)\|_{L^\infty(\mathbb{R}_+)} &\le \sup_{x\in\mathbb{R}_+}\left|e^{-t}u_0(x)\right|\\ &\quad + \sup_{x\in\mathbb{R}_+}\left|\int_0^t e^{-(t-s)}\int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds\right|\\ &\quad\quad + \sup_{x\in\mathbb{R}_+}\left|\int_0^t e^{-(t-s)}\int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds\right|\\ &\le \sup_{x\in\mathbb{R}_+}|u_0(x)|\\ &\quad + \sup_{x\in\mathbb{R}_+}\int_0^t e^{-(t-s)}\int_{\mathbb{R}_+} J(x-y)|w(y,s)|\,dy\,ds\\ &\quad\quad + \sup_{x\in\mathbb{R}_+}\int_0^t e^{-(t-s)}\int_{\mathbb{R}_-} J(x-y)|g(y,s)|\,dy\,ds\\ &\le C_1 + \sup_{x\in\mathbb{R}_+}\int_0^t e^{-(t-s)}\|w(s)\|_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+} J(x-y)\,dy\,ds\\ &\quad\quad + C_0\sup_{x\in\mathbb{R}_+}\int_0^t e^{-(t-s)}\int_{\mathbb{R}_-} J(x-y)\,dy\,ds\\ &\le C_1 + \int_0^t e^{-(t-s)}\|w(s)\|_{L^\infty(\mathbb{R}_+)}\,ds + C_0\int_0^t e^{-(t-s)}\,ds\\ &\le C_1 + (1-e^{-T})(\|w\|_{\mathcal{X}_T} + C_0) < \infty\end{aligned}$$

Thus, $\mathcal{M}w(t) \in L^\infty(\mathbb{R}_+)$ for each $t \in [0,T]$. To show that $\mathcal{M}w$ is continuous, let

$t_0 \in (0,T)$ and $t > t_0$. Set

$$I_+(x) = \int_0^{t_0} e^{-(t_0-s)} \int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds - \int_0^{t} e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds,$$
$$I_-(x) = \int_0^{t_0} e^{-(t_0-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds - \int_0^{t} e^{-(t-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds,$$

for $x \in \mathbb{R}_+$. We write

$$\begin{aligned} I_+(x) &= \int_0^{t_0} \left(e^{-(t_0-s)} - e^{-(t-s)}\right) \int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds \\ &\quad - \int_{t_0}^{t} e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds, \\ I_-(x) &= \int_0^{t_0} \left(e^{-(t_0-s)} - e^{-(t-s)}\right) \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds \\ &\quad - \int_{t_0}^{t} e^{-(t-s)} \int_{\mathbb{R}_-} J(x-y)g(y,s)\,dy\,ds, \end{aligned}$$

for $x \in \mathbb{R}_+$. Then, we have

$$\begin{aligned} |I_+(x)| &\le \left|\int_0^{t_0} \left(e^{-(t_0-s)} - e^{-(t-s)}\right) \int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds\right| \\ &\quad + \left|\int_{t_0}^{t} e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)w(y,s)\,dy\,ds\right| \\ &\le \|w\|_{\mathcal{X}_T} \int_0^{t_0} \left(e^{-(t_0-s)} - e^{-(t-s)}\right) \int_{\mathbb{R}_+} J(x-y)\,dy\,ds \\ &\quad + \|w\|_{\mathcal{X}_T} \int_{t_0}^{t} e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)\,dy\,ds \\ &\le \|w\|_{\mathcal{X}_T} \int_0^{t_0} \left(e^{-(t_0-s)} - e^{-(t-s)}\right)\,ds + \|w\|_{\mathcal{X}_T} \int_{t_0}^{t} e^{-(t-s)}\,ds \\ &= \|w\|_{\mathcal{X}_T} \left(1 - e^{t_0-t} - e^{-t_0} + e^{-t}\right) + \|w\|_{\mathcal{X}_T} \left(1 - e^{t_0-t}\right) \\ &\le \|w\|_{\mathcal{X}_T} \left[2\left(1 - e^{t_0-t}\right) + \left(e^{-t_0} - e^{-t}\right)\right] \end{aligned}$$

for all $x \in \mathbb{R}_+$. Similarly, we get $|I_-(x)| \le C_0 \left[2\left(1 - e^{t_0-t}\right) + \left(e^{-t_0} - e^{-t}\right)\right]$ for all

$x \in \mathbb{R}_+$. By the triangle inequality, we have

$$\begin{aligned}
\|\mathcal{M}w(t_0) - \mathcal{M}w(t)\|_{L^\infty(\mathbb{R}_+)} &\leq \sup_{x\in\mathbb{R}_+} \left|(e^{-t_0} - e^{-t})u_0(x)\right| + \sup_{x\in\mathbb{R}_+} |I_+(x)| + \sup_{x\in\mathbb{R}_+} |I_-(x)| \\
&\leq C_1 \left(e^{-t_0} - e^{-t}\right) \\
&\quad + \|w\|_{\mathcal{X}_T} \left[2\left(1 - e^{t_0 - t}\right) + \left(e^{-t_0} - e^{-t}\right)\right] \\
&\quad + C_0 \left[2\left(1 - e^{t_0 - t}\right) + \left(e^{-t_0} - e^{-t}\right)\right] \\
&= 2\left(\|w\|_{\mathcal{X}_T} + C_0\right)\left(1 - e^{t_0 - t}\right) \\
&\quad + (C_0 + C_1)\left(e^{-t_0} - e^{-t}\right) \to 0,
\end{aligned}$$

as $t \to t_0$. For $t < t_0$, we can show that $\mathcal{M}w(t) \to \mathcal{M}w(t_0)$ using the same argument. Therefore, $\mathcal{M}w \in \mathcal{X}_T$ as desired.

Finally, we show that $\mathcal{M}$ is a contraction on $\mathcal{X}_T$ for small T. Let $w_1, w_2 \in \mathcal{X}_T$. We have

$$\begin{aligned}
\|\mathcal{M}w_1(t) - \mathcal{M}w_2(t)\|_{L^\infty} &= \sup_{x\in\mathbb{R}_+} \left|\int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)\left(w_1(y,s) - w_2(y,s)\right) dy\, ds\right| \\
&\leq \sup_{x\in\mathbb{R}_+} \int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)\left|w_1(y,s) - w_2(y,s)\right| dy\, ds \\
&\leq \sup_{x\in\mathbb{R}_+} \int_0^t e^{-(t-s)} \|w_1(s) - w_2(s)\|_{L^\infty(\mathbb{R}_+)} \int_{\mathbb{R}_+} J(x-y)\, dy\, ds \\
&\leq \sup_{x\in\mathbb{R}_+} \int_0^t e^{-(t-s)} \|w_1(s) - w_2(s)\|_{L^\infty(\mathbb{R}_+)}\, ds \\
&= \int_0^t e^{-(t-s)} \|w_1(s) - w_2(s)\|_{L^\infty(\mathbb{R}_+)}\, ds \\
&\leq \|w_1 - w_2\|_{\mathcal{X}_T} \int_0^t e^{-(t-s)}\, ds \\
&\leq (1 - e^{-T})\|w_1 - w_2\|_{\mathcal{X}_T},
\end{aligned}$$

Taking the supremum over all $t \in [0, T]$,

$$\|\mathcal{M}w_1 - \mathcal{M}w_2\|_{\mathcal{X}_T} \leq (1 - e^{-T})\|w_1 - w_2\|_{\mathcal{X}_T},$$

hence by choosing $T < 1$, $\mathcal{M}$ is a contraction on $\mathcal{X}_T$. It follows by the Banach fixed point theorem that $\mathcal{M}$ has a unique fixed point, which is a unique solution for (2.2) on the time interval $[0, T]$.

Observe that $T \in (0, 1)$ can be chosen independently of the initial condition. So by applying the same argument to the operator arisen from the right hand side in the semigroup property (2.3), we can extend the solution to time intervals $[0, 2T]$, $[0, 3T]$, and so forth. Therefore we obtain a unique solution for (2.2) on the time interval $[0, \infty)$. □

Remark 3.2. If u_0 satisfies $u_0(x) \to 0$ as $x \to \infty$, then using the fact that the space of continuous functions vanishing at infinity is closed in $C(\mathbb{R}_+) \cap L^\infty(\mathbb{R})$, then the solution in the preceding theorem also satisfies $u(x, t) \to 0$ as $x \to \infty$.

3.2 Comparison principle

Comparison principle is a very useful tool in studying nonlocal dispersal equation.

Theorem 3.3 (Comparison principle)**.** *Suppose that u_1 is a subsolution to (2.1) and u_2 a supersolution to (2.1). If $u_1(x, 0) \le u_2(x, 0)$ on $\mathbb{R}_+$, then $u_1(x, t) \le u_2(x, t)$ for all $(x, t) \in \mathbb{R}_+ \times (0, \infty)$.*

Proof. Fix (x, t) in the domain. Denote $w = (u_1 - u_2)_+$, the positive part of $u_1 - u_2$, and define $r(t) = \|w(t)\|_{L^\infty(\mathbb{R}_+)}$. By (2.4) and (2.5), we have

$$\begin{aligned}
u_1(x,t) - u_2(x,t) &\le u_1(x,0) - u_2(x,0) \\
&\qquad + \int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)(u_1(y,s) - u_2(y,s))\, dy\, ds \\
&\le \int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)(u_1(y,s) - u_2(y,s))\, dy\, ds \\
&\le \int_0^t \int_{\mathbb{R}_+} J(x-y) w(y,s)\, dy\, ds.
\end{aligned}$$

Since J and w are nonnegative, we get

$$\begin{aligned} w(x,t) &\leq \left(\int_0^t \int_{\mathbb{R}_+} J(x-y) w(y,s)\, dy\, ds \right)_+ \\ &= \int_0^t \int_{\mathbb{R}_+} J(x-y) w(y,s)\, dy\, ds. \end{aligned}$$

Taking the L^∞-norm with respect to x, we have

$$\begin{aligned} r(t) &\leq \sup_{x \in \mathbb{R}_+} \left| \int_0^t \int_{\mathbb{R}_+} J(x-y) w(y,s)\, dy\, ds \right| \\ &\leq \sup_{x \in \mathbb{R}_+} \int_0^t \int_{\mathbb{R}_+} J(x-y) |w(y,s)|\, dy\, ds \\ &\leq \sup_{x \in \mathbb{R}_+} \int_0^t r(s) \int_{\mathbb{R}_+} J(x-y)\, dy\, ds \\ &\leq \sup_{x \in \mathbb{R}_+} \int_0^t r(s)\, ds \\ &= \int_0^t r(s)\, ds \end{aligned}$$

for all $t \geq 0$. Using Gronwall's inequality Lemma 2.3, we obtain that $r \equiv 0$. Hence $u_1 \leq u_2$. □

Lemma 3.4. *$-C_0 - C_1$ is a subsolution to (2.1) and $C_0 + C_1$ is a supersolution to (2.1).*

Proof. We consider

$$\begin{aligned} e^{-t} u_0(x) &+ \int_0^t e^{-(t-s)} \int_{\mathbb{R}_+} J(x-y)(C_0 + C_1)\, dy\, ds + \int_0^t e^{-(t-s)} \int_{\mathbb{R}_-} J(x-y) g(y,s)\, dy\, ds \\ &\leq C_1 e^{-t} + (C_0 + C_1) \int_0^t e^{-(t-s)} \int_{\mathbb{R}} J(x-y)\, dy\, ds \\ &\leq C_1 e^{-t} + (C_0 + C_1) \int_0^t e^{-(t-s)}\, ds \\ &\leq C_1 e^{-t} + (C_0 + C_1)(1 - e^{-t}) \\ &\leq C_1 e^{-t} + C_0 + C_1(1 - e^{-t}) = C_0 + C_1. \end{aligned}$$

Thus, $C_0 + C_1$ is a supersolution to (2.1). By a similar argument, we can show that $-C_0 - C_1$ is a subsolution to (2.1). $\square$

Lemma 3.5. *Suppose that u is a solution of (1.1). Then,*

$$-C_0 - C_1 \leq u(x,t) \leq C_0 + C_1$$

for $(x,t) \in \mathbb{R}_+ \times [0,\infty)$.

Proof. By Lemma 3.4, we have $u_1 = -C_0 - C_1$ satisfies (2.4) and $u_2 = C_0 + C_1$ satisfies (2.5). Applying Theorem 3.3, we conclude that $u_1 \leq u \leq u_2$ as desired. $\square$

CHAPTER IV
STATIONARY PROBLEM

In this chapter, we study the following stationary problem for establishing the long-time behavior of solution in Chapter V:

$$\int_{\mathbb{R}} J(x-y)\phi(y)\,dy - \phi(x) = 0 \quad \text{in } \mathbb{R}_+, \tag{4.1}$$

with $\phi = 0$ on $\mathbb{R}_-$. We show that $\phi \equiv 0$. Denote

$$J * \phi(x) = \int_{\mathbb{R}} J(x-y)\phi(y)\,dy.$$

Theorem 4.1. *Suppose that $\phi_1, \phi_2 \in C(\mathbb{R})$, $\phi_1(x) - \phi_2(x) \to 0$ as $x \to \infty$ and bounded on $\mathbb{R}_+$. If*

$$J * \phi_1(x) - \phi_1(x) \geq J * \phi_2(x) - \phi_2(x) \quad \text{in } \mathbb{R}_+ \tag{4.2}$$

and $\phi_1 \leq \phi_2$ in $\mathbb{R}_-$, then $\phi_1 \leq \phi_2$ in $\mathbb{R}_+$.

Proof. By considering $\phi := (\phi_1 - \phi_2)_+$, From (4.2), we get

$$\int_{\mathbb{R}} J(x-y)(\phi_1(y) - \phi_2(y))\,dy \geq \phi_1(x) - \phi_2(x).$$

Since J is nonnegative, we have

$$\begin{aligned}\int_{\mathbb{R}} J(x-y)(\phi_1(y) - \phi_2(y))_+\,dy &\geq \left(\int_{\mathbb{R}} J(x-y)(\phi_1(y) - \phi_2(y))\,dy\right)_+ \\ &\geq (\phi_1(x) - \phi_2(x))_+.\end{aligned}$$

This means $J * \phi \geq \phi$. It suffices to prove the following assertion: if $J * \phi \geq \phi$, where $\phi \equiv 0$ in $\mathbb{R}_-$, $\phi \geq 0$ is continuous, and $\phi(x) \to 0$ as $x \to \infty$, then $\phi \equiv 0$ on $\mathbb{R}_+$.

Suppose the desired conclusion is not true, so $M := \sup_{\mathbb{R}} \phi > 0$ is attained at some $x_1 > 0$. Let $\varepsilon > 0$ and define $A_\varepsilon = \{x \in \mathbb{R} \mid \phi(x) \leq M - \varepsilon\}$. Then,

$$\begin{aligned} M = \phi(x_1) &\leq \int_{\mathbb{R}} J(x_1 - y)\phi(y)\,dy \\ &= \int_{A_\varepsilon} J(x_1 - y)\phi(y)\,dy + \int_{\mathbb{R}\setminus A_\varepsilon} J(x_1 - y)\phi(y)\,dy \\ &\leq (M - \varepsilon)\int_{A_\varepsilon} J(x_1 - y)\,dy + M\int_{\mathbb{R}\setminus A_\varepsilon} J(x_1 - y)\,dy \\ &= M\int_{\mathbb{R}} J(x_1 - y)\,dy - \varepsilon\int_{A_\varepsilon} J(x_1 - y)\,dy \\ &= M - \varepsilon\int_{A_\varepsilon} J(x_1 - y)\,dy, \end{aligned}$$

which implies $\int_{A_\varepsilon} J(x_1 - y)dy = 0$. This is true for all $\varepsilon > 0$. Passing $\varepsilon \to 0$, we obtain $\int_{\mathbb{R}} J(x_1 - y)dy = 0$, a contradiction. □

Corollary 4.2. *Assume that ϕ is a continuous solution of (4.1) and $\phi(x) \to 0$ as $x \to \infty$. Then, $\phi = 0$ in $\mathbb{R}$.*

Proof. Observe that $0 = J * 0 - 0$. By Theorem 4.1, we get that $\phi \geq 0$ and $\phi \leq 0$, hence $\phi \equiv 0$. □

CHAPTER V
LONG-TIME BEHAVIOR OF SOLUTIONS

Now, we prove the last main result of this work.

Theorem 5.1. *Assume that the Hypothesis 1.5 holds. Let u be the solution of (1.1). Then,*

$$\lim_{t\to\infty} u(x,t) = \theta_*$$

locally uniformly in $\mathbb{R}$.

Proof. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[0,\infty)$ such that $t_n \to \infty$ as $n \to \infty$. By Lemma 3.5, we have $\{u(x,t_n)\}_{n=1}^{\infty}$ is uniformly bounded. Let $H > 0$.

Claim 1. $\{u(x,t_n)\}_{n=1}^{\infty}$ is uniformly equicontinuous for $x \in [0,H]$.

Proof of Claim 1. Let $x_1, x_2 \in [0,H]$ and $\delta := |x_1 - x_2|$. By (2.2), we get

$$\begin{aligned} u(x_1,t_n) - u(x_2,t_n) = e^{-t_n}&[u_0(x_1) - u_0(x_2)] \\ &+ \int_0^{t_n} e^{-(t_n-s)} \int_{\mathbb{R}_+} [J(x_1-y) - J(x_2,y)]u(y,s)\,dy\,ds \\ &+ \int_0^{t_n} e^{-(t_n-s)} \int_{\mathbb{R}_-} [J(x_1-y) - J(x_2-y)]g(y,s)\,dy\,ds. \end{aligned}$$

Using Hypothesis 1.5, we have

$$|u(x_1,t_n) - u(x_2,t_n)| \le |u_0(x_1) - u_0(x_2)| + (C_0 + C_1) \int_{\mathbb{R}} |J(x_1-y) - J(x_2-y)|\,dy.$$

By the triangle inequality, changing variables $z = x_2 - y$, and applying Lemma 2.4, we have

$$\int_{\mathbb{R}_-} |J(x_1-y) - J(x_2-y)|\,dy \le \int_{\mathbb{R}} |J(x_1-x_2+z) - J(z)|dz \to 0$$

as $\delta \to 0$. Since u_0 in uniformly continuous on $[0, H]$, we get $|u_0(x_1) - u_0(x_2)| \to 0$ as well. Therefore, the claim is true. $\square$

Now, we apply Lemma 2.7 to get a subsequence $\{u(x, t_{n_k})\}$ and a continuous function $\theta : [0, H] \to \mathbb{R}$ such that $u(x, t_{n_k}) \to \theta(x)$ as $k \to \infty$, uniformly for $x \in [0, H]$. By increasing H and passing to subsequences (diagonal argument), we can extend θ to be $\theta : [0, \infty) \to \mathbb{R}$ and get a susequence, still denoted by $\{u(x, t_{n_k})\}$, such that

$$\lim_{k \to \infty} u(x, t_{n_k}) = \theta(x) \tag{5.1}$$

locally uniformly for $x \in \mathbb{R}_+$. Using Remark 3.2, one can deduce that $\theta(x) \to \theta_*$ as $x \to \infty$.

Fix $x \in \mathbb{R}_+$. Substitute $u(x, t_{n_k})$ in (2.2) to obtain $u(x, t_{n_k}) = e^{-t_{n_k}} u_0(x) + I_k + S_k$, where

$$I_k = \int_0^{t_{n_k}} e^{-(t_{n_k} - s)} \int_{\mathbb{R}_+} J(x-y) u(y, s) dy ds,$$
$$S_k = \int_0^{t_{n_k}} e^{-(t_{n_k} - s)} \int_{\mathbb{R}_-} J(x-y) g(y, s) dy ds.$$

We can express

$$I_k = \frac{F(t_{n_k})}{e^{t_{n_k}}} \quad \text{with} \quad F'(t_{n_k}) = e^{t_{n_k}} \int_{\mathbb{R}_+} J(x-y) u(y, t_{n_k}) dy$$
$$S_k = \frac{G(t_{n_k})}{e^{t_{n_k}}} \quad \text{with} \quad G'(t_{n_k}) = e^{t_{n_k}} \int_{\mathbb{R}_-} J(x-y) g(y, t_{n_k}) dy.$$

In view of the L'Hospital's rule [9], we need to prove the following claim:

Claim 2. As $k \to \infty$, we have

$$\int_{\mathbb{R}_+} J(x-y) u(y, t_{n_k}) dy \to \int_{\mathbb{R}_+} J(x-y) \theta(y)\, dy,$$
$$\int_{\mathbb{R}_-} J(x-y) g(y, t_{n_k}) dy \to \theta_* \int_{\mathbb{R}_-} J(x-y) dy.$$

Proof of Claim 2. By (5.1), we have $J(x-y)u(y,t_{n_k}) \to J(x-y)\theta(y)$ for each $y \in \mathbb{R}_+$, as $k \to \infty$. Also, by Lemma 3.5, $|J(x-y)u(y,t_{n_k})| \le (C_0+C_1)J(x-y) \in L^1(\mathbb{R}_+)$. Thus, we obtain by Lemma 2.5 that

$$\int_{\mathbb{R}_+} J(x-y)u(y,t_{n_k})dy \to \int_{\mathbb{R}_+} J(x-y)\theta(y)dy$$

Similarly, we get

$$\int_{\mathbb{R}_-} J(x-y)g(y,t_{n_k})\,dy \to \theta_* \int_{\mathbb{R}_-} J(x-y)dy.$$

These prove the claim. □

Applying **Claim 2** and the above consideration, we have

$$u(x,t_{n_k}) = e^{-t_{n_k}}u_0(x) + I_k + S_k \to \int_{\mathbb{R}_+} J(x-y)\theta(y)dy + \theta_* \int_{\mathbb{R}_-} J(x-y)dy,$$

Thus,

$$\theta(x) = \int_{\mathbb{R}_+} J(x-y)\theta(y)dy + \theta_* \int_{\mathbb{R}_-} J(x-y)dy. \tag{5.2}$$

This is true for all $x \in \mathbb{R}_+$. Setting $k(x) = \theta(x) - \theta_*$ for $x \in \mathbb{R}$, From (5.2), we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}_+} J(x-y)\theta(y)dy + \theta_* \int_{\mathbb{R}_-} J(x-y)dy - \theta(x) \\
&= \int_{\mathbb{R}_+} J(x-y)k(y)dy + \theta_* \int_{\mathbb{R}_+} J(x-y)dy + \theta_* \int_{\mathbb{R}_-} J(x-y)dy - \theta(x) \\
&= \int_{\mathbb{R}_+} J(x-y)k(y)dy + \theta_* \int_{\mathbb{R}} J(x-y)dy - \theta(x) \\
&= \int_{\mathbb{R}_+} J(x-y)k(y)dy + \theta_* - \theta(x) \\
&= \int_{\mathbb{R}} J(x-y)k(y)dy - k(x)
\end{aligned}$$

in $\mathbb{R}_+$. Observe that $k(x) = 0$ for $x \in \mathbb{R}_-$ and $k(x) \to 0$ as $x \to \infty$. By Corollary

4.2, we conclude that $k \equiv 0$. Hence, $\theta(x) = \theta_*$ is constant. Therefore, the theorem is proved. $\square$

CHAPTER VI
CONCLUSION AND DISCUSSION

We studied a nonlocal dispersal equation (1.1) in the case that J has non-compact support, the long-time behavior of the solution is proved. The proof is complete under the initial value and the boundary value in the Hypothesis 1.5. Furthermore, we investigated a comparison principle and a stationary problem, which are an important role to prove the long-time behavior of the solution.

We paid attention to the kernel J for the nonlocal dispersal equation. In the future of our work, we may expand a kernel or a model for a nonlocal problem in studying the long-time behavior of a solution.

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