CHAPTER IV

INVERTIBLE MATRICES OVER A SEMIFIELD

It is very well-known that a square matrix A over a field F is invertible over F if and only if $\det(A) \neq 0$. By the definition of semifields, every field is a semifield. The aim of this chapter is to characterize invertible matrices over a semifield which is not a field. Also, we study invertible matrices over a semifield of nonnegative real numbers under the usual multiplication.

The following semirings are semifields which are not fields : $(\mathbb{R}^+\cup\{0\},+,\cdot)$, $(\mathbb{Q}^+\cup\{0\},+,\cdot)$, $(\mathbb{R}^+\cup\{0\},\max,\cdot)$, $(\mathbb{Q}^+\cup\{0\},\max,\cdot)$ where + and $\cdot$ are the usual addition and multiplication of real numbers, respectively.

<u>Theorem 4.1</u>. Let S be a semifield which is not a field and A a square matrix over S. Then the matrix A is invertible over S if and only if every row and every column of A has exactly one nonzero element.

<u>Proof</u> : By Theorem 1.4, 0 is the only additively invertible element of S. Since S is a semifield, $S\setminus\{0\}$ is closed under multiplication, so $xy = 0$ in S implies that $x = 0$ or $y = 0$. By Theorem 1.5, the matrix A is invertible over S if and only if every row and every column of A has exactly one nonzero element and every nonzero element of A is a multiplicatively invertible element of S. But every nonzero element of S is multiplicatively invertible in S

since S is a semifield, it then follows that the matrix A is invertible over S if and only if every row and every column of A has exactly one nonzero element. #

The four examples of semifields given above are semifields of nonnegative real numbers under the usual multiplication which are not fields. However, we can prove by Theorem 1.6 that there exists an operation $\oplus$ on $\mathbb{Q}^+\cup\{0\}$ such that $(\mathbb{Q}^+\cup\{0\},\oplus,\cdot)$ is a field where $\cdot$ is the usual multiplication. By Theorem 1.6, there exists an operation $*$ on $\mathbb{N}\cup\{0\}$ such that $(\mathbb{N}\cup\{0\},*,\cdot)$ is a ring where $\cdot$ is the usual multiplication. Define $\oplus$ on $\mathbb{Q}^+\cup\{0\}$ by

$$\frac{m}{n}\oplus\frac{p}{q}=\frac{mq*pn}{nq}$$

for all $m,n,p,q \in \mathbb{N}\cup\{0\}$, $n \neq 0$, $q \neq 0$. To show that $\oplus$ is well-defined, let $\frac{m}{n}=\frac{m'}{n'}$ and $\frac{p}{q}=\frac{p'}{q'}$ where $m,m',p,p' \in \mathbb{N}\cup\{0\}$ and $n,n',q,q' \in \mathbb{N}$. Then $mn'=m'n$ and $pq'=p'q$, so $mn'qq'=m'nqq'$ and $nn'pq'=nn'p'q$. Thus $n'q'(mq*pn)=n'q'mq*n'q'pn=m'nqq'*nnp'q=nq(m'q'*p'n')$ and hence

$$\frac{m}{n}\oplus\frac{p}{q}=\frac{mq*pn}{nq}=\frac{m'q'*p'n'}{n'q'}=\frac{m'}{n'}\oplus\frac{p'}{q'}.$$

It is straightforward to verify that the operation $\oplus$ is associative on $\mathbb{Q}^+\cup\{0\}$, the usual multiplication is distributive over the operation $\oplus$ on $\mathbb{Q}^+\cup\{0\}$ and $0\oplus x = x\oplus 0 = x$ for all $x \in \mathbb{Q}^+\cup\{0\}$. Because the operation $*$ is commutative on $\mathbb{N}\cup\{0\}$, the operation $\oplus$ is commutative on $\mathbb{Q}^+\cup\{0\}$. It remains to show that for each $x \in \mathbb{Q}^+\cup\{0\}$, $x\oplus y = 0$ for some $y \in \mathbb{Q}^+\cup\{0\}$. Let $\frac{m}{n} \in \mathbb{Q}^+\cup\{0\}$ where $m \in \mathbb{N}\cup\{0\}$ and $n \in \mathbb{N}$. Since $(\mathbb{N}\cup\{0\},*)$ is a group, $m*p = p*m = 0$ for some $p \in \mathbb{N}\cup\{0\}$. Then $\frac{p}{n} \in \mathbb{Q}^+\cup\{0\}$ and

$$\frac{m}{n}\oplus\frac{p}{n}=\frac{mn*pn}{nn}=\frac{m*p}{n}=\frac{0}{n}=0.$$

Therefore $(\mathbb{Q}^+\cup\{0\},\oplus,\cdot)$ is a field.

We shall prove in the next proposition that for any semifield S of nonnegative real numbers under the usual multiplication, if S is a field, then the characteristic of S is 2, and if $1\oplus 1 = 0$ where $\oplus$ is the addition of the semifield S, then S is a field.

Proposition 4.2. Let S be a semifield of nonnegative real numbers under the usual multiplication. Then S is a field if and only if $1\oplus 1 = 0$ where $\oplus$ is the addition of the semifield S.

Proof : Assume that S is a field. Then $1\oplus x = 0$ for some $x \in S$. Thus $0 = x0 = x(1\oplus x) = x\oplus x^2$ which implies that $x^2 = 1$ because $(S,\oplus)$ is a group. Since $x^2 = 1$ and $x \geqslant 0$, $x = 1$. Hence $1\oplus 1 = 0$.

For the converse, assume that $1\oplus 1 = 0$. Then $x\oplus x = x(1\oplus 1) = x0 = 0$ for all $x \in S$, so $(S,\oplus)$ is a group. Hence S is a field. #

The next theorem related to invertible matrices over any semifield of nonnegative real numbers under the usual multiplication follows easily from the well-known result mentioned at the beginning of this chapter, Theorem 4.1 and Proposition 4.2.

Theorem 4.3. Let S be a semifield of nonnegative real numbers under the usual multiplication and A a square matrix over S. Suppose that the addition of S is $\oplus$. Then the following statements hold :

(1) If $1\oplus 1 = 0$, then the matrix A is invertible over S if and only if $\det^+ A \neq \det^- A$.

(2) If $1 \oplus 1 \neq 0$, then the matrix A is invertible over S if and only if every row and every column of A has exactly one nonzero element.