Chapter 1

Statements of the results

Fix a positive real number t. A Gaussian measure μ_t on $\mathbb{C}^d$ is defined by

$$d\mu_t(z) = \frac{1}{(\pi t)^d} e^{-|z|^2/t}\, dz$$

where $z = (z_1, \ldots, z_d)$ and $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$. The Segal-Bargmann space, denoted by $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, is the space of all holomorphic functions on $\mathbb{C}^d$ which are square-integrable with respect to the Gaussian measure μ_t. Denote by $SO(d, \mathbb{C})$ the special complex orthogonal group. i.e., the group of all $d \times d$ matrices A with entries in $\mathbb{C}$ such that $A^t = A^{-1}$ and $\det A = 1$. Let $\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ be the space of all $SO(d, \mathbb{C})$-invariant holomorphic functions on $\mathbb{C}^d$, and let

$$\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})} = \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \cap L^2(\mathbb{C}^d, \mu_t).$$

Then it is a closed subspace of the Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, and hence is a Hilbert space. In this work, we find an orthonormal basis and the reproducing kernel of this space. We do this by expressing the space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ as a space of holomorphic functions on $\mathbb{C}$ which are square-integrable with respect to some non-Gaussian measure. The latter space is easier to work with. So we will find its orthonormal basis and the reproducing kernel, and then we transform everything back to $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ by a unitarily equivalent map.

The main results of this work are as follows:

1. The following set

$$\left\{ \frac{(z,z)^n}{\left(t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)!\right)^{1/2}} \right\}_{n=0}^{\infty}$$

forms an orthonormal basis of $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$.

2. The reproducing kernel for the space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ is given by

$$K(z,w) = \sum_{n=0}^{\infty} \frac{(z,z)^n \, \overline{(w,w)^n}}{t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)!}.$$

3. We have the pointwise bound

$$|F(z)|^2 \leq \sum_{n=0}^{\infty} \frac{|(z,z)|^{2n}}{t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)!} ||F||^2$$

for any $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$.