Chapter 3

Holomorphic function spaces

In this chapter, we provide some standard results that will be referred to in the next chapter. First, we discuss a reproducing kernel, which exists on any Hilbert space such that each pointwise evaluation is bounded. After that we consider the space of holomorphic functions which are square-integrable with respect to the weight α. We will see that this space also has a reproducing kernel.

Definition 3.1. Let X be a nonempty set. Let H be a Hilbert space of functions defined on X such that the pointwise evaluation

$$\Phi_z : H \to \mathbb{F}, \quad f \mapsto f(z),$$

is continuous for each $z \in X$. Then for each $z \in X$, Φ_z is a bounded linear functional on H. So there exists a unique function $\phi_z \in H$ such that

$$f(z) = \Phi_z(f) = \langle \phi_z, f \rangle$$

for all $f \in H$. Define a function $K : X \times X \to \mathbb{F}$ by

$$K(w, z) = \phi_z(w) \quad \text{for all } w, z \in X.$$

K is called the **reproducing kernel** of H and H is said to be a **reproducing kernel Hilbert space**.

Theorem 3.2. *Let H be a reproducing kernel Hilbert space. Then the reproducing kernel K of H has the following properties:*

(i) $K(w,z) = \overline{K(z,w)}$ *for all* $z, w \in X$.

(ii) For all $z \in X$,

$$|F(z)|^2 \le K(z,z)\,||F||^2,$$

and the constant $K(z,z)$ *is optimal in the sense that for each* $z \in X$ *there exists a non-zero* $F_z \in H$ *for which equality holds.*

(iii) Given any $z \in X$, *if* $\psi_z \in H$ *satisfies*

$$F(z) = \langle \psi_z, F \rangle$$

for all $F \in H$, *then* $\psi_z(w) = K(w,z)$ *for any* $w \in X$.

Proof. (i) Let $z, w \in X$. Then

$$K(w,z) = \phi_z(w) = \Phi_w(\phi_z) = \langle \phi_w, \phi_z \rangle.$$

Similarly, $K(z,w) = \langle \phi_z, \phi_w \rangle$. Hence $K(w,z) = \overline{K(z,w)}$ for all $z, w \in X$.

(ii) Let $z \in X$. Then

$$||\phi_z||^2 = \langle \phi_z, \phi_z \rangle = \Phi_z(\phi_z) = \phi_z(z) = K(z,z).$$

For any $F \in H$,

$$|F(z)|^2 = |\Phi_z(F)|^2 = |\langle \phi_z, F \rangle|^2 \le ||\phi_z||^2 ||F||^2.$$

Hence, $|F(z)|^2 \le K(z,z)||F||^2$, for all $F \in H$.

(iii) Let $z \in X$. Assume that $\phi_z \in H$ satisfies

$$F(z) = \langle \phi_z, F \rangle \quad \text{for all } F \in H.$$

By the definition of the reproducing kernel and (i), we have

$$F(z) = \langle K(\cdot, z), F \rangle = \langle \overline{K(z, \cdot)}, F \rangle \quad \text{for all } F \in H.$$

Hence, $\langle \overline{K(z, \cdot)} - \phi_z, F \rangle = 0$ for all $F \in H$. Thus $\overline{K(z, \cdot)} - \phi_z = 0$. which implies that $\overline{\phi_z(w)} = K(z, w)$ for all $w \in X$.

□

Theorem 3.3. *Let H be a reproducing kernel Hilbert space of functions defined on a nonempty set X. Let $\{e_n\}_{n=0}^{\infty}$ be any orthonormal basis of H. Then for all $z, w \in X$,*

$$\sum_{n=0}^{\infty} \left| e_n(z)\, \overline{e_n(w)} \right| < \infty$$

and

$$K(z, w) = \sum_{n=0}^{\infty} e_n(z)\, \overline{e_n(w)}.$$

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of H. For any $F \in H$, by Parseval's identity,

$$\sum_{n=0}^{\infty} \left| \langle F, e_n \rangle \right|^2 = ||F||^2.$$

By the Schwarz's inequality, we have for any $f, g \in H$.

$$\sum_{n=0}^{\infty} \left| \langle f, e_n \rangle \langle e_n, g \rangle \right| \leq ||f|| ||g||.$$

Taking $f = \phi_z$ and $g = \phi_w$ we get

$$\sum_{n=0}^{\infty} \left| e_n(z) \overline{e_n(w)} \right| = \sum_{n=0}^{\infty} \left| \langle \phi_z, e_n \rangle \langle e_n, \phi_w \rangle \right| \leq ||\phi_z|| ||\phi_w|| < \infty.$$

Now think of the partial sum of $\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$ as a functions of w with z fixed.

Then the series is orthogonal and

$$\begin{aligned}\sum_{n=0}^{\infty} \|e_n(z)\,\overline{e_n}\|^2 &= \sum_{n=0}^{\infty} \|\langle \phi_z, e_n\rangle e_n\|^2 \\ &= \sum_{n=0}^{\infty} \left|\langle \phi_z, e_n\rangle\right|^2 \|e_n\|^2 \\ &= \sum_{n=0}^{\infty} \left|\langle \phi_z, e_n\rangle\right|^2 \\ &= \|\phi_z\|^2 < \infty.\end{aligned}$$

Hence, $\sum_{n=0}^{\infty} e_n(z)\overline{e_n}$ is absolutely convergent in H, which implies that $\sum_{n=0}^{\infty} e_n(z)\overline{e_n}$ is convergent in H. For any $F \in H$, we have that for all $z \in X$

$$\begin{aligned}F(z) &= \langle \phi_z, F\rangle \\ &= \sum_{n=0}^{\infty} \langle \phi_z, e_n\rangle\langle e_n, F\rangle \\ &= \sum_{n=0}^{\infty} e_n(z)\langle e_n, F\rangle \\ &= \sum_{n=0}^{\infty} \langle \overline{e_n(z)}e_n, F\rangle \\ &= \left\langle \sum_{n=0}^{\infty} \overline{e_n(z)}e_n, F\right\rangle.\end{aligned}$$

By Theorem 3.2(iii), $\overline{\sum_{n=0}^{\infty} \overline{e_n(z)}e_n(w)} = K(z,w)$ for all $z, w \in X$. Hence, for all $z, w \in X$

$$K(z,w) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}.$$

□

Next, we review some basic concepts of a holomorphic function space. All of these results appeared in [H2].

Definition 3.4. Let d be a positive integer and U a nonempty open set in $\mathbb{C}^d$. A function $f\colon U \to \mathbb{C}$ is **holomorphic** if f is continuous and holomorphic in each variable with the other variables fixed. Let $\mathcal{H}(U)$ be the space of holomorphic functions on U.

Definition 3.5. Let α be a continuous strictly-positive function on a nonempty open set U in $\mathbb{C}^d$. Denote by $\mathcal{H}L^2(U, \alpha)$ the space of L^2-holomorphic functions with respect to the weight α. In other words,

$$\mathcal{H}L^2(U, \alpha) = \left\{ F \in \mathcal{H}(U) \;\middle|\; \int_U |F(z)|^2 \, \alpha(z) \, dz < \infty \right\}.$$

Theorem 3.6. *(i) $\mathcal{H}L^2(U, \alpha)$ is a closed subspace of $L^2(U, \alpha)$, which implies that it is a Hilbert space.*

(ii) The pointwise evaluation $\phi_z \colon F \mapsto F(z)$ is continuous for each $z \in U$. Hence, $\mathcal{H}L^2(U, \alpha)$ is a reproducing kernel Hilbert space.

Definition 3.7. The **Segal-Bargmann space** is the holomorphic function space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, where

$$\mu_t(z) = (\pi t)^{-d} \, e^{-|z|^2/t}.$$

Here, $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$ and t is a positive number.

Theorem 3.8. *The reproducing kernel of $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ is given by*

$$K(z, w) = e^{\langle w, z \rangle / t}.$$

Hence, we have the pointwise bound

$$|F(z)|^2 \leq e^{|z|^2/t} \|F\|^2$$

for any $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$.