CHAPTER II.

Feynman's Path Integrals

The best places to find out about path integrals is in Feynman's paper [(4)]. Our approach is not to use path integrals as a way of arriving at quantum mechanics, although Feynman has used this point in his book with Hibbs [(4)]. We assume knowledge of quantum mechanics and deduce the path integrals formalism [(12)] from it. This gets us into the subject quickly.

II.1 Defining the Path Integrals.

The wave function of a non-relativistic spinless particle in one dimension evolves according to Schroedinger's equation

$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \qquad (\text{II}.1)$$

$$H = T + V = \frac{p^2}{2m} + V = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V. \qquad (\text{II}.2)$$

Our interest is in the propagator K or Green's function G which satisfies the equation

$$\left(H - i\hbar \frac{\partial}{\partial t} \right) G(t_b, t_a) = -i\hbar 1 \delta(t_b - t_a) \qquad (\text{II}.3)$$

in operator notation. In coordinate space this is written for the propagator as

$$\left(H - i\hbar \frac{\partial}{\partial t} \right) K(x_b, x_a; t_b, t_a) = -i\hbar\, \delta(x_b - x_a)\, \delta(t_b - t_a). \qquad (\text{II}.4)$$

G and K are related by

$$K(x_b, x_a; t_b, t_a) = \langle x_b | G(t_b, t_a) | x_a \rangle . \qquad (\text{II}.5)$$

Knowing $G(t_b, t_a)$ means having a solution to the time dependent Schroedinger's equation in the sense that if $\psi(t_a)$ is the state of the system at the time t_a, $\psi(t_b)$, given by

$$\psi(t_b) = G(t_b, t_a)\,\psi(t_a) \qquad (\text{II}.6)$$

is the state at the time t_b. For the time independent H an operator solution of Eq. (II.3) can immediately be written . as

$$G(t_b, t_a) = \Theta(t_b - t_a)\exp\left\{-\frac{i}{\hbar}H(t_b - t_a)\right\} \qquad (\text{II}.7)$$

where $\Theta(t_b - t_a)$ is the step function. Since H is assumed to be time independent we can, without loss of generality, take $t_a = 0$ and $t_b = t$. Then for $t > 0$ we can

$$K(x_b, x_a; t) = \langle x_b | \exp\left\{-\frac{i}{\hbar}Ht\right\} | x_a \rangle \qquad (\text{II}.8)$$

where the argument $t_a = 0$ has been deleted.

The path integrals arise from the fact that

$$e^{A} = \left(e^{A/N}\right)^{N} . \qquad (\text{II}.9)$$

Letting $\lambda = it/\hbar$ yields

$$K(x_b, x_a; t) = \langle x_b | e^{-\lambda(T+V)/N} \cdots e^{-\lambda(T+V)/N} | x_a \rangle \qquad (\text{II}.10)$$

with the products in the bracket taken N times. Now we make use of a fundamental fact about the exponential of two operators, namely

$$e^{-\lambda(T+V)/N} = e^{-\lambda T/N} e^{-\lambda V/N} + O\left(\frac{\lambda^2}{N^2}\right). \qquad (\text{II}.11)$$

This can be proved easily * and in a power series expansion the coeffient of the $\frac{\lambda^2}{N^2}$ term is

$$A = \frac{1}{2}[V, T]_c \,.$$

In subsequent manipulation we assume that the $O(\frac{1}{N^2})$ term is well behaved, that is stays bounded when applied to states, and so on. For reasonable potentials this assumption is justisfied. More is said on this topic in the appendix A.

What we are now arriving for is to replace the term

$$\left[e^{-\lambda(T+V)/N} \right]^N = \left[e^{-\lambda T/N} e^{-\lambda V/N} + O\left(\frac{1}{N^2}\right)\right]^N \qquad (\text{II}.12)$$

by the term

$$\left[e^{-\lambda T/N} e^{-\lambda V/N} \right]^N. \qquad (\text{II}.13)$$

For real numbers (rather than operators), this replacement is a

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* : An expression is conveniently generated by looking at derivative of $\exp(\lambda T/N)\exp(-\lambda(T+V)/N)\exp(\lambda V/N)$.

reflection of a fundamental fact about the exponential. For operators a bit of care is required, and the trick is to express the difference of Eqs. (II.12) and (II.13) is a peculiar way;

$$\left[e^{-\lambda T/N}e^{-\lambda V/N}\right]^N - \left[e^{-\lambda(T+V)/N}\right]^N$$
$$= \left[e^{-\lambda T/N}e^{-\lambda V/N} - e^{-\lambda(T+V)/N}\right]\left[e^{-\lambda(T+V)/N}\right]^{N-1}$$
$$+\left[e^{-\lambda T/N}e^{-\lambda V/N}\right]\left[e^{-\lambda T/N}e^{-\lambda V/N} - e^{-\lambda(T+V)/N}\right]\left[e^{-\lambda(T+V)/N}\right]^{N-2}$$
$$+\cdots+\left[e^{-\lambda T/N}e^{-\lambda V/N}\right]^{N-1}\left[e^{-\lambda T/N}e^{-\lambda V/N} - e^{-\lambda(T+V)/N}\right]. \tag{II.14}$$

Eq. (II.14) is an identity. It contains N terms, each of which has the factor $\left\{\exp(-\lambda T/N)\exp(-\lambda V/N) - \exp(-\lambda(T+V)/N)\right\}$ which by Eq. (II.11) is of order $\frac{1}{N^2}$. Hence in the limit $N\to\infty$ the difference is zero. In appendix A, mention is made for various finer points in the estimate.

We have therefore justified the replacement of Eq. (II.10) by

$$K(x_b,x_a;t) = \lim_{N\to\infty} \langle x_b|\left\{e^{-\lambda T/N}e^{-\lambda V/N}\right\}^N|x_a\rangle \tag{II.15}$$

From here, getting the path integrals is just a few easy steps. The identity operator, in the form

$$\int dx_j\,|x_j\rangle\langle x_j| \qquad ; \quad j=1,2,\ldots,N-1. \tag{II.16}$$

is inserted between each terms in the product in Eq. (II.15), yielding

$$K(x_b, x_a; t) = \lim_{N\to\infty} \int dx_1 \cdots \int dx_{N-1} \prod_{j=0}^{N-1} \langle x_{j+1} | e^{-\lambda T/N} e^{-\lambda V/N} | x_j \rangle \qquad (\text{II}.17)$$

for convenient we have taken $x_a = x_0$, $x_b = x_N$. The multiplication operator V is diagonal in coordinate space so that

$$\exp\left(-\lambda V/N\right) | x_j \rangle = | x_j \rangle \exp\left(-\lambda V(x_j)/N\right) . \qquad (\text{II}.18)$$

Next we require coordinate space matrix elements of $e^{-\lambda T/N}$ between states $\langle s |$ and $| f \rangle$, and to obtain these we insert a complete set of momentum states

$$1 = \int dp \, | p \rangle \langle p |$$

with

$$\langle p | f \rangle = \frac{1}{(2\pi\hbar)^{1/2}} \exp\left(- i p f / \hbar \right) . \qquad (\text{II}.19)$$

This requires

$$\begin{aligned} \langle s | e^{-\lambda T/N} | f \rangle &= \int dp \, \langle s | e^{-\lambda T/N} | p \rangle \langle p | f \rangle \\ &= \int dp \, \exp\left(-\lambda p^2/2mN\right) \langle s | p \rangle \langle p | f \rangle \\ &= \frac{1}{2\pi\hbar} \int dp \, \exp\left(- \frac{\lambda p^2}{2mN}\right) \exp\left(i p (s-f)/\hbar\right) . \end{aligned} \qquad (\text{II}.20)$$

This is the gaussian integration . The general formula is

$$\int_{-\infty}^{\infty} dy \exp\left(- a y^2 + b y\right) = \left(\frac{\pi}{a}\right)^{1/2} \exp\left(b^2/4a\right) . \qquad (\text{II}.21)$$

More detials of this formula is contained in appendix B. Using Eq. (II.21), Eq. (II.20) becomes

$$\langle s|\exp(-\lambda T/N)|f\rangle = \left[\frac{mN}{2\lambda\pi\hbar^2}\right]^{1/2} \exp\left(-\frac{mN\,(s-f)^2}{2\lambda\hbar^2}\right). \quad (\text{II}.22)$$

Eqs. (II.18) and (II.22) are inserted into Eq. (II.17) to yield

$$K(x_b,x_a;t) = \lim_{N\to\infty} \int dx_1 \cdots \int dx_{N-1} \left[\frac{mN}{2\lambda\pi\hbar^2}\right]^{1/2} \times \prod_{j=0}^{N-1} \exp\left\{-\frac{mN\,(x_{j+1}-x_j)^2}{2\lambda\hbar^2} - \frac{V(x_j)}{N}\right\}. \quad (\text{II}.23)$$

Now let $\epsilon = t/N = \hbar\lambda/iN$ and combine the exponentials in Eq. (II.23);

$$K(x_b,x_a;t) = \lim_{N\to\infty} \int dx_1 \cdots \int dx_{N-1} \left[\frac{m}{2\pi i\epsilon\hbar}\right]^{1/2} \times \exp\left\{\frac{i}{\hbar}\prod_{j=0}^{N-1}\left[\frac{m}{2\epsilon}(x_{j+1}-x_j)^2 - \epsilon V(x_j)\right]\right\}. \quad (\text{II}.24)$$

Eq. (II.24) is the path integrals expression for the propagator. A few words are in order , however, on why this is called a " path integrals " or " sum over histories ".

Imagine that the points $x_a, x_1, x_2, \cdots, x_{N-1}, x_b$ are connected by lines. Then we have a broken lines path from x_a to x_b. The sum in the exponential of Eq. (II.24) can be integrated as a Riemann sum of a certain integral along that path:

$$\prod_{j=0}^{N-1}\left[\frac{m}{2\epsilon}(x_{j+1}-x_j)^2 - \epsilon V(x_j)\right] \sim \int_0^t \left\{\frac{m}{2}\left[\frac{d}{d\tau}X(\tau)\right]^2 - V(x)\right\} d\tau. \quad (\text{II}.25)$$

The integrand in Eq.(II.25) is well known in classical mechanics. It is just the lagrangian

$$\mathcal{L} = \frac{m}{2}\left[\frac{d}{d\tau}x(\tau)\right]^2 - V(x(\tau)) \qquad (\text{II}.26)$$

of the classical system which when quantized has the hamiltonian Eq. (II.2). Furthermore, the action

$$S = \int_0^t \mathcal{L}\{x(\tau)\}\,d\tau \qquad (\text{II}.27)$$

is no less prominant an object in classical mechanics. The argument in Eq. (II.24) is thus $\frac{iS}{\hbar}$, with S evaluated along the broken line path connecting $x_a, x_1, \ldots, x_{N-1}, x_b$.

The integral over the quantitis $x_1, x_2, \ldots, x_{N-1}$ can be interpreted as summing over all possible broken line paths connecting x_a and x_b and Eq. (II.24) has become

$$K(x_b, x_a; t) = C \sum_{\substack{\text{over all} \\ \text{paths from } x_a \text{ to } x_b.}} \exp\left\{\frac{i}{\hbar} S[x(\tau)]\right\}, \qquad (\text{II}.28)$$

where C is called normalized constant and defined to be

$$C = \left[\frac{m}{2\pi i\hbar t}\right]^{1/2} \qquad (\text{II}.29)$$

from Eq. (II.24). A final cosmetic expression on Eq. (II.24) is to be written as

$$K(a,b) = \int_{x_a}^{x_b} \mathcal{D}[x(\tau)] \exp\left\{\frac{i}{\hbar} S[x(\tau)]\right\}, \qquad (\text{II}.30)$$

where the notation $\mathcal{D}[x(\tau)]$ means the mathematical measure of the integration variable $x(\tau)$. We call this expression[4] " Feynman's path integration ".

II.2 Gaussian Integration

The simplest path integrals are those in which all the varibles appear up to the second degree in the exponent. In quantum mechanics this corresponds to a case in which the action S involves the path $x(\tau)$ up to and including the second order.

To illustrate how the method work in such a case, consider a particle whose Lagrangian has the form

$$\mathcal{L}\{\dot{x}, x; \tau\} = a(\tau)\dot{x}^2 + b(\tau)\dot{x}x + c(\tau)x^2 + d(\tau)\dot{x} + e(\tau)x + f(\tau) . \qquad (\text{II}.31)$$

The action is the integral of this function with respect to the time between two end points. We wish to determine

$$K(a,b) = \int_a^b \mathcal{D}[x(\tau)] \exp\left\{ \int_0^t \mathcal{L}\{\dot{x}, x; \tau\} d\tau \right\} , \qquad (\text{II}.32)$$

the integral is over all paths which go from $(x_a, 0)$ to (x_b, t).

Of course , it is possible to carry out this integral over all paths in the way which was first described in section I.1. But we shall not go through this tedious calculation, since we can determine the most important characteristics of the propagator in the following manner.

Let $x_{cl}(\tau)$ be the classical path between the specified end points . This is the path which is an extremum for the action S . In the notation we have been using

$$S_{cl}(a,b) = S\{x_{cl}(\tau)\} . \qquad (\text{II}.33)$$

We can represent $x(\tau)$ in terms of $x_{cl}(\tau)$ and a new varible $x'(\tau)$;

$$x(\tau) = x_{cl}(\tau) + x'(\tau) . \qquad (\text{II}.34)$$

That is to say, instead of defining a point on the path by its distance $x(\tau)$ from an arbitary coordinate axis, we measure the deviation $x'(\tau)$ from the calssical path $x_{cl}(\tau)$.

At each τ the variable $x(\tau)$ and $x'(\tau)$ differ by the constant $x_{cl}(\tau)$. Therefore, clearly, $dx_i = dx'_i$ for each specific point τ_i in the subdivision of time. In general, we may say $\mathcal{D}[x(\tau)] = \mathcal{D}[x'(\tau)]$.

The integral for the action can be written

$$S\{x(\tau)\} = S\{x_{cl}(\tau) + x'(\tau)\} \qquad (\text{II}.35)$$

This can be easily shown that we can write

$$S\{x(\tau)\} = S\{x_{cl}(\tau)\} + \int_0^t \left[a(\tau)\dot{x}'^2 + b(\tau)\dot{x}'x' + c(\tau)x'^2 \right] d\tau . \qquad (\text{II}.36)$$

The integrals over all paths does not depend upon the classical path, so that the propagator can be written

$$K(a,b) = \exp\left\{\frac{i}{\hbar} S(a,b)\right\}$$

$$\times \int_0^0 \mathcal{D}[x'(\tau)] \exp\left\{\frac{i}{\hbar}\int_0^t \left[a(\tau)\dot{x}'^2 + b(\tau)\dot{x}'x + c(\tau)x'^2 \right] d\tau\right\}. \qquad (\text{II}\cdot 37)$$

Since all paths $x'(\tau)$ start from and return to the point $x'(\tau)=0$, the integrals over all paths can be a function of time at the end points. This means that the propagator can be written as

$$K(a,b) = F(t,0)\exp\left\{\frac{i}{\hbar} S(a,b)\right\}. \qquad (\text{II}\cdot 38)$$

So that $K(a,b)$ is determined except for a function of t.

It follows from the works of Van Vleck (9) and Pauli (8) that for the local problem, Eq. (II·38) can also be written as

$$K(a,b) = \left[\frac{1}{2\pi i\hbar}\right]^{\frac{1}{2}} \left\{ \frac{\partial^2 S(a,b)}{\partial x_a \partial x_b} \right\}^{\frac{1}{2}} \exp\left\{\frac{i}{\hbar} S(a,b)\right\}. \qquad (\text{II}\cdot 39)$$

And in n dimensions coordinate space this formula is generalized to be

$$K(a,b) = \left[\frac{1}{2\pi i\hbar}\right]^{n/2} \left\{ \det\left[\frac{\partial^2 S(a,b)}{\partial \mathbf{x}_a \partial \mathbf{x}_b}\right] \right\}^{\frac{1}{2}} \exp\left\{\frac{i}{\hbar} S(a,b)\right\}, \qquad (\text{II}\cdot 40)$$

where $\mathbf{X}(\tau)$ is the n- dimensional coordinate space variable. A details of the evaluations of this result will appear in appendix C.

However, the propagator of Eqs. (II·39) or (II·40) are satified for the system which contain the local potential only. They cannot be used for the nonlocal potential.

II.3 Application to the Local Problems.

It is of interest to examine some simple problems of the quadratic Lagrangian system. The simplest one is of the free particle, the Lagrangian of this system is given by

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 \quad . \qquad (\text{II}\cdot 41)$$

By the routine calculation of the classical solution $x_{cl}(\tau)$ of the equation of motion of this system , the action function can be evaluated to be

$$S_{cl}(a,b) = \frac{m}{2t}(x_b - x_a)^2 \quad . \qquad (\text{II}\cdot 42)$$

Using Van Vleck-Pauli's results in Eq. (II·39), the propagator[12] becomes

$$K(a,b) = \left[\frac{m}{2\pi i\hbar t}\right]^{1/2} \exp\left\{\frac{im}{2\hbar t}(x_b - x_a)^2\right\} . \qquad (\text{II}\cdot 43)$$

Another problem is of the free electron moving in two dimensions under the influence of a constant magnetic field which presented in perpendicular direction to the plane of electronic motion. The Lagrangian of such a system, with the symmetric gauge of the magnetic field, $\vec{A} = (-yB, xB, 0)$, is given to be

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m}{2}\Omega(x\dot{y} - y\dot{x}) \; . \qquad (\text{II}\cdot 44)$$

where $\Omega = \dfrac{eB}{mc}$, the cyclotron frequency.

Applying Hamilton's theorem[13] to this lagrangian in both directions and evaluate the classical solutions $X_{cl}(\tau)$ and $Y_{cl}(\tau)$ for the corresponding equations. The action function can be calculated to be

$$S_{cl}(a,b) = \frac{m\Omega}{4} \cot\left(\frac{\Omega t}{2}\right)\left\{(x_b - x_a)^2 + (y_b - y_a)^2\right\}$$

$$+ \frac{m\Omega}{2}(x_a y_b - x_b y_a) . \qquad (\text{II}\cdot 45)$$

Using Van Vleck-Pauli's result in Eq. (II·39), the propagator becomes

$$K(a,b) = \left[\frac{m}{2\pi i \hbar t}\right]\left[\frac{\Omega t}{2\sin(\Omega t/2)}\right]$$

$$\times \exp\left\{\frac{im}{2\hbar}\left[\frac{\Omega}{2}\cot\left(\frac{\Omega}{2}t\right)\left\{(x_b - x_a)^2 + (y_b - y_a)^2\right\}\right.\right.$$

$$\left.\left. + \Omega(x_a y_b - x_b y_a)\right]\right\} . \qquad (\text{II}\cdot 46)$$

The last problem we will study here is of the simple harmonic oscillator . The lagrangian of this system is

$$\mathcal{L} = \frac{m}{2}\left[\dot{x}^2 - \omega^2 x^2\right] . \qquad (\text{II}\cdot 47)$$

The classical action function of this problem can be evaluated systematically and it becomes

$$S_{cl}(a,b) = \frac{m\omega}{2\sin(\omega t)}\left[\cos(\omega t)\left\{x_b^2 + x_a^2\right\} - 2x_b x_a\right] . \qquad (\text{II}\cdot 48)$$

Using Van Vleck-Pauli's result in Eq. (II·39) , the propagator becomes

$$K(a,b) = \left[\frac{m\omega}{2\pi i\hbar \sin(\omega t)}\right]^{1/2}$$
$$\times \exp\left\{\frac{im\omega}{2\hbar \sin(\omega t)}\left[\cos(\omega t)\left\{x_b^2 + x_a^2\right\} - 2x_b x_a\right]\right\}. \tag{II.49}$$

In next section we will take attention on the problem which contain nonlocal potential. The simplest one is known as the nonlocal harmonic oscillator.

II.4 Nonlocal Harmonic Oscillator.

Path integral theory of the nonlocal harmonic oscillator was first done by Bezak (14). The Lagrangian is given by

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 - \frac{m\nu^2}{4t}\int_0^t \left[x(\tau) - x(\sigma)\right]^2 d\sigma , \tag{II.50}$$

and its corresponding action function $S[x(\tau)]$ is

$$S[x(\tau)] = \int_0^t \frac{m}{2}\dot{x}^2(\tau)d\tau - \frac{m\nu^2}{4t}\int_0^t\int_0^t \left[x(\tau) - x(\sigma)\right]^2 d\sigma d\tau . \tag{II.51}$$

And the path integral expression for the propagator is written to be

$$K(a,b) = \int_a^b \mathcal{D}[x(\tau)] \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2}\int_0^t \dot{x}^2(\tau)d\tau - \frac{m\nu^2}{4t}\int_0^t\int_0^t \left[x(\tau) - x(\sigma)\right]^2 d\sigma d\tau\right]\right\} . \tag{II.52}$$

As is shown by Bezak (15) and Sa-yakanit (10) that this propagator cannot be written in the form of Eqs. (II.34) or (II.40) . But this problem takes the Lagrangian in quadratic form, so that it can be written in the form of Eq.(II.38). However, following Papadopoulos's idea we can transform the nonlocal problem into the local one and Eqs. (II.39) and (II.40) are satisfy.

At first step towards our attention we express the action function appear in Eq. (II.51) by

$$S\{x(\tau)\} = \frac{m}{2}\int_0^t \left[\dot{x}^2 - \nu^2 x^2\right] d\tau + \frac{m\nu^2}{2t}\left\{\int_0^t x(\tau)\, d\tau\right\}^2. \qquad (\text{II}.53)$$

Inserting this action function into the expression of the propagator in Eq. (II.52), our path integral takes the form

$$K(a,b) = \int_a^b \mathcal{D}[x(\tau)] \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2}\int_0^t \left[\dot{x}^2 - \nu^2 x^2\right] d\tau + \frac{m\nu^2}{2t}\left\{\int_0^t x(\tau)\, d\tau\right\}^2\right]\right\}. \qquad (\text{II}.54)$$

We can generate the path integration through the averaging by a linear exponential functional involving an auxiliary random force F independent of the time τ. More explicitely we have

$$K(a,b) = \left[\frac{it}{2\pi\hbar m\nu^2}\right]^{1/2} \int_{-\infty}^{\infty} dF \exp\left\{\frac{-it}{2\hbar m\nu^2} F^2\right\} K_0(a,b) \qquad (\text{II}.55)$$

where

$$K_0(a,b) = \int_a^b \mathcal{D}[x(\tau)] \exp\left\{\frac{i}{\hbar}\int_0^t \left[\frac{m}{2}\left[\dot{x}^2 - \nu^2 x^2\right] + Fx\right] d\tau\right\}. \qquad (\text{II}.56)$$

The propagator of Eq. (II.56) is of the force harmonic oscillator and this can be found in the literature [(16) (17)]. So that the propagator of the nonlocal harmonic oscillator is obtained after inserting the propagator for the force harmonic oscillator into Eq. (II.56) and performing the F - integration. The propagator becomes [10].

$$K(a,b) = \left[\frac{m}{2\pi i\hbar t} \right] \left[\frac{\nu t}{2\sin(\nu t/2)} \right]^2$$

$$\times \exp\left\{ \frac{im\nu}{4\hbar} \cot(\nu t/2) \left[x_b - x_a \right]^2 \right\} . \qquad (\text{II}.57)$$