Chapter III

ERRORS IN SYSTEM IDENTIFICATION

3.1 Introduction

There are many possible error sources[5] in practical experimental process for the determination of the impulse response of a linear system. The block diagram of a practical experimental process is illustrated in Fig. 6.

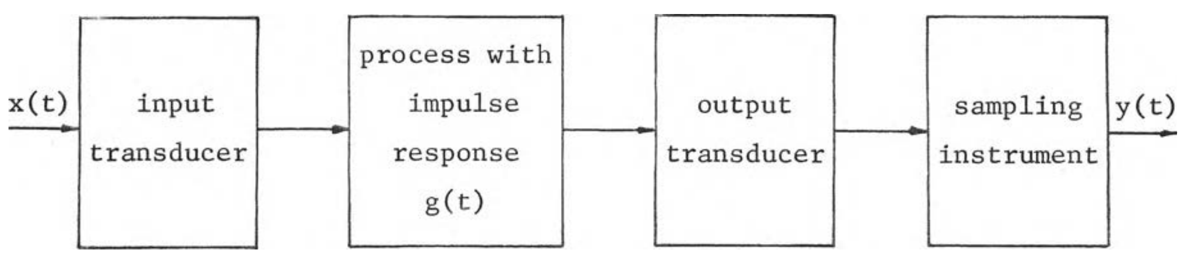

Fig. 6 A practical experimental process

For the discrete cross-correlation technique described in the previous chapter, the errors may be conveniently grouped into two parts, these depend on whether they are of systematic or random origin[14].

3.1.1 Systematic error

The sources of systematic error are summarised as follows:

(a) the autocorrelation function of the b.m.l.s. input signal;

(b) the number of the sampling points in the output measurement which depends on the time derivatives of the system impulse response;

(c) the d.c. bias in the b.m.l.s. input signal;

(d) the input and the output transducers.

3.1.2 Random errors

Random errors are due to uncontrollable fluctuations in the system and test-equipment environments. There are two cases of random errors to be considered. These errors are distinguished by their power spectrum relative to the system frequency response and are summarised as follows:

(a) the random noise whose power spectrum is uniform throughout the bandwidth of the system frequency response;

(b) the drift which is occurred in the output signal is the low-order polynomial of the time t with unknown coefficients.

3.2 Technique for Error Reduction

The error analyses[14,15] are introduced that the systematic errors can be made suitably small by appropriately choosing the b.m.l.s. length N and time-bit interval Δt. The output measurement should be sampled after applying the b.m.l.s. input signal at least one period which should be at least five times the dominant time constant of the system impulse response.

The autocorrelation function of the b.m.l.s. causes the errors due to the derivative terms of the impulse response and the d.c. offset in the autocorrelation function. This d.c. offset is inversely proportional to N, but the mean square errors due to wide-band noise[14] are proportional to N. Thus, the error due to the d.c. offset in the autocorrelation function cannot be made suitably small by choosing the large value of N. A correction to this error will be described in the next section. The methods of minimising the other errors are also discussed in Section 3.4 and Section 3.5.

3.3 Error due to D.C. Offset in Autocorrelation Function

The d.c. offset in the autocorrelation function of the b.m.l.s. is the term $-a^2/N$ in Eqns. (6) and (11). This causes the error expression $\frac{a^2}{N}\Delta t\sum_{j=0}^{N-1} g(j\Delta t)$ in Eqn.(35) and the error expression $\frac{a^2}{mN}\Delta t\sum_{j=0}^{N-1}\sum_{r=0}^{m-1} g(j\Delta t+\frac{r}{m}\Delta t)$ in Eqn. (40). The method for the determination of the impulse response when the d.c. offset in the autocorrelation function is reduced is derived.

Let A be the system steady-state gain and defined as

$$A = \int_0^T g(s)ds \qquad (44)$$

For the discrete method, the system steady-state gain can be written in the form

$$A = \Delta t\sum_{j=0}^{N-1} g(j\Delta t) \qquad (45)$$

To evaluate the error expression $\frac{a^2}{N}\Delta t\sum_{j=0}^{N-1} g(j\Delta t)$ in Eqn. (35), all derivative terms of the impulse response are first neglected. Thus, Eqn. (35) becomes

$$\begin{aligned} \phi_{xy}(i\Delta t) &= \frac{a^2(N+1)}{2N}\Delta t g(i\Delta t)-\frac{a^2}{N}A && \text{for } i = 0 \\ &= \frac{a^2(N+1)}{N}\Delta t g(i\Delta t)-\frac{a^2}{N}A && \text{otherwise} \end{aligned} \qquad (46)$$

Applying Trapezoidal rule to Eqns. (44) and (45), and the impulse response decays to zero within time period $N\Delta t$, we obtain

$$A = \Delta t\left[\frac{1}{2}g(0)+\sum_{j=1}^{N-1} g(j\Delta t)\right] \tag{47}$$

Taking the summation over one period on both sides of Eqn. (46), we have

$$\sum_{j=0}^{N-1} \emptyset_{xy}(j\Delta t) = \frac{a^2(N+1)}{N}\Delta t\left[\frac{1}{2}g(0)+\sum_{j=1}^{N-1} g(j\Delta t)\right]-\frac{a^2}{N} AN$$

$$= \frac{a^2}{N} A \tag{48}$$

This is the error due to the d.c. offset in the autocorrelation function of the b.m.l.s. From Eqn. (46), the discrete impulse response of the system can be determined as

$$g(i\Delta t) = \frac{2N}{a^2(N+1)\Delta t}\left[\emptyset_{xy}(i\Delta t)+\sum_{j=0}^{N-1} \emptyset_{xy}(j\Delta t)\right] \quad \text{for } i = 0$$

$$= \frac{N}{a^2(N+1)\Delta t}\left[\emptyset_{xy}(i\Delta t)+\sum_{j=0}^{N-1} \emptyset_{xy}(j\Delta t)\right] \quad \text{otherwise} \tag{49}$$

For the new method in correlation technique described in Section 2.5, the system steady-state gain is

$$A = \frac{1}{m}\Delta t \sum_{j=0}^{N-1} \sum_{r=0}^{m-1} g(j\Delta t+\frac{r}{m}\Delta t) \tag{50}$$

In the same manner, the error due to the d.c. offset in the autocorrelation of the b.m.l.s. can be evaluated as

$$\frac{a^2}{N} A = \frac{1}{m}\sum_{j=0}^{N-1} \sum_{r=0}^{m-1} \emptyset_{xy}(j\Delta t+\frac{r}{m}\Delta t) \tag{51}$$

To avoid the error due to the d.c. offset in the autocorrelation function of the b.m.l.s., the method of shifting the autocorrelation function will be used. (See also Appendix B.) Let the input, denoted by x(t) in Fig. 7a, be a b.m.l.s. whose two states are +a and -a and

$\overline{x}(t)$ denote the b.m.l.s. whose two state are +a and 0 as shown in Fig. 7b.

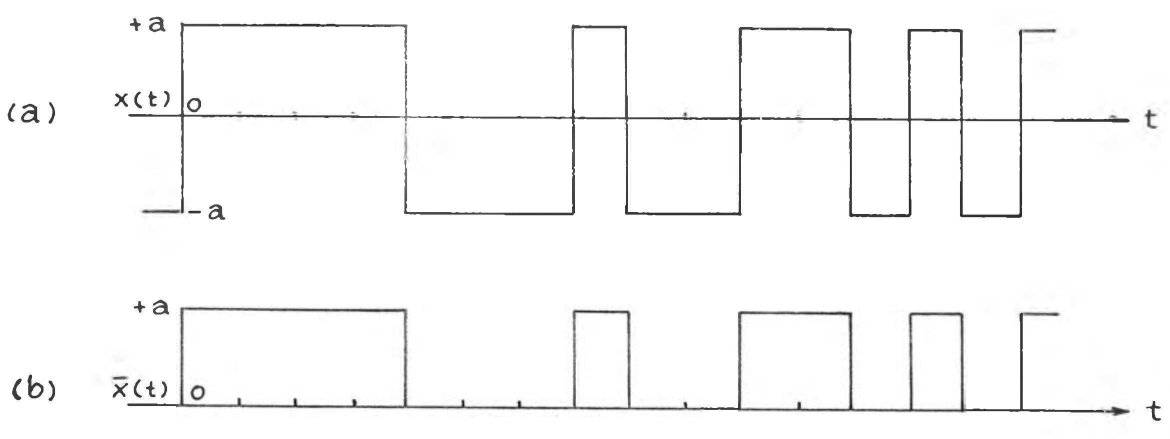

Fig. 7 A typical waveform of x(t) and the corresponding waveform of $\overline{x}(t)$

From Eqn. (12), the system output due to the input signal x(t) is

$$y(i\Delta t+\frac{1}{2}\Delta t) = \Delta t\sum_{j=0}^{\infty} g(j\Delta t)x(i\Delta t+\frac{1}{2}\Delta t-j\Delta t) \quad (52)$$

The cross-correlation between the output y(t) and the b.m.l.s. $\overline{x}(t)$ is

$$\phi_{\overline{x}y}(i\Delta t) = \Delta t\sum_{j=0}^{N-1} g(j\Delta t)\phi_{x\overline{x}}(i\Delta t-j\Delta t) \quad (53)$$

where $\phi_{x\overline{x}}(i\Delta t)$ is the cross-correlation between x(t) and $\overline{x}(t)$. (See also Appendix B.)

Thus, we have

$$\phi_{\overline{x}y}(i\Delta t) = \frac{a^2(N+1)}{2N}\Delta t\sum_{j=0}^{N-1} g(j\Delta t)\delta_r(i\Delta t-j\Delta t) \quad (54)$$

where $\delta_r(i\Delta t)$ is the unit rectangular pulse of pulse width Δt.

The above equation can be rewritten, in the similar way as the derivation of Eqn. (24), as

$$\phi_{\bar{x}y}(i\Delta t) = \frac{a^2(N+1)}{4N}\Delta t\Big[g(0)+\sum_{j=1}^{\infty}\frac{(\Delta t)^j}{2^j(j+1)!}g^{(j)}(0)\Big] \qquad \text{for } i = 0$$

$$= \frac{a^2(N+1)}{2N}\Delta t\Big[g(i\Delta t)+\sum_{j=1}^{\infty}\frac{(\Delta t)^{2j}}{2^{2j}(2j+1)!}g^{(2j)}(i\Delta t)\Big] \quad \text{otherwise} \qquad (55)$$

This method may be used for the determination of the impulse response by the new method of correlation technique described in Section 2.5.

3.4 Error due to Derivative Terms

The error due to the derivative terms of the system impulse response occurs in the determination of the impulse response because the discrete autocorrelation of the input signal is a rectangular pulse. Since the sampled points of the cross-correlation function $\phi_{xy}(t)$ obtained by the discrete cross-correlation method or the new method of correlation technique are uniformly spaced, the difficulty of reduction the error due to the derivative terms can be overcome by the use of digital computer. The method of removal the derivative terms of the impulse response is presented below.

From Eqns. (35) and (39), we may write

$$g(\tfrac{i}{m}\Delta t) = K(\tfrac{i}{m}\Delta t)-\sum_{j=1}^{\infty}\frac{(\Delta t/m)^j}{2^j(j+1)!}g^{(j)}(0) \qquad \text{for } i = 0$$

$$= K(\tfrac{i}{m}\Delta t)-\sum_{j=1}^{\infty}\frac{(\Delta t/m)^{2j}}{2^{2j}(2j+1)!}g^{(2j)}(\tfrac{i}{m}\Delta t) \quad \text{otherwise} \qquad (56)$$

When the value of $m = 1$, and $K(\frac{i}{m}\Delta t)$ denotes the first expression in the right hand side of Eqn. (35), then Eqn. (56) represents Eqn. (35). When the value of $m > 1$, and $K(\frac{i}{m}\Delta t)$ denotes $g_e(j\Delta t+\frac{r}{m}\Delta t)$ evaluated by

Eqns. (40), (41), (42) and (43), then Eqn. (56) represents Eqn. (39). As a first approximation, the coefficients of the derivative terms are negligible. The first approximate impulse response denoted by $g_1(\frac{i}{m}\Delta t)$ is equivalent to $K(\frac{i}{m}\Delta t)$. The series expression of the derivative terms can be calculated from the value of the first approximate impulse response. (See Appendix C.)

Thus, for the case $2 \leqslant i \leqslant mN-3$ as an example, the second approximate impulse response is

$$g_2(\frac{i}{m}\Delta t) = K(\frac{i}{m}\Delta t)+0.1010417g_1(\frac{i}{m}\Delta t)-0.0534722\{g_1(\frac{i-1}{m}\Delta t)+g_1(\frac{i+1}{m}\Delta t)\}$$
$$+0.0029514\{g_1(\frac{i-2}{m}\Delta t)+g_1(\frac{i+2}{m}\Delta t)\} \quad (57)$$

In general, the n+1 th. approximate impulse response can be expressed in term of the n th. approximate impulse response as

$$g_{n+1}(0) = K(0)+0.435162g_n(0)-0.8004541g_n(\frac{1}{m}\Delta t)+0.6470756g_n(\frac{2}{m}\Delta t)$$
$$-0.406789g_n(\frac{3}{m}\Delta t)+0.148358g_n(\frac{4}{m}\Delta t)-0.0233519g_n(\frac{5}{m}\Delta t) \quad (58)$$

$$g_{n+1}(\frac{1}{m}\Delta t) = K(\frac{1}{m}\Delta t)-0.0387153g_n(0)+0.0715278g_n(\frac{1}{m}\Delta t)$$
$$-0.0239583g_n(\frac{2}{m}\Delta t)-0.0118056g_n(\frac{3}{m}\Delta t)+0.0025148g_n(\frac{4}{m}\Delta t) \quad (59)$$

$$g_{n+1}(\frac{i}{m}\Delta t) = K(\frac{i}{m}\Delta t)+0.1010417g_n(\frac{i}{m}\Delta t)-0.0534722\{g_n(\frac{i-1}{m}\Delta t)+g_n(\frac{i+1}{m}\Delta t)\}$$
$$+0.0029514\{g_n(\frac{i-2}{m}\Delta t)+g_n(\frac{i+2}{m}\Delta t)\} \quad (60)$$

$$g_{n+1}(\frac{mN-2}{m}\Delta t) = K(\frac{mN-2}{m}\Delta t)-0.0446181g_n(\frac{mN-1}{m}\Delta t)+0.0951389g_n(\frac{mN-2}{m}\Delta t)$$
$$-0.059375g_n(\frac{mN-3}{m}\Delta t)+0.0118056g_n(\frac{mN-4}{m}\Delta t)$$
$$-0.0029514g_n(\frac{mN-5}{m}\Delta t) \quad (61)$$

$$g_{n+1}(\frac{mN-1}{m}\Delta t) = K(\frac{mN-1}{m}\Delta t)-0.1220486g_n(\frac{mN-1}{m}\Delta t)+0.3631944g_n(\frac{mN-2}{m}\Delta t)$$

$$-0.3989583g_n(\frac{mN-3}{m}\Delta t)+0.1965278g_n(\frac{mN-4}{m}\Delta t)$$

$$-0.0387153g_n(\frac{mN-5}{m}\Delta t) \quad (62)$$

This process may be repeated until the value of $g_{n+1}(\frac{i}{m}\Delta t)$ is very close to $g_n(\frac{i}{m}\Delta t)$ for every value of $i = 0,1,2,\ldots,mN-1$. Then $g_{n+1}(\frac{i}{m}\Delta t)$ is the good approximate value of the system impulse response.

This process can also be applied to remove the derivative terms of the impulse response when the shifted autocorrelation function is used

3.5 Error due to Polynomial Drift in Output

The error due to the polynomial drift in the output signal is one of the random errors described in Section 3.1.2. When the polynomial drift in the output signal is considered, the output signal can be expressed as

$$y(t) = \int_0^T g(s)x(t-s)ds+\sum_{j=0}^{\infty} d_j t^j \quad (63)$$

where d_j is the coefficient of the time variable of power j.

Let A be the system steady-state gain and defined as

$$A = \int_0^T g(s)ds = \frac{1}{m}\Delta t\sum_{j=0}^{N-1}\sum_{r=0}^{m-1} g(j\Delta t+\frac{r}{m}\Delta t) \quad (64)$$

Applying Trapezoidal rule as in Eqn. (47) to Eqn. (64), we obtain

$$A = \frac{1}{m}\Delta t\sum_{j=0}^{N-1}\sum_{r=0}^{m-1} g(j\Delta t+\frac{r}{m}\Delta t)-\frac{1}{2}g(0) \quad (65)$$

A technique for determining the parameters d_j of the polynomial drift terms hinges on the fact that the integral of a b.m.l.s. taken over one complete period is known, $a\Delta t$. The integral of the measured output signal over one complete period can be expressed as

$$\int_{T+i\Delta t}^{2T+i\Delta t} y(t)dt = \int_0^T g(s)ds \int_0^T x(t)dt + \sum_{j=0}^{\infty} \int_{T+i\Delta t}^{2T+i\Delta t} d_j t^j dt$$

$$= Aa\Delta t + d_0 T + \sum_{j=1}^{\infty} d_j \left[\frac{(2T+i\Delta t)^{j+1} - (T+i\Delta t)^{j+1}}{j+1}\right] \qquad (66)$$

where $i = 0,1,2,\ldots\ldots$

For a b.m.l.s., $T = N\Delta t$, thus Eqn. (66) becomes

$$\frac{1}{m}\Delta t \sum_{j=N+i}^{2N+i-1} \sum_{r=0}^{m-1} y(j\Delta t + \frac{r+1/2}{m}\Delta t) = Aa\Delta t + d_0 N\Delta t$$

$$+ \sum_{j=1}^{\infty} d_j \left[\frac{(2N+i)^{j+1} - (N+i)^{j+1}}{j+1}\right](\Delta t)^{j+1} \qquad (67)$$

It can be seen that, for $i = 0,1,2,\ldots,k$, Eqn. (67) provides $k+1$ simultaneous linear equations. If q is the highest order of the polynomial drift to be eliminated, the parameters d_j, where $j = 0,1,2,\ldots,q$, can be solved from $q+1$ simultaneous equations obtained by substituting $i = 0,1,2,\ldots,q$ into Eqn. (67). The value of q may be chosen to meet the accuracy in the estimation of the impulse response.

Consider the case when $q = 2$ for an example of a low-order polynomial $d_0+d_1t+d_2t^2$. Thus, for $i = 0,1,2$, we have

$$\frac{1}{m}\Delta t \sum_{j=N}^{2N-1} \sum_{r=0}^{m-1} y(j\Delta t + \frac{r+1/2}{m}\Delta t) = N\Delta t(\frac{Aa}{N} + d_0) + \frac{(2N)^2 - N^2}{2}(\Delta t)^2 d_1$$

$$+ \frac{(2N)^3 - N^3}{3}(\Delta t)^3 d_2 \qquad (68)$$

$$\frac{1}{m}\Delta t\sum_{j=N+1}^{2N}\sum_{r=0}^{m-1} y(j\Delta t+\frac{r+1/2}{m}\Delta t) = N\Delta t(\frac{Aa}{N}+d_0)+\frac{(2N+1)^2-(N+1)^2}{2}(\Delta t)^2 d_1 + \frac{(2N+1)^3-(N+1)^3}{3}(\Delta t)^3 d_2 \qquad (69)$$

$$\frac{1}{m}\Delta t\sum_{j=N+2}^{2N+1}\sum_{r=0}^{m-1} y(j\Delta t+\frac{r+1/2}{m}\Delta t) = N\Delta t(\frac{Aa}{N}+d_0)+\frac{(2N+2)^2-(N+2)^2}{2}(\Delta t)^2 d_1 + \frac{(2N+2)^3-(N+2)^3}{3}(\Delta t)^3 d_2 \qquad (70)$$

If the value of N, Δt and the summations in the left hand side of Eqns. (68), (69) and (70) are known, the values of $\frac{Aa}{N}+d_0$, d_1, and d_2 can be evaluated by solving these three simultaneous equations. The new output signal can be obtained by subtracting the output polynomial drift from Eqn. (63) as

$$y(t) = \int_0^T g(s)x(t-s)ds-\frac{Aa}{N} \qquad (71)$$

The cross-correlation between the input signal and the above output signal is

$$\begin{aligned}\phi_{xy}(i\Delta t+\frac{\ell}{m}\Delta t) &= \frac{1}{m}\Delta t\sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t)\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t-\frac{j}{m}\Delta t)-\frac{Aa}{N}\cdot\frac{1}{N}\sum_{j=0}^{N-1}x(j\Delta t)\\ &= \frac{1}{m}\Delta t\sum_{j=0}^{mN-1} g(\frac{j}{m}\Delta t)\phi_{xx}(i\Delta t+\frac{\ell}{m}\Delta t-\frac{j}{m}\Delta t)-\frac{Aa^2}{N^2} \qquad (72)\end{aligned}$$

From Eqns. (36), (38), and (64), then Eqn. (72) can be rewritten as

$$\begin{aligned}\phi_{xy}(i\Delta t+\frac{\ell}{m}\Delta t) &= \frac{a^2(N+1)}{mN}\Delta t\Big[\sum_{r=0}^{\ell} g_e(\frac{r}{m}\Delta t)-\frac{1}{2}g_e(0)\Big]-\frac{Aa^2}{N}-\frac{Aa^2}{N^2} \quad \text{for } i=0\\ &= \frac{a^2(N+1)}{mN}\Delta t\sum_{r=0}^{m-1} g_e(i\Delta t+\frac{r}{m}\Delta t)-\frac{Aa^2}{N}-\frac{Aa^2}{N^2} \quad \text{for } i\neq 0 \qquad (73)\end{aligned}$$

where $g_e(i\Delta t+\frac{r}{m}\Delta t)$ is previously defined and expressed by Eqn. (39).

Thus, we have

$$g_e(0) = \frac{2mN}{a^2(N+1)\Delta t}\left[\phi_{xy}(0)+\frac{Aa^2}{N}+\frac{Aa^2}{N^2}\right] \quad (74)$$

$$g_e(\frac{\ell}{m}\Delta t) = \frac{mN}{a^2(N+1)\Delta t}\left[\phi_{xy}(\frac{\ell}{m}\Delta t)-\phi_{xy}(\frac{\ell-1}{m}\Delta t)\right] \quad \text{for } \ell = 1,2,\ldots,m-1 \quad (75)$$

$$g_e(\Delta t) = \frac{mN}{a^2(N+1)\Delta t}\left[\phi_{xy}(\Delta t)-\phi_{xy}(\frac{m-1}{m}\Delta t)\right]+\frac{1}{2}g_e(0) \quad (76)$$

$$g_e(i\Delta t+\frac{\ell}{m}\Delta t) = \frac{mN}{a^2(N+1)\Delta t}\left[\phi_{xy}(i\Delta t+\frac{\ell}{m}\Delta t)-\phi_{xy}(i\Delta t+\frac{\ell-1}{m}\Delta t)\right]$$
$$+g_e(i\Delta t-\Delta t+\frac{\ell}{m}\Delta t) \quad \text{otherwise} \quad (77)$$

From the set of the above equations, it is seen that the error due to the system steady-state gain, A , is included in the values of $g_e(i\Delta t+\frac{\ell}{m}\Delta t)$ when $\ell = 0$ and $i = 0,1,2,\ldots,N-1$. Since the system impulse response decays to zero within the time period $N\Delta t$ and the coefficients of the derivative terms are first neglected, then the first approximate impulse response is

$$g_1(i\Delta t+\frac{\ell}{m}\Delta t) = g_e(i\Delta t+\frac{\ell}{m}\Delta t)-g_e\{(N-1)\Delta t+\frac{\ell}{m}\Delta t\} \quad \text{for } \ell = 0,\ i = 0,1,\ldots,N-1$$
$$= g_e(i\Delta t+\frac{\ell}{m}\Delta t) \quad \text{otherwise} \quad (78)$$

Now applying the iteration method for determination of the impulse response, the step by step is described as follows:

1 st. Step Calculate the value of A from the previous approximate impulse response by using Eqn. (65).

2 nd. Step Calculate $g_e(\frac{i}{m}\Delta t)$ which is equivalent to $K(\frac{i}{m}\Delta t)$ in Eqn. (56) by using Eqns. (74), (75), (76), and (77).

3 rd. Step Remove the error due to the derivative terms of the impulse response by using Eqns. (58), (59), (60), (61), and (62).

4 th. Step Compare the new approximate values of the impulse response with the former approximate values point by point. If one of the differences between the corresponding points is not in the allowed range, the 1 st. to the 3 rd. step must be repeated again.

It is suitable to use digital computer to estimate the discrete impulse response from the discrete cross-correlation function by applying this iteration process to reduce the errors described in the previous sections.

This process is applied to the ordinary discrete cross-correlation method when $m = 1$, and is applied to the new method of correlation technique when $m > 1$.

When the systematic error due to the d.c. bias in the b.m.l.s. input signal is considered, the steady-state output is

$$y(t) = \int_0^T g(s)x(t-s)ds + c\int_0^T g(s)ds + \sum_{j=0}^{\infty} d_j t^j$$

$$= \int_0^T g(s)x(t-s)ds + cA + \sum_{j=0}^{\infty} d_j t^j \qquad (79)$$

where c is the d.c. bias in the b.m.l.s. input signal and A is the system steady-state gain.

It can be seen that the value of cA is eliminated at the same time as d_0.