CHAPTER III

SEMILATTICE CONGRUENCES AND FACTORIZABLE INVERSE SEMIGROUPS

Let S be a factorizable inverse semigroup. It is shown in this chapter that every congruence on E(S) can be extended to a semilattice congruence on S if and only if S is a semilattice of groups. Moreover, it is also shown that if any such extension of a given congruence on E(S) exists, then it is unique.

The following theorem which has been proved by Chen and Hsieh in [3] shows various properties of a factorizable inverse semigroup.

3.1 <u>Theorem</u> [3]. Let S be an inverse semigroup. If S is factorizable as GE, then the following hold :

(1) S = EG.

(2) S has an identity 1 which is the identity of G.

(3) G is the group of units of S.

(4) For any g, h $\in$ G and e, f $\in$ E(S), ge = hf implies e = f.

(5) E = E(S), the set of all idempotents of S.

A homomorphic image of a semilattice is clearly a semilattice. Therefore, every congruence on a semilattice S is a semilattice congruence on S.

Let an inverse semigroup S be factorizable as GE(S). Then every congruence on E(S) is a semilattice congruence on E(S). A following interesting problem is raised to be solved : Can every congruence cn E(S) be extended to a semilattice congruence on S ?

The answer is "No" as shown in the following example :

<u>Example</u>. Let $X = \{a, b\}$, and I_X be the symmetric inverse semigroup on the set X. Let 0 and 1 be the zero and the identity of I_X; respectively, and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ be one-to-one partial transformations on X defined by $\Delta\alpha_1 = \nabla\alpha_1 = \{a\}$, $\Delta\alpha_2 = \nabla\alpha_2 = \{b\}$, $\Delta\alpha_3 = \{a\}$, $\nabla\alpha_3 = \{b\}$, $\Delta\alpha_4 = \{b\}$, $\nabla\alpha_4 = \{a\}$ and $\Delta\alpha_5 = \nabla\alpha_5 = \{a, b\}$ such that $a\alpha_5 = b$, $b\alpha_5 = a$. Then $I_X = \{0, 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and the multiplicative table is as follows :

$\cdot$	0	1	α_1	α_2	α_3	α_4	α_5
0	0	0	0	0	0	0	0
1	0	1	α_1	α_2	α_3	α_4	α_5
α_1	0	α_1	α_1	0	α_3	0	α_3
α_2	0	α_2	0	α_2	0	α_4	α_4
α_3	0	α_3	0	α_3	0	α_1	α_1
α_4	0	α_4	α_4	0	α_2	0	α_2
α_5	0	α_5	α_4	α_3	α_2	α_1	1

The permutation group on X, G_X is $\{1, \alpha_5\}$ and $E(I_X) = \{0, 1, \alpha_1, \alpha_2\}$. Since X is finite, I_X is factorizable [3, Corollary of Theorem 3.1] and so by Theorem 3.1, $I_X = G_X E(I_X)$. Let $i_{E(I_X)}$ be the identity congruence on $E(I_X)$. Suppose $i_{E(I_X)}$ can be extended to a semilattice congruence ρ on I_X. Because $\alpha_4\alpha_3 = \alpha_2$, $\alpha_3\alpha_4 = \alpha_1$ and ρ is a semilattice congruence on I_X, it follows that $\alpha_2\rho = (\alpha_4\alpha_3)\rho = (\alpha_3\alpha_4)\rho = \alpha_1\rho$. But $\alpha_1, \alpha_2 \in E(I_X)$.

Then $(\alpha_1, \alpha_2) \in \rho \cap (E(I_X) \times E(I_X))$. Since $\rho \cap (E(I_X) \times E(I_X)) = i_{E(I_X)}$, $\alpha_1 = \alpha_2$ which is a contradiction. #

The first theorem of this chapter shows necessary and sufficient conditions of a factorizable inverse semigroup S such that every congruence on E(S) can be extended to a semilattice congruence on S. The following lemmas are required :

3.2 Lemma. Let ρ be a semilattice congruence on a factorizable semigroup S which factors as GE(S). Then for any $g \in G$, $e \in E(S)$, $(ge)\rho = e\rho = (eg)\rho$. Hence, $S/\rho = \{e\rho / e \in E(S)\}$.

Proof : Let $g \in G$ and $e \in E(S)$. Since ρ is a semilattice congruence on S and G is a subgroup of S, G is contained in a single ρ-class of S. Then $G \subseteq f\rho$ where f is the identity of G. By Lemma 1.1, f is a left identity of S, so fe = e. Therefore

$(ge)\rho = (g\rho)(e\rho) = (f\rho)(e\rho) = (fe)\rho = e\rho$ and

$(eg)\rho = e\rho g\rho = g\rho e\rho = (ge)\rho = e\rho$. #

Let an inverse semigroup S be factorizable as GE(S). By Theorem 3.1(4), for each $x \in S$, there exists a unique $e \in E(S)$ such that x = ge for some $g \in G$, such e will be denoted by e_x. Then the map $x \to e_x$ $(x \in S)$ is a map from S onto E(S) and $e_f = f$ for all $f \in E(S)$.

3.3 Lemma. Let S be a factorizable inverse semigroup as GE(S). If Ge = eG for all $e \in E(S)$, then for all $x, y \in S$, $e_x e_y = e_{xy}$; in particular, $e_{x^2} = e_x$ for all $x \in S$.

Proof : Let $x, y \in S$. Then $x = ge_x$ and $y = he_y$ for some $g, h \in G$. Then $xy = ge_x he_y$. By assumption, $Ge_x = e_x G$, so $e_x h = h' e_x$ for some $h' \in G$. Hence $xy = gh' e_x e_y = gh'(e_x e_y)$. But $gh' \in G$ and $e_x e_y \in E(S)$. By Theorem 3.1(4), $e_x e_y = e_{xy}$.

Next, let $x \in S$. From the above proof, $e_x e_x = e_{x^2}$. But $e_x \in E(S)$, so $e_x e_x = e_x$. Hence $e_{x^2} = e_x$. #

3.4 Theorem. Let S be a factorizable inverse semigroup as GE(S). Then any congruence on E(S) can be extended to a semilattice congruence on S if and only if $eG = Ge$ for all $e \in E(S)$.

Proof : Assume that any semilattice congruence on E(S) can be extended to a semilattice congruence on S. To show $eG = Ge$ for all $e \in E(S)$, let $e \in E(S)$. Let $x \in Ge$. Then $x = ge$ for some $g \in G$. By Theorem 3.1(1), $x = fh$ for some $h \in G$, $f \in E(S)$. Thus $ge = fh$. Let $i_{E(S)}$ be the identity congruence on E(S). By assumption, there exists a semilattice congruence ρ on S such that $\rho \cap (E(S) \times E(S)) = i_{E(S)}$. By Lemma 3.2, $(ge)\rho = e\rho$ and $(fh)\rho = f\rho$. Therefore $e\rho = (ge)\rho = (fh)\rho = f\rho$. It follows that $(e, f) \in \rho \cap (E(S) \times E(S))$. But $\rho \cap (E(S) \times E(S)) = i_{E(S)}$, so $e = f$. Thus $x = fh = eh \in eG$. This proves $Ge \subseteq eG$. Similarly, we can show that $eG \subseteq Ge$. Hence $eG = Ge$.

Conversely, assume that for any $e \in E(S)$, $eG = Ge$. Let ρ be a congruence on E(S). Let $\bar{\rho}$ be the relation on S defined as follows :

$$x\bar{\rho}y \text{ if and only if } e_x \rho e_y.$$

Because ρ is an equivalence relation on E(S), it is clearly seen that $\bar{\rho}$ is an equivalence relation on S. To show $\bar{\rho}$ is compatible, let x, y, z $\in$ S such that $x\bar{\rho}y$. Then $e_x \rho e_y$. Because ρ is a congruence on E(S) and $e_z \in$ E(S), $e_z e_x \rho e_z e_y$ and $e_x e_z \rho e_y e_z$. Since Ge = eG for all e $\in$ E(S), by Lemma 3.3, we have $e_z e_x = e_{zx}$, $e_z e_y = e_{zy}$, $e_x e_z = e_{xz}$ and $e_y e_z = e_{yz}$. Hence $e_{zx} \rho e_{zy}$ and $e_{xz} \rho e_{yz}$. Therefore $zx\bar{\rho}zy$ and $xz\bar{\rho}yz$.

Hence $\bar{\rho}$ is a congruence on S.

Because $e_f = f$ for all f $\in$ E(S), it follows that for any f, f' $\in$ E(S), (f, f') $\in \bar{\rho}$ if and only if (f, f') = $(e_f, e_{f'}) \in \rho$. Hence $\bar{\rho} \cap (E(S) \times E(S)) = \rho$.

To show $\bar{\rho}$ is a semilattice congruence on S, let x, y $\in$ S. Since S is an inverse semigroup and $e_x, e_y \in$ E(S), $e_x e_y = e_y e_x$. Thus, by Lemma 3.3, $e_{xy} = e_x e_y = e_y e_x = e_{yx}$. Also, by Lemma 3.3, $e_{x^2} = e_x$. But ρ is reflexive on E(S), so $e_{xy} \rho e_{yx}$ and $e_{x^2} \rho e_x$. Thus $xy\bar{\rho}yx$ and $x^2\bar{\rho}x$.

Hence the theorem is completely proved. #

3.5 Corollary. Let S be a factorizable inverse semigroup. If the identity congruence on E(S) can be extended to a semilattice congruence on S, then every congruence on E(S) can be extended to a semilattice congruence on S.

Proof : Let S be factorizable as GE(S). From the first part of the proof of Theorem 3.4, it is shown that if the identity congruence on E(S) can be extended to a semilattice congruence on S, Then $eG = Ge$ for all $e \in E(S)$. From Theorem 3.4, if $eG = Ge$ for all $e \in E(S)$, then every congruence on $E(S)$ can be extended to a semilattice congruence on S.

Therefore, the corollary is proved. #

The Green's relations $\mathcal{L}$ and $\mathcal{R}$ on any semigroup S are right compatible and left compatible; respectively. Then, if $\mathcal{L} = \mathcal{R}$ on a semigroup S, then $\mathcal{H} = \mathcal{L} = \mathcal{R}$ is a congruence on S.

Let S be an inverse semigroup. Then every $\mathcal{L}$-class of S and every $\mathcal{R}$-class of S contains exactly one idempotent. Then $S = \bigcup_{e \in E(S)} L_e = \bigcup_{e \in E(S)} R_e$ which are disjoint union. Suppose that $L_e = R_e$ for all $e \in E(S)$. Then $\mathcal{H} = \mathcal{L} = \mathcal{R}$ is a congruence on S, and moreover, $S = \bigcup_{e \in E(S)} H_e$ which is a disjoint union of groups.

It has been proved by Chen and Hsieh in [3] that if an inverse semigroup is factorizable as $GE(S)$, then $L_e = Ge$ and $R_e = eG$ for all $e \in E(S)$.

Suppose that S is a factorizable inverse semigroup which factors as $GE(S)$, and assume that $\mathcal{H}$ is a congruence on S. To show $Ge = eG$ for all $e \in E(S)$, let $e \in E(S)$. Since $G = H_1$ where 1 is the identity of S, $g \mathcal{H} 1$ for all $g \in G$. Because $\mathcal{H}$ is a congruence on S, $ge \mathcal{H} e$ and $eg \mathcal{H} e$ for all $g \in G$. Hence $Ge \subseteq H_e$ and $eG \subseteq H_e$. But $Ge = L_e \supseteq H_e$ and $eG = R_e \supseteq H_e$. Therefore $Ge = H_e = eG$. This proves that if $\mathcal{H}$ is a congruence on S, then $Ge = eG$ for all $e \in E(S)$ and $H_e = Ge$ for all $e \in E(S)$.

Therefore, we have

3.6 Lemma. Let S be an inverse semigroup which factors as GE(S). Then $\mathcal{H}$ is a congruence on S if and only if $H_e = Ge = eG$ for all $e \in E(S)$.

From Lemma 3.6 and Theorem 3.4, the following proposition is directly obtained :

3.7 Proposition. Let S be a factorizable inverse semigroup. Then every congruence on E(S) can be extended to a semilattice congruence on S if and only if the Green's relation $\mathcal{H}$ on S is a congruence on S.

Recall that a semigroup S is said to be a semilattice of groups if there exists a semilattice, Y such that $S = \bigcup_{\alpha \in Y} G_\alpha$ is a disjoint union, where G_α's are subgroups of S, and $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ for all α, $\beta \in Y$. Therefore, a semigroup S is a semilattice of groups if and only if S has a semilattice congruence ρ such that each ρ-class forms a subgroup of S.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. For each $\alpha \in Y$, let e_α denote the identity of the group G_α. Because for each $\alpha \in Y$, G_α is a maximal subgroup of S, $G_\alpha = H_{e_\alpha}$ for all $\alpha \in Y$. Let a, b, c $\in$ S such that $a \mathcal{H} b$. Then there exist $\alpha, \beta \in Y$ such that a, b $\in G_\alpha$ and $c \in G_\beta$. Thus ac, bc, ca, cb $\in G_{\alpha\beta}$ and hence $ac \mathcal{H} bc$ and $ca \mathcal{H} cb$. Therefore $\mathcal{H}$ is a congruence on S and for each $\alpha \in Y$, G_α is an $\mathcal{H}$-class of S.

We show in the next theorem that a factorizable inverse semigroup S has the following property : Every congruence on E(S) can be extended to a semilattice congruence on S if and only if S is a semilattice of groups.

3.8 Theorem. Let S be a factorizable inverse semigroup. Then every congruence on E(S) can be extended to a semilattice congruence on S if and only if S is a semilattice of groups.

Proof : Because S is a factorizable inverse semigroup S = GE(S) where G is the group of units of S.

Assume that every congruence on E(S) can be extended to a semilattice congruence on S. By Proposition 3.7, $\mathcal{H}$ is a congruence on S. By Lemma 3.6, $H_e = Ge = eG$ for all $e \in S$. Because $S = \bigcup_{e \in E(S)} Ge$ and $Ge = H_e$ for all $e \in E(S)$, every $\mathcal{H}$-class of S is a subgroup of S. To show $\mathcal{H}$ is a semilattice congruence on S, let a, $b \in S$. Then $a \in H_e$, $b \in H_f$ for some e, $f \in E(S)$. Since H_e is a subgroup of S and $a \in H_e$, $a^2 \in H_e$. Thus $a \mathcal{H} a^2$. Because $a \mathcal{H} e$, $b \mathcal{H} f$ and $\mathcal{H}$ is a congruence on S, it follows that $ab \mathcal{H} ef$ and $ba \mathcal{H} fe$. But ef = fe. Then $ab \mathcal{H} ba$. This proves $\mathcal{H}$ is a semilattice congruence on S and each $\mathcal{H}$-class is a subgroup of S. Hence, S is a semilattice of groups.

Conversely, if S is a semilattice of groups, then $\mathcal{H}$ is a congruence on S, and hence, by Proposition 3.7, every congruence on E(S) can be extended to a semilattice congruence on S. #

Recall that in any semigroup S, the minimum semilattice congruence on S always exists and it is the intersection of all semilattice congruences on S.

3.9 Proposition. Let S be a factorizable inverse semigroup. If the identity congruence on E(S) can be extended to a semilattice congruence on S, then the extension is the minimum semilattice congruence on S.

Proof : Let S be factorizable as GE(S). Let ρ be a semilattice congruence on S which is an extension of $i_{E(S)}$, where $i_{E(S)}$ is the identity congruence on E(S). Let η be the minimum semilattice congruence on S. Then $\eta \subseteq \rho$.

Let $(x, y) \in \rho$. Because $S = GE(S)$, $x = ge$ and $y = hf$ for some $g, h \in G$, $e, f \in E(S)$. By Lemma 3.2, $x\rho = e\rho$ and $y\rho = f\rho$ and so $(e, f) \in \rho$. Because $e, f \in E(S)$ and $\rho \cap (E(S) \times E(S)) = i_{E(S)}$, $(e, f) \in i_{E(S)}$ and hence $e = f$. Thus $(e, f) \in \eta$. But $x\eta = e\eta$ and $y\eta = f\eta$ by Lemma 3.2. Hence $(x, y) \in \eta$.

Therefore $\eta = \rho$. #

We end this chapter by showing that for any factorizable inverse semigroup S, for any given congruence ρ on E(S), if a semilattice congruence on S extending ρ exists, then the extension is unique.

3.10 Theorem. Let S be a factorizable inverse semigroup as GE(S). and ρ be a congruence on E(S). If ρ can be extended to a semilattice congruence on S, then the extension is unique.

Proof : Let $\bar{\rho}$ and $\hat{\rho}$ be semilattice congruences on S which are extensions of ρ. To show $\bar{\rho} = \hat{\rho}$, let $(x, y) \in \bar{\rho}$. Because

$S = GE(S)$, $x = ge$ and $y = hf$ for some $g, h \in G$, $e, f \in E(S)$. By Lemma 3.2, $x\bar{\rho} = e\bar{\rho}$, $y\bar{\rho} = f\bar{\rho}$, $x\hat{\rho} = e\hat{\rho}$ and $y\hat{\rho} = f\hat{\rho}$. Then $(e, f) \in \bar{\rho}$ since $(x, y) \in \bar{\rho}$. But $\bar{\rho} \cap (E(S) \times E(S)) = \rho$, and so $(e, f) \in \rho$. Since $\rho = \hat{\rho} \cap (E(S) \times E(S))$, $(e, f) \in \hat{\rho}$. But $x\hat{\rho} = e\hat{\rho}$ and $y\hat{\rho} = f\hat{\rho}$. Hence $(x, y) \in \hat{\rho}$. Therefore $\bar{\rho} \subseteq \hat{\rho}$. Similarly, we can show that $\hat{\rho} \subseteq \bar{\rho}$. Thus, $\bar{\rho} = \hat{\rho}$.

Hence, the theorem is completely proved. #