CHAPTER IV

TRANSFORMATION SEMIGROUPS

The purpose of this section is to study the factorizabilities of full transformation semigroups and partial transformation semigroups. It is shown that the full transformation semigroup on a set X is factorizable if and only if X is finite and the partial transformation semigroup on a set X is factorizable if and only if X is finite.

Throughout this chapter for any set X, let G_X, I_X, T_X and $\mathcal{T}_X$ denote the permutation group, the symmetric inverse semigroup, the partial transformation semigroup and the full transformation semigroup on the set X; respectively. Then, for any set X, $I_X \subseteq T_X$, $\mathcal{T}_X \subseteq T_X$ and G_X is the group of units of T_X, also of I_X and of $\mathcal{T}_X$. For any set X, let 0 and 1 denote the zero and the identity of T_X; respectively.

Partial transformation semigroups and full transformation semigroups are regular semigroups. Moreover, for any set X, $\alpha \in T_X$, $\alpha \in E(T_X)$ if and only if $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$, and $\alpha \in E(I_X)$ if and only if α is the identity map on $\Delta\alpha$.

Let S be a semigroup. For $a \in S$, let the map $\rho_a : S \to S$ defined by $x\rho_a = xa$ for all $x \in S$. If a, $b \in S$, then for each $x \in S$, $x\rho_a\rho_b = (xa)b = x(ab) = x\rho_{ab}$. Therefore the set $\{\rho_a \mid a \in S\}$ is a subsemigroup of $\mathcal{T}_S$. Let $\psi : S \to \mathcal{T}_S$ be defined by $a\psi = \rho_a$ for all $a \in S$. Then ψ is a homomorphism from S into $\mathcal{T}_S$. Suppose S has a left identity e. Then, if a, $b \in S$ such that $\rho_a = \rho_b$, we have $a = ea = e\rho_a = e\rho_b = eb = b$. It follows that ψ is an isomorphism.

From Lemma 1.1, every factorizable semigroup has a left identity. Hence the following proposition follows :

4.1 Proposition. Every factorizable semigroup can be embedded in a full transformation semigroup.

It has been proved by Chen and Hsieh in [3] that for any set X, the symmetric inverse semigroup I_X is factorizable if and only if X is finite. Hence, for any finite set X, $I_X = G_X E(I_X)$ by Theorem 3.1(3) and (5). Using this result, we show in the two following theorems that for any set X, the partial transformation semigroup on X is factorizable if and only if X is finite; and the full transformation semigroup on X is factorizable if and only if X is finite.

4.2 Theorem. Let X be a set. Then the partial transformation semigroup on the set X, T_X, is factorizable if and only if X is finite.

Proof : Assume T_X is factorizable. Because G_X is the group of units of T_X, from Theorem 1.7, $T_X = G_X E$ for some subset E of $E(T_X)$. Then $T_X = G_X E(T_X)$. To show X is a finite set, suppose not. Let $a \in X$. Because X is infinite, $|X - \{a\}| = |X|$. Let α be a one-to-one map from $X - \{a\}$ onto X. Then $\alpha \in T_X$. Since $T_X = G_X E(T_X)$, $\alpha = \beta\gamma$ for some $\beta \in G_X$, $\gamma \in E(T_X)$. It follows that

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$X = \nabla\alpha = \nabla\beta\gamma = (\nabla\beta \cap \Delta\gamma)\gamma = (X \cap \Delta\gamma)\gamma = (\Delta\gamma)\gamma = \nabla\gamma$. Because $\gamma \in E(T_X)$, $x^\gamma = x$ for all $x \in \nabla\gamma$. But $\nabla\gamma = X$. It follows that γ is the identity map on X. Hence $\alpha = \beta \in G_X$, it is a contradiction since $\Delta\alpha \neq X$.

Conversely, assume X is finite. Then I_X is factorizable and $I_X = G_X E(I_X)$. To show that $T_X = G_X E(T_X)$, let $\alpha \in T_X$. For each $a \in \nabla\alpha$, choose one element from the set $a\alpha^{-1}$ and call it x_a. Let $K = \{x_a \mid a \in \nabla\alpha\}$. Then $K \subseteq \Delta\alpha$, $x_a \neq x_b$ if $a, b \in \nabla\alpha$, $a \neq b$, and $|K| = |\nabla\alpha|$. Define the map β from K onto $\nabla\alpha$ by $x_a\beta = a$ for all $a \in \nabla\alpha$. Therefore $\beta \in I_X$ and $\nabla\alpha = \nabla\beta$. Because $I_X = G_X E(I_X)$, $\beta = \lambda\gamma$ for some $\lambda \in G_X$, $\gamma \in E(I_X)$. Define the map $\bar{\gamma}$ from $(\Delta\alpha)\lambda$ into X as follows :

$$x\bar{\gamma} = \begin{cases} x\gamma & \text{if } x \in (\Delta\alpha)\lambda \cap \Delta\gamma \\ (x\lambda^{-1})\alpha & \text{if } x \in (\Delta\alpha)\lambda - \Delta\gamma \end{cases}$$

First, we show $\nabla\bar{\gamma} \subseteq \nabla\gamma$; that is, to show $((\Delta\alpha)\lambda)\bar{\gamma} \subseteq \nabla\gamma$, let $x \in (\Delta\alpha)\lambda$. If $x \in \Delta\gamma$, then $x\bar{\gamma} = x\gamma \in \nabla\gamma$. Assume $x \notin \Delta\gamma$. Then $x\bar{\gamma} = (x\lambda^{-1})\alpha \in \nabla\alpha$. Since $\nabla\alpha = \nabla\beta$, there exists $y \in \Delta\beta$ such that $(x\lambda^{-1})\alpha = y\beta$. But $\beta = \lambda\gamma$, so $(x\lambda^{-1})\alpha = y\lambda\gamma = (y\lambda)\gamma \subseteq \nabla\gamma$. Hence $x\bar{\gamma} \in \nabla\gamma$. Therefore $\nabla\bar{\gamma} \subseteq \nabla\gamma$.

Next, we show that $\nabla\bar{\gamma} \subseteq \Delta\bar{\gamma}$. Let $x \in \nabla\bar{\gamma}$. Because $\nabla\bar{\gamma} \subseteq \nabla\gamma$ and $\nabla\gamma = \Delta\gamma$, $x \in \Delta\gamma$. Therefore $x\lambda^{-1} \in (\Delta\gamma)\lambda^{-1} = (X \cap \Delta\gamma)\lambda^{-1} = (\nabla\lambda \cap \Delta\gamma)\lambda^{-1} = \Delta\lambda\gamma = \Delta\beta \subseteq \Delta\alpha$ and hence $x = (x\lambda^{-1})\lambda \in (\Delta\alpha)\lambda = \Delta\bar{\gamma}$. To show that $y\bar{\gamma} = y$ for all $y \in \nabla\bar{\gamma}$, let $y \in \nabla\bar{\gamma}$. Because $\nabla\bar{\gamma} \subseteq \Delta\bar{\gamma}$, $y \in \Delta\bar{\gamma} = (\Delta\alpha)\lambda$. Since $\nabla\bar{\gamma} \subseteq \nabla\gamma$, $y \in \nabla\gamma$. But $\nabla\gamma = \Delta\gamma$, so $y \in \Delta\gamma$. Therefore $y \in (\Delta\alpha)\lambda \cap \Delta\gamma$. Hence $y\bar{\gamma} = y\gamma = y$. This proves $\bar{\gamma} \in E(T_X)$.

We claim that $\alpha = \lambda\bar{\gamma}$. Since $\nabla\lambda = X$ and $\Delta\bar{\gamma} = (\Delta\alpha)\lambda$, $\Delta\lambda\bar{\gamma} = (\nabla\lambda \cap \Delta\bar{\gamma})\lambda^{-1} = (X \cap \Delta\bar{\gamma})\lambda^{-1} = (\Delta\bar{\gamma})\lambda^{-1} = ((\Delta\alpha)\lambda)\lambda^{-1} = \Delta\alpha$. Next, let $x \in \Delta\lambda\bar{\gamma} = \Delta\alpha$. Then $x\lambda \in (\Delta\alpha)\lambda = \Delta\bar{\gamma}$. If $x\lambda \in \Delta\gamma$, then $x \in (\Delta\gamma)\lambda^{-1} = (X \cap \Delta\gamma)\lambda^{-1} = (\nabla\lambda \cap \Delta\gamma)\lambda^{-1} = \Delta\beta$, which implies that $x\alpha = x\beta = x\lambda\gamma = (x\lambda)\bar{\gamma}$ since $x\lambda \in \Delta\gamma$. Assume $x\lambda \notin \Delta\gamma$. Then $(x\lambda)\bar{\gamma} = ((x\lambda)\lambda^{-1})\alpha = x\alpha$. Therefore $\alpha = \lambda\bar{\gamma}$.

Hence the proof of Theorem is completely proved. #

4.3 <u>Theorem</u>. Let X be any set. The full transformation semigroup on X, $\mathcal{T}_X$, is a factorizable semigroup if and only if X is finite.

<u>Proof</u> : Assume $\mathcal{T}_X$ is a factorizable semigroup. Because G_X is the group of units of $\mathcal{T}_X$, by Theorem 1.7, $\mathcal{T}_X = G_X E$ for some subset E of $E(\mathcal{T}_X)$. Then $\mathcal{T}_X = G_X E(\mathcal{T}_X)$. Suppose that X is infinite. Let $a \in X$. Then there exists a one-to-one map α from X onto $X - \{a\}$. Therefore $\alpha \in \mathcal{T}_X$ but $\alpha \notin G_X$. Because $\mathcal{T}_X = G_X E(\mathcal{T}_X)$, $\alpha = \beta\gamma$ for some $\beta \in G_X$, $\gamma \in E(\mathcal{T}_X)$. Since $\alpha = \beta\gamma$, $\nabla\alpha \subseteq \nabla\gamma$. But $\nabla\alpha = X - \{a\}$. Then either $\nabla\gamma = X$ or $\nabla\gamma = X - \{a\}$. If $\nabla\gamma = X$, then γ is the identity map on X which implies $\alpha = \beta \in G_X$, a contradiction. Assume $\nabla\gamma = X - \{a\}$. Then $a\gamma \neq a$. Since $\nabla\beta = X$, there exist $b, c \in X$ such that $b\beta = a$ and $c\beta = a\gamma$. It then follows that $b\alpha = b\beta\gamma = a\gamma = a\gamma\gamma = (a\gamma)\gamma = (c\beta)\gamma = c\alpha$. But α is a one-to-one map. Then $b = c$, so $b\beta = c\beta$ which implies that $a = a\gamma$. It is a contradiction. Hence X is a finite set.

Conversely, assume X is a finite set. By Theorem 4.2, T_X is factorizable and so $T_X = G_X E(T_X)$. To show $\mathcal{T}_X = G_X E(\mathcal{T}_X)$, let $\alpha \in \mathcal{T}_X$. Then $\alpha \in T_X$. Because $T_X = G_X E(T_X)$, $\alpha = \beta\gamma$ for some $\beta \in G_X$, $\gamma \in E(T_X)$.

Then $\gamma = \beta^{-1}\alpha$. But $\Delta\beta = \nabla\beta = X$ and $\Delta\alpha = X$. Therefore $\Delta\gamma = \Delta(\beta^{-1}\alpha) = (\nabla\beta^{-1} \cap \Delta\alpha)\beta = (X \cap X)\beta = X$. Hence $\gamma \in E(\mathcal{T}_X)$, and so $\alpha = \beta\gamma \in G_X E(\mathcal{T}_X)$. This show that $\mathcal{T}_X = G_X E(\mathcal{T}_X)$ as required. #

Let X be a set. It is clearly seen that $T_X = G_X$ if and only if $X = \phi$, and $\mathcal{T}_X = G_X$ if and only if $|X| \leq 1$. Therefore by Theorem 4.2, 4.3 and Corollary 2.6, we have the following corollary :

4.4 <u>Corollary</u>. Let X be a finite set. Then T_X is η-simple if and only if $X = \phi$; and $\mathcal{T}_X$ is η-simple if and only if $|X| \leq 1$.

For any finite set X, the symmetric inverse semigroup on the set X, I_X, is factorizable. It is clearly seen that $I_X = G_X$ if and only if $X = \phi$. By Corollary 2.7, we also have the following : For a finite set X, I_X is η-simple if and only if $X = \phi$.

For the case of infinite sets, I_X and T_X can be η-simple, as shown in the following : Let X be a denumerable set. Then X can be written as $X = \{x_1, x_2, x_3, \ldots\}$, $x_i \neq x_j$ if $i \neq j$. Let α and β be maps on X defined by $x_i\alpha = x_{2i}$ and $x_i\beta = x_{2i+1}$ for all $i \in \{1, 2, 3, \ldots\}$. Then $\alpha, \beta \in I_X \subseteq T_X$ but $\alpha \notin G_X$ and $\beta \notin G_X$ Moreover, $\alpha\alpha^{-1} = \beta\beta^{-1}$ which is the identity map on X. Let η and η' be the minimum semilattice congruences on T_X and I_X; respectively. By lemma 2.2, 1η and $1\eta'$ are the smallest filters of T_X containing 1 and of I_X containing 1; respectively, where 1 denotes the identity of T_X. Thus $\alpha\alpha^{-1} = \beta\beta^{-1} \in 1\eta$ and $\alpha\alpha^{-1} = \beta\beta^{-1} \in 1\eta'$. Then $\alpha, \alpha^{-1}, \beta, \beta^{-1} \in 1\eta$ and $\alpha, \alpha^{-1}, \beta, \beta^{-1} \in 1\eta'$. Because 1η and $1\eta'$ are subsemigroups of T_X and I_X; respectively, $0 = \alpha\beta^{-1} \in 1\eta$ and $0 = \alpha\beta^{-1} \in 1\eta'$ where 0 denotes the zero of T_X. Thus $0\eta = 1\eta$ and $0\eta' = 1\eta'$. Therefore for any $\lambda \in T_X$, $\lambda\eta = (\lambda 1)\eta = (\lambda\eta)(1\eta) = (\lambda\eta)(0\eta) = (\lambda 0)\eta = 0\eta$, and similarly, for any $\lambda' \in I_X$, $\lambda'\eta' = 0\eta'$.

Hence, T_X and I_X are η-simple.