สมบัติตั้งดีแล้วในวงกว้างของสมการปฏิกิริยา-การแพร่ซึ่งมีเคอร์เนลไม่เป็นฟังก์ชันเชิงรัศมี

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GLOBAL WELL-POSEDNESS OF REACTION-DIFFUSION EQUATIONS WITH NON-RADIAL KERNEL FUNCTION

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$$\begin{cases} \partial_t u - \Delta u = u\left(f(u) - \alpha \int_{\mathbb{R}^n} g(x,y)\, u(y,t)\, dy\right) & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

โดยที่ f เป็นฟังก์ชันที่หาอนุพันธ์แล้วต่อเนื่องบน $\mathbb{R}$ และ $\alpha > 0$ เป็นค่าคงตัว และ $u_0 \geq 0$ เป็นฟังก์ชันต่อเนื่อง และ g ไม่เป็นฟังก์ชันเชิงรัศมี ในการวิจัยนี้เราได้สร้างหลักการเปรียบเทียบที่สมนัยกัน จากนั้นเราพิสูจน์การมีจริงและการมีเพียงหนึ่งเดียวเฉพาะที่ด้วยวิธีการลำดับทำซ้ำทางเดียว สุดท้ายเราได้กำหนดเงื่อนไขที่เพียงพอบางประการของพจน์ไม่เชิงเส้นเฉพาะที่ ที่รับรองการมีจริงและการมีเพียงหนึ่งเดียวของผลเฉลยวงกว้างได้

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In this thesis, we study the following reaction-diffusion equation:

$$\begin{cases} \partial_t u - \Delta u = u\left(f\left(u\right) - \alpha \int_{\mathbb{R}^n} g\left(x,y\right) u\left(y,t\right) dy \right) & x \in \mathbb{R}^n,\ t > 0, \\ u\left(x,0\right) = u_0\left(x\right) & x \in \mathbb{R}^n, \end{cases}$$

where f is a continuously differentiable function on $\mathbb{R}$, $\alpha > 0$ is a constant, $u_0 \geq 0$ is a continuous function and g is non-radial kernel function. We establish the corresponding comparison principle. Then, we prove the local existence and uniqueness result based on the monotone iteration sequence technique. Finally, certain sufficient conditions on the local nonlinear term that guarantee the existence and uniqueness of global solution are specified.

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CONTENTS

CHAPTER I
INTRODUCTION

In recent years, there have been many studies on reaction-diffusion equations involving a nonlocal term of convolution type of the form:

$$\begin{cases} \partial_t u - \Delta u = u\left(f\left(u\right) - \alpha \int_{\mathbb{R}^n} g\left(x-y\right) u\left(y,t\right) dy \right) & x \in \mathbb{R}^n,\ t>0, \\ u\left(x,0\right) = u_0\left(x\right) & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is an unknown real valued function defined on $\mathbb{R}^n \times [0,\infty)$, f and g are given functions, α is a positive constant and u_0 is the initial condition of u with

$$\int_{\mathbb{R}^n} g(x)dx = 1.$$

These problems are used, for example, to study the dynamics of population with competition between individuals of the same species.

In 1989, Britton [2] introduced (1.1) with $f(u) = 1 + au - bu^2$ and $\alpha = 1 + a - b$ for $0 < b < 1 + a$ and $a > 0$, to model a single biological population that derives some competitive advantage from local aggregation (represented by au) within the capacity of the local environment (represented by $-bu^2$). In this model, the population also compete among themself through the depletion of resource in a neighborhood of its original position (represented by the convolution term).

Later, Britton [3] carried out a linear stability analysis for the uniform steady-state solution of (1.1) and then investigated the bifurcation from this uniform steady-state.

In 2001, Gourley et al. [8] and in 2004, Billingham [1] studied travelling wavefront solutions of (1.1) when $n = 1$ and assumed that the kernel $g(x) = \frac{1}{2}e^{-|x|}$ by using numerical and asymptotic techniques.

In 2008, Deng [4] established a comparison principle and employed the technique of monotone sequences to get the existence and uniqueness of solutions for (1.1), and analyze the long-time behavior of solutions when $n = 1$.

In 2014, Sun [11] proved the existence and uniqueness of positive solutions for a nonlocal dispersal population model by the monotone iteration sequences technique.

In 2015, Deng and Wu [5] extended the results of [4] in 2008 to the case of arbitrary $n \geq 1$ and established a comparison principle and then constructed monotone sequences to show the existence and uniqueness of solution for (1.1), and analyzed the global stability.

In this work, motivated by Deng and Wu [5], we extend the study to the reaction-diffusions equation of a more general form:

$$\begin{cases} \partial_t u - \Delta u = u\left(f\left(u\right) - \alpha \int_{\mathbb{R}^n} g\left(x,y\right) u\left(y,t\right) dy \right) & x \in \mathbb{R}^n,\ t > 0, \\ u\left(x,0\right) = u_0\left(x\right) & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Here, g needs not be radially symmetric, that is, we do not require $g(x,y) = g(x-y)$; more importantly, our nonlinear local term f can behave more irregularly in the sense that $f(u)$ can be positive for all $u \geq 0$. The condition on f is a bit more general than that of Deng and Wu [5], where in [5] it was assumed that $f(u) \leq 0$ for all u sufficiently large which allows the global existence result to be easily established. As will be shown in Chapter IV, the positivity of $f(u)$ for large u imposes a nontrivial difficulty. Throughout this work, we assume that g satisfies the following conditions:

(i) g is a nonnegative continuous function on $\mathbb{R}^n \times \mathbb{R}^n$;

(ii) there exits a constant $C_g > 0$ such that

$$\int_{\mathbb{R}^n} g(x,y)dy \le C_g \quad \text{for all} \quad x \in \mathbb{R}^n;$$

(iii) $\nabla_x g$ is a bounded continuous function on $\mathbb{R}^n \times \mathbb{R}^n$ with $\nabla_x g(x,\cdot)$ is integrable for all $x \in \mathbb{R}^n$.

This thesis is organized into four chapters as follows. In Chapter II, we introduce some notions, definitions and preliminaries that will be useful. Next, in Chapter III, we establish our fundamental result, the comparison principle and uniqueness of solution. Next, by constructing monotone sequences of coupled upper and lower solutions, we can prove the basic (local) existence of solutions to the problem (1.2). Finally, the global existence of solutions will be obtained in the last chapter under various conditions on the local nonlinear term f. To prove this result, we shall study the maximal solution to certain initial value problem. It will be followed that the solutions obtained are bounded by the maximal solutions. Thus, we also get the asymptotic behavior of solutions.

CHAPTER II
PRELIMINARIES

In this chapter, we introduce some notations, definitions, and preliminaries which will be used in this work. For more details, the reader can consult any textbooks in partial differential equations (PDEs) or analysis see [6] , [7], [10].

2.1 Notation

Let X be a measurable space and $0 < T \leq \infty$.

(i) $Q_T = \mathbb{R}^n \times (0, T)$ and $\Sigma_T = \mathbb{R}^n \times [0, T)$.

(ii) For $1 \leq p < \infty$, the space $L^p(X)$ consists of the Lebesgue measurable functions $u : X \to \mathbb{R}$ such that

$$\int_X |u|^p dx < \infty.$$

The L^p norm of $u \in L^p(X)$ is defined by $\|u\|_{L^p(X)} = \left\{\int_X |u|^p dx\right\}^{1/p}$.

(iii) The space $L^\infty(X)$ consists of the Lebesgue measurable functions $u : X \to \mathbb{R}$ such that

$$\|u\|_{L^\infty(X)} := \sup_{x \in X} |u(x)| < \infty.$$

(iv) $C^{2,1}(Q_T)$ denotes the space of all functions that are twice continuously differentiable in x and continuously differentiable in t, for all $(x, t) \in Q_T$.

(v) If X is also a topological space, $C(X)$ denotes the space of continuous functions on X.

(vi) For $k \geq 1$, $C^k(X)$ denotes the set of all functions whose derivatives of up to k are all continuous in X.

(vii) $C_b(X)$ denotes the space of all bounded continuous functions on X.

(viii) The space $C([0,T); C_b(X))$ consists of the continuous function $u : [0,T) \to C_b(X)$.

2.2 Basic Theory

Theorem 2.1 (Minkowski's inequality)**.** *Assume that* $1 \leq p \leq \infty$ *and* $u, v \in L^p(X)$*. Then,*

$$\|u + v\|_{L^p(X)} \leq \|u\|_{L^p(X)} + \|v\|_{L^p(X)}.$$

Theorem 2.2 (Gronwall's inequality, the integral form)**.** *Suppose that a continuous functions* $u : [0,T] \to [0,\infty)$ *satisfies*

$$u(t) \leq C_1 + C_2 \int_0^t u(s)ds \quad \textit{for all} \quad t \in [0,T]$$

where $C_1, C_2 \geq 0$ *are constants. Then,*

$$u(t) \leq C_1(1 + C_2 t e^{C_2 t}) \quad \textit{for all} \quad t \in [0,T].$$

Theorem 2.3 (Dominated Convergence Theorem)**.** *Let* $\{f_k\}_{k=0}^{\infty}$ *be a sequence of real-valued measurable functions on* X *such that*

(i) $f(x) = \lim_{k\to\infty} f_k(x)$ *exists for all* $x \in X$*, and*

(ii) there exists a function $g \in L^1(X)$ *such that*

$$|f_k(x)| \leq g(x) \quad \textit{for all} \quad k \in \mathbb{N} \quad \textit{and} \quad x \in X.$$

Then,

$$\lim_{k\to\infty}\int_X f_k dx = \int_X \lim_{k\to\infty} f_k dx.$$

Theorem 2.4 (Contraction Mapping Theorem or Banach Fixed Point Theorem)**.** *Let X be a complete metric space and $\mathcal{A} : X \to X$ a contraction mapping. Then, $\mathcal{A}$ has a unique fixed point $x \in X$.*

Theorem 2.5 (Young's inequality)**.** *If $1 \le p \le \infty$, $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, then*

$$\|f * g\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)},$$

where the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy \quad \text{for } x \in \mathbb{R}^n.$$

2.3 Coupled Upper and Lower Solutions

Definition 2.6. A classical solution u of the problem (1.2) is a function which satisfies the problem (1.2) and $u \in C^{2,1}(Q_T)$.

Definition 2.7 (Mild Solution)**.** A function u is called a mild solution of the problem (1.2) if $u \in C([0,T); C_b(\mathbb{R}^n))$ for some $0 < T \le \infty$ and u satisfies the equation

$$u(x,t) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-s)u(x,s)\Big(f(u(x,s)) - \alpha(g\star u(x,s))\Big)ds \qquad (2.1)$$

at each $x \in \mathbb{R}^n$, $0 \le t < T$, where

$$g\star u(x,t) = \int_{\mathbb{R}^n} g(x,y)u(y,t)dy$$

and the operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t-s)v(x,s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s)v(y,s)dy = \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} v(y,s)dy.$$

Note that Φ is the fundamental solution of the heat equation (or the heat kernel) given by

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} \quad (x \in \mathbb{R}^n, t > 0).$$

Remark 2.8.

(i) If u is a solution of (1.2) for some $0 < T < \infty$, then one often call u is a local (in time) solution of (1.2).

(ii) On the other hand, if u is a solution of (1.2) with $T = \infty$, then u is called a global (in time) solution of (1.2).

(iii) If there exists $0 < T < \infty$ such that

$$\lim_{t \to T^-} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = \infty,$$

then u is called a blow-up solution of (1.2).

Theorem 2.9 ([10])**.** *Let $F, \nabla_x F \in C_b(\mathbb{R}^n \times (0,\infty))$, $u_0 \in C_b(\mathbb{R}^n)$ and u be defined by*

$$u(x,t) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-s)F(x,s)ds \tag{2.2}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Then, $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$ and satisfies

$$\partial_t u - \Delta u = F \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

and for any $x_0 \in \mathbb{R}^n$, $\lim_{(x,t)\to(x_0,0)} u(x,t) = u_0(x_0)$.

Lemma 2.10. *Fix $T > 0$. The operator $\mathcal{G}(t)$ satisfies*

$$\|\mathcal{G}(t)v\|_{L^\infty(\mathbb{R}^n)} \leq \|v\|_{L^\infty(\mathbb{R}^n)}$$

for $v \in L^\infty(\mathbb{R}^n)$ and $0 \leq t \leq T$.

Proof. When $t = 0$, we have $\mathcal{G} = id$. Thus, the desired estimate holds trivially. Assume that $t > 0$. We have

$$\|\mathcal{G}(t)v\|_{L^\infty(\mathbb{R}^n)} = \left\| \int_{\mathbb{R}^n} \Phi(\cdot - y, t)v(y)dy \right\|_{L^\infty(\mathbb{R}^n)}.$$

Applying Theorem 2.5 (Young's inequality) , we get

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} \Phi(\cdot - y, t)v(y)dy \right\|_{L^\infty(\mathbb{R}^n)} &\leq \|\Phi(\cdot, t)\|_{L^1(\mathbb{R}^n)}\|v\|_{L^\infty(\mathbb{R}^n)} \\ &= \|v\|_{L^\infty(\mathbb{R}^n)}, \end{aligned}$$

because $\int_{\mathbb{R}^n} \Phi(x,t)dx = 1$ for all $t > 0$. □

Definition 2.11. Functions $\overline{u}$ and $\underline{u}$ are called *coupled upper and lower classical solutions* of (1.2) on Q_T, respectively, if they satisfy the following conditions:

(i) $\overline{u}, \underline{u} \in C^{2,1}(Q_T) \cap C_b(\Sigma_T)$;

(ii) $\underline{u}(x, 0) \leq u_0(x) \leq \overline{u}(x, 0)$ in $\mathbb{R}^n$;

(iii) For any $(x, t) \in Q_T$,

$$\begin{cases} \partial_t\overline{u} - \Delta\overline{u} \geq \overline{u}\Big(f(\overline{u}) - \alpha g \star \underline{u}\Big), \\ \partial_t\underline{u} - \Delta\underline{u} \leq \underline{u}\Big(f(\underline{u}) - \alpha g \star \overline{u}\Big). \end{cases} \tag{2.3}$$

Definition 2.12. Functions $\overline{u}$ and $\underline{u}$ are called *coupled upper and lower mild solutions* of (1.2) on Q_T, respectively, if they satisfy the following conditions:

(i) $\overline{u}$, $\underline{u} \in C([0, T); C_b(\mathbb{R}^n))$ and $\overline{u}$, $\underline{u} \in L^\infty(Q_T)$;

(ii) $\underline{u}(x,0) \leq u_0(x) \leq \overline{u}(x,0)$ in $\mathbb{R}^n$;

(iii) For any $(x,t) \in Q_T$,

$$\begin{cases} \overline{u}(x,t) \geq \mathcal{G}(t)\overline{u}(x,0) + \int_0^t \mathcal{G}(t-s)\overline{u}(x,s)\Big(f(\overline{u}(x,s)) - \alpha(g \star \underline{u}(x,s))\Big)ds, \\ \underline{u}(x,t) \leq \mathcal{G}(t)\underline{u}(x,0) + \int_0^t \mathcal{G}(t-s)\underline{u}(x,s)\Big(f(\underline{u}(x,s)) - \alpha(g \star \overline{u}(x,s))\Big)ds. \end{cases}$$

We summarize the hypotheses for this work here.

(H1) g is a nonnegative continuous function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying:

(i) There exists a positive constant $C_g > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} g(x,y)dy \leq C_g;$$

(ii) $\nabla_x g$ is a bounded continuous function on $\mathbb{R}^n \times \mathbb{R}^n$ with $\nabla_x g(x,\cdot) \in L^1(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$.

(H2) u_0 is a nonnegative continuous function on $\mathbb{R}^n$ with $u_0 \in L^\infty(\mathbb{R}^n)$.

(H3) f is continuously differentiable on $\mathbb{R}$. Further properties of f will be provided when needed.

Lemma 2.13. *Suppose that* (H3) *holds and let $v = v(x,t)$ and $w = w(x,t)$ be bounded functions on Q_T. Then, there exists a function θ which lies between v and w at each point such that*

$$vf(v) - wf(w) = \Big(f(\theta) + \theta f^{'}(\theta)\Big)(v-w).$$

Proof. Since v and w are bounded, there exists $M > 0$ such that

$$|v| \leq M \quad \text{and} \quad |w| \leq M.$$

Setting $\Psi(\varepsilon) = \varepsilon f(\varepsilon)$ for $\varepsilon \in [-M, M]$.

Clearly, Ψ is continuous and differentiable on $[-M, M]$ and we have

$$\Psi^{'}(\varepsilon) = f(\varepsilon) + \varepsilon f^{'}(\varepsilon).$$

Applying the Mean Value Theorem, there exists $\theta = (1-\lambda)w + \lambda v$ for $0 < \lambda < 1$ such that

$$\Psi(v) - \Psi(w) = \Psi^{'}(\theta)(v - w).$$

This implies the desired result. □

Proposition 2.14. *Assume that* (H1)-(H3) *hold. Let* $u, \nabla_x u \in C_b(\mathbb{R}^n \times (0, \infty))$ *and be defined by* (2.1) *for* $(x,t) \in \mathbb{R}^n \times [0, \infty)$. *Then,* $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$ *and satisfies*

$$\partial_t u - \Delta u = u\Big(f\,(u) - \alpha g \star u\Big). \tag{2.4}$$

Proof. Setting $F(x,t) = u\Big(f\,(u) - \alpha g \star u\Big)$. Since u and $\nabla_x u$ are bounded, there exist $M, M' > 0$ such that

$$|u| \leq M \quad \text{and} \quad |\nabla_x u| \leq M'.$$

By (H1), there exist $C_g, C_g' > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} g(x,y)dy \leq C_g \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla_x g(x,y)|dy \leq C_g' \quad \text{for all} \quad x \in \mathbb{R}^n.$$

By (H3), we obtain that $f(\varepsilon)$ and $f'(\varepsilon)$ are bounded for $\varepsilon \in [-M, M]$, i.e. there exist $L, L' > 0$ such that

$$|f(\varepsilon)| \leq L \quad \text{and} \quad |f'(\varepsilon)| \leq L'.$$

We then estimate F by

$$\begin{aligned}
\|F\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} &\le \|uf(u)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} + \|\alpha u(g\star u)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\\
&\le M\|f(u)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} + \alpha M\|g\star u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\\
&\le ML + \alpha M^2 \left\|\int_{\mathbb{R}^n} g(\cdot,y)dy\right\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\\
&= ML + \alpha C_g M^2 < \infty
\end{aligned}$$

which implies that F is a bounded continuous function on $\mathbb{R}^n\times(0,\infty)$.

Next, we have

$$\begin{aligned}
\triangledown_x F &= u\,\triangledown_x \Big(f\left(u\right) - \alpha g\star u\Big) + \left(\triangledown_x u\right)\Big(f\left(u\right) - \alpha g\star u\Big)\\
&= u\Big(\triangledown_x f\left(u\right) - \alpha\left(\triangledown_x g\star u\right)\Big) + \left(\triangledown_x u\right)\Big(f\left(u\right) - \alpha g\star u\Big)\\
&= u\Big(f'(u)\left(\nabla_x u\right) - \alpha\left(\triangledown_x g\star u\right)\Big) + \left(\triangledown_x u\right)\Big(f\left(u\right) - \alpha g\star u\Big).
\end{aligned}$$

By the assumption, we get $\nabla_x F$ is a continuous function on $\mathbb{R}^n\times(0,\infty)$.

We then estimate $\nabla_x F$ by

$$\begin{aligned}
&\|\triangledown_x F\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\\
&\quad\le \|u\left(f'(u)\nabla_x u - \alpha\left(\triangledown_x g\star u\right)\right)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} + \|\left(\triangledown_x u\right)\left(f\left(u\right) - \alpha g\star u\right)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\\
&\quad\le \|u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\Big(\|f'(u)\left(\nabla_x u\right)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} + \|\alpha\left(\triangledown_x g\star u\right)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\Big)\\
&\qquad + \|\triangledown_x u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\Big(\|f(u)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} + \|\alpha\left(g\star u\right)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\Big)\\
&\quad\le M\Big(M'L' + \alpha M\left\|\int_{\mathbb{R}^n}\nabla_x g(\cdot,y)dy\right\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\Big)\\
&\qquad + M'\left(L + \alpha M\left\|\int_{\mathbb{R}^n} g(\cdot,y)dy\right\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\right)\\
&\quad\le M\left(M'L' + \alpha M C_g'\right) + M'(L + \alpha C_g M) < \infty.
\end{aligned}$$

Then, we get that $\triangledown_x F$ is a bounded continuous function on $\mathbb{R}^n\times(0,\infty)$. By Theorem 2.9, we conclude that $u\in C^{2,1}(\mathbb{R}^n\times(0,\infty))$ and satisfies (2.4). $\square$

Lemma 2.15. *Assume that* $u \in C^{2,1}(Q_T) \cap C_b(\Sigma_T)$ *satisfies*

$$\begin{cases} \partial_t u - \Delta u + c(x,t)u \leq 0 \quad in \quad Q_T \\ u(x,0) \leq 0 \quad on \quad \mathbb{R}^n \end{cases}$$

and c *is bounded function in* Q_T. *Then,* $u \leq 0$ *on* Σ_T.

Proof. Since c is bounded, there exists $\delta > 0$ such that $\|c\|_{L^\infty(Q_T)} \leq \delta$.
Let $w = e^{-\delta t}u$. Then, w satisfies

$$\begin{cases} \partial_t w - \Delta w + (\delta + c)w \leq 0 \quad \text{in} \quad Q_T \\ w(x,0) \leq 0 \quad \text{on} \quad \mathbb{R}^n. \end{cases} \tag{2.5}$$

To show that $w \leq 0$ in Q_T, we assume that $w > 0$ somewhere in Q_T. We then let $v = \frac{w}{1+|x|^2+\gamma t}$ be an auxiliary function where $\gamma > 0$ is a constant to be specified. From (2.5) and the fact that $\Delta(fg) = (\Delta f)g + 2\,(\nabla f)\,(\nabla g) + f\Delta g$, we get

$$\begin{cases} \Big(1 + |x|^2 + \gamma t\Big)\Big(v_t - \Delta v + (\delta + c(x,t))v\Big) \\ \qquad\qquad -4\sum_{i=1}^n x_i v_{x_i} + (\gamma - 2n)v \leq 0 \quad \text{in} \quad Q_T \\ v(x,0) \leq 0 \quad \text{on} \quad \mathbb{R}^n. \end{cases} \tag{2.6}$$

Since $v(x,0) \leq 0$ on $\mathbb{R}^n$ and w is bounded on Q_T,

$$\lim_{|x|\to\infty} v(x,t) = 0 \quad \text{for each} \quad 0 \leq t \leq T.$$

Thus, v attains its positive maximum $v_{\max}$ at (x^*, t^*) in Q_T. Then, at the point (x^*, t^*), we have $v_t \geq 0$, $\nabla v = 0$ and $\Delta v \leq 0$. Hence, by (2.6) we have

$$(\delta + c)\Big(1 + |x^*|^2 + \gamma t^*\Big)v_{\max} + (\gamma - 2n)v_{\max} \leq 0. \tag{2.7}$$

Since $\delta + c \geq 0$ and $v_{\max} > 0$, it follows by choosing γ large enough that (2.7) is a contradiction. Hence, $w \leq 0$ in Q_T. Since $w = e^{-\delta t}u$, $u \leq 0$ in Σ_T. □

CHAPTER III

MAIN RESULT I

In this chapter, we prove the comparison principle for coupled upper and lower mild solutions of (1.2) and derive the local existence and uniqueness of solutions for (1.2).

3.1 Comparison Principle

In this section, we establish a comparison principle for mild solutions of (1.2) under the Definition 2.12.

Theorem 3.1. *Suppose that* (H1)-(H3) *hold. Let* $\overline{u}$ *and* $\underline{u}$ *be a pair of nonnegative coupled upper and lower mild solutions of* (1.2)*, respectively, and* $\nabla_x \overline{u}, \nabla_x \underline{u} \in C_b(Q_T)$*. Then,* $\overline{u} \geq \underline{u}$ *in* Σ_T*.*

Proof. The proof of this theorem is essentially that of Deng and Wu [5]. We present it here for completeness. First, let us consider $w = \underline{u} - \overline{u}$. Then, for any $(x,t) \in Q_T$, w satisfies

$$
\begin{aligned}
w(x,t) &\leq \mathcal{G}(t)\underline{u}(x,0) + \int_0^t \mathcal{G}(t-s)\underline{u}(x,s)\Big(f(\underline{u}(x,s)) - \alpha(g \star \overline{u}(x,s))\Big)ds \\
&\qquad - \mathcal{G}(t)\overline{u}(x,0) - \int_0^t \mathcal{G}(t-s)\overline{u}(x,s)\Big(f(\overline{u}(x,s)) - \alpha(g \star \underline{u}(x,s))\Big)ds \\
&= \mathcal{G}(t)\Big(\underline{u}(x,0) - \overline{u}(x,0)\Big) \\
&\qquad + \int_0^t \mathcal{G}(t-s)\Big(\underline{u}(x,s)f(\underline{u}(x,s)) - \overline{u}(x,s)f(\overline{u}(x,s))\Big)ds \\
&\qquad - \alpha\int_0^t \mathcal{G}(t-s)\Big(\underline{u}(x,s)(g \star \overline{u}(x,s)) - \overline{u}(x,s)(g \star \underline{u}(x,s))\Big)ds \\
&= \mathcal{G}(t)w(x,0) \\
&\qquad + \int_0^t \mathcal{G}(t-s)\Big(\underline{u}(x,s)f(\underline{u}(x,s)) - \overline{u}(x,s)f(\overline{u}(x,s))\Big)ds
\end{aligned}
$$

$$-\alpha \int_0^t \mathcal{G}(t-s)\Big(\underline{u}(x,s)(g\star\overline{u}(x,s)) - \overline{u}(x,s)(g\star\underline{u}(x,s))\Big)ds.$$

We define $\tilde{w}$ to be the solution of the following integral equation:

$$\tilde{w} = \mathcal{G}(t)\tilde{w}_0 + \int_0^t \mathcal{G}(t-s)\Big(\underline{u}f(\underline{u}) - \overline{u}f(\overline{u}) - \alpha\underline{u}(g\star\overline{u}) + \alpha\overline{u}(g\star\underline{u})\Big)ds \tag{3.1}$$

with $\tilde{w}_0 = w(x,0)$. It follows that

$$w \le \tilde{w} \quad \text{in} \quad \Sigma_T. \tag{3.2}$$

We then claim that $\tilde{w}$ is a classical solution of the following problem:

$$\begin{cases} \partial_t\tilde{w} - \Delta\tilde{w} = \underline{u}f(\underline{u}) - \overline{u}f(\overline{u}) - \alpha\underline{u}(g\star\overline{u}) + \alpha\overline{u}(g\star\underline{u}) \quad \text{in} \quad Q_T \\ \tilde{w}(x,0) = w(x,0) \quad \text{on} \quad \mathbb{R}^n. \end{cases} \tag{3.3}$$

Letting $F(x,t) = \underline{u}f(\underline{u}) - \overline{u}f(\overline{u}) - \alpha\underline{u}(g\star\overline{u}) + \alpha\overline{u}(g\star\underline{u})$. Then,

$$\begin{aligned} \nabla_x F &= \underline{u}\left(\nabla_x f(\underline{u})\right) + \left(\nabla_x\underline{u}\right)f(\underline{u}) - \overline{u}\left(\nabla_x f(\overline{u})\right) - \left(\nabla_x\overline{u}\right)f(\overline{u}) \\ &\qquad - \alpha\underline{u}\left(\nabla_x g\star\overline{u}\right) - \alpha\left(\nabla_x\underline{u}\right)\left(g\star\overline{u}\right) + \alpha\overline{u}\left(\nabla_x g\star\underline{u}\right) + \alpha\left(\nabla_x\overline{u}\right)\left(g\star\underline{u}\right) \\ &= \underline{u}f'(\underline{u})\left(\nabla_x\underline{u}\right) + \left(\nabla_x\underline{u}\right)f(\underline{u}) - \overline{u}f'(\overline{u})\left(\nabla_x\overline{u}\right) - \left(\nabla_x\overline{u}\right)f(\overline{u}) \\ &\qquad - \alpha\underline{u}(\nabla_x g\star\overline{u}) - \alpha\left(\nabla_x\underline{u}\right)\left(g\star\overline{u}\right) + \alpha\overline{u}(\nabla_x g\star\underline{u}) + \alpha\left(\nabla_x\overline{u}\right)\left(g\star\underline{u}\right). \end{aligned}$$

By the assumption, we get $\nabla_x F$ is a continuous function on Q_T. Since $\overline{u}$ and $\underline{u}$ are nonnegative and $\overline{u}$, $\underline{u}$, $\nabla_x\overline{u}$ and $\nabla_x\overline{u}$ are bounded, there exist $M, M' > 0$ such that

$$0 \le \overline{u}, \underline{u} \le M \quad \text{and} \quad |\nabla_x\overline{u}|, |\nabla_x\underline{u}| \le M' \quad \text{on} \quad Q_T. \tag{3.4}$$

By (H3), it follows that $f(\varepsilon)$ and $f'(\varepsilon)$ are bounded for $\varepsilon \in [0, M]$, i.e., there exist

$L, L' > 0$ such that

$$|f(\varepsilon)| \leq L \quad \text{and} \quad |f'(\varepsilon)| \leq L'. \tag{3.5}$$

By (H1), there exist $C_g, C'_g > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} g(x,y)dy \leqslant C_g \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla_x g(x,y)|dy \leq C'_g \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{3.6}$$

We then estimate F and $\nabla_x F$ by

$$\begin{aligned}
\|F\|_{L^\infty(Q_T)} &\leq \|\underline{u} f(\underline{u})\|_{L^\infty(Q_T)} + \|\overline{u} f(\overline{u})\|_{L^\infty(Q_T)} + \|\alpha \underline{u}(g \star \overline{u})\|_{L^\infty(Q_T)} + \|\alpha \overline{u}(g \star \underline{u})\|_{L^\infty(Q_T)} \\
&\leq 2M \left(L + \alpha M \left\| \int_{\mathbb{R}^n} g(\cdot,y)dy \right\|_{L^\infty(Q_T)} \right) \\
&\leq 2M(L + \alpha C_g M) < \infty
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla_x F\|_{L^\infty(Q_T)} &\leq \|\underline{u} f'(\underline{u}) \left(\nabla_x \underline{u}\right) \|_{L^\infty(Q_T)} + \| \left(\nabla_x \underline{u}\right) f(\underline{u})\|_{L^\infty(Q_T)} \\
&\quad + \|\overline{u} f'(\overline{u}) \left(\nabla_x \overline{u}\right) \|_{L^\infty(Q_T)} + \| \left(\nabla_x \overline{u}\right) f(\overline{u})\|_{L^\infty(Q_T)} \\
&\quad + \|\alpha \underline{u} \left(\nabla_x g \star \overline{u}\right) \|_{L^\infty(Q_T)} + \|\alpha \left(\nabla_x \underline{u}\right) \left(g \star \overline{u}\right) \|_{L^\infty(Q_T)} \\
&\quad + \|\alpha \overline{u} \left(\nabla_x g \star \underline{u}\right) \|_{L^\infty(Q_T)} + \|\alpha \left(\nabla_x \overline{u}\right) \left(g \star \underline{u}\right)\|_{L^\infty(Q_T)} \\
&\leq 2(MM'L' + M'L) \\
&\quad + \alpha \Big(M^2 \left\| \int_{\mathbb{R}^n} \nabla_x g(\cdot,y)dy \right\|_{L^\infty(Q_T)} + M'M \left\| \int_{\mathbb{R}^n} g(\cdot,y)dy \right\|_{L^\infty(Q_T)} \Big) \\
&\leq 2(MM'L' + M'L + \alpha M^2 C'_g + \alpha M' C_g M) < \infty.
\end{aligned}$$

This implies that F and $\nabla_x F$ are bounded functions on Q_T. By Theorem 2.9, we conclude that $\tilde{w} \in C^{2,1}(Q_T)$ with $T \leq \infty$ and satisfies (3.3) Moreover, from (3.4) we get that

$$-2M \leq w \leq 2M \quad \text{on} \quad \Sigma_T.$$

From (3.4)-(3.6), we obtain

$$
\begin{aligned}
|\varepsilon f'(\varepsilon)+f(\varepsilon)-\alpha g\star \underline{u}| &\le |f(\varepsilon)+\theta f^{'}(\varepsilon)|+|\alpha g\star \underline{u}| \\
&\le L+ML'+\alpha M\left|\int_{\mathbb{R}^n} g(x,y)dy\right| \\
&\le ML'+L+\alpha C_g M =: \delta
\end{aligned}
$$

or equivalently,

$$
\delta \pm (\varepsilon f'(\varepsilon)+f(\varepsilon)-\alpha g\star \underline{u}) \ge 0 \quad \text{for} \quad 0\le \varepsilon \le M.
$$

From (3.2) and (3.3), it follows that for any $(x,t)\in Q_T$

$$
\begin{aligned}
\partial_t \tilde{w}-\Delta \tilde{w} &= \underline{u}f(\underline{u}) - \overline{u}f(\overline{u})-\alpha \underline{u}(g\star \overline{u})+\alpha \overline{u}(g\star \underline{u}) \\
&= \Big(\underline{u}f(\underline{u})-\overline{u}f(\overline{u})\Big)-\alpha \underline{u}\,(g\star \overline{u})+\alpha \underline{u}\,(g\star \underline{u})-\alpha \underline{u}\,(g\star \underline{u})+\alpha \overline{u}\,(g\star \underline{u}) \\
&= \Big(\theta f^{'}(\theta)+f(\theta)\Big)(\underline{u}-\overline{u})+\alpha \underline{u}\,(g\star(\underline{u}-\overline{u}))-\alpha(\underline{u}-\overline{u})\,(g\star \underline{u}) \\
&= \Big(\theta f^{'}(\theta)+f(\theta)-\alpha g\star \underline{u}\Big)w+\alpha \underline{u}\,(g\star w) \\
&\le \Big(\theta f^{'}(\theta)+f(\theta)-\alpha g\star \underline{u}\Big)w+\alpha \underline{u}\,(g\star \tilde{w}) \\
&= \Big(\theta f^{'}(\theta)+f(\theta)-\alpha g\star \underline{u}+\delta\Big)w-\delta w+\alpha \underline{u}\,(g\star \tilde{w}) \\
&\le \Big(\theta f^{'}(\theta)+f(\theta)-\alpha g\star \underline{u}+\delta\Big)\tilde{w}+\delta(-w)+\alpha \underline{u}\,(g\star \tilde{w}) \\
&\le \Big(\theta f^{'}(\theta)+f(\theta)-\alpha g\star \underline{u}+\delta\Big)\tilde{w}+2\delta M+\alpha \underline{u}\,(g\star \tilde{w})\,,
\end{aligned}
$$

where in the third equality we have used Lemma 2.13 and θ is between $\underline{u}$ and $\overline{u}$. We also note that

$$
\tilde{w}(x,0)=w(x,0)=\underline{u}(x,0)-\overline{u}(x,0)\le 0 \quad \text{on} \quad \mathbb{R}^n.
$$

To study the problem (3.3), we set $c(x,t)=\theta f^{'}(\theta)+f(\theta)-\alpha g\star \underline{u}$.

Observe that $\delta \pm c(x,t) \geq 0$. Thus, $\tilde{w}$ satisfies

$$\begin{cases} \partial_t \tilde{w} - \Delta \tilde{w} \leq (c(x,t)+\delta)\tilde{w} + 2\delta M + \alpha \underline{u}\,(g \star \tilde{w}) \quad \text{in} \quad Q_T \\ \tilde{w}\,(x,0) \leq 0 \quad \text{on} \quad \mathbb{R}^n. \end{cases} \tag{3.7}$$

Now, we let $u = e^{-2\delta t}\tilde{w}$. Then, by (3.7), we have

$$e^{2\delta t}\,(2\delta u + u_t - \Delta u) \leq (c(x,t)+\delta)e^{2\delta t}u + \alpha \underline{u}\left(g \star e^{2\delta t}u\right) + 2M\delta \quad \text{in} \quad Q_T$$

and

$$u(x,0) = w(x,0) \leq 0 \quad \text{on } \mathbb{R}^n.$$

Thus, u satisfies

$$\begin{cases} u_t - \Delta u + (\delta - c)u \leq \alpha \underline{u}\,(g \star u) + 2M\delta e^{-2\delta t} \quad \text{in} \quad Q_T \\ u(x,0) \leq 0 \quad \text{on} \quad \mathbb{R}^n. \end{cases} \tag{3.8}$$

Now, we claim that $u \leq 0$ in Q_{T_0} where $T_0 = \min\left\{T, \frac{R}{4M(\alpha R C_g + 2\delta)}\right\}$ for some $R > 0$. We let $v = \frac{u}{1+|x|^2+\gamma t}$ be an auxiliary function where $\gamma > 0$ is a constant to be specified. By (3.8) and the fact that $\Delta(fg) = (\Delta f)g + 2\,(\nabla f)\,(\nabla g) + f\Delta g$, we get

$$\begin{cases} \left(1+|x|^2+\gamma t\right)\left(v_t - \Delta v + (\delta - c(x,t))v\right) - 4\sum_{i=1}^n x_i v_{x_i} + (\gamma - 2n)v \\ \qquad\qquad \leq \alpha \underline{u}\,(g \star u) + 2M\delta e^{-2\delta t} \quad \text{in} \quad Q_{T_0} \\ v(x,0) \leq 0 \quad \text{on} \quad \mathbb{R}^n. \end{cases} \tag{3.9}$$

Assume that $u > 0$ somewhere in Q_{T_0}. Since u is bounded, we have

$$u_{\sup} = \sup_{(x,t)\in Q_{T_0}} u(x,t) > 0.$$

By the definition of $u_{\rm sup}$ and (3.8), there exists $(x^\star, t^\star)$ in Q_{T_0} such that

$$u(x^\star, t^\star) \geq \frac{u_{\rm sup}}{2}. \tag{3.10}$$

Since $v(x,0) \leq 0$ on $\mathbb{R}^n$ and u is bounded on Q_{T_0},

$$\lim_{|x|\to\infty} v(x,t) = 0 \quad \text{for each} \quad 0 \leq t \leq T_0.$$

Thus, v attains its positive maximum $v_{\max}$ at $(\overline{x}, \overline{t})$ in Q_{T_0}. Thus, by (3.10) we have that

$$v_{\max} = \max_{(x,t)\in Q_{T_0}} \frac{u(x,t)}{1+|x|^2+\gamma t} \geq \frac{u(x^\star,t^\star)}{1+|x^\star|^2+\gamma t^\star} \geq \frac{u_{\rm sup}}{2(1+|x^\star|^2+\gamma t^\star)}.$$

Since $v_{\max} > 0$, we obtain

$$\frac{u_{\rm sup}}{v_{\max}} \leq 2\Big(1+|x^\star|^2+\gamma t^\star\Big). \tag{3.11}$$

Moreover, at the point $(\overline{x}, \overline{t})$ we have $v_t \geq 0$, $\nabla v = 0$ and $\Delta v \leq 0$. Hence, by (3.9) we have at $(\overline{x}, \overline{t})$

$$\begin{aligned}(\delta - c)\Big(1+|\overline{x}|^2+\gamma\overline{t}\Big)v_{\max} + (\gamma-2n)v_{\max} &\leq \alpha\underline{u}(\overline{x},\overline{t})\int_{\mathbb{R}^n} g(\overline{x},y)u(y,\overline{t})dy + 2M\delta e^{-2\delta t} \\ &\leq \alpha\underline{u}(\overline{x},\overline{t})u_{\rm sup}\int_{\mathbb{R}^n} g(\overline{x},y)dy + 2\delta M.\end{aligned}$$

Since $\delta - c \geq 0$ and $v_{\max} > 0$, it follows that

$$(\gamma-2n)v_{\max} \leq \alpha\underline{u}(\overline{x},\overline{t})u_{\rm sup}\int_{\mathbb{R}^n} g(\overline{x},y)dy + 2\delta M.$$

Hence, by (H1), (3.4) and (3.11) we get that

$$\begin{aligned}\gamma - 2n &\leq \alpha\underline{u}(\overline{x},\overline{t})\frac{u_{\rm sup}}{v_{\max}}\int_{\mathbb{R}^n} g(\overline{x},y)dy + \frac{2\delta M}{v_{\max}} \\ &\leq 2\alpha\underline{u}(\overline{x},\overline{t})\Big(1+|x^\star|^2+\gamma t^\star\Big)\int_{\mathbb{R}^n} g(\overline{x},y)dy + \frac{4\delta M\Big(1+|x^\star|^2+\gamma t^\star\Big)}{u_{sup}}\end{aligned}$$

$$\le 2\alpha M\Big(1+|x^\star|^2+\gamma T_0\Big)\int_{\mathbb{R}^n} g(\overline{x},y)dy + \frac{4\delta M\Big(1+|x^\star|^2+\gamma T_0\Big)}{u_{sup}}$$
$$\le 2\alpha M\Big(1+|x^\star|^2+\gamma T_0\Big)\sup_{\overline{x}\in\mathbb{R}^n}\int_{\mathbb{R}^n} g(\overline{x},y)dy + \frac{4\delta M\Big(1+|x^\star|^2+\gamma T_0\Big)}{u_{sup}}$$
$$\le 2\alpha M\Big(1+|x^\star|^2+\gamma T_0\Big)C_g + \frac{4\delta M\Big(1+|x^\star|^2+\gamma T_0\Big)}{u_{sup}}$$
$$= \Big(1+|x^\star|^2\Big)\Big(2\alpha M C_g + \frac{4\delta M}{u_{sup}}\Big) + 2MT_0\gamma\Big(\alpha C_g + \frac{2\delta}{u_{sup}}\Big).$$

This implies that

$$\Big(1-2MT_0\Big(\alpha C_g+\frac{2\delta}{u_{sup}}\Big)\Big)\gamma \le \Big(1+|x^\star|^2\Big)\Big(2\alpha M C_g+\frac{4\delta M}{u_{sup}}\Big)+2n. \tag{3.12}$$

Since $x^\star$ is independent of γ, it follows by taking γ large enough that (3.12) is impossible. Thus, we get a contradiction. Therefore, $u \le 0$ in Q_{T_0} where $T_0 = \min\Big\{T, \frac{R}{4M(\alpha R C_g+2\delta)}\Big\}$ with $R = u_{\text{sup}}$. Since $u = e^{-2\delta t}\tilde{w}$, $w \le \tilde{w} \le 0$ in Σ_{T_0} which means that $\underline{u} \le \overline{u}$ in Σ_{T_0}.

If $T > T_0$ we can continue the process using $t = T_0$ as the initial condition and obtain $\underline{u} \ge \overline{u}$ on $[T_0, 2T_0]$ and so forth. □

As a direct consequence of Theorem 3.1 , we have

Corollary 3.2. *Suppose that* (H1)-(H3) *hold. Let* $\overline{u}$ *and* $\underline{u}$ *be a pair of nonnegative coupled upper and lower classical solutions of* (1.2)*, respectively, and* $\nabla_x\overline{u}, \nabla_x\underline{u} \in C_b(Q_T)$*. Then,* $\overline{u} \ge \underline{u}$ *in* Σ_T*.*

Theorem 3.3. *Suppose that* (H1)-(H3) *hold. Let* $\overline{u}$ *and* $\underline{u}$ *be a pair of nonnegative coupled upper and lower classical solutions of* (1.2)*, respectively. Then,* $\overline{u} \ge \underline{u}$ *in* Σ_T*.*

Proof. The proof uses the same argument as in Theorem 3.1. □

3.2 Local Existence and Uniqueness

For this section, we prove the local existence and uniqueness of solution for the problem (1.2). We first introduce the uniqueness result.

Theorem 3.4. *Suppose that (H1)-(H3) hold. Then, problem* (1.2) *has at most one bounded solution.*

Proof. Suppose that u_1 and u_2 are two bounded solutions of the problem (1.2) in Q_T. Then, for each $i \in \{1, 2\}$, the solution u_i satisfies

$$u_i(x,t) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-s)u_i(x,s)\left(f(u_i(x,s)) - \alpha(g\star u_i(x,s))\right) ds.$$

We let $v = u_1 - u_2$. Then, for any $(x,t) \in Q_T$, v satisfies

$$\begin{aligned} v(x,t) &= \int_0^t \mathcal{G}(t-s)u_1(x,s)\Big(f(u_1(x,s)) - \alpha(g\star u_1(x,s))\Big) ds \\ &\quad - \int_0^t \mathcal{G}(t-s)u_2(x,s)\left(f(u_2(x,s)) - \alpha(g\star u_2(x,s))\right) ds \\ &= \int_0^t \mathcal{G}(t-s)\left(u_1(x,s)f(u_1(x,s)) - u_2(x,s)f(u_2(x,s))\right) ds \\ &\quad - \alpha\int_0^t \mathcal{G}(t-s)\Big(u_1(x,s)\left(g\star u_1(x,s)\right) - u_2(x,s)\left(g\star u_2(x,s)\right)\Big) ds \\ &=: I + II. \end{aligned}$$

First, we consider I. By Lemma (2.13), we get

$$I = \int_0^t \mathcal{G}(t-s)\Big(f(\theta(x,s)) + \theta f'(\theta(x,s))\Big)v(x,s)ds \tag{3.13}$$

where θ is between u_1 and u_2. Next, we consider II. Since

$$\begin{aligned} u_1(g\star u_1) - u_2(g\star u_2) &= u_1(g\star u_1) - u_1(g\star u_2) + u_1(g\star u_2) - u_2(g\star u_2) \\ &= u_1(g\star(u_1 - u_2)) + (u_1 - u_2)(g\star u_2) \\ &= u_1(g\star v) + v(g\star u_2), \end{aligned}$$

we obtain

$$II = -\alpha \int_0^t \mathcal{G}(t-s)\Big(u_1(x,s)(g\star v(x,s)) + v(x,s)(g\star u_2(x,s))\Big)ds. \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\begin{aligned} v(x,t) = &\int_0^t \mathcal{G}(t-s)\Big(f(\theta(x,s)) + \theta f^{'}(\theta(x,s))\Big)v(x,s)ds \\ &\quad - \alpha \int_0^t \mathcal{G}(t-s)\Big(u_1(x,s)(g\star v(x,s)) + v(x,s)(g\star u_2(x,s))\Big)ds \\ = &\int_0^t \mathcal{G}(t-s)\Big(f(\theta(x,s)) + \theta f^{'}(\theta(x,s)) - \alpha g\star u_2(x,s)\Big)v(x,s)ds \\ &\quad - \alpha \int_0^t \mathcal{G}(t-s)\Big(u_1(x,s)(g\star v(x,s))\Big)ds. \end{aligned}$$

Since u_1 and u_2 are bounded, there exists $M > 0$ such that

$$|u_1|, |u_2| \leq M \quad \text{and thus,} \quad |\theta| \leq M. \tag{3.15}$$

By (H1) and (H3), we get that $f(\theta)$ and $f^{'}(\theta)$ are bounded for $\theta \in [-M, M]$, i.e., there exist $L_1, L_2 > 0$ such that

$$|f(\theta)| \leq L \text{ and } \left|f^{'}(\theta)\right| \leq L',$$

and hence,

$$\begin{aligned} \left|f(\theta) + \theta f^{'}(\theta) - \alpha g \star u_2\right| &\leq \left|f(\theta) + \theta f^{'}(\theta)\right| + |\alpha g \star u_2| \\ &\leq |f(\theta)| + \left|\theta f^{'}(\theta)\right| + \alpha M \left|\int_{\mathbb{R}^n} g(x,y)dy\right| \\ &\leq L + ML' + \alpha M C_g =: R. \end{aligned} \tag{3.16}$$

By Lemma 2.10, (3.15) and (3.16), we estimate that

$$\begin{aligned} &\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \left\|\int_0^t \mathcal{G}(t-s)\left(f(\theta(\cdot,s)) + \theta f^{'}(\theta(\cdot,s)) - \alpha g\star u_2(\cdot,s)\right) v(\cdot,s)ds\right\|_{L^\infty(\mathbb{R}^n)} \end{aligned}$$

$$
\begin{aligned}
&\quad + \left\| \alpha \int_0^t \mathcal{G}(t-s)\left(u_1(\cdot,s)(g\star v(\cdot,s))\right) ds \right\|_{L^\infty(\mathbb{R}^n)} \\
&\le \int_0^t \left\| \mathcal{G}(t-s)\left(f(\theta(\cdot,s)) + \theta f'(\theta(\cdot,s)) - \alpha g\star u_2(\cdot,s)\right) v(\cdot,s) \right\|_{L^\infty(\mathbb{R}^n)} ds \\
&\quad + \alpha \int_0^t \left\| \mathcal{G}(t-s)\left(u_1(\cdot,s)(g\star v(\cdot,s))\right) \right\|_{L^\infty(\mathbb{R}^n)} ds \\
&\le \int_0^t \left\| \left(f(\theta(\cdot,s)) + \theta f'(\theta(\cdot,s)) - \alpha g\star u_2(\cdot,s)\right) v(\cdot,s) \right\|_{L^\infty(\mathbb{R}^n)} ds \\
&\quad + \alpha \int_0^t \left\| u_1(\cdot,s)(g\star v(\cdot,s)) \right\|_{L^\infty(\mathbb{R}^n)} ds \\
&\le R \int_0^t \|v(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds + \alpha M \int_0^t \|g\star v(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds \\
&\le R \int_0^t \|v(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds + \alpha M \int_0^t \left\| \int_{\mathbb{R}^n} g(\cdot,y) dy \right\|_{L^\infty(\mathbb{R}^n)} \|v(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds \\
&= \left(R + \alpha M \left\| \int_{\mathbb{R}^n} g(\cdot,y) dy \right\|_{L^\infty(\mathbb{R}^n)} \right) \int_0^t \|v(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds \\
&\le (R + \alpha M C_g) \int_0^t \|v(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds
\end{aligned}
$$

for all $t \in (0,T)$. Thus, by applying Theorem 2.2 (Gronwall's inequality) we get that

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = 0 \text{ for } t \in (0,T).$$

Since v is continuous, $v \equiv 0$. That is $u_1 \equiv u_2$ in Σ_T. $\square$

Theorem 3.5. *Suppose that* (H1)-(H3) *hold and let* $\underline{u}$ *and* $\overline{u}$ *be a pair of nonnegative lower and upper classical solutions of* (1.2)*, respectively. Then, the problem* (1.2) *has a unique solution* u *in* Q_T*, and* u *satisfies*

$$\underline{u} \le u \le \overline{u} \quad \text{in} \quad \Sigma_T.$$

Proof. By Definition 2.11, we have $\overline{u}$ and $\underline{u}$ are bounded and thus, there exists $M > 0$ such that

$$0 \le \underline{u} \le M \quad \text{and} \quad 0 \le \overline{u} \le M \quad \text{in} \quad \Sigma_T. \tag{3.17}$$

By (H3), f and f' are bounded on $[0, M]$, i.e., there exist $L, L' > 0$ such that

$$|f(\varepsilon)| \leq L \quad \text{and} \quad |f^{'}(\varepsilon)| \leq L' \quad \text{for} \quad 0 \leq \varepsilon \leq M. \tag{3.18}$$

Then,

$$\left|f(\varepsilon) + \varepsilon f^{'}(\varepsilon)\right| \leq L + ML' < 2\,(L + ML') =: \beta,$$

or equivalently,

$$\varepsilon f^{'}(\varepsilon) + f(\varepsilon) + \beta > 0 \quad \text{for} \quad 0 \leq \varepsilon \leq M.$$

We divide the proof into three steps.

Step 1. Denote $\underline{u}^0 = \underline{u}$ and $\overline{u}^0 = \overline{u}$.

We then construct sequences $\left\{\underline{u}^k\right\}_{k=0}^{\infty}$ and $\left\{\overline{u}^k\right\}_{k=0}^{\infty}$ by iteratively solving the following linear problem. For $k \in \mathbb{N}$, let $\underline{u}^k$ and $\overline{u}^k$ satisfy

$$\begin{cases} \partial_t \underline{u}^k - \Delta \underline{u}^k = \underline{u}^{k-1} f\left(\underline{u}^{k-1}\right) - \alpha \underline{u}^k \left(g \star \overline{u}^{k-1}\right) - \beta(\underline{u}^k - \underline{u}^{k-1}) & \text{in} \quad Q_T \\ \quad \underline{u}^k\,(x, 0) = u_0\,(x) \quad \text{on} \quad \mathbb{R}^n \end{cases} \tag{3.19}$$

and

$$\begin{cases} \partial_t \overline{u}^k - \Delta \overline{u}^k = \overline{u}^{k-1} f\left(\overline{u}^{k-1}\right) - \alpha \overline{u}^k \left(g \star \underline{u}^{k-1}\right) - \beta(\overline{u}^k - \overline{u}^{k-1}) & \text{in} \quad Q_T \\ \quad \overline{u}^k\,(x, 0) = u_0\,(x) \quad \text{on} \quad \mathbb{R}^n. \end{cases} \tag{3.20}$$

Then, $\underline{u}^k$ and $\overline{u}^k$ are well-defined, since the existence and uniqueness of (3.19) and (3.20) are guaranteed by [9]. We also note that $\underline{u}^k$ and $\overline{u}^k$ are classical solutions.

Moreover, we consider the linear problem (3.19) and let

$$\mathcal{A}_k(t) := \Delta - F_k(x, t)I$$

be a linear operator where $F_k(x, t) \equiv \alpha\left(g \star \overline{u}^{k-1}\right) + \beta$ and I represents the identity operator, i.e., $Iv = v$. Then, we can write (3.19) as

$$\begin{cases} \partial_t \underline{u}^k - \mathcal{A}_k(t)\underline{u}^k = \underline{u}^{k-1} f\left(\underline{u}^{k-1}\right) + \beta \underline{u}^{k-1} & \text{in} \quad Q_T \\ \underline{u}^k(x,0) = u_0(x) \quad \text{on} \quad \mathbb{R}^n. \end{cases}$$

Therefore, the solution of problem (3.19) satisfies

$$\underline{u}^k(x,t) = S_k(t,0)u_0 + \int_0^t S_k(t,s)\Big(\underline{u}^{k-1}(x,s)\left(f\left(\underline{u}^{k-1}(x,s)\right) + \beta\right)\Big)ds$$

where the solution operator S_k is given by

$$S_k(t_2,t_1) = e^{\int_{t_1}^{t_2} \mathcal{A}_k(\tau)d\tau} = e^{(t_2-t_1)\Delta - \int_{t_1}^{t_2} F_k(x,\tau)d\tau} \quad \text{and} \quad F_k(x,t) \equiv \alpha\left(g \star \overline{u}^{k-1}\right) + \beta.$$

We can check that the operator S_k has the following properties:

(i) $S_k(t,t) = I$;

(ii) $S_k(t,r)S_k(r,0) = e^{(t-r)\Delta - \int_r^t F_k(x,\tau)d\tau} e^{r\Delta - \int_0^r F_k(x,\tau)d\tau} = e^{t\Delta - \int_0^t F_k(x,\tau)d\tau} = S_k(t,0)$;

(iii) $\dfrac{\partial S_k(t,r)}{\partial t} = e^{(t-r)\Delta - \int_r^t F_k(x,\tau)d\tau}\left(\Delta - F_k(x,t)I\right) = S_k(t,r)\mathcal{A}_k(t)$.

Furthermore, we use these properties of S_k to get that

$$\begin{aligned} \frac{d\underline{u}^k}{dt} &= \frac{\partial S_k(t,0)}{\partial t}u_0 + \underline{u}^{k-1}\left(f\left(\underline{u}^{k-1}\right) + \beta\right) + \int_0^t \frac{\partial S_k(t,s)}{\partial t}\Big(\underline{u}^{k-1}\left(f\left(\underline{u}^{k-1}\right) + \beta\right)\Big)ds \\ &= \mathcal{A}_k(t)\Big(S_k(t,0)u_0 + \int_0^t S_k(t,s)\Big(\underline{u}^{k-1}\left(f\left(\underline{u}^{k-1}\right) + \beta\right)\Big)ds\Big) + \underline{u}^{k-1}\left(f\left(\underline{u}^{k-1}\right) + \beta\right) \\ &= \mathcal{A}_k(t)u^k + \underline{u}^{k-1}\left(f\left(\underline{u}^{k-1}\right) + \beta\right) \end{aligned}$$

and $\underline{u}^k(x,0) = S_k(0,0)u_0 = u_0$. By (3.17) and (3.18), we get

$$0 \le F_1(x,t) \equiv \alpha\left(g \star \overline{u}^0\right) + \beta \le \alpha C_g M + \beta$$

and then for $v \in L^\infty(Q_T)$ and $t_1 \le t_2$

$$\|S_1(t_2,t_1)v\|_{L^\infty(Q_T)} = \|e^{(t_2-t_1)\Delta - \int_{t_1}^{t_2} F_1(x,\tau)d\tau}v\|_{L^\infty(Q_T)} \le \|e^{(t_2-t_1)\Delta}v\|_{L^\infty(Q_T)} \le \|v\|_{L^\infty(Q_T)}$$

because $e^{(t_2-t_1)\Delta}v = \mathcal{G}(t_2-t_1)v$. Therefore, for $k=1$

$$\begin{aligned}|\underline{u}^1| &\le |e^{t\Delta - \int_0^t F_1(x,\tau)d\tau}u_0| + \int_0^t \left|e^{(t-s)\Delta - \int_s^t F_1(x,\tau)d\tau}\Big(\underline{u}^0\left(f\left(\underline{u}^0\right)+\beta\right)\Big)\right|ds \\ &\le M + M(L+\beta)t\end{aligned}$$

which yields

$$\|\underline{u}^1\|_{L^\infty(Q_T)} \le M + (L+\beta)T < \infty.$$

This shows that $\underline{u}^1$ is bounded on Σ_T. Similarly, for the linear problem (3.20), we obtain $\overline{u}^1$ is bounded on Σ_T.

Step 2. We show that $\{\underline{u}^k\}_{k=0}^\infty$ and $\{\overline{u}^k\}_{k=0}^\infty$ satisfy the following monotone property:

$$\underline{u}^0 \le \underline{u}^k \le \underline{u}^{k+1} \le \overline{u}^{k+1} \le \overline{u}^k \le \overline{u}^0 \quad \text{in} \quad \Sigma_T \tag{3.21}$$

for $k \in \mathbb{N}$.

Let us begin to show that (3.21) holds if $k=1$, i.e.,

$$\underline{u}^0 \le \underline{u}^1 \le \overline{u}^1 \le \overline{u}^0 \quad \text{in} \quad \Sigma_T. \tag{3.22}$$

Step 2.1 To show that

$$\underline{u}^0 \le \underline{u}^1 \quad \text{in} \quad \Sigma_T,$$

let $v = \underline{u}^0 - \underline{u}^1$. Then, v satisfies

$$\begin{aligned}\partial_t v - \Delta v &= (\partial_t \underline{u}^0 - \Delta \underline{u}^0) \ - (\partial_t \underline{u}^1 - \Delta \underline{u}^1) \\ &= (\partial_t \underline{u}^0 - \Delta \underline{u}^0) \ - \Big(\underline{u}^0 f(\underline{u}^0) - \alpha \underline{u}^1\left(g \star \overline{u}^0\right) - \beta(\underline{u}^1 - \underline{u}^0)\Big) \\ &= \Big(\partial_t \underline{u} - \Delta \underline{u} - \underline{u} f(\underline{u})\Big) + \alpha \underline{u}^1\left(g \star \overline{u}^0\right) - \beta(\underline{u}^0 - \underline{u}^1) \\ &\le -\alpha \underline{u}\left(g \star \overline{u}\right) + \alpha \underline{u}^1\left(g \star \overline{u}^0\right) - \beta(\underline{u}^0 - \underline{u}^1)\end{aligned}$$

$$
\begin{aligned}
&= -\alpha \underline{u}^0 \left(g \star \overline{u}^0\right) + \alpha \underline{u}^1 \left(g \star \overline{u}^0\right) - \beta(\underline{u}^0 - \underline{u}^1) \\
&= -\alpha(\ \underline{u}^0 - \underline{u}^1)\ \left(g \star \overline{u}^0\right) - \beta(\underline{u}^0 - \underline{u}^1) \\
&= -\Big(\alpha g \star \overline{u}^0 + \beta\Big)(\ \underline{u}^0 - \underline{u}^1) \\
&= -\Big(\alpha g \star \overline{u}^0 + \beta\Big) v \quad \text{in} \quad Q_T
\end{aligned}
$$

and

$$
v(x,0) = \underline{u}^0(x,0) - \underline{u}^1(x,0) = \underline{u}(x,0) - u_0(x) \leq 0 \qquad \text{on} \quad \mathbb{R}^n.
$$

Thus, by Lemma 2.15, we obtain $v \leq 0$ in Σ_T which means that

$$
\underline{u}^0 \leq \underline{u}^1 \quad \text{in} \quad \Sigma_T.
$$

Step 2.2 We show that

$$
\overline{u}^1 \leq \overline{u}^0 \quad \text{in} \quad \Sigma_T.
$$

Let $w = \overline{u}^1 - \overline{u}^0$. Then, w satisfies

$$
\begin{aligned}
\partial_t w - \Delta w &= (\partial_t \overline{u}^1 - \Delta \overline{u}^1)\ - (\partial_t \overline{u}^0 - \Delta \overline{u}^0) \\
&= \Big(\overline{u}^0 f(\overline{u}^0) - \alpha \overline{u}^1 \left(g \star \underline{u}^0\right) - \beta(\overline{u}^1 - \overline{u}^0)\Big) - (\partial_t \overline{u}^0 - \Delta \overline{u}^0) \\
&= -\Big(\partial_t \overline{u} - \Delta \overline{u} - \overline{u} f(\overline{u})\Big) - \alpha \overline{u}^1 \left(g \star \underline{u}^0\right) - \beta(\overline{u}^1 - \overline{u}^0) \\
&\leq \alpha \overline{u}\,(g \star \underline{u}) - \alpha \overline{u}^1 \left(g \star \underline{u}^0\right) - \beta(\overline{u}^1 - \overline{u}^0) \\
&= \alpha \overline{u}^0 \left(g \star \underline{u}^0\right) - \alpha \overline{u}^1 \left(g \star \underline{u}^0\right) - \beta(\overline{u}^1 - \overline{u}^0) \\
&= -\alpha(\ \overline{u}^1 - \overline{u}^0) \left(g \star \underline{u}^0\right) - \beta(\overline{u}^1 - \overline{u}^0) \\
&= -\Big(\alpha g \star \underline{u}^0 + \beta\Big)(\ \overline{u}^1 - \overline{u}^0) \\
&= -\Big(\alpha g \star \underline{u}^0 + \beta\Big) w \quad \text{in} \quad Q_T
\end{aligned}
$$

and

$$w(x,0) = \overline{u}^1(x,0) - \overline{u}^0(x,0) = u_0(x) - \overline{u}(x,0) \leq 0 \qquad \text{on} \quad \mathbb{R}^n.$$

Thus, by Lemma 2.15, we obtain $w \leq 0$ in Σ_T and hence,

$$\overline{u}^1 \leq \overline{u}^0 \quad \text{in} \quad \Sigma_T.$$

Step 2.3 Now, we claim that

$$\underline{u}^1 \leq \overline{u}^1 \quad \text{in} \quad \Sigma_T.$$

Let $u^1 = \underline{u}^1 - \overline{u}^1$. Then, u^1 satisfies

$$\begin{aligned}
&\partial_t u^1 - \Delta u^1 \\
&\quad = (\partial_t \underline{u}^1 - \Delta \underline{u}^1) - (\partial_t \overline{u}^1 - \Delta \overline{u}^1) \\
&\quad = \Big(\underline{u}^0 f(\underline{u}^0) - \alpha \underline{u}^1 (g \star \overline{u}^0) - \beta(\underline{u}^1 - \underline{u}^0)\Big) - \Big(\overline{u}^0 f(\overline{u}^0) - \alpha \overline{u}^1 (g \star \underline{u}^0) - \beta(\overline{u}^1 - \overline{u}^0)\Big) \\
&\quad = \Big(\underline{u}^0 f(\underline{u}^0) - \overline{u}^0 f(\overline{u}^0)\Big) - \beta(\underline{u}^1 - \underline{u}^0) + \beta(\overline{u}^1 - \overline{u}^0) - \alpha \underline{u}^1 (g \star \overline{u}^0) + \alpha \overline{u}^1 (g \star \underline{u}^0) \\
&\quad = \Big(\tilde{\theta} f'(\tilde{\theta}) + f(\tilde{\theta})\Big)(\underline{u}^0 - \overline{u}^0) - \beta(\underline{u}^1 - \overline{u}^1) + \beta(\underline{u}^0 - \overline{u}^0) \\
&\qquad\quad - \alpha \underline{u}^1 (g \star \overline{u}^0) + \alpha \overline{u}^1 (g \star \overline{u}^0) - \alpha \overline{u}^1 (g \star \overline{u}^0) + \alpha \overline{u}^1 (g \star \underline{u}^0) \\
&\quad = \Big(\tilde{\theta} f'(\tilde{\theta}) + f(\tilde{\theta}) + \beta\Big)(\underline{u}^0 - \overline{u}^0) - \beta(\underline{u}^1 - \overline{u}^1) \\
&\qquad\quad - \alpha(\underline{u}^1 - \overline{u}^1)\left(g \star \overline{u}^0\right) + \alpha \overline{u}^1 g \star (\underline{u}^0 - \overline{u}^0) \\
&\quad \leq \Big(\tilde{\theta} f'(\tilde{\theta}) + f(\tilde{\theta}) + \beta\Big)(\underline{u}^0 - \overline{u}^0) - \beta u^1 - \alpha u^1 (g \star \overline{u}^0) \\
&\quad \leq -\Big(\alpha g \star \overline{u}^0 + \beta\Big) u^1 \quad \text{in} \quad Q_T
\end{aligned}$$

and

$$u^1(x,0) = \underline{u}^1(x,0) - \overline{u}^1(x,0) = u_0 - u_0 \leq 0 \qquad \text{on} \quad \mathbb{R}^n,$$

where $\tilde{\theta}$ is between $\underline{u}^0$ and $\overline{u}^0$. By Lemma 2.15, we get that $u^1 \leq 0$ in Σ_T and hence,

$$\underline{u}^1 \leq \overline{u}^1 \quad \text{in} \quad \Sigma_T.$$

From Steps 2.1-2.3, we conclude that (3.22) holds.

Step 2.4 Now, we show that $\underline{u}^1$ and $\overline{u}^1$ are coupled lower and upper classical solutions of (1.2), respectively. From (3.22), we obtain that

$$\begin{aligned}
\partial_t \underline{u}^1 - \Delta \underline{u}^1 &= \underline{u}^0 f(\underline{u}^0) - \alpha \underline{u}^1 (g \star \overline{u}^0) - \beta(\underline{u}^1 - \underline{u}^0) \\
&\leq \underline{u}^0 f(\underline{u}^0) - \alpha \underline{u}^1 (g \star \overline{u}^1) - \beta(\underline{u}^1 - \underline{u}^0) \\
&= \underline{u}^0 f(\underline{u}^0) - \underline{u}^1 f(\underline{u}^1) + \underline{u}^1 f(\underline{u}^1) - \alpha \underline{u}^1 (g \star \overline{u}^1) - \beta(\underline{u}^1 - \underline{u}^0) \\
&= (\underline{\theta} f^{'}(\underline{\theta}) + f(\underline{\theta}) + \beta)(\underline{u}^0 - \underline{u}^1) + \underline{u}^1 f(\underline{u}^1) - \alpha \underline{u}^1 (g \star \overline{u}^1) \\
&\leq \underline{u}^1 f(\underline{u}^1) - \alpha \underline{u}^1 (g \star \overline{u}^1) \quad \text{in} \quad Q_T
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \overline{u}^1 - \Delta \overline{u}^1 &= \overline{u}^0 f(\overline{u}^0) - \alpha \overline{u}^1 (g \star \underline{u}^0) - \beta(\overline{u}^1 - \overline{u}^0) \\
&\geq \overline{u}^0 f(\overline{u}^0) - \alpha \overline{u}^1 (g \star \underline{u}^1) - \beta(\overline{u}^1 - \overline{u}^0) \\
&= \overline{u}^0 f(\overline{u}^0) - \overline{u}^1 f(\overline{u}^1) + \overline{u}^1 f(\overline{u}^1) - \alpha \overline{u}^1 (g \star \underline{u}^1) - \beta(\overline{u}^1 - \overline{u}^0) \\
&= (\overline{\theta} f^{'}(\overline{\theta}) + f(\overline{\theta}) + \beta)(\overline{u}^0 - \overline{u}^1) + \overline{u}^1 f(\overline{u}^1) - \alpha \overline{u}^1 (g \star \underline{u}^1) \\
&\geq \overline{u}^1 f(\overline{u}^1) - \alpha \overline{u}^1 (g \star \underline{u}^1) \quad \text{in} \quad Q_T,
\end{aligned}$$

where $\underline{\theta}$ is between $\underline{u}^0$ and $\underline{u}^1$, and $\overline{\theta}$ is between $\overline{u}^0$ and $\overline{u}^1$.

Next, we assume that for some $k > 1$, $\underline{u}^k$ and $\overline{u}^k$ are coupled lower and upper classical solutions of (1.2), respectively. Continuing the above process, we can show that

$$\underline{u}^k \leq \underline{u}^{k+1} \leq \overline{u}^{k+1} \leq \overline{u}^k \quad \text{in} \quad \Sigma_T$$

and we have that $\underline{u}^{k+1}$ and $\overline{u}^{k+1}$ are also coupled lower and upper classical solutions of (1.2), respectively. By induction, we conclude that

$$\underline{u}^0 \leq \underline{u}^k \leq \underline{u}^{k+1} \leq \overline{u}^{k+1} \leq \overline{u}^k \leq \overline{u}^0 \quad \text{in} \quad \Sigma_T$$

for $k \in \{0, 1, 2, \ldots\}$. This shows that

$$\underline{u}^k \leq M \quad \text{and} \quad \overline{u}^k \leq M \quad \text{for all} \quad k \geq 0$$

Step 3. We show the existence and uniqueness of solution (1.2).
Since the sequences $\left\{\underline{u}^k\right\}_{k=0}^{\infty}$ and $\left\{\overline{u}^k\right\}_{k=0}^{\infty}$ are monotone and bounded, there exist two functions $\underline{U}$ and $\overline{U}$ such that

$$\lim_{k\to\infty} \underline{u}^k = \underline{U} \quad \text{and} \quad \lim_{k\to\infty} \overline{u}^k = \overline{U}$$

pointwise on Q_T. Since $\underline{u}^k \leq \overline{u}^k$ for all $k \geq 1$, we take $k \to \infty$ to obtain

$$\underline{U} \leq \overline{U} \quad \text{in } Q_T,$$

and hence,

$$\underline{u}^0 \leq \underline{U} \leq \overline{U} \leq \overline{u}^0 \quad \text{in } Q_T.$$

We now show that $\underline{U} = \overline{U}$. Let $W = \overline{U} - \underline{U}$. Since $\underline{U} \leq \overline{U}$, $W \geq 0$. By the Duhamel's principle, we get that the solutions of problems (3.19) and (3.20) are given by the integral representations

$$\underline{u}^k = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\Big(\underline{u}^{k-1} f\left(\underline{u}^{k-1}\right) - \alpha \underline{u}^k \left(g \star \overline{u}^{k-1}\right) - \beta\left(\underline{u}^k - \underline{u}^{k-1}\right)\Big) ds$$

and

$$\overline{u}^k = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\Big(\overline{u}^{k-1} f\left(\overline{u}^{k-1}\right) - \alpha \overline{u}^k \left(g \star \underline{u}^{k-1}\right) - \beta\left(\overline{u}^k - \overline{u}^{k-1}\right)\Big) ds.$$

Since for each $k \geq 1$

$$\begin{aligned}
&\Big|\Phi(x-y,t-s)\Big(\underline{u}^{k-1} f\left(\underline{u}^{k-1}\right) - \alpha \underline{u}^{k}\left(g \star \overline{u}^{k-1}\right) - \beta\left(\underline{u}^{k} - \underline{u}^{k-1}\right)\Big)\Big| \\
&\qquad \leq (ML + \alpha M^2 C_g + 2\beta M)\Phi(x-y,t-s) =: g(x-y,t-s)
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\Phi(x-y,t-s)\Big(\overline{u}^{k-1} f\left(\overline{u}^{k-1}\right) - \alpha \overline{u}^{k}\left(g \star \underline{u}^{k-1}\right) - \beta\left(\overline{u}^{k} - \overline{u}^{k-1}\right)\Big)\Big| \\
&\qquad \leq (ML + \alpha M^2 C_g + 2\beta M)\Phi(x-y,t-s) =: g(x-y,t-s),
\end{aligned}$$

we obtain

$$\int_0^t \int_{\mathbb{R}^n} g(x-y,t-s)dyds = (ML + \alpha M^2 C_g + 2\beta M)t < \infty.$$

Applying Theorem 2.3 (Dominated Convergent Theorem), we get that

$$\begin{aligned}
\underline{U} &= \mathcal{G}(t)u_0 + \lim_{k\to\infty}\int_0^t \mathcal{G}(t-s)\Big(\underline{u}^{k-1} f\left(\underline{u}^{k-1}\right) - \alpha \underline{u}^{k}\left(g \star \overline{u}^{k-1}\right) - \beta\left(\underline{u}^{k} - \underline{u}^{k-1}\right)\Big)ds \\
&= \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\Big(\underline{U} f\left(\underline{U}\right) - \alpha \underline{U}\left(g \star \overline{U}\right) - \beta\left(\underline{U} - \underline{U}\right)\Big)ds \\
&= \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\underline{U}\Big(f\left(\underline{U}\right) - \alpha g \star \overline{U}\Big)ds,
\end{aligned}$$

and

$$\begin{aligned}
\overline{U} &= \mathcal{G}(t)u_0 + \lim_{k\to\infty}\int_0^t \mathcal{G}(t-s)\Big(\overline{u}^{k-1} f\left(\overline{u}^{k-1}\right) - \alpha \overline{u}^{k}\left(g \star \underline{u}^{k-1}\right) - \beta\left(\overline{u}^{k} - \overline{u}^{k-1}\right)\Big)ds \\
&= \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\Big(\overline{U} f\left(\overline{U}\right) - \alpha \overline{U}\left(g \star \underline{U}\right) - \beta\left(\overline{U} - \overline{U}\right)\Big)ds \\
&= \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\overline{U}\Big(f\left(\overline{U}\right) - \alpha g \star \underline{U}\Big)ds.
\end{aligned}$$

Next, we claim that $\underline{U}$ and $\overline{U}$ are continuous on $[0.T)$. For any $t, t' \in [0,T)$, we have

$$
\begin{aligned}
&\|\underline{U}(\cdot,t)-\underline{U}(\cdot,t')\|_{L^\infty(\mathbb{R}^n)}\\
&\le \left\|\mathcal{G}(t)u_0-\mathcal{G}(t')u_0\right\|_{L^\infty(\mathbb{R}^n)}\\
&\quad+\left\|\int_0^t \mathcal{G}(t-s)\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)ds\right.\\
&\qquad\left.-\int_0^{t'} \mathcal{G}(t'-s)\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)ds\right\|_{L^\infty(\mathbb{R}^n)}\\
&=\left\|\{\mathcal{G}(t)-\mathcal{G}(t')\}\,u_0\right\|_{L^\infty(\mathbb{R}^n)}\\
&\quad+\left\|\int_0^{t'} \mathcal{G}(t-s)\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)ds\right.\\
&\qquad+\int_{t'}^{t} \mathcal{G}(t-s)\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)ds\\
&\qquad\left.-\int_0^{t'} \mathcal{G}(t'-s)\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)ds\right\|_{L^\infty(\mathbb{R}^n)}\\
&=\left\|\{\mathcal{G}(t)-\mathcal{G}(t')\}\,u_0\right\|_{L^\infty(\mathbb{R}^n)}\\
&\quad+\left\|\int_0^{t'} \{\mathcal{G}(t-s)-\mathcal{G}(t'-s)\}\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)ds\right.\\
&\qquad\left.+\int_{t'}^{t} \mathcal{G}(t-s)\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)ds\right\|_{L^\infty(\mathbb{R}^n)}\\
&\le \left\|\{\mathcal{G}(t)-\mathcal{G}(t')\}u_0\right\|_{L^\infty(\mathbb{R}^n)}\\
&\quad+\int_0^{t'}\left\|\{\mathcal{G}(t-s)-\mathcal{G}(t'-s)\}\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)\right\|_{L^\infty(\mathbb{R}^n)}ds\\
&\quad+\int_{t'}^{t}\left\|\mathcal{G}(t-s)\underline{U}(\cdot,s)\Big(f\left(\underline{U}(\cdot,s)\right)-\alpha g\star\overline{U}(\cdot,s)\Big)\right\|_{L^\infty(\mathbb{R}^n)}ds.
\end{aligned}
$$

Taking $t\to t'$, we obtain

$$\|\underline{U}(\cdot,t)-\underline{U}(\cdot,t')\|_{L^\infty(\mathbb{R}^n)}\to 0.$$

This implies that $\underline{U}$ is continuous on $[0,T)$. Similarly, we can show that $\overline{U}$ is continuous on $[0,T)$. Then, W satisfies

$$\begin{aligned}
W &= \int_0^t \mathcal{G}(t-s)\left[\overline{U}f\left(\overline{U}\right) - \underline{U}f\left(\underline{U}\right) - \alpha\overline{U}\left(g\star\underline{U}\right) + \alpha\underline{U}\left(g\star\overline{U}\right)\right] ds \\
&= \int_0^t \mathcal{G}(t-s)\Big[\left(\rho f^{'}(\rho) + f(\rho)\right)\left(\overline{U} - \underline{U}\right) \\
&\qquad - \alpha\overline{U}\left(g\star\underline{U}\right) + \alpha\underline{U}\left(g\star\underline{U}\right) - \alpha\underline{U}\left(g\star\underline{U}\right) + \alpha\underline{U}\left(g\star\overline{U}\right)\Big] ds \\
&= \int_0^t \mathcal{G}(t-s)\left[\left(\rho f^{'}(\rho) + f(\rho)\right)W - \alpha W\left(g\star\underline{U}\right) + \alpha\underline{U}\left(g\star W\right)\right] ds \quad \text{in} \quad Q_T,
\end{aligned}$$

where ρ is between $\overline{U}$ and $\underline{U}$. We then estimate

$$\begin{aligned}
&\|W(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \\
&= \Bigg\| \int_0^t \mathcal{G}(t-s)\Big[\left(\rho f^{'}(\rho) + f(\rho)\right)W(\cdot,s) \\
&\qquad -\alpha W(\cdot,s)\left(g\star\underline{U}(\cdot,s)\right) + \alpha\underline{U}(\cdot,s)\left(g\star W(\cdot,s)\right)\Big] ds\Bigg\|_{L^\infty(\mathbb{R}^n)} \\
&\le \int_0^t \Big\|\mathcal{G}(t-s)\Big[\left(\rho f^{'}(\rho) + f(\rho)\right)W(\cdot,s) \\
&\qquad -\alpha W(\cdot,s)\left(g\star\underline{U}(\cdot,s)\right) + \alpha\underline{U}(\cdot,s)\left(g\star W(\cdot,s)\right)\Big]\Big\|_{L^\infty(\mathbb{R}^n)} ds \\
&\le \int_0^t \Big\|\mathcal{G}(t-s)\left(\rho f^{'}(\rho) + f(\rho) - \alpha g\star\underline{U}(\cdot,s)\right)W(\cdot,s)\Big\|_{L^\infty(\mathbb{R}^n)} ds \\
&\quad + \int_0^t \Big\|\mathcal{G}(t-s)\alpha\underline{U}(\cdot,s)\left(g\star W(\cdot,s)\right)\Big\|_{L^\infty(\mathbb{R}^n)} ds \\
&\le \int_0^t \Big\|\left(\rho f^{'}(\rho) + f(\rho) - \alpha g\star\underline{U}(\cdot,s)\right)W(\cdot,s)\Big\|_{L^\infty(\mathbb{R}^n)} ds \\
&\qquad + \int_0^t \Big\|\alpha\underline{U}(\cdot,s)\left(g\star W(\cdot,s)\right)\Big\|_{L^\infty(\mathbb{R}^n)} ds \\
&\le (\beta + \alpha M C_g)\int_0^t \|W(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds + \alpha M\int_0^t \|g\star W(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds \\
&\le (\beta + \alpha M C_g)\int_0^t \|W(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds + \alpha M C_g\int_0^t \|W(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds \\
&= (2\alpha M C_g + \beta)\int_0^t \|W(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} ds.
\end{aligned}$$

By Theorem 2.2 (Gronwall's inequality), we obtian

$$\|W(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = 0 \quad \text{for} \quad t \in [0,T).$$

Since W is continuous, $W \equiv 0$ which implies that $\overline{U} = \underline{U}$ in Σ_T. Therefore, for any $(x,t) \in Q_T$

$$\begin{aligned}\underline{U}(x,t) &= \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-s)\underline{U}(x,s)\Big(f\left(\underline{U}(x,s)\right) - \alpha \int_{\mathbb{R}^n} g \star \overline{U}(x,s)\Big)ds \\ &= \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-s)\underline{U}(x,s)\Big(f\left(\underline{U}(x,s)\right) - \alpha \int_{\mathbb{R}^n} g \star \underline{U}(x,s)\Big)ds.\end{aligned}$$

We then define $u = \overline{U} = \underline{U}$ and hence, we can find that u is a solution of (1.2) and satisfies

$$\underline{u} \leq u \leq \overline{u} \quad in \quad \Sigma_T.$$

Moreover, we conclude that u is the unique bounded solution of (1.2), by Theorem 3.4. □

CHAPTER IV
MAIN RESULT II

In this chapter, our aim is to prove the global existence result of problem (1.2). We begin this chapter by studying a nonlinear ordinary differential equation.

4.1 Nonlinear Ordinary Differential Equation

Lemma 4.1. *Assume that* $\varphi := \varphi(t)$ *is a differentiable function which is a solution of the following problem:*

$$\begin{cases} \dfrac{d\varphi}{dt} = \varphi f(\varphi) \quad \textit{for} \quad 0 < t < T, \\ \varphi(0) = \varphi_0, \end{cases} \tag{4.1}$$

where φ_0 *is a constant. Then,* φ *satisfies*

$$\varphi(t) = \varphi_0 + \int_0^t \varphi(s) f(\varphi(s)) ds \quad \textit{for} \quad 0 \leq t < T. \tag{4.2}$$

Proof. Integrating both sides of (4.1) with respect to s, from 0 to t, we obtain

$$\int_0^t \frac{d\varphi}{ds} ds = \varphi(t) - \varphi(0) = \int_0^t \varphi(s) f(\varphi(s)) ds,$$

and hence, φ satisfies (4.2). □

Theorem 4.2 (Local Existence and Uniqueness). *Assume that the following assumptions hold:*

(i) There exists $M > 0$ *such that* $|f(\varphi)| \leq M$ *for all* $\varphi \in C([0, T])$,

(ii) f is Lipschitz-continuous on $\mathbb{R}$, i.e., there exists a constant $L > 0$ such that

$$|f(u) - f(v)| \leq L|u - v|$$

for all $u, v \in \mathbb{R}$.

Let $R > 0$ be a constant. If

$$T \leq \min\left\{\frac{R}{M(R + \|\varphi_0\|)}, \frac{1}{2L(R + \|\varphi_0\| + M/L)}\right\},$$

then the problem (4.1) *has a unique solution on the interval* $[0, T]$.

Proof. Let $X = C([0, T])$ be the Banach space of all continuous functions φ on $[0, T]$ with the norm

$$\|\varphi\| = \sup_{0 \leq t \leq T} |\varphi(t)| < \infty.$$

For $R > 0$, let

$$\overline{B}_R(\varphi_0) = \{\varphi \in X : \|\varphi - \varphi_0\| \leq R\}.$$

Then, $\overline{B}_R(\varphi_0)$ is a complete metric space, since it is a closed subset of the Banach space $(X, \|\cdot\|)$. We define a map $\mathcal{A} : \overline{B}_R(\varphi_0) \to X$ as follows

$$\mathcal{A}\varphi(t) = \varphi_0 + \int_0^t \varphi(s) f(\varphi(s)) ds.$$

First, we show that $\mathcal{A}$ is well-defined on $\overline{B}_R(\varphi_0)$, that is $\mathcal{A}\varphi \in X$ for $\varphi \in \overline{B}_R(\varphi_0)$. For $\varphi \in \overline{B}_R(\varphi_0) \subset X$, we have

$$\begin{aligned} |\mathcal{A}\varphi(t)| &\leq |\varphi_0| + \int_0^t |\varphi(s) f(\varphi(s))| ds \\ &\leq |\varphi_0| + M \int_0^t |\varphi(s)| ds \leq |\varphi_0| + M\|\varphi\| t, \end{aligned}$$

and thus,

$$\|\mathcal{A}\varphi(t)\| \leq \|\varphi_0\| + M\|\varphi\|T < \infty.$$

Moreover, for any $t, t' \in [0, T]$, we have

$$\|\mathcal{A}\varphi(t) - \mathcal{A}\varphi(t')\| = \left\| \int_0^t \varphi(s)f(\varphi(s))ds - \int_0^{t'} \varphi(s)f(\varphi(s))ds \right\| \longrightarrow 0$$

as $t' \to t$. Then, $\mathcal{A}$ is well-defined on $\overline{B}_R(\varphi_0)$.

Next, we show that $\mathcal{A}$ is a self-map on $\overline{B}_R(\varphi_0)$. For $\varphi \in \overline{B}_R(\varphi_0)$, we have

$$\begin{aligned} |\mathcal{A}\varphi(t) - \varphi_0| &\leq \int_0^t |\varphi(s)f(\varphi(s))|ds \\ &\leq M \int_0^t |\varphi(s)|ds \\ &\leq M \int_0^t \|\varphi\|ds \\ &= M \int_0^t \|\varphi - \varphi_0 + \varphi_0\|ds \\ &\leq M(\|\varphi - \varphi_0\| + \|\varphi_0\|)t, \end{aligned}$$

and we have by the assumption on T that

$$\|\mathcal{A}\varphi(t) - \varphi_0\| \leq M(R + \|\varphi_0\|)T \leq R.$$

Finally, we show that $\mathcal{A}$ is a contraction map. For any $\varphi_1, \varphi_2 \in B_R(\varphi_0)$, we have

$$\begin{aligned} &|\mathcal{A}\varphi_1 - \mathcal{A}\varphi_2| \\ &\quad \leq \int_0^t |\varphi_1(s)f(\varphi_1(s)) - \varphi_2(s)f(\varphi_2(s))|ds \\ &\quad = \int_0^t |\varphi_1(s)f(\varphi_1(s)) - \varphi_1(s)f(\varphi_2(s)) + \varphi_1(s)f(\varphi_2(s)) - \varphi_2(s)f(\varphi_2(s))|ds \end{aligned}$$

$$\le \int_0^t \left(|\varphi_1(s)f(\varphi_1(s)) - \varphi_1(s)f(\varphi_2(s))| + |\varphi_1(s)f(\varphi_2(s)) - \varphi_2(s)f(\varphi_2(s))| \right) ds$$
$$\le \int_0^t \left(|\varphi_1(s)||f(\varphi_1(s)) - f(\varphi_2(s))| + |\varphi_1(s) - \varphi_2(s)||f(\varphi_2(s))| \right) ds$$
$$\le \int_0^t |\varphi_1(s) - \varphi_2(s)| \left(L|\varphi_1(s)| + M \right) ds$$
$$\le \|\varphi_1 - \varphi_2\| \int_0^t \left(L\|\varphi_1\| + M \right) ds$$
$$= \|\varphi_1 - \varphi_2\| L \left(\|\varphi_1 - \varphi_0 + \varphi_0\| + \frac{M}{L} \right) t$$
$$\le \|\varphi_1 - \varphi_2\| L \left(\|\varphi_1 - \varphi_0\| + \|\varphi_0\| + \frac{M}{L} \right) t$$
$$\le \|\varphi_1 - \varphi_2\| L \left(R + \|\varphi_0\| + \frac{M}{L} \right) t$$

and hence,

$$\|\mathcal{A}\varphi_1 - \mathcal{A}\varphi_2\| \le \|\varphi_1 - \varphi_2\| L \left(R + \|\varphi_0\| + \frac{M}{L} \right) T \le \frac{1}{2}\|\varphi_1 - \varphi_2\|.$$

By Theorem 2.4 (Contraction Mapping Theorem), $\mathcal{A}$ has a unique fixed point, i.e., there exists exactly one $\varphi^* \in \overline{B}_R(\varphi_0)$ such that

$$\mathcal{A}\varphi^* = \varphi^*,$$

that is

$$\varphi^*(t) = \varphi_0 + \int_0^t \varphi^*(s) f(\varphi^*(s)) ds.$$

This means that $\varphi^*(t)$ is the unique solution of the problem (4.1). □

Lemma 4.3. *Assume that $\varphi \in C^1([0, T))$ satisfies*

$$\varphi(t) = \varphi_0 + \int_0^t \varphi(s) f(\varphi(s)) ds.$$

Then, φ is a solution of the problem (4.1).

Proof. Substituting $t = 0$, we get $\varphi(t) = \varphi_0$. Since $\varphi(t)$ is continuous, the integrand $\varphi(s)f(\varphi(s))$ is also a continuous function. By the fundamental theorem of calculus, we obtain

$$\frac{d\varphi}{dt} = \varphi f(\varphi).$$

This implies the desired result. □

Next, we study the global solutions of the problem (4.1) and an upper estimate of the global solutions are given.

Lemma 4.4. *Assume that $\varphi \in C^1([0,T))$ is a positive solution of the problem* (4.1) *and the integral*

$$\int_{\varphi_0}^{\infty} \frac{d\eta}{\eta f(\eta)} = \infty. \tag{4.3}$$

Then, φ can be extended to a global solution of (4.1) *and furthermore,*

$$\varphi(t) = H^{-1}(t),$$

where

$$H(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\eta}{\eta f(\eta)} \quad \textit{for} \quad \varphi_0 \leq \varphi$$

and H^{-1} is the inverse function of H.

Proof. Let φ be a positive solution of the problem (4.1) for $t \in [0,T)$.

First, we show that φ is a global solution, i.e., $T = \infty$. Assume by contradiction that $\varphi(t)$ blows up at finite time, i.e.

$$T < \infty \quad \text{with} \quad \lim_{t \to T^-} |\varphi(t)| = \infty.$$

Since $\varphi(t) > 0$, $\lim_{t\to T^-} \varphi(t) = \infty$. We can rewrite (4.1) as

$$\frac{1}{\varphi(t)f(\varphi(t))}\frac{d\varphi}{dt} = 1.$$

Integrating both sides from 0 to t, we obtain

$$\int_0^t \frac{1}{\varphi(s)f(\varphi(s))}\frac{d\varphi}{ds}ds = t \quad \text{for} \quad 0 \leq t < T.$$

Letting $\eta = \varphi(s)$, we get

$$\int_{\varphi_0}^{\varphi(t)} \frac{d\eta}{\eta f(\eta)} = t \quad \text{for} \quad 0 \leq t < T. \tag{4.4}$$

Passing to the limit as $t \to T^-$, we get

$$\infty = \int_{\varphi_0}^{\infty} \frac{d\eta}{\eta f(\eta)} = T < \infty$$

which is a contraction. Thus, φ is defined on $[0, \infty)$. In addition, from (4.4) we have

$$H(\varphi) = \int_{\varphi_0}^{\varphi(t)} \frac{d\eta}{\eta f(\eta)} = t.$$

Observe that H is non-decreasing and thus, H^{-1} exists. Then,

$$\varphi(t) = H^{-1}(t).$$

The proof is completed. □

Lemma 4.5. *Assume that $\varphi \in C^1([0,T))$ is a positive solution of the problem* (4.1) *and there exists a function F such that $f(\varphi) \leq F(\varphi)$ with the integral*

$$\int_{\varphi_0}^{\infty} \frac{d\eta}{\eta F(\eta)} = \infty. \tag{4.5}$$

Then, φ *can be extended to a global solution of* (4.1) *and furthermore,*

$$\varphi(t) \leq \Psi^{-1}(t),$$

where

$$\Psi(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\eta}{\eta F(\eta)} \quad \textit{for} \quad \varphi_0 \leq \varphi$$

and Ψ^{-1} *is the inverse function of* Ψ*.*

Proof. The proof is entirely the same as that of Lemma 4.4. □

For the rest of this work, we consider the following typical types of the nonlinear function f on $[0, \infty)$. Let $K > 0$ be a constant.

(H4) f is a positive decreasing function such that

$$f(0) \leq K \text{ and } \lim_{s\to\infty} f(s) = 0.$$

(H5) (Exponential function)

$$f(s) = Ke^{-\epsilon s} \text{ for some } \epsilon > 0.$$

(H6) (Power function)

$$f(s) = \tfrac{K}{(1+s)^{\beta}} \text{ for some } \beta > 0.$$

Proposition 4.6. *Assume that* (H4) *holds and let* $\varphi \in C^1([0, T))$ *be a positive solution of the problem* (4.1)*. Then, the solution* φ *of* (4.1) *satisfies*

$$\varphi(t) \leq \varphi_0 e^{tK} \quad \textit{for all} \quad t \geq 0$$

Proof. By (H4), there exists $K > 0$ such that $f(\varphi) \leq K$ for all $\varphi \geq 0$ and we obtain that

$$\varphi(t) \leq \varphi_0 + K \int_0^t \varphi(s) ds.$$

By Theorem 2.2 (Gronwall' s inequlity), we get

$$\varphi(t) \leq \varphi_0 e^{tK} \quad \text{for all} \quad t \in [0, T).$$

We can check that the condition (4.5) holds. By Lemma 4.5, we obtain that φ must be a global solution and

$$\varphi(t) \leq \varphi_0 e^{tK} \quad \text{for all} \quad t \geq 0.$$

The proof is completed. □

Proposition 4.7. *Assume that* (H5) *holds and let* $\varphi \in C^1([0, T))$ *be a positive solution of the problem* (4.1)*. For each* $k \in \mathbb{N}$*, there exists a positive constant* C_k *that depends on a positive integer* k *such that the solution* φ *satisfies*

$$\varphi(t) \leq C_k(1+t)^{1/k} \quad \textit{for all} \quad t \geq 0$$

and

$$\lim_{t \to \infty} \varphi(t) = \infty.$$

Proof. By (H5), we have

$$\frac{d\varphi}{dt} = \varphi K e^{-\epsilon\varphi} \quad \text{for some} \quad \epsilon > 0.$$

By Taylor series, we know that

$$e^{\epsilon\varphi} = 1 + \epsilon\varphi + \frac{(\epsilon\varphi)^2}{2!} + \frac{(\epsilon\varphi)^3}{3!} + \cdots.$$

Since $\varphi > 0$, we have

$$1 + \frac{(\epsilon\varphi)^k}{k!} \leq e^{\epsilon\varphi} \text{ for all } k \geq 1.$$

Then,

$$\frac{d\varphi}{dt} \leq K\varphi \left(\frac{k!}{k! + (\epsilon\varphi)^k} \right). \tag{4.6}$$

We can rewrite (4.6) as

$$\frac{k! + (\epsilon\varphi)^k}{\varphi} \frac{d\varphi}{dt} \leq Kk!.$$

Integrating both sides from 0 to t, we obtain

$$\int_0^t \frac{k! + (\epsilon\varphi)^k}{\varphi} \frac{d\varphi}{ds} ds \leq Kk!t.$$

Letting $\eta = \varphi(s)$, we get

$$\int_{\varphi_0}^{\varphi(t)} \frac{(\epsilon\eta)^k}{\eta} d\eta \leq \int_{\varphi_0}^{\varphi(t)} \frac{k! + (\epsilon\eta)^k}{\eta} d\eta \leq Kk!t, \tag{4.7}$$

where $\varphi_0 = \varphi(0)$. Therefore, we have that

$$\int_{\varphi_0}^{\infty} \frac{(\epsilon\eta)^k}{\eta Kk!} d\eta = \frac{\epsilon^k}{Kk!} \int_{\varphi_0}^{\infty} \eta^{k-1} d\eta = \left(\frac{\epsilon^k}{Kk!} \right) \frac{\eta^k}{k} \bigg|_{\varphi_0}^{\infty} = \infty,$$

which implies that the condition (4.5) holds. By Lemma 4.5 , we obtain that φ must be a global solution. Moreover, from (4.7) we have

$$\frac{(\epsilon\varphi(t))^k - (\epsilon\varphi_0)^k}{k} = \frac{(\epsilon\eta)^k}{k} \bigg|_{\varphi_0}^{\varphi(t)} \leq Kk!t$$

and hence,

$$\begin{aligned} (\epsilon\varphi(t))^k &\leq Kkk!t + (\epsilon\varphi_0)^k \\ (\varphi(t))^k &\leq \left(\frac{Kkk!}{\epsilon^k} \right) t + \varphi_0^k. \end{aligned}$$

Thus, for all $t \geq 0$

$$\varphi(t) \leq \left[\left(\frac{Kkk!}{\epsilon^k} \right) t + \varphi_0^k \right]^{1/k} \leq C_k(1+t)^{1/k},$$

where $C_k = \sqrt[k]{\max\left\{\frac{Kkk!}{\epsilon^k}, \varphi_0^k\right\}}$.

Now, we claim that $\lim_{t\to\infty} \varphi(t) = \infty$. Assume that there exists a constant $M > 0$ such that

$$0 \leq \varphi(t) \leq M \;\text{ for all }\; t \geq 0.$$

Setting $\kappa = \min_{\eta\in[0,M]} \eta K e^{-\epsilon\eta}$, we obtain

$$\kappa \leq \frac{d\varphi}{dt} = \varphi K e^{-\epsilon\varphi}$$

and hence,

$$\kappa t \leq \varphi(t) \leq M.$$

Taking $t \to \infty$, we get

$$\infty = \lim_{t\to\infty} \kappa t \leq \lim_{t\to\infty} \varphi(t) \leq M,$$

a contradiction. □

Proposition 4.8. *Assume that* (H6) *holds and let* $\varphi \in C^1([0,T))$ *be a positive solution of the problem* (4.1). *Then, there exists a positive constant* C *such that the solution* φ *satisfies*

$$\varphi(t) \leq C(1+t)^{1/\beta} \quad \textit{for all} \quad t \geq 0$$

and

$$\lim_{t\to\infty} \varphi(t) = \infty.$$

Proof. By (H6), we have

$$\frac{d\varphi}{dt} \leq \frac{K\varphi}{(1+\varphi)^\beta} \quad \text{for some} \quad \beta > 0.$$

Since

$$\varphi^\beta \leq (1+\varphi)^\beta \quad \text{for some} \quad \beta > 0,$$

we obtain

$$\frac{d\varphi}{dt} \leq \frac{K\varphi}{\varphi^\beta} = \frac{K}{\varphi^{\beta-1}}. \tag{4.8}$$

We can rewrite (4.8) as

$$\varphi^{\beta-1}\frac{d\varphi}{dt} \leq K.$$

Integrating both sides from 0 to t, we obtain

$$\int_0^t \varphi^{\beta-1}\frac{d\varphi}{ds}ds \leq Kt.$$

Letting $\eta = \varphi(s)$, we get

$$\int_{\varphi_0}^{\varphi(t)} \eta^{\beta-1} d\eta \leq Kt,$$

where $\varphi_0 = \varphi(0)$. We check that the condition (4.5) holds. By Lemma 4.5, we get that φ must be a global solution. Moreover, we have

$$\frac{(\varphi(t))^\beta - {\varphi_0}^\beta}{\beta} = \left.\frac{\eta^\beta}{\beta}\right|_{\varphi_0}^{\varphi(t)} \leq Kt,$$

and thus,

$$(\varphi(t))^\beta \leq \beta K t + \varphi_0^\beta.$$

Then,

$$\varphi(t) \leq (\beta K t + \varphi_0^\beta)^{1/\beta} \leq C(1+t)^{1/\beta}$$

for $t \geq 0$ with $C = \left(\max\left\{\beta K, \varphi_0^\beta\right\}\right)^{1/\beta}$.

By an argument similar as the proof of Proposition 4.7, we can prove that $\lim_{t\to\infty} \varphi(t) = \infty$. □

4.2 Global Solution

Theorem 4.9. *Assume that* (H1)-(H4) *hold. Then, the problem* (1.2) *has a unique global solution satisfying*

$$0 \leq u(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} e^{tK} \quad \textit{for all} \quad (x,t) \in \mathbb{R}^n \times [0,\infty),$$

where K is a positive constant.

Proof. Let $\overline{u} := \varphi(t) \in C^1([0,T))$ be the positive solution of the following problem:

$$\begin{cases} \dfrac{d\varphi}{dt} & = \varphi f(\varphi) \quad \text{for} \quad 0 < t < T, \\ \varphi(0) & = \|u_0\|_{L^\infty(\mathbb{R}^n)} \end{cases}$$

and $\underline{u} = 0$. It is clear that $\overline{u}, \underline{u} \in C^{2,1}(Q_T) \cap C_b(\Sigma_T)$ and

$$\underline{u}(x,0) = 0 \leq u_0(x) \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} = \overline{u}(x,0) \quad \text{on} \quad \mathbb{R}^n.$$

Furthermore, for any $(x,t) \in Q_T$

$$\partial_t \underline{u} - \Delta \underline{u} - \underline{u}\Big(f(\underline{u}) - \alpha g \star \overline{u}\Big) \leq 0$$

and

$$\partial_t \overline{u} - \Delta \overline{u} - \overline{u}\Big(f\left(\overline{u}\right) - \alpha g \star \underline{u}\Big) = \frac{d\varphi}{dt} - \varphi f(\varphi) \geq 0.$$

Then, $\underline{u} = 0$ and $\overline{u} = \varphi$ are coupled lower and upper classical solutions of (1.2) . By Theorem 3.5, we get that the problem (1.2) has a unique solution u such that

$$0 \leq u(x,t) \leq \varphi(t) \quad \text{for} \ \ (x,t) \in Q_T.$$

By Proposition 4.6, we have that φ is a global solution and hence,

$$0 \leq u(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} e^{tK} \quad \text{for all} \ \ t \geq 0.$$

The proof is completed. □

Theorem 4.10. *Assume that* (H1)-(H3) *and* (H5) *hold. Then, the problem* (1.2) *has a unique global solution satisfying*

$$0 \leq u(x,t) \leq C_k (1+t)^{1/k} \ \ \textit{for all} \ \ \ (x,t) \in \mathbb{R}^n \times [0, \infty),$$

where C_k is a positive constant that depends on a positive integer k.

Proof. The proof uses the same argument as in Theorem 4.9. □

Theorem 4.11. *Assume that* (H1)-(H3) *and* (H6) *hold. Then, the problem* (1.2) *has a unique global solution satisfying*

$$0 \leq u(x,t) \leq C (1+t)^{1/\beta} \ \ \textit{for all} \ \ \ (x,t) \in \mathbb{R}^n \times [0, \infty),$$

where C and β are positive constant.

Proof. The proof uses the same argument as in Theorem 4.9. □

Remark 4.12.

(i) In addition we notice that either (H5) or (H6) implies (H4).

(ii) We see that the assumption (H4) gives exponential growth for global solution u, but (H5) and (H6) give polynomial growth solution.

(iii) Compare to Deng and Wu [5], we can find a constant $K > 0$ such that the global solution u of the problem (1.2) satisfying

$$0 \leq u(x,t) \leq K \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times [0,\infty)$$

if there exists $r > 0$ such that $f(r) = 0$ and $f(\eta) < 0$ for $\eta > r$.

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