สูตรรูปแบบปิดสำหรับการตั้งราคาตัวอย่างแบบวิยุตของโมเมนต์สวอปส์บนกระบวนการอิโตหนึ่งมิติ

นายกิตติศักดิ์ ชุมพงศ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์และวิทยาการคณนา
ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2561

CLOSED-FORM FORMULA FOR PRICING DISCRETELY-SAMPLED MOMENT SWAPS ON ONE-DIMENSIONAL ITO PROCESS

Mr. Kittisak Chumpong

A Dissertation Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy Program in Applied Mathematics and

Computational Science

Department of Mathematics and Computer Science

Faculty of Science

Chulalongkorn University

Academic Year 2018

Dissertation Title	CLOSED-FORM FORMULA FOR PRICING DISCRETELY-SAMPLED MOMENT SWAPS ON ONE-DIMENSIONAL ITO PROCESS
By	Mr. Kittisak Chumpong
Field of Study	Applied Mathematics and Computational Science
Thesis Advisor	Assistant Professor Khamron Mekchay, Ph.D.
Thesis Co-advisor	Associate Professor Sanae Rujivan, Ph.D.
	Nopporn Thamrongrat, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

. . . . Dean of the Faculty of Science
(Professor Polkit Sangvanich, Ph.D.)

DISSERTATION COMMITTEE

. . . . Chairman
(Associate Professor Anusorn Chonwerayuth, Ph.D.)

. . . . Advisor
(Assistant Professor Khamron Mekchay, Ph.D.)

. . . . Co-advisor
(Associate Professor Sanae Rujivan, Ph.D.)

. . . . Co-advisor
(Nopporn Thamrongrat, Ph.D.)

. . . . Examiner
(Assistant Professor Jiraphan Suntornchost, Ph.D.)

. . . . Examiner
(Assistant Professor Boonyarit Intiyot, Ph.D.)

. . . . External Examiner
(Anuwat Tangthanawatsakul, Ph.D.)

กิตติศักดิ์ ชุมพงศ์ : สูตรรูปแบบปิดสำหรับการตั้งราคาตัวอย่างแบบวิยุตของโมเมนต์สวอปส์บนกระบวนการอิโตหนึ่งมิติ. (CLOSED-FORM FORMULA FOR PRICING DISCRETELY-SAMPLED MOMENT SWAPS ON ONE-DIMENSIONAL ITO PROCESS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.คำรณ เมฆฉาย, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : รศ.ดร.เสน่ห์ รุจิวรรณ, และ ดร.นพพร ธรรมรงค์รัตน์ 73 หน้า.

โมเมนต์สวอปส์คือสัญญาล่วงหน้าที่สำคัญบนโมเมนต์ที่สูงขึ้นของผลตอบแทนแบบลอการิทึมของสินค้าอ้างอิงที่ระบุไว้ ซึ่งมีบทบาทสำคัญในการป้องกันความเสี่ยงของตลาดในรูปแบบต่างๆ สวอปส์ของความแปรปรวน ความเบ้ และความโด่ง เป็นตัวอย่างของโมเมนต์สวอปส์ที่ใช้ซื้อขายแลกเปลี่ยนในตลาด ณ ปัจจุบัน งานนี้ให้สูตรการตั้งราคาที่ง่ายและนักลงทุนนำไปใช้ได้สะดวกสำหรับโมเมนส์สวอปส์ของการสุ่มตัวอย่างแบบวิยุตภายใต้กระบวนการอิโต ได้แก่ แบบจำลองส่วนขยายแบล็ค-โชลส์สำหรับราคาหุ้น และแบบจำลองของชวาร์ซสำหรับราคาสินค้าโภคภัณฑ์ สูตรการตั้งราคาถูกตรวจสอบหัวข้อที่น่าสนใจต่อไป และสุดท้ายเราดำเนินการจำลองแบบมอนติคาร์โลของการเฟ้นสุ่มเพื่อรับรองความถูกต้องของสูตรการตั้งราคาและแสดงตัวอย่างเชิงตัวเลขตรวจสอบความไวของพารามิเตอร์

ภาควิชา	คณิตศาสตร์และ	ลายมือชื่อนิสิต ....
	วิทยาการคอมพิวเตอร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก ....
สาขาวิชา	คณิตศาสตร์ประยุกต์	ลายมือชื่อ อ.ที่ปรึกษาร่วม ....
	และวิทยาการคณนา	ลายมือชื่อ อ.ที่ปรึกษาร่วม ....
ปีการศึกษา	2561	

5872805923 : MAJOR APPLIED MATHEMATICS AND COMPUTATIONAL SCIENCE

KEYWORDS : MOMENT SWAPS / DISCRETE SAMPLING / ITO PROCESS / BLACK-SCHOLES MODEL / TIME-DEPENDENT PARAMETERS / SCHWARTZ MODEL

KITTISAK CHUMPONG : CLOSED-FORM FORMULA FOR PRICING DISCRETELY-SAMPLED MOMENT SWAPS ON ONE-DIMENSIONAL ITO PROCESS. ADVISOR : ASST. PROF. KHAMRON MEKCHAY, Ph.D., DISSERTATION COADVISOR : ASSO. PROF. SANAE RUJIVAN, Ph.D., and NOPPORN THAMRONGRAT, Ph.D., 73 pp.

Moment swaps are essentially forward contracts on realized higher moments of log-returns of a specified underlying asset, which play an important role in protection against different kinds of market shocks. Variance, skewness, and kurtosis swaps are examples of moment swaps currently traded in markets. To facilitate market practitioners, this work provides a simple and easy-to-use pricing formula of moment swaps on discrete sampling under the Itô process, extended Black-Scholes model for stock prices and Schwartz model for commodity prices. Furthermore, the interesting topics of the fair prices are also investigated. Finally, Monte Carlo simulations are performed to support the accuracy of the pricing formulas and numerical examples are provided to check the sensitivity of the parameters.

Department :	Mathematics and	Student's Signature ....
	Computer Science	Advisor's Signature ....
Field of Study :	Applied Mathematics and	Co-advisor's Signature ....
	Computational Science	Co-advisor's Signature ....
Academic Year :	2018	

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my thesis advisor, Assistant Professor Khamron Mekchay, Ph.D., and co-advisor, Associate Professor Sanae Rujivan, Ph.D. and Nopporn Thamrongrat, Ph.D., for their willingness to sacrifice their time to suggest and advise me along the way of my Ph.D. studies. Moreover, not only for coaching me research but also for broadening my academic vision.

In addition, I am greatly indebted to my thesis committees: Associate Professor Anusorn Chonwerayuth, Ph.D., Assistant Professor Jiraphan Suntornchost, Ph.D., Assistant Professor Boonyarit Intiyot, Ph.D. and Anuwat Tangthanawatsakul, Ph.D. Their suggestions and comment are sincere appreciation. I feel very thankful to all of my teachers of the Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University for my knowledge and skills.

In particular, I have been financially supported by the Chulalongkorn University.

Finally, I feel very thankful to my beloved parents, sister and girlfriend for their love, support, understanding and encouragement. Furthermore, I would like to thank my dear friends and my colleagues in the AMCS program for their useful advice, helpful comments and friendship throughout my graduate study. This thesis would not have been possibly completed without them. Thank you.

CONTENTS

CHAPTER I

INTRODUCTION

Financial markets, including capital and derivatives markets, are worldwide exchanges for small and large businesses to raise capital and hedge against different types of risks. Capital markets include stock and bond markets, and derivatives markets include futures and options markets. Investors may invest in these markets directly through banks and online stockbrokers and indirectly through mutual funds and pension funds. Derivatives have become increasingly important in finance. Futures and options are actively traded on many exchanges throughout the world. Many different types of derivatives are entered into by financial institutions, fund managers, and corporate treasurers. Derivatives are added to bond issues, used in executive compensation plans, embedded in capital investment opportunities, used to transfer risks in mortgages from the original lenders to investors, and so on. Derivatives are securities whose value is determined by an underlying asset (stocks, commodities, equities, interest rates and currencies) on which it is based. Therefore the underlying asset determines the price and if the price of the asset changes, the derivative changes along with it. The purpose of derivatives is to give producers and manufacturers the possibility to hedge risks. There are three types of derivative market: forward/futures, options and swaps.

Moment swaps are essentially forward contracts on the realized higher moments of the log-returns of a specified underlying asset. More specifically, their payoff is a function of powers of the (daily) log-returns of the underlying asset at certain pre-specified discretely sampled points. According to recent studies by Schoutens [17] and Rompolis and Tzavalis [13], moment swaps play such an important role in financial markets to cover different kinds of market shocks. Speculators trade variance swaps (second order moment swaps) as an easy way to gain exposure to future levels of variance, and they may need to hedge against their portfolio volatility risk. Skewness swaps (third order moment swaps) provide protection against changes in the symmetry of the underlying distribution. Kurtosis swaps (fourth order moment swaps) provide protection against unexpected occurrences of very large jumps or changes in the tail behavior of the underlying distribution. These studies suggest that using variance and higher-moment swaps to hedge European options gives better performance compared with traditional delta hedging strategies. Therefore, it is meaningful to define and price higher-moment swaps to hedge the existing skewness and kurtosis risks.

As a result of the increasing trading activities of variance swaps, Zhu and Lian [23], [24] obtain a closed-form formula under Heston stochastic volatility model for the underlying asset price process by solving a coupled system of partial differential equations. However, Zhu and Lian's results [23, 24] are still too complicated for facilitating market practitioners. Rujivan and Zhu [16, 15] simplified the formulas of [23, 24] by employing the dimension-reduction technique. Zheng and Kwok [22] also extended Zhu and Lian's results [24] to price variance swaps under the stochastic volatility models with simultaneous jumps in the asset price where the variance processe relies on the availability of the analytical expression of the joint moment generating function of the underlying processe. Moreover, Rujivan [14] presented a simple closed-form formula for pricing discretely sampled gamma swaps based on Heston stochastic volatility model. Recently, Weraprasertsakun and Rujivan [19] presented an analytical approach for pricing discretely-sampled variance swaps on commodities described by the Schwartz model. Due to the launching of CBOE Skew Index (SKEW) to measure the skewness risk in the financial market by the Chicago Board Options Exchange (CBOE) in 2011, the study of skewness and kurtosis risks is tremendous growth. Neuberger [11] studied a set of tools to improve the measurement of the skewness of asset returns. Kozhan et al. [7] measured the skew risk premium in the equity index market through the skewness swap. Zhao et al. [21] approximated the skewness and kurtosis swap contracts. Zhang et al. [20] studied the skewness of stock returns under the Heston model. For study on moment swaps, Schoutens [17] defined higher-moment swaps using daily log-returns for the realized moments, and claimed that moment swaps can protect against incorrectly estimated skewness or kurtosis and Rompolis and Tzavalis [13] suggested perfect hedging strategies of contingent claims under stochastic volatility and/or random jumps of the underlying asset price. However, Schoutens [17] and Rompolis and Tzavalis [13] did not derive an exact pricing formula for moment swaps.

In this thesis, an analytical method is derived to price the discretely-sampled moment swaps introduced by Schoutens [17]. The study begins by considering a probability space $(\Omega, \mathcal{F}, Q)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and a risk-neutral probability measure Q. The dynamics of the underlying asset price S_t is assumed to follow the Black-Scholes and Schwartz model, described by the Itô process.

The thesis is organized into five chapters. Chapter 2 provides some basic knowledges in Itô process, moment swaps and methods using in this research. Our analytical approach for obtaining the fair price of moment swaps and its interesting topic based on the Black-Scholes and Schwartz model are presented in chapter 3 and 4, respectively. Finally, in chapter 5, we conclude the result of thesis and compare the fair price of moment swaps for underlying asset described by the Black-Scholes and Schwartz model.

CHAPTER II

PRELIMINARIES

In this chapter, we review the concept of Itô process and introduce the moment swaps, which are used to investigate the closed-form formula. The chapter is divided into three sections: Itô process, Moment swaps, and Methods.

2.1 Itô Process

Itô process is a branch of stochastic process that operates on differential equation. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to Brownian motion. It is used to model systems that behave randomly. The Itô process has been widely applied in financial mathematics and economics to model the stock price, commodity price, interest rates, etc. More details on Itô process can be found in [4], [6], [9] and [12].

Definition 2.1. A **stochastic process** $\{X_t\}_{t\geq 0}$ is a family of random variables $X_t : \Omega \times [0,\infty) \to \mathbb{R}$ with the continuous map $t \mapsto X_t(\omega)$ for each $\omega \in \Omega$.

Definition 2.2. A **Brownian motion** $\{W_t\}_{t\geq 0}$ is a stochastic process that satisfies the following:

1. W_t is continuous and $W_0 = 0$ a.s.,
2. W_t has independent increments,
3. The increment $\Delta W_t = W_{t+\Delta t} - W_t$ is normally distributed with zero mean and variance Δt, $\Delta W_t \sim N(0, \Delta t)$.

Definition 2.3. An **Itô process** is a stochastic process $\{X_t\}_{t\geq 0}$ that can be written in the form

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{2.1}$$

where μ and σ are known as **drift** and **diffusion terms**, respectively. The integral $\int_0^t \sigma(s, X_s)dW_s$ is called the **Itô integral**. It is usual to rewrite (2.1) in differential form or stochastic differential equation (SDE)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{2.2}$$

Theorem 2.4 (Itô lemma)**.** *Let S_t be an Itô process given by*

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t.$$

Let $g(t, s) \in C^{1,2}([0, \infty) \times \mathbb{R})$. Then

$$X_t = g(t, S_t)$$

is again an Itô process, and

$$dX_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial s}dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial s^2}(dS_t)^2$$

where $(dS_t)^2$ is computed according to the rules

$$(dt)^2 = dtdW_t = dW_tdt = 0 \text{ and } (dW_t)^2 = dt.$$

Lemma 2.5. *According to Theorem 2.4, for $X_t = g(t, S_t)$ and $Y_t = h(t, X_t)$ such that $h(t, x) \in C^{1,2}([0, \infty) \times \mathbb{R})$, then Y_t is also an Itô process with*

$$\begin{aligned} dY_t = &\left[\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\frac{\partial g}{\partial t} + \mu(t, S_t)\frac{\partial h}{\partial x}\frac{\partial g}{\partial s} + \frac{1}{2}\sigma^2(t, S_t)\frac{\partial h}{\partial x}\frac{\partial^2 g}{\partial s^2} + \frac{1}{2}\sigma^2(t, S_t)\frac{\partial^2 h}{\partial x^2}\left(\frac{\partial g}{\partial s}\right)^2\right]dt \\ &+ \sigma(t, S_t)\frac{\partial h}{\partial x}\frac{\partial g}{\partial s}dW_t. \end{aligned}$$

Proof. From Theorem 2.4,

$$\begin{aligned}
dX_t &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial s}dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial s^2}(dS_t)^2 \\
&= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial s}\Big[\mu(t,S_t)dt + \sigma(t,S_t)dW_t\Big] + \frac{1}{2}\frac{\partial^2 g}{\partial s^2}\Big[\mu(t,S_t)dt + \sigma(t,S_t)dW_t\Big]^2 \\
&= \frac{\partial g}{\partial t}dt + \mu(t,S_t)\frac{\partial g}{\partial s}dt + \sigma(t,S_t)\frac{\partial g}{\partial s}dW_t + \frac{1}{2}\frac{\partial^2 g}{\partial s^2}\mu^2(t,S_t)(dt)^2 \\
&\quad + \frac{\partial^2 g}{\partial s^2}\mu(t,S_t)\sigma(t,S_t)dtdW_t + \frac{1}{2}\frac{\partial^2 g}{\partial s^2}\sigma^2(t,S_t)(dW_t)^2 \\
&= \frac{\partial g}{\partial t}dt + \mu(t,S_t)\frac{\partial g}{\partial s}dt + \sigma(t,S_t)\frac{\partial g}{\partial s}dW_t + \frac{1}{2}\frac{\partial^2 g}{\partial s^2}\sigma^2(t,S_t)dt \\
&= \left[\frac{\partial g}{\partial t} + \mu(t,S_t)\frac{\partial g}{\partial s} + \frac{1}{2}\sigma^2(t,S_t)\frac{\partial^2 g}{\partial s^2}\right]dt + \sigma(t,S_t)\frac{\partial g}{\partial s}dW_t.
\end{aligned}$$

Applying Theorem 2.4 to $Y_t = h(t, X_t)$, we get

$$\begin{aligned}
dY_t &= \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(dX_t)^2 \\
&= \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial x}\left[\left[\frac{\partial g}{\partial t} + \mu(t,S_t)\frac{\partial g}{\partial s} + \frac{1}{2}\sigma^2(t,S_t)\frac{\partial^2 g}{\partial s^2}\right]dt + \sigma(t,S_t)\frac{\partial g}{\partial s}dW_t\right] \\
&\quad + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\left[\left[\frac{\partial g}{\partial t} + \mu(t,S_t)\frac{\partial g}{\partial s} + \frac{1}{2}\sigma^2(t,S_t)\frac{\partial^2 g}{\partial s^2}\right]dt + \sigma(t,S_t)\frac{\partial g}{\partial s}dW_t\right]^2 \\
&= \left[\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\frac{\partial g}{\partial t} + \mu(t,S_t)\frac{\partial h}{\partial x}\frac{\partial g}{\partial s} + \frac{1}{2}\sigma^2(t,S_t)\frac{\partial h}{\partial x}\frac{\partial^2 g}{\partial s^2} + \frac{1}{2}\sigma^2(t,S_t)\frac{\partial^2 h}{\partial x^2}\left(\frac{\partial g}{\partial s}\right)^2\right]dt \\
&\quad + \sigma(t,S_t)\frac{\partial h}{\partial x}\frac{\partial g}{\partial s}dW_t.
\end{aligned}$$

□

Itô process is usually used to describe the stock and commodity prices, the Black-Scholes and Schwartz model, respectively.

2.1.1 The Extended Black-Scholes Model

The Black-Scholes (BS) model is one of the most important concepts in modern financial theory. It was developed in 1973 by Fisher Black, Robert Merton and Myron Scholes and is still widely used to described the stock. The dynamics of the stock price

S_t is assumed to follow the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{2.3}$$

where r is the risk-free interest rate, σ is the volatility of the stock prices and W_t is a Brownian motion on the probability space $(\Omega, \mathcal{F}, Q)$ [1]. Moreover, Merton [10] suggested that the Black-Scholes model (2.3) also holds for time dependent risk-free interest rate $r(t)$ and volatility $\sigma(t)$, which hereinafter refer as the extended Black-Scholes (EBS) model,

$$dS_t = r(t)S_t dt + \sigma(t)S_t dW_t. \tag{2.4}$$

In addition to the BS model, the assumption of time-dependent parameters provides flexibility to describe the possible events, politically or economically, that may occur in different time.

2.1.2 The Schwartz Model

In 1997, Schwartz [18] described the spot commodity price, denoted by S_t, follows the SDE,

$$dS_t = \kappa (\mu - \ln S_t) S_t dt + \sigma S_t dW_t, \tag{2.5}$$

where μ is the long-run mean, κ is the speed of the reversion, σ is the volatility of the commodity prices, and W_t is a Brownian motion on the probability space $(\Omega, \mathcal{F}, Q)$. This dynamic of commodity prices is different from those of equities, interest rates, or currencies, but similar to physically produced, transported, stored and consumed. It is natural to expect that they should be treated differently from financial security markets.

2.2 Moment Swaps

Moment swaps are essentially forward contracts on the realized higher moments of returns of a specified underlying asset, which play an important role in protection against different kinds of market shocks rapid changes of prices. Variance, skewness, and kurtosis swaps are examples of moment swaps traded in derivative markets. In literature, methods of calculating realized moment are classified into two categories: continuous and discrete sampling. The continuous sampling one has greatly increased the mathematical tractability. The discrete sampling is divided based on two different definitions, the actual return-based realized moment and the log-return realized moment [17]. The annualized realized m^{th}-moment, $m \geq 2$, in terms of discrete sampling over the contract life $[0, T]$ for a maturity time $T > 0$ on an underlying asset S_t is

$$MOMS^{(m)} = N' \times \sum_{i=1}^{N} \ln^m \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right)$$

where S_{t_i} are the closing prices of the underlying asset observed at times t_i, for $i = 0, 1, ..., N$, and N' is the nominal amount, $N' = \frac{AF}{N}$ when AF is the annualized factor for converting to annualized higher moments. If the sampling frequency is calculated daily, then $AF = 252$, assuming that there are 252 trading days in one year; if weekly, then $AF = 52$; and if monthly, then $AF = 12$. Typically, $T = \dfrac{N}{AF}$ with equally-spaced discrete observations $\Delta t = t_i - t_{i-1} > 0$, for $i = 1, 2, ..., N$. The annualized factor becomes $AF = \dfrac{N}{T} = \dfrac{1}{\Delta t}$, and the typical formula for the measure of realized m^{th}-moment is

$$MOMS^{(m)} = \frac{1}{T} \sum_{i=1}^{N} \ln^m \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) = \frac{1}{T} \sum_{i=1}^{N} \left(X_{t_i} - X_{t_{i-1}} \right)^m \tag{2.6}$$

where $X_t := \ln S_t$, a log price process.

In a risk-neutral world, the value of an m^{th}-moment swap at time t, denoted by V_t, is the expected present value of the future payoff

$$V_t = E_t^Q \left[e^{-\int_t^T r(s) ds} (MOMS^{(m)} - K^m) L \right]$$

where K^m is the annualized delivery price for the m^{th}-moment swap and L is the notional amount of the swap. The value of V_t should be zero at the beginning of contract because both parties pay zero cost to enter into a forward contract. Therefore, the fair delivery price of the m^{th}-moment swap when $V_0 = 0$, is

$$K^m = E_0^Q[MOMS^{(m)}]. \tag{2.7}$$

The valuation problem for an m^{th}-moment swap is reduced to calculating the conditional expectation of the realized m^{th}-moment (2.6) in the risk-neutral world which are solved by the method of next section.

2.3 Methods for computing conditional moments

This section introduces two well-known techniques for obtaining the conditional expectation of Itô process, i.e., the Monte Carlo simulation and the Feynman-Kac formula.

2.3.1 Monte Carlo Simulations

Monte Carlo (MC) simulations are the method for calculating conditional expectation based on probability simulations by repeated random sampling. MC simulations are often used practically in many areas such as finance, engineer, project management, manufacturing, environment and other forecasting models because it is easy to implement even with a complicated stochastic model. However, in practice the accuracy of MC simulations is a trade-off with computational time. In general, there are many well-known MC simulations such as Euler-Maruyama (EM) scheme, Milstein Scheme ([3], [6]). In this thesis, the conditional expectations of the log price process, $X_t = \ln S_t$,

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{2.8}$$

is calculated via MC simulation by EM scheme to compare with the obtained closed-form formula. The simplest way is EM discretization for the log price process (2.8) on the time interval $[0,T]$,

$$X_{t_j}(\omega) = X_{t_{j-1}}(\omega) + \mu(t_j, X_{t_j})\Delta t + \sigma(t_j, X_{t_j})\sqrt{\Delta t} Z_{t_j}(\omega), \tag{2.9}$$

where $\omega \in \Omega, \Delta t = \frac{T}{M_e}$, $t_j = j\Delta t, j = 0, 1, ..., M_e$, M_e is a positive integer representing the number of time steps used in the discretization, and Z_{t_j} is the standard normal random variable.

To compute the realized m^{th}-moment defined in (2.6), we set $M_e = N$ for simplicity and this gives us the approximate of X_{t_i} at the observation time $t_i, i = 1, 2, ..., N$. Next, we introduce an approximate of $K^m(T,N)$ obtained by MC simulations as

$$K^m(T,N;N_p) := \frac{\sum_{p=1}^{N_p} \left(\frac{1}{T} \sum_{i=1}^{N} \left(X_{t_i}(\omega_p) - X_{t_{i-1}}(\omega_p) \right)^m \right)}{N_p},$$

for $\omega_p \in \Omega$ and $p = 1, 2, ..., N_p$, where N_p is the number of sample paths used in MC simulations.

2.3.2 Feynman-Kac Theorem

The Feynman-Kac formula, named after Richard Feynman and Mark Kac, establishes a link between the solutions of partial differential equations and the conditional expectations of Itô processes [12]. This technique is the quite efficient in terms of computational times when compared with MC simulations for finding conditional expectations, especially when high accuracy is required.

Theorem 2.6 (Feynman-Kac Formula). *Suppose that X_t follows the Itô process*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where W_t is a Brownian motion. If $u(t,x) \in C^{1,2}([0,T] \times K)$, for a compact support

$K \subseteq \mathbb{R}$, follows the PDE

$$\frac{\partial u}{\partial t} + \mu(t,x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 u}{\partial x^2} - V(t,x)u = 0, \tag{2.10}$$

subject to the terminal condition $u(T,x) = f(x)$ with bounded below V. Then,

$$u(t,x) = E\left[e^{-\int_t^T V(r,X_r)dr} f(X_T) \mid X_t = x\right]. \tag{2.11}$$

Moreover, u is unique, i.e., if $w(t,x) \in C^{1,2}([0,T]\times K)$ and solves (2.10) with the terminal conditional, then $w(t,x) = u(t,x)$ for all $t \in [0,T]$ and $x \in K$.

Example 2.7. Consider the problem of pricing of a forward contract on stock described by BS model (2.3). Let f be a forward contract on the underlying where forward payoff at maturity time T is $f(S_T)$. The value of the forward contract at time t is

$$u(t,s) := E[e^{-r^*(T-t)} f(S_T) \mid S_t = s],$$

for $0 \leq t \leq T$. This conditional expectation is of the form that occurs in the Feynman-Kac formula (2.11) with constant $V(r, S_r) = r^*$. Therefore, $u(t,s)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + rs\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} - r^* u = 0$$

with the terminal condition $u(T,s) = f(s)$. This is the Black-Scholes-Merton partial differential equation.

Example 2.8. For $t \geq 0$, consider the discount process

$$D(t) = e^{-\int_0^t r_t dt}$$

defined on the interest rate process r_t,

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t.$$

For a zero-coupon bond that pays \$1 at time T, the value of the bond at time $t \in [0, T]$ is given by

$$v(t,r) := E[e^{-\int_t^T r_u du} \mid r_t = r].$$

for $0 \leq t \leq T$. Using the Feynman-Kac formula (2.11), by uniqueness, $v(t,r)$ satisfies the partial differential equation

$$\frac{\partial v}{\partial t} + \mu(t,r)\frac{\partial v}{\partial r} + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 v}{\partial r^2} - rv = 0$$

with the terminal condition $v(T,r) = 1$.

In this thesis, the Feynman-Kac formula is applied to find conditional moment by solving the solution of PDE associated with EBS model (2.4), which is equivalent to solving the system of ordinary differential equations (ODEs). Moreover, the efficiency of this technique is investigated through the computational times by comparing with MC simulations.

CHAPTER III

CLOSED-FORM FORMULA FOR PRICING MOMENT SWAPS UNDER THE EXTENDED BLACK-SCHOLES MODEL

This chapter provides a simple and easy-to-use pricing formula for moment swaps based on discrete sampling under the EBS model (2.4) and BS model (2.3) for the underlying stock prices. The formulas are obtained by using Feynman-Kac theorem and combinatorial techniques. Moreover, some interesting observations of the fair prices are presented.

In section 3.1, we obtain the analytical formula for the conditional moment, which is used for deriving the fair price in section 3.2. The positivity and the relation of the fair prices for moment swaps are provided in sections 3.3 and 3.4, respectively. Finally, in section 3.5, we conduct Monte Carlo simulations to provide a verification of the correctness of the pricing formula and demonstrate with numerical examples to show the sensitivity of the parameters and the relations of fair prices.

3.1 Conditional Moments

This section presents the closed-form formula for k^{th}-conditional moment of the EBS model (2.4) and the BS model (2.3) in the two conditions of parameters.

Theorem 3.1. *Suppose that $k \geq 2$ is an integer and S_t follows the EBS model in* (2.4). *We set $X_t = \ln S_t$ and $\Delta t_i = t_i - t$ for all $i = 1, 2, \dots, N$. If $r(t)$, $\sigma(t) > 0$ are integrable on $[t_{i-1}, t_i]$ in which $r(t) - \frac{1}{2}\sigma^2(t)$ is not a zero function on $[t_{i-1}, t_i]$ then*

$$E^Q_{t_{i-1}}[X_t^k] = E^Q[X_t^k \mid X_{t_{i-1}} = x] = \sum_{j=0}^{k} x^{k-j} A_j(\Delta t_i; t_i, k) \tag{3.1}$$

for all $t \in [t_{i-1}, t_i]$ *and* $x \in (-\infty, \infty)$, *where we define* $x^0 := 1$ *for all* $x \in (-\infty, \infty)$ *and*

$$A_0(\Delta t_i; t_i, k) = 1, \tag{3.2}$$

$$A_1(\Delta t_i; t_i, k) = k \int_0^{\Delta t_i} \left(r(t_i - \eta) - \frac{1}{2}\sigma^2(t_i - \eta) \right) d\eta, \tag{3.3}$$

$$\begin{aligned} A_j(\Delta t_i; t_i, k) = (k - (j-1)) \int_0^{\Delta t_i} \left(r(t_i - \eta) - \frac{1}{2}\sigma^2(t_i - \eta) \right) A_{j-1}(\eta; t_i, k) d\eta \\ + \frac{1}{2} (k - (j-2)) (k - (j-1)) \int_0^{\Delta t_i} \sigma^2(t_i - \eta) A_{j-2}(\eta; t_i, k) d\eta \end{aligned} \tag{3.4}$$

for $j = 2, 3, \ldots, k$.

Proof. We let $g(t, s) = \ln s, h(t, x) = x^k$ and note that

$$\frac{dg}{dt} = 0, \frac{dg}{ds} = \frac{1}{s}, \frac{d^2 g}{ds^2} = -\frac{1}{s^2}, \frac{dh}{dt} = 0, \frac{dh}{dx} = kx^{k-1}, \frac{d^2 h}{dx^2} = k(k-1)x^{k-2}.$$

This implies from Lemma 2.5 to the transformation $X_t = \ln S_t$ and $Y_t = X_t^k$ that Y_t follows the Ito process

$$\begin{aligned} dY_t &= \left[r(t)kX_t^{k-1} - \frac{1}{2}\sigma^2(t)kX_t^{k-1} + \frac{1}{2}k(k-1)\sigma^2(t)X_t^{k-2} \right] dt + k\sigma(t)X_t^{k-1} dW_t \\ &= \left[\left(r(t) - \frac{1}{2}\sigma^2(t) \right) kX_t^{k-1} + \frac{1}{2}k(k-1)\sigma^2(t)X_t^{k-2} \right] dt + k\sigma(t)X_t^{k-1} dW_t \\ &= \left[\left(r(t) - \frac{1}{2}\sigma^2(t) \right) kY_t^{1-\frac{1}{k}} + \frac{1}{2}k(k-1)\sigma^2(t)Y_t^{1-\frac{2}{k}} \right] dt + k\sigma(t)Y_t^{1-\frac{1}{k}} dW_t. \end{aligned} \tag{3.5}$$

Consider a real-valued function defined by

$$U_i^{(k)}(y, t) := E^Q[Y_t \mid Y_{t_{i-1}} = y], \tag{3.6}$$

for all $(y, t) \in \mathbb{R} \times [t_{i-1}, t_i)$. Applying the Feynman-Kac formula (2.10) to (3.5) and (3.6), we have that $U_i^{(k)}$ satisfies the PDE

$$\frac{\partial U_i^{(k)}}{\partial t}+\left[\left(r(t)-\frac{1}{2}\sigma^2(t)\right)ky^{1-\frac{1}{k}}+\frac{1}{2}k(k-1)\sigma^2(t)y^{1-\frac{2}{k}}\right]\frac{\partial U_i^{(k)}}{\partial y}$$
$$+\frac{1}{2}\left[k\sigma(t)y^{1-\frac{1}{k}}\right]^2\frac{\partial^2 U_i^{(k)}}{\partial y^2}=0 \tag{3.7}$$

subject to the terminal condition

$$U_i^{(k)}(y,t_i)=y \tag{3.8}$$

for all $(y,t)\in\mathbb{R}\times[t_{i-1},t_i)$. Let $\tau=t_i-t$. We solve the PDE (3.7) subject to the terminal condition (3.8) by assuming that the solution can be written in the form

$$U_i^{(k)}(y,t)=\sum_{j=0}^{k}y^{1-\frac{j}{k}}A_j(\tau;t_i,k) \tag{3.9}$$

where $A_j(\tau;t_i,k)$ is the function depend on τ, t_i and k for $j=0,1,\ldots,k$. Calculating all partial derivatives of $U_i^{(k)}$ in (3.7) by using the solution form (3.9) yields

$$\frac{\partial U_i^{(k)}}{\partial t}=-\left(\sum_{j=0}^{k}y^{1-\frac{j}{k}}\frac{dA_j}{d\tau}\right), \tag{3.10}$$

$$\frac{\partial U_i^{(k)}}{\partial y}=\sum_{j=0}^{k-1}\left(1-\frac{j}{k}\right)y^{-\frac{j}{k}}A_j, \tag{3.11}$$

$$\frac{\partial^2 U_i^{(k)}}{\partial y^2}=\sum_{j=0}^{k-1}\left(1-\frac{j}{k}\right)\left(-\frac{j}{k}\right)y^{-\frac{j}{k}-1}A_j. \tag{3.12}$$

Inserting (3.10)–(3.12) into (3.7), we obtain

$$
\begin{aligned}
0 = &- \left[\sum_{j=0}^{k} y^{1-\frac{j}{k}} \frac{dA_j}{d\tau} \right] \\
&+ \left[\left(r(t_i - \tau) - \frac{1}{2}\sigma^2(t_i - \tau) \right) k y^{1-\frac{1}{k}} + \frac{1}{2} k(k-1)\sigma^2(t_i - \tau) y^{1-\frac{2}{k}} \right] \left[\sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) y^{-\frac{j}{k}} A_j \right] \\
&+ \left[\frac{1}{2} k^2 \sigma^2 (t_i - \tau) y^{2-\frac{2}{k}} \right] \left[\sum_{j=1}^{k-1} \left(1 - \frac{j}{k}\right) \left(-\frac{j}{k}\right) y^{-\frac{j}{k}-1} A_j \right] \\
=: &-A_1 + A_2 + A_3 + A_4,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \sum_{j=0}^{k} y^{1-\frac{j}{k}} \frac{dA_j}{d\tau}, \\
A_2 &= \left(r(t_i - \tau) - \frac{1}{2}\sigma^2(t_i - \tau) \right) k \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) A_j y^{1-\frac{(j+1)}{k}}, \\
A_3 &= \frac{1}{2} k(k-1)\sigma^2(t_i - \tau) \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) A_j y^{1-\frac{(j+2)}{k}}, \\
A_4 &= \frac{1}{2} k^2 \sigma^2 (t_i - \tau) \sum_{j=1}^{k-1} \left(1 - \frac{j}{k}\right) \left(-\frac{j}{k}\right) A_j y^{1-\frac{(j+2)}{k}}.
\end{aligned}
$$

To find coefficients A_j, we will collect the coefficients of $y^{1-\frac{j}{k}}$ for $j = 0, 1, \ldots, k$ by considering A_1, A_2, A_3, A_4. First, we separate A_1 into three terms as $A_1 =: A_{1,1} + A_{1,2} + A_{1,3}$, where

$$
A_{1,1} = y \frac{dA_0}{d\tau}, \quad A_{1,2} = y^{1-\frac{1}{k}} \frac{dA_1}{d\tau}, \quad A_{1,3} = \sum_{j=2}^{k} y^{1-\frac{j}{k}} \frac{dA_j}{d\tau}.
$$

Next, considering $A_2 =: A_{2,1} + A_{2,2}$, where

$$
\begin{aligned}
A_{2,1} &= \left(r(t_i - \tau) - \frac{1}{2}\sigma^2(t_i - \tau) \right) k A_0 y^{1-\frac{1}{k}}, \\
A_{2,2} &= \left(r(t_i - \tau) - \frac{1}{2}\sigma^2(t_i - \tau) \right) k \sum_{j=2}^{k} \left(1 - \frac{(j-1)}{k}\right) A_{j-1} y^{1-\frac{j}{k}}
\end{aligned}
$$

with shifting index j in to $j-1$. Then, we investigate $A_3 =: A_{3,1} + A_{3,2}$, where

$$A_{3,1} = \frac{1}{2}k(k-1)\sigma^2(t_i-\tau)\sum_{j=2}^{k}\left(1-\frac{(j-2)}{k}\right)A_{j-2}y^{1-\frac{j}{k}},$$
$$A_{3,2} = \frac{1}{2}(k-1)\sigma^2(t_i-\tau)A_{k-1}y^{-\frac{1}{k}}$$

with shifting index j in to $j-2$. Finally, we divided $A_4 =: A_{4,1} + A_{4,2}$, where

$$A_{4,1} = \frac{1}{2}k^2\sigma^2(t_i-\tau)\sum_{j=2}^{k}\left(1-\frac{(j-2)}{k}\right)\left(-\frac{(j-2)}{k}\right)A_{j-2}y^{1-\frac{j}{k}},$$
$$A_{4,2} = -\frac{1}{2}(k-1)\sigma^2(t_i-\tau)A_{k-1}y^{-\frac{1}{k}}$$

with shifting index j into $j-2$. Since $A_{3,2} + A_{4,2} = 0$, we have

$$A_{1,1} + A_{1,2} + A_{1,3} = A_{2,1} + \left(A_{2,2} + A_{3,1} + A_{4,1}\right).$$

By collecting the coefficients of $y^{1-\frac{j}{k}}$ for $j = 0, 1, \ldots, k$, this implies a system of ODEs that

$$\frac{dA_0}{d\tau} = 0, \tag{3.13}$$
$$\frac{dA_1}{d\tau} = k\left(r(t_i-\tau) - \frac{1}{2}\sigma^2(t_i-\tau)\right)A_0, \tag{3.14}$$
$$\frac{dA_j}{d\tau} = (k-(j-1))\left(r(t_i-\tau) - \frac{1}{2}\sigma^2(t_i-\tau)\right)A_{j-1} + \frac{1}{2}\left(k-(j-2)\right)\left(k-(j-1)\right)\sigma^2(t_i-\tau)A_{j-2} \tag{3.15}$$

for $j = 2, 3, ..., k$, subject to the initial conditions derived from the terminal condition (3.8) as

$$A_0(0; t_i, k) = 1 \quad \text{and} \quad A_j(0; t_i, k) = 0 \quad \text{for} \quad j = 1, 2, \ldots, k. \tag{3.16}$$

The solution of (3.13), (3.14) and (3.15) subject to the initial conditions (3.16) can be found by integration as expressed in (3.2), (3.3), and (3.4), respectively. This completes the proof of the theorem. □

Next, we investigate the relation of parameters in case that $r(t) - \frac{1}{2}\sigma^2(t)$ is a zero function on $[0, T]$ which can be obtained by the following corollary.

Corollary 3.2. *Suppose that $k \geq 2$ is an integer and S_t follows the EBS model in (2.4). We set $X_t = \ln S_t$ and $\Delta t_i = t_i - t$ for all $i = 1, 2, \dots, N$. If $r(t)$, $\sigma(t) > 0$ are integrable on $[t_{i-1}, t_i]$ in which $r(t) - \frac{1}{2}\sigma^2(t)$ is a zero function on $[t_{i-1}, t_i]$ then*

$$E^Q_{t_{i-1}}[X_t^k] = E^Q[X_t^k \mid X_{t_{i-1}} = x] = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} x^{k-2j} A_{2j}(\Delta t_i; t_i, k)$$

for all $t \in [t_{i-1}, t_i]$ and $x \in (-\infty, \infty)$, where we define $x^0 := 1$ for all $x \in (-\infty, \infty)$ and

$$A_0(\Delta t_i; t_i, k) = 1, \tag{3.17}$$

$$A_2(\Delta t_i; t_i, k) = \frac{1}{2} k(k-1) \int_0^{\Delta t_i} \sigma^2(t_i - \eta) d\eta, \tag{3.18}$$

$$A_{2j}(\Delta t_i; t_i, k) = \frac{1}{2^j} \left(\prod_{r=0}^{2j-1} (k - r) \right) \int_0^{\Delta t_i} \int_0^{\eta_j} \cdots \int_0^{\eta_2} \sigma^2(t_i - \eta_1) \cdots \sigma^2(t_i - \eta_j) d\eta_1 \cdots d\eta_j \tag{3.19}$$

for $j = 2, 3, \dots, \lfloor \frac{k}{2} \rfloor$.

Proof. Since $r(t) - \frac{1}{2}\sigma^2(t)$ is a zero function $[t_{i-1}, t_i]$, we can reduce $A_1(\Delta t_i; t_i, k)$ and $A_j(\Delta t_i; t_i, k)$ defined as (3.3) and (3.4) to the form

$$A_j(\Delta t_i; t_i, k) = \begin{cases} 0, & j \text{ odd}, \\ \frac{1}{2}\left(k - (j-2)\right)\left(k - (j-1)\right) \int_0^{\Delta t_i} \sigma^2(t_i - \eta) A_{j-2}(\eta; t_i, k) d\eta, & j \text{ even}, \end{cases}$$

for $j = 1, 2, \dots, k$. This proof is complete. □

Next, we consider the BS model where r and $\sigma > 0$ are constants with $r \neq \dfrac{1}{2}\sigma^2$. From Theorem 3.1, the ODEs (3.2)-(3.4) subject to the initial conditions (3.16) can be solved analytically as proposed in the following theorem.

Theorem 3.3. *Suppose that S_t follows the BS model in* (2.3) *such that $r \neq \frac{1}{2}\sigma^2$ and $k \geq 1$ is an integer. Then, the solution of ODEs* (3.13)–(3.15) *subject to the initial conditions* (3.16) *can be expressed as*

$$A_j(\tau; t_i, k) = \frac{k!}{(k-j)!} \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(j-2n)!} \left(r - \frac{1}{2}\sigma^2 \right)^{j-2n} \tau^{j-n} \tag{3.20}$$

for $\tau \geq 0$ and $j = 0, 1, ..., k$.

Proof. It suffices to show that $A_j(\tau; t_i, k)$ satisfies the ODEs (3.13)-(3.15) subject to the initial conditions (3.16) for all $j = 0, 1, ..., k$ with $r(t) = r$ and $\sigma(t) = \sigma$. We begin to consider in case of $j = 0, 1, 2$.

$$A_0(\tau; t_i, k) = 1 = \frac{k!}{(k-0)!} \sum_{n=0}^{\lfloor \frac{0}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(0-2n)!} \left(r - \frac{1}{2}\sigma^2 \right)^{0-2n} \tau^{0-n},$$

$$A_1(\tau; t_i, k) = k\tau \left(r - \frac{1}{2}\sigma^2 \right) = \frac{k!}{(k-1)!} \sum_{n=0}^{\lfloor \frac{1}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(1-2n)!} \left(r - \frac{1}{2}\sigma^2 \right)^{1-2n} \tau^{1-n}$$

and

$$\begin{aligned}
A_2(\tau; t_i, k) &= (k-1) \int_0^\tau \left(r - \frac{1}{2}\sigma^2 \right) A_1(\eta; t_i, k) d\eta + \frac{1}{2}k(k-1) \int_0^\tau \sigma^2 A_0(\eta; t_i, k) d\eta \\
&= (k-1) \int_0^\tau k \left(r - \frac{1}{2}\sigma^2 \right)^2 \eta d\eta + \frac{1}{2}k(k-1) \int_0^\tau \sigma^2 d\eta \\
&= \frac{k!}{(k-2)!} \left[\frac{1}{2} \left(r - \frac{1}{2}\sigma^2 \right)^2 \tau^2 + \frac{1}{2}\sigma^2 \tau \right] \\
&= \frac{k!}{(k-2)!} \sum_{n=0}^{\lfloor \frac{2}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(2-2n)!} \left(r - \frac{1}{2}\sigma^2 \right)^{2-2n} \tau^{2-n},
\end{aligned}$$

this show that the solution of ODEs (3.13)-(3.15) can be written in the form (3.20) when $j = 0, 1, 2$.

For other j, we divide it into two cases. If $j = 2\ell + 1$ for $\ell \in \left\{1, 2, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor \right\}$, we have

$$
\begin{aligned}
& A_{2\ell+1}(\tau; t_i, k) \\
&= (k-2\ell)\left(r - \frac{1}{2}\sigma^2\right)\int_0^\tau A_{2\ell}(\eta; t_i, k)d\eta \\
&\quad + \frac{1}{2}\sigma^2 (k-(2\ell-1))(k-2\ell)\int_0^\tau A_{2\ell-1}(\eta; t_i, k)d\eta \\
&= (k-2\ell)\left(r - \frac{1}{2}\sigma^2\right)\int_0^\tau \left(\frac{k!}{(k-2\ell)!}\sum_{n=0}^{\lfloor \frac{2\ell}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(2\ell-2n)!}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell-2n} \eta^{2\ell-n}\right) d\eta \\
&\quad + \frac{1}{2}\sigma^2 (k-(2\ell-1))(k-2\ell) \\
&\quad \int_0^\tau \left(\frac{k!}{(k-(2\ell-1))!}\sum_{n=0}^{\lfloor \frac{2\ell-1}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!((2\ell-1)-2n)!}\left(r - \frac{1}{2}\sigma^2\right)^{(2\ell-1)-2n} \eta^{(2\ell-1)-n}\right) d\eta \\
&= \frac{k!}{(k-(2\ell+1))!}\left[\sum_{n=0}^{\lfloor \frac{2\ell}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(2\ell-2n)!(2\ell-n+1)}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell-2n+1} \tau^{2\ell-n+1}\right. \qquad (3.21) \\
&\quad \left. + \sum_{n=0}^{\lfloor \frac{2\ell-1}{2} \rfloor} \frac{\sigma^{2n+2}}{2^{n+1} n!((2\ell-1)-2n)!(2\ell-n)}\left(r - \frac{1}{2}\sigma^2\right)^{(2\ell-1)-2n} \tau^{2\ell-n}\right] \\
&= \frac{k!}{(k-(2\ell+1))!}\left[A_1 + A_2 + A_3\right], \qquad (3.22)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \frac{1}{(2\ell+1)!}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell+1} \tau^{2\ell+1}, \qquad (3.23) \\
A_2 &= \sum_{n=1}^{\lfloor \frac{2\ell}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(2\ell-2n)!(2\ell-n+1)}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell-2n+1} \tau^{2\ell-n+1}, \qquad (3.24) \\
A_3 &= \sum_{n=0}^{\lfloor \frac{2\ell-1}{2} \rfloor} \frac{\sigma^{2n+2}}{2^{n+1} n!((2\ell-1)-2n)!(2\ell-n)}\left(r - \frac{1}{2}\sigma^2\right)^{(2\ell-1)-2n} \tau^{2\ell-n}.
\end{aligned}
$$

Then, we shift index from n into $n-1$ of A_3 to get that

$$
A_3 = \sum_{n=1}^{\lfloor \frac{2\ell-1}{2} \rfloor + 1} \frac{\sigma^{2n}}{2^n (n-1)!(2\ell-2n+1)!(2\ell-n+1)}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell-2n+1} \tau^{2\ell-n+1}. \qquad (3.25)
$$

From (3.24) and (3.25),

$$
\begin{aligned}
A_2 + A_3 = & \sum_{n=1}^{\lfloor \frac{2\ell+1}{2} \rfloor} \left[\frac{\sigma^2}{2^n n!(2\ell-2n)!(2\ell-n+1)} + \frac{\sigma^2}{2^n (n-1)!(2\ell-2n+1)!(2\ell-n+1)} \right] \\
& \left(r - \frac{1}{2}\sigma^2 \right)^{2\ell-2n+1} \tau^{2\ell-n+1} \\
= & \sum_{n=1}^{\lfloor \frac{2\ell+1}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!((2\ell+1)-2n)!} \left(r - \frac{1}{2}\sigma^2 \right)^{(2\ell+1)-2n} \tau^{(2\ell+1)-n}. \qquad (3.26)
\end{aligned}
$$

We conclude form (3.22), (3.23) and (3.26),

$$
A_{2\ell+1}(\tau; t_i, k) = \frac{k!}{(k-(2\ell+1))!} \sum_{n=0}^{\lfloor \frac{2\ell+1}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!((2\ell+1)-2n)!} \left(r - \frac{1}{2}\sigma^2 \right)^{(2\ell+1)-2n} \tau^{(2\ell+1)-n}, \qquad (3.27)
$$

which can be written in the form of (3.20) when set $j = 2\ell + 1$ in this case.

If $j = 2\ell + 2$ for $\ell \in \left\{1, 2, \ldots, \left\lfloor \frac{k-2}{2} \right\rfloor\right\}$, we have

$$\begin{aligned}
&A_{2\ell+2}(\tau; t_i, k) \\
&= (k - (2\ell+1))\left(r - \frac{1}{2}\sigma^2\right)\int_0^\tau A_{2\ell+1}(\eta; t_i, k)d\eta \\
&\quad + \frac{1}{2}\sigma^2\,(k-2\ell)\,(k-(2\ell+1))\int_0^\tau A_{2\ell}(\eta; t_i, k)d\eta \\
&= (k - (2\ell+1))\left(r - \frac{1}{2}\sigma^2\right) \\
&\quad \int_0^\tau \left(\frac{k!}{(k-(2\ell+1))!}\sum_{n=0}^{\left\lfloor\frac{2\ell+1}{2}\right\rfloor}\frac{\sigma^{2n}}{2^n n!(2\ell+1-2n)!}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell+1-2n}\eta^{2\ell+1-n}\right)d\eta \\
&\quad + \frac{1}{2}\sigma^2\,(k-2\ell)\,(k-(2\ell+1)) \\
&\quad \int_0^\tau \left(\frac{k!}{(k-2\ell)!}\sum_{n=0}^{\left\lfloor\frac{2\ell}{2}\right\rfloor}\frac{\sigma^{2n}}{2^n n!(2\ell-2n)!}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell-2n}\eta^{2\ell-n}\right)d\eta \\
&= \frac{k!}{(k-(2\ell+2))!}\left[\sum_{n=0}^{\left\lfloor\frac{2\ell+1}{2}\right\rfloor}\frac{\sigma^{2n}}{2^n n!(2\ell+1-2n)!(2\ell+2-n)}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell+2-2n}\tau^{2\ell+2-n}\right. \\
&\quad \left. + \sum_{n=0}^{\left\lfloor\frac{2\ell}{2}\right\rfloor}\frac{\sigma^{2n+2}}{2^{n+1}n!(2\ell-2n)!(2\ell+1-n)}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell-2n}\tau^{2\ell+1-n}\right] \\
&=: \frac{k!}{(k-(2\ell+2))!}\left[B_1 + B_2 + B_3 + B_4\right] \qquad (3.28)
\end{aligned}$$

where

$$B_1 = \frac{1}{(2\ell+2)!}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell+2}\tau^{2\ell+2}, \qquad (3.29)$$

$$B_2 = \sum_{n=1}^{\left\lfloor\frac{2\ell+1}{2}\right\rfloor}\frac{\sigma^{2n}}{2^n n!(2\ell+1-2n)!(2\ell+2-n)}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell+2-2n}\tau^{2\ell+2-n}, \qquad (3.30)$$

$$B_3 = \sum_{n=0}^{\left\lfloor\frac{2\ell}{2}\right\rfloor-1}\frac{\sigma^{2n+2}}{2^{n+1}n!(2\ell-2n)!(2\ell+1-n)}\left(r - \frac{1}{2}\sigma^2\right)^{2\ell-2n}\tau^{2\ell+1-n},$$

$$B_4 = \frac{\sigma^{2\ell+2}}{2^{\ell+1}(\ell+1)!}\tau^{\ell+1}. \qquad (3.31)$$

Shifting index n into $n-1$ of B_3, we obtain that

$$B_3 = \sum_{n=1}^{\lfloor \frac{2\ell}{2} \rfloor} \frac{\sigma^{2n}}{2^n (n-1)!(2\ell-2n+2)!(2\ell+2-n)} \left(r - \frac{1}{2}\sigma^2\right)^{2\ell+2-2n} \tau^{2\ell+2-n} \tag{3.32}$$

From (3.30) and (3.32),

$$\begin{aligned} &B_2 + B_3 \\ &= \sum_{n=1}^{\lfloor \frac{2\ell+1}{2} \rfloor} \left[\frac{\sigma^{2n}}{2^n n!(2\ell+1-2n)!(2\ell+2-n)} + \frac{\sigma^{2n}}{2^n (n-1)!(2\ell-2n+2)!(2\ell+2-n)} \right] \\ &\quad \left(r - \frac{1}{2}\sigma^2\right)^{2\ell+2-2n} \tau^{2\ell+2-n} \\ &= \sum_{n=1}^{\lfloor \frac{2\ell+1}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!((2\ell+2)-2n)!} \left(r - \frac{1}{2}\sigma^2\right)^{(2\ell+2)-2n} \tau^{(2\ell+2)-n}. \end{aligned} \tag{3.33}$$

We conclude from (3.28), (3.29), (3.31) and (3.33),

$$A_{2\ell+2}(\tau; t_i, k) = \frac{k!}{(k-(2\ell+2))!} \sum_{n=0}^{\lfloor \frac{2\ell+2}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!((2\ell+2)-2n)!} \left(r - \frac{1}{2}\sigma^2\right)^{(2\ell+2)-2n} \tau^{(2\ell+2)-n},$$

which can be written in the form of (3.20) when set $j = 2\ell+2$ in this case. □

The next corollary is a special case of Corollary 3.2 when $r = \dfrac{1}{2}\sigma^2$ described by BS model.

Corollary 3.4. *Suppose that S_t follows the BS model in (2.3) such that $r = \frac{1}{2}\sigma^2$ and $k \geq 1$ is an integer. Then, the solution of ODEs (3.17)–(3.19) can be express as*

$$A_{2j}(\tau; t_i, k) = \begin{cases} 1, & j = 0, \\ \left(\displaystyle\prod_{r=0}^{2j-1} (k-r)\right) \dfrac{\sigma^{2j}}{2^j j!} \tau^j, & j = 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor, \end{cases} \tag{3.34}$$

for $\tau \geq 0$.

Proof. It suffices to show that $A_{2j}(\tau; t_i, k)$ satisfies (3.17)-(3.19) for all $j = 1, 2, ..., \lfloor \frac{k}{2} \rfloor$

with $r(t) = r$ and $\sigma(t) = \sigma$. We begin to consider in case of $j = 1$.

$$A_2(\tau; t_i, k) = \frac{1}{2}k(k-1)\sigma^2\tau.$$

If $j = 2, \dots, \lfloor \frac{k}{2} \rfloor$, we write the counterpart form of (3.19) as

$$\begin{aligned} A_{2j}(\tau; t_i, k) &= \frac{1}{2}\left(k - (2j-2)\right)\left(k - (2j-1)\right)\int_0^\tau \sigma^2 A_{2j-2}(\eta; t_i, k)d\eta \\ &= \frac{1}{2}\left(k - (2j-2)\right)\left(k - (2j-1)\right)\int_0^\tau \sigma^2 \left(\left(\prod_{r=0}^{2j-3}(k-r)\right)\frac{\sigma^{2(j-1)}}{2^{j-1}(j-1)!}\tau^{j-1}\right)d\eta \\ &= \left(\prod_{r=0}^{2j-1}(k-r)\right)\frac{\sigma^{2j}}{2^j j!}\tau^j. \end{aligned}$$

This show that (3.34) holds for $j = 0, 1, \dots, \lfloor \frac{k}{2} \rfloor$. □

3.2 Pricing Moment Swaps

This section derives analytical formulas for pricing discretely-sampled moment swaps under the EBS model (2.4) and the BS model (2.3) based on two cases of condition on parameters, $r(t) - \frac{1}{2}\sigma^2(t) \neq 0$ or $r(t) - \frac{1}{2}\sigma^2(t) = 0$ for all t.

The following lemmas will be used to derive the fair delivery price of moment swaps under the EBS model (2.4).

Lemma 3.5. *Let $\tau, \zeta \in \mathbb{R}$, $j \in \mathbb{N} \cup \{0\}$ and let $A_j(\tau; \zeta, k_1), A_j(\tau; \zeta, k_2)$ be the sequence of function defined as* (3.2), (3.3) *and* (3.4). *Then,*

$$A_j(\tau; \zeta, k_1) = \frac{k_1!}{(k_1-j)!}\frac{(k_2-j)!}{k_2!}A_j(\tau; \zeta, k_2) \tag{3.35}$$

for all $k_1, k_2 \in \{j, j+1, ...\}$.

Proof. We shall prove the lemma by using the strong induction principle. First, we will to investigate that (3.35) holds for $j = 0, 1, 2$. Hence,

$$A_0(\tau;\zeta,k_1) = 1 = A_0(\tau;\zeta,k_2),$$

$$\begin{aligned}
A_1(\tau;\zeta,k_1) &= k_1 \int_0^\tau \left(r(\zeta-\eta) - \frac{1}{2}\sigma^2(\zeta-\eta) \right) d\eta \\
&= \frac{k_1!}{(k_1-1)!} \left(\frac{k_2!}{k_2!}\right) \int_0^\tau \left(r(\zeta-\eta) - \frac{1}{2}\sigma^2(\zeta-\eta) \right) d\eta \\
&= \frac{k_1!}{(k_1-1)!} \frac{(k_2-1)!}{k_2!} k_2 \int_0^\tau \left(r(\zeta-\eta) - \frac{1}{2}\sigma^2(\zeta-\eta) \right) d\eta \\
&= \frac{k_1!}{(k_1-1)!} \frac{(k_2-1)!}{k_2!} A_1(\tau;\zeta,k_2)
\end{aligned}$$

and

$$\begin{aligned}
A_2(\tau;\zeta,k_1) &= (k_1-1) \int_0^\tau \left(r(\zeta-\eta) - \frac{1}{2}\sigma^2(\zeta-\eta) \right) A_1(\eta;\zeta,k_1) d\eta \\
&\quad + \frac{1}{2}k_1(k_1-1) \int_0^\tau \sigma^2(\zeta-\eta) A_0(\eta;\zeta,k_1) \\
&= (k_1-1) \int_0^\tau \left(r(\zeta-\eta) - \frac{1}{2}\sigma^2(\zeta-\eta) \right) \left(\frac{k_1!}{(k_1-1)!} \frac{(k_2-1)!}{k_2!} A_1(\eta;\zeta,k_2) \right) d\eta \\
&\quad + \frac{1}{2}k_1(k_1-1) \int_0^\tau \sigma^2(\zeta-\eta) A_0(\eta;\zeta,k_2) d\eta \\
&= \frac{k_1!}{(k_1-2)!} \frac{(k_2-2)!}{k_2!} (k_2-1) \int_0^\tau \left(r(\zeta-\eta) - \frac{1}{2}\sigma^2(\zeta-\eta) \right) A_1(\eta;\zeta,k_2) d\eta \\
&\quad + k_1(k_1-1) \frac{1}{k_2(k_2-1)} \frac{1}{2} k_2(k_2-1) \int_0^\tau \sigma^2(\zeta-\eta) A_0(\eta;\zeta,k_2) d\eta \\
&= \frac{k_1!}{(k_1-2)!} \frac{(k_2-2)!}{k_2!} (k_2-1) \int_0^\tau \left(r(\zeta-\eta) - \frac{1}{2}\sigma^2(\zeta-\eta) \right) A_1(\eta;\zeta,k_2) d\eta \\
&\quad + \frac{k_1!}{(k_1-2)!} \frac{(k_2-2)!}{k_2!} \frac{1}{2} k_2(k_2-1) \int_0^\tau \sigma^2(\zeta-\eta) A_0(\eta;\zeta,k_2) d\eta \\
&= \frac{k_1!}{(k_1-2)!} \frac{(k_2-2)!}{k_2!} A_2(\tau;\zeta,k_2).
\end{aligned}$$

Let $n \in \mathbb{N}$. We assume that (3.35) holds for $j = 0, 1, \ldots, n$. From (3.4), we separate to two terms as

$$A_{n+1}(\tau;\zeta,k_1) = A'_{n+1}(\tau;\zeta,k_1) + A''_{n+1}(\tau;\zeta,k_1),$$

where

$$A'_{n+1}(\tau;\zeta,k_1) = (k_1-n)\int_0^\tau \left(r(\zeta-\eta)-\frac{1}{2}\sigma^2(\zeta-\eta)\right)A_n(\eta;\zeta,k_1)d\eta,$$
$$A''_{n+1}(\tau;\zeta,k_1) = \frac{1}{2}(k_1-(n-1))(k_1-n)\int_0^\tau \sigma^2(\zeta-\eta)A_{n-1}(\eta;\zeta,k_1)d\eta.$$

By the hypothesis for $k_1, k_1 \geq n$, using (3.35) with $j=n$ and $j=n-1$ gives

$$\begin{aligned}
&A'_{n+1}(\tau;\zeta,k_1)\\
&=(k_1-n)\int_0^\tau \left(r(\zeta-\eta)-\frac{1}{2}\sigma^2(\zeta-\eta)\right)A_n(\eta;\zeta,k_1)d\eta,\\
&=(k_1-n)\int_0^\tau \left(r(\zeta-\eta)-\frac{1}{2}\sigma^2(\zeta-\eta)\right)\left(\frac{k_1!}{(k_1-n)!}\frac{(k_2-n)!}{k_2!}A_n(\eta;\zeta,k_2)\right)d\eta,\\
&=\frac{k_1!}{(k_1-(n+1))!}\frac{(k_2-(n+1))!}{k_2!}(k_2-n)\int_0^\tau \left(r(\zeta-\eta)-\frac{1}{2}\sigma^2(\zeta-\eta)\right)A_n(\eta;\zeta,k_2)d\eta
\end{aligned}$$

and

$$\begin{aligned}
&A'_{n+1}(\tau;\zeta,k_1)\\
&=\frac{1}{2}(k_1-(n-1))(k_1-n)\int_0^\tau \sigma^2(\zeta-\eta)A_{n-1}(\eta;\zeta,k_1)d\eta\\
&=\frac{1}{2}(k_1-(n-1))(k_1-n)\int_0^\tau \sigma^2(\zeta-\eta)\left(\frac{k_1!}{(k_1-(n-1))!}\frac{(k_2-(n-1))!}{k_2!}A_{n-1}(\eta;\zeta,k_2)\right)d\eta\\
&=\frac{k_1!}{(k_1-(n+1))!}\frac{(k_2-(n+1))!}{k_2!}\frac{1}{2}(k_2-(n-1))(k_2-n)\int_0^\tau \sigma^2(\zeta-\eta)A_{n-1}(\eta;\zeta,k_2)d\eta,
\end{aligned}$$

respectively. Therefore, from (3.4),

$$A_{n+1}(\tau;\zeta,k_1) = \frac{k_1!}{(k_1-(n+1))!}\frac{(k_2-(n+1))!}{k_2!}A_{n+1}(\tau;\zeta,k_2). \tag{3.36}$$

This show that (3.35) holds for $j=n+1$, hence, it is true for all $j\in\mathbb{N}$. □

Lemma 3.6. *For $0\leq j\leq m-1$, we have*

$$\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}\frac{k!}{(k-j)!}=0.$$

Proof. By combinatorial techniques, we obtain that

$$\begin{aligned}
\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}\frac{k!}{(k-j)!} &= \sum_{k=j}^{m}(-1)^{m-k}\frac{m!}{(m-k)!k!}\frac{k!}{(k-j)!} && \text{(by combination)}\\
&= \sum_{k=j}^{m}(-1)^{m-k}\frac{m!}{(m-k)!(k-j)!}\\
&= \sum_{k=j}^{m}(-1)^{m-k}\binom{m-j}{k-j}\frac{m!}{(m-j)!}\\
&= \sum_{k=0}^{m-j}(-1)^{m-(k+j)}\binom{m-j}{k}\frac{m!}{(m-j)!} && \text{(by shifting index)}\\
&= \frac{m!}{(m-j)!}(-1)^{m-j}\sum_{k=0}^{m-j}\binom{m-j}{k}(-1)^{k}\\
&= \frac{m!}{(m-j)!}(-1)^{m-j}\left(1+(-1)\right)^{m-j} && \text{(by binomial theorem)}\\
&= 0.
\end{aligned}$$

□

In the following theorem, we derive the fair delivery price of the m^{th}-moment swap under the EBS model (2.4) by utilizing Theorem 3.1, Lemma 3.5 and Lemma 3.6.

Theorem 3.7. *Suppose that S_t follows the EBS model* (2.4) *and $m \geq 2$ is an integer. Then, the fair delivery price of the m^{th}-moment swap under the EBS model* (2.4) *in which $r(t) - \frac{1}{2}\sigma^2(t)$ is not a zero function on $[0, T]$, denoted by K^m_{EBS}, can be expressed as*

$$K^m_{\text{EBS}}(T,N) = \frac{1}{T}\sum_{i=1}^{N} A_m(\Delta t; t_i, m) \tag{3.37}$$

where $\Delta t = \frac{T}{N}, t_i = i\Delta t, i = 0, 1, ..., N,$ and $A_m(\Delta t; t_i, m)$ are defined in (3.2)–(3.4).

Proof. From (2.6) and (2.7), we have

$$K^m_{\text{EBS}}(T,N) = E^Q_0\left[\frac{1}{T}\sum_{i=1}^{N}\left(X_{t_i} - X_{t_{i-1}}\right)^m\right] = \frac{1}{T}\sum_{i=1}^{N}E^Q_0\left[\left(X_{t_i} - X_{t_{i-1}}\right)^m\right]$$

It suffices to show that

$$E_0^Q\left[\left(X_{t_i}-X_{t_{i-1}}\right)^m\right]=A_m(\Delta t;t_i,m). \tag{3.38}$$

Hence,

$$\begin{aligned}
&E_0^Q\left[\left(X_{t_i}-X_{t_{i-1}}\right)^m\right]\\
&=E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}X_{t_{i-1}}^{m-k}X_{t_i}^{k}\right] && \text{(by binomial theorem)}\\
&=E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}X_{t_{i-1}}^{m-k}E_{t_{i-1}}^Q\left[X_{t_i}^{k}\right]\right] && \text{(by Tower property)}\\
&=E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}X_{t_{i-1}}^{m-k}\left(\sum_{j=0}^{k}A_j(\Delta t;t_i,k)X_{t_{i-1}}^{k-j}\right)\right] && \text{(by Theorem 3.1)}\\
&=E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\sum_{j=0}^{k}A_j(\Delta t;t_i,k)X_{t_{i-1}}^{m-j}\right]\\
&=E_0^Q\left[\sum_{j=0}^{m}\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}A_j(\Delta t;t_i,k)X_{t_{i-1}}^{m-j}\right] && \text{(by rearrangement of summation)}\\
&=E_0^Q\left[\sum_{j=0}^{m}\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}\left(\frac{k!}{(k-j)!j!}A_j(\Delta t;t_i,j)\right)X_{t_{i-1}}^{m-j}\right] && \text{(by Lemma 3.5)}\\
&=E_0^Q\left[\sum_{j=0}^{m}\left(\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}\frac{k!}{(k-j)!}\right)\frac{1}{j!}A_j(\Delta t;t_i,j)X_{t_{i-1}}^{m-j}\right]\\
&=E_0^Q\left[\sum_{j=0}^{m-1}\left(\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}\frac{k!}{(k-j)!}\right)\frac{1}{j!}A_j(\Delta t;t_i,j)X_{t_{i-1}}^{m-j}+A_m(\Delta t;t_i,m)\right]\\
&=E_0^Q\left[A_m(\Delta t;t_i,m)\right] && \text{(by Lemma 3.6)}\\
&=A_m(\Delta t;t_i,m). && \text{(by deterministic function)}
\end{aligned}$$

□

We can derive the next corollary by using Corollary 3.2 with Theorem 3.7.

Corollary 3.8. *Suppose that S_t follows the EBS model* (2.4) *and $m \geq 2$ is an integer. Then, the fair delivery price of the m^{th}-moment swap under the EBS model* (2.4) *in which*

$r(t)-\frac{1}{2}\sigma^2(t)$ is a zero function on $[0,T]$, denoted by $K^m_{\mathrm{EBS}^*}$, can be expressed as

$$K^m_{\mathrm{EBS}^*}(T,N)=\begin{cases}0, & m \text{ odd},\\ \dfrac{1}{T}\displaystyle\sum_{i=1}^{N} A_m(\Delta t; t_i, m), & m \text{ even},\end{cases}$$

where $\Delta t=\frac{T}{N}, t_i=i\Delta t, i=0,1,...,N$, and $A_m(\Delta t; t_i, m)$ are defined in (3.17)–(3.19).

Proof. From Theorem 3.7 and utilizing (3.17), (3.18) and (3.19) in Corollary 3.2. This completes the proof. □

Applying Theorem 3.3 and Theorem 3.7, the fair delivery price of moment swaps under the BS model (2.3) when $r \neq \dfrac{1}{2}\sigma^2$ can be deduced as follows.

Theorem 3.9. *Suppose that S_t follows the BS model (2.3) such that $r \neq \frac{1}{2}\sigma^2$ and $m \geq 2$ is an integer. Then, the fair delivery price of the m^{th}-moment swap under the BS model (2.3), denoted by K^m_{BS}, can be expressed as*

$$K^m_{\mathrm{BS}}(T,N)=\sum_{n=0}^{\lfloor \frac{m}{2} \rfloor}\frac{m!}{2^n n!(m-2n)!}\sigma^{2n}\left(r-\frac{1}{2}\sigma^2\right)^{m-2n}(\Delta t)^{m-n-1} \tag{3.39}$$

where $\Delta t=\frac{T}{N}$. In particular, the fair delivery prices of variance, skewness, and kurtosis swaps under the BS model (2.3) can be expressed as

$$K^2_{\mathrm{BS}}(T,N)=\left(r-\frac{\sigma^2}{2}\right)^2\frac{T}{N}+\sigma^2,$$
$$K^3_{\mathrm{BS}}(T,N)=\left(r-\frac{\sigma^2}{2}\right)^3\frac{T^2}{N^2}+3\sigma^2\left(r-\frac{\sigma^2}{2}\right)\frac{T}{N},$$
$$K^4_{\mathrm{BS}}(T,N)=\left(r-\frac{\sigma^2}{2}\right)^4\frac{T^3}{N^3}+6\sigma^2\left(r-\frac{\sigma^2}{2}\right)^2\frac{T^2}{N^2}+3\sigma^4\frac{T}{N},$$

respectively.

Proof. Utilizing (3.20) in Theorem 3.3, we have

$$A_m(\Delta t; t_i, m) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(m-2n)!} \left(r - \frac{1}{2}\sigma^2\right)^{m-2n} (\Delta t)^{m-n}.$$

This implies from Theorem 3.7 that

$$\begin{aligned} K_{\text{BS}}^m(T, N) &= \frac{1}{T} \sum_{i=1}^{N} A_m(\Delta t; t_i, m) \qquad (3.40) \\ &= \frac{1}{T} \sum_{i=1}^{N} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(m-2n)!} \left(r - \frac{1}{2}\sigma^2\right)^{m-2n} (\Delta t)^{m-n} \\ &= \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(m-2n)!} \left(r - \frac{1}{2}\sigma^2\right)^{m-2n} (\Delta t)^{m-n-1}. \end{aligned}$$

□

From Corollary 3.4 and Theorem 3.7, we can derive the fair delivery price of moment swaps under the BS model (2.3) when $r = \dfrac{1}{2}\sigma^2$ can be deduced as follows.

Theorem 3.10. *Suppose that S_t follows the BS model* (2.3) *such that $r = \frac{1}{2}\sigma^2$ and $m \geq 2$ is an integer. Then, the fair delivery price of the m^{th}-moment swap under the BS model* (2.3)*, denoted by $K_{\text{BS}^*}^m$, can be expressed as*

$$K_{\text{BS}^*}^m(T, N) = \begin{cases} 0, & m \text{ odd}, \\ \left(\displaystyle\prod_{r=0}^{m-1} (m-r)\right) \dfrac{\sigma^m}{2^{\frac{m}{2}} (\frac{m}{2})!} \Delta t^{\frac{m}{2}-1}, & m \text{ even}, \end{cases}$$

where $\Delta t = \frac{T}{N}$.

Proof. Obviously, $K_{\text{BS}^*}^m(T, N) = 0$ when m is odd. Utilizing (3.34) in Corollary 3.4, we have

$$A_m(\Delta t; t_i, m) = \left(\prod_{r=0}^{m-1} (m-r)\right) \frac{\sigma^m}{2^{\frac{m}{2}} (\frac{m}{2})!} \Delta t^{\frac{m}{2}}$$

This implies from Theorem 3.7 that

$$\begin{aligned} K^m_{\mathrm{BS}^*}(T,N) &= \frac{1}{T}\sum_{i=1}^{N} A_m(\Delta t; t_i, m) \\ &= \frac{1}{T}\sum_{i=1}^{N}\left(\prod_{r=0}^{m-1}(m-r)\right)\frac{\sigma^m}{2^{\frac{m}{2}}(\frac{m}{2})!}\Delta t^{\frac{m}{2}} \\ &= \left(\prod_{r=0}^{m-1}(m-r)\right)\frac{\sigma^m}{2^{\frac{m}{2}}(\frac{m}{2})!}\Delta t^{\frac{m}{2}-1}. \end{aligned}$$

□

3.3 Positivity of Validated Solution

The construction of the formula for pricing moment swaps under the EBS model (2.4) presents some interesting discussions in terms of the validity of the solution. The purpose of such an examination is to ensure the fundamental assumptions that the fair delivery price of a moment swap is finite and strictly positive for a given set of parameters determined from market data.

Theorem 3.11. *According to Theorem* 3.7*, if the parameter functions* $r(t)$*,* $\sigma(t) > 0$ *are integrable and satisfy*

$$r(t) - \frac{1}{2}\sigma^2(t) > 0 \tag{3.41}$$

for all $t \in [0, T]$*. Then,*

$$0 < K^m_{\mathrm{EBS}}(T,N) < \infty \tag{3.42}$$

for all integer $m \geq 2$*.*

Proof. Since the function $r(t) - \frac{1}{2}\sigma^2(t)$ is integrable on $[0, T]$, we can compute the coefficient functions $A_j(\Delta t; t_i, m)$ for all $i = 1, ..., N$, and $j = 1, 2, .., m$ by (3.3) and (3.4).

This implies that $A_m(\Delta t; t_i, m)$ are bounded for all $i = 1, \ldots, N$. Hence,

$$K_{\text{EBS}}^m(T, N) = \frac{1}{T} \sum_{i=1}^{N} A_m(\Delta t; t_i, m) < \infty.$$

Similarly, from (3.3) and (3.4), it follows that $A_j(\Delta t; t_i, m) > 0$ for all $i = 1, ..., N$, and $j = 1, 2, .., m$ by the positive condition (3.41). □

3.4 Comparison of Fair Delivery Prices

This subsection provides a comparison theorem for the fair delivery prices of different moment swaps under the BS model (2.3). The following theorem demonstrates that trading variance swaps is more expensive than trading any higher moment swaps.

Theorem 3.12. *According to Theorem* 3.9, *we suppose that* $r > \frac{1}{2}\sigma^2$ *and* m, n *are integers such that* $2 \leq n < m - 1$. *Then,*

$$K_{\text{BS}}^m(T, N) < K_{\text{BS}}^n(T, N) \tag{3.43}$$

for $\frac{T}{N} \in (0, \tau_{m,n}^*)$ *where* $\tau_{m,n}^*$ *is the smallest positive root of a polynomial function of degree* $m - n + \lfloor \frac{n}{2} \rfloor$ *with respect to* τ *defined by*

$$P_{m-n+\lfloor \frac{n}{2} \rfloor}(s) := \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{m,j} s^{m-(n+j)+\lfloor \frac{n}{2} \rfloor} - \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} C_{n,j} s^{\lfloor \frac{n}{2} \rfloor - j},$$

$$C_{\ell,j} := \frac{\ell!}{2^j j!(\ell - 2j)!} \sigma^{2j} \left(r - \frac{1}{2}\sigma^2 \right)^{\ell - 2j} \tag{3.44}$$

for $\ell = m, n$. *In particular,*

$$K_{\text{BS}}^m(T, N) = K_{\text{BS}}^n(T, N) \tag{3.45}$$

when $\frac{T}{N} = \tau_{m,n}^*$.

Proof. Recall from (3.39),

$$K_{\mathrm{BS}}^{m}(T,N)=\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\frac{m!}{2^j j!(m-2j)!}\sigma^{2j}\left(r-\frac{1}{2}\sigma^2\right)^{m-2j}\tau^{m-j-1}=:\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}C_{m,j}\tau^{m-j-1}$$

and

$$K_{\mathrm{BS}}^{n}(T,N)=\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!}{2^j j!(n-2j)!}\sigma^{2j}\left(r-\frac{1}{2}\sigma^2\right)^{n-2j}\tau^{n-j-1}=:\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}C_{n,j}\tau^{n-j-1}$$

one can derive the following relation

$$\begin{aligned}K_{\mathrm{BS}}^{m}(T,N)-K_{\mathrm{BS}}^{n}(T,N)&=\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}C_{m,j}\tau^{m-j-1}-\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}C_{n,j}\tau^{n-j-1}\\&=\tau^{n-\lfloor \frac{n}{2}\rfloor-1}\left(\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}C_{m,j}\tau^{m-(n+j)+\lfloor \frac{n}{2}\rfloor}-\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}C_{n,j}\tau^{\lfloor \frac{n}{2}\rfloor-j}\right)\\&=:\tau^{n-\lfloor \frac{n}{2}\rfloor-1}P_{m-n+\lfloor \frac{n}{2}\rfloor}(\tau)\end{aligned}$$

for $\tau=\dfrac{T}{N}$. To obtain (3.43), we shall show that $\lim_{s\to 0^+}P_{m-n+\lfloor \frac{n}{2}\rfloor}(s)<0$ and note that

$$\begin{aligned}&P_{m-n+\lfloor \frac{n}{2}\rfloor}(s)\\&=\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}C_{m,j}s^{m-(n+j)+\lfloor \frac{n}{2}\rfloor}-\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}C_{n,j}s^{\lfloor \frac{n}{2}\rfloor-j}\\&=C_{m,0}s^{m-n+\lfloor \frac{n}{2}\rfloor}+\sum_{j=1}^{\lfloor \frac{m}{2}\rfloor-1}C_{m,j}s^{m-(n+j)+\lfloor \frac{n}{2}\rfloor}+C_{m,\lfloor \frac{m}{2}\rfloor}s^{m-(n+\lfloor \frac{m}{2}\rfloor)+\lfloor \frac{n}{2}\rfloor}\\&\quad-\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor-1}C_{n,j}s^{\lfloor \frac{n}{2}\rfloor-j}-C_{n,\lfloor \frac{n}{2}\rfloor}.\end{aligned}\tag{3.46}$$

Since $2 \leq n < m-1$ and $r > \frac{1}{2}\sigma^2$, the limit can be deduced from

$$\begin{aligned}\lim_{s\to 0^+} P_{m-n+\lfloor \frac{n}{2} \rfloor}(s) &= -C_{n,\lfloor \frac{n}{2} \rfloor} \\ &= -\frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}\lfloor \frac{n}{2} \rfloor!(n-2\lfloor \frac{n}{2} \rfloor)!}\sigma^{2\lfloor \frac{n}{2} \rfloor}\left(r-\frac{1}{2}\sigma^2\right)^{n-2\lfloor \frac{n}{2} \rfloor} \\ &< 0. \end{aligned} \tag{3.47}$$

Next, we consider the coefficient of the highest order (3.46) as $s^{m-n+\lfloor \frac{n}{2} \rfloor}$. We note from (3.44) that $C_{m,0} = (r - \dfrac{1}{2}\sigma^2)^m > 0$ and this implies

$$\lim_{s\to\infty} P_{m-n+\lfloor \frac{n}{2} \rfloor}(s) = \infty. \tag{3.48}$$

From (3.47) and (3.48), we immediately obtain that $P_{m-n+\lfloor \frac{n}{2} \rfloor}(s)$ has at least one positive root by the Intermediate Value Theorem and the continuity of $P_{m-n+\lfloor \frac{n}{2} \rfloor}(s)$. We let $\tau^*_{m,n}$ be the smallest positive root. Therefore, (3.43) and (3.45) hold for $\frac{T}{N} \in (0, \tau^*_{m,n})$ and $\dfrac{T}{N} = \tau^*_{m,n}$, respectively. □

Corollary 3.13. *According to Theorem* 3.12, *if* $r > \frac{1}{2}\sigma^2$ *then* (3.43) *and* (3.45) *hold for all integers* m, n *such that* m *is odd and* $2 \leq n < m$.

Proof. The proof is complete following the fact that when m is odd, (3.47) and (3.48) hold for $2 \leq n < m$. □

Corollary 3.14. *According to Theorem* 3.12, *if* $r > \frac{3}{2}\sigma^2$ *then* (3.43) *and* (3.45) *hold for all integers* m, n *such that* $2 \leq n < m$.

Proof. Since $r > \frac{3}{2}\sigma^2 > \frac{1}{2}\sigma^2$. Thus, we have the following facts: (i) (3.47) and (3.48) hold for $2 \leq n < m-1$ from Theorem 3.12 and (ii) (3.47) and (3.48) hold for m is odd and $2 \leq n < m$ from Corollary 3.13. Next, we suffices to consider $\lim\limits_{s\to 0^+} P_{m-n+\lfloor \frac{n}{2} \rfloor}(s)$ under the case that m is even and $n = m-1$. The limit can be deduced from (3.44), (3.46) that

$$
\begin{aligned}
\lim_{s\to 0^+} P_{m-n+\lfloor \frac{n}{2} \rfloor}(s) &= C_{m,\lfloor \frac{m}{2} \rfloor} - C_{n,\lfloor \frac{n}{2} \rfloor} \\
&= C_{2h,h} - C_{2h-1,h-1} \\
&= \frac{(2h)!}{2^h h!}\sigma^{2h} - \frac{(2h-1)!}{2^{h-1}(h-1)!}\sigma^{2h-2}\left(r - \frac{1}{2}\sigma^2\right) \\
&= \frac{(2h)!}{2^h h!}\sigma^{2h-2}\left(\sigma^2 - \left(r - \frac{1}{2}\sigma^2\right)\right) \\
&= -\frac{(2h)!}{2^h h!}\sigma^{2h-2}\left(r - \frac{3}{2}\sigma^2\right) \\
&< 0. \qquad (3.49)
\end{aligned}
$$

where $m = 2h$ for some positive integer h. Using (3.47) and (3.49), we now obtain (3.43) and (3.45) for $\frac{T}{N} \in (0, \tau^*_{m,n})$ and $\frac{T}{N} = \tau^*_{m,n}$, respectively. □

3.5 Numerical Results and Discussions

In this section, numerical examples are presented to demonstrating the correctness of our closed-form formulas (3.37) and (3.39). We compare the results obtained from our formulas and those from MC simulations. Although theoretically there would be no need to discuss the accuracy of the closed-form formulas and present numerical results, some comparisons with the MC simulations provide a sense of verification for the newly found solutions. This is particularly so for some market practitioners who are very used to MC simulations and would not trust analytical solutions that may contain algebraic errors, unless they have seen numerical evidence of such a comparison.

Example 3.15 (Comparison to MC simulations)**.** In this example, we confirm our closed-form formula (3.28) by comparing with MC simulations. The parameters used in the experiment are $N = 252$, and for various $T = 0.1, 0.2, \ldots, 1.0$. The testing is taken on the EBS with the parameter functions $r(t) = 0.075 + 0.05t$ and $\sigma(t) = \sqrt{0.03 + 0.02t}$ satisfying the condition (3.41). The comparisons for $m = 2, 3, 4$ as displayed in Figure 3.1.

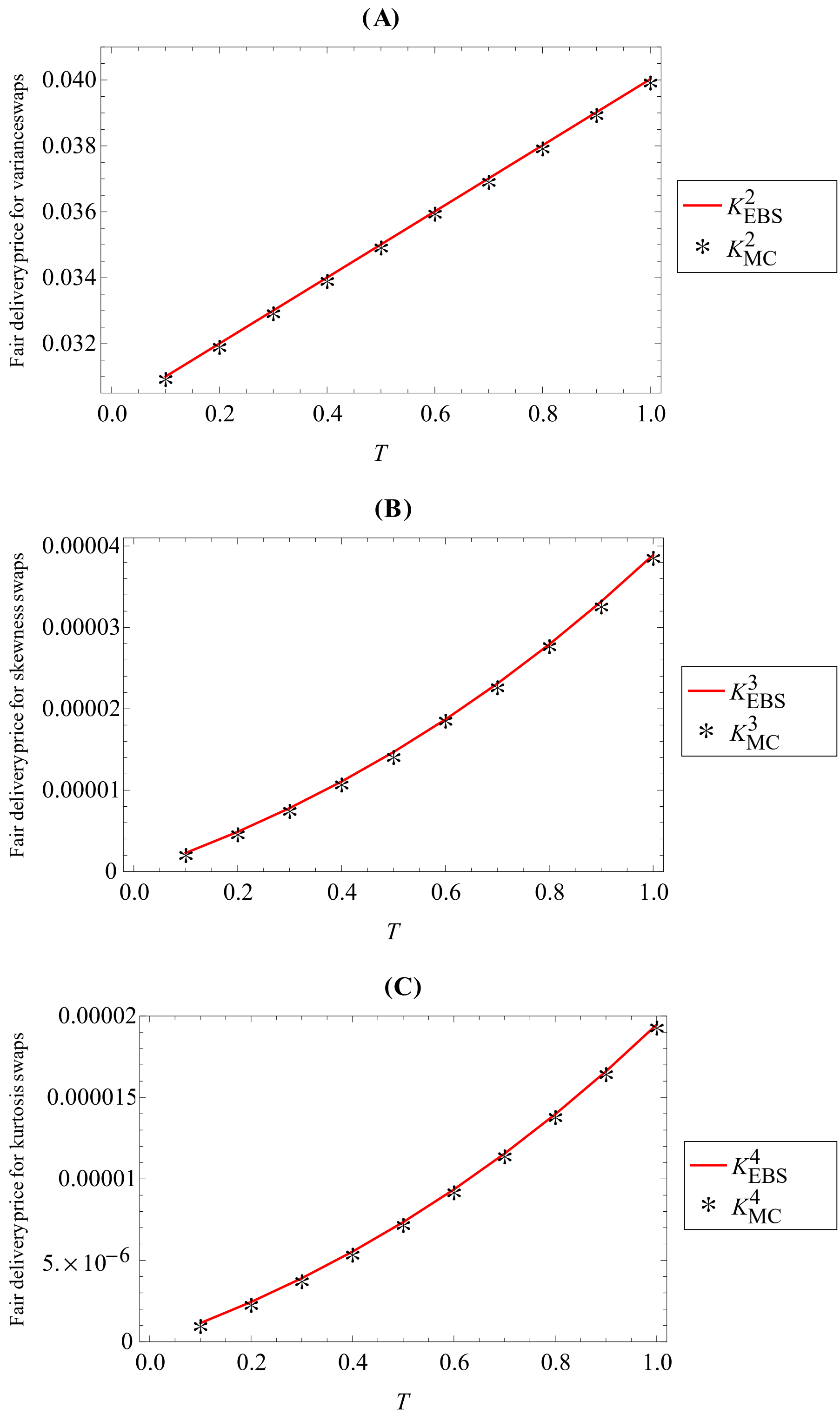

Figure 3.1: Comparisons of fair delivery prices from the closed-form solution K^m_{EBS} and the MC simulations for pricing K^m_{MC}: (A) variance swaps, (B) skewness swaps, and (C) kurtosis swaps

Figure 3.1 shows that the results from the closed-form solution and the MC simulations perfectly match, illustrating that the closed-form formula does not contain any algebraic errors and practitioners can confidently use the formula for pricing moment swaps.

In addition to the comparisons in Figure 3.1, we define the percentage relative error (ε^m) from using MC simulations by

$$\varepsilon^m(T,N;N_p) := \left|\frac{K^m_{\text{EBS}}(T,N) - K^m_{MC}(T,N;N_p)}{K^m_{\text{EBS}}(T,N)}\right| \times 100\%,$$

to measure the levels of accuracy which is shown in the Table 3.1 for $N_p = 10,000, 30,000, 50,000$, and $T = 1$.

m^{th}	K^m_{MC}			K^m_{EBS}
moment	N_p	$\varepsilon^m(\%)$	Comp. (s)	Comp. (s)
	$10,000$	0.074	6403.919	
$m=2$	$30,000$	0.053	19840.584	0.406
	$50,000$	0.033	34406.980	
	$10,000$	4.421	6861.916	
$m=3$	$30,000$	1.970	20313.791	3.609
	$50,000$	1.024	33784.318	
	$10,000$	0.169	6314.332	
$m=4$	$30,000$	0.087	18831.897	10.297
	$50,000$	0.050	31651.802	

Table 3.1: Percentage relative errors ε^m and computational times (Comp.) of MC simulations for pricing variance swaps ($m = 2$), skewness swaps ($m = 3$) and kurtosis swaps ($m = 4$) for $N_p = 10,000, 30,000$, and $50,000$, comparing with computational times of the closed-form formula

Table 3.1 confirms in addition that the results from the closed-form formula and the

MC simulations match with high accuracy with very small ε^m for all cases of m and N_p, the highest ε^m is 4.4% when $m = 3$ and $N_p = 10,000$. Moreover, the accuracy for MC simulations is improved when N_p increases, trade-off with increasing in computational times. The experiment showed that the computational time from closed-form formula is extremely faster than that from MC simulations, around 600 times faster.

Example 3.16 (Sensitivity of parameters)**.** In this study, we investigate the sensitivity of fair prices for moment swaps ($m = 2, 3, 4$) based on small changes of parameters $r(t) = r_0 + r_1 t$ and $\sigma(t) = \sqrt{\sigma_0 + \sigma_1 t}$ in the EBS. Here, we use the same parameters provided in Example 3.15 with $r_0 = 0.075$, $r_1 = 0.050$, $\sigma_0 = 0.030$, and $\sigma_1 = 0.020$. To check the sensitivity of each parameter separately, the change of fair price is computed corresponding to the change of one parameter while the other three parameters are fixed. The sensitivity is measured based on the percentage relative errors of the fair price K^m_{EBS} and parameter ΔP, defined by

$$\Delta P := \left| \frac{P - P'}{P} \right| \times 100\%, \Delta K^m_{\text{EBS}}(P, P') := \left| \frac{K^m_{\text{EBS}}(P) - K^m_{\text{EBS}}(P')}{K^m_{\text{EBS}}(P)} \right| \times 100\%,$$

with fixed $T = 1$ and $N = 252$. The results are shown in Tables 3.2–3.3.

P	P'	$\Delta P(\%)$	$\Delta K^2_{\text{EBS}}(P,P')(\%)$	$\Delta K^3_{\text{EBS}}(P,P')(\%)$	$\Delta K^4_{\text{EBS}}(P,P')(\%)$
r_0	$r'_0 = 1.02r_0$	2	2.402×10^{-3}	1.838	4.800×10^{-3}
	$r'_0 = 1.04r_0$	4	4.848×10^{-3}	3.675	9.688×10^{-3}
	$r'_0 = 1.06r_0$	6	7.339×10^{-3}	5.513	1.463×10^{-2}
	$r'_0 = 1.08r_0$	8	9.875×10^{-3}	7.350	1.973×10^{-2}
	$r'_0 = 1.10r_0$	10	1.246×10^{-2}	9.188	2.488×10^{-2}
r_1	$r'_1 = 1.02r_1$	2	8.625×10^{-4}	0.664	1.852×10^{-3}
	$r'_1 = 1.04r_1$	4	1.732×10^{-3}	1.327	3.718×10^{-3}
	$r'_1 = 1.06r_1$	6	2.607×10^{-3}	1.991	5.600×10^{-3}
	$r'_1 = 1.08r_1$	8	3.490×10^{-3}	2.654	7.494×10^{-3}
	$r'_1 = 1.10r_1$	10	4.379×10^{-3}	3.318	9.404×10^{-3}
σ_0	$\sigma'_0 = 1.02\sigma_0$	2	1.499	1.096	2.958
	$\sigma'_0 = 1.04\sigma_0$	4	2.997	2.181	5.960
	$\sigma'_0 = 1.06\sigma_0$	6	4.496	3.255	9.006
	$\sigma'_0 = 1.08\sigma_0$	8	5.994	4.318	12.095
	$\sigma'_0 = 1.10\sigma_0$	10	7.493	5.370	15.229
σ_1	$\sigma'_1 = 1.02\sigma_1$	2	0.500	0.397	1.063
	$\sigma'_1 = 1.04\sigma_1$	4	0.999	0.792	2.133
	$\sigma'_1 = 1.06\sigma_1$	6	1.499	1.186	3.210
	$\sigma'_1 = 1.08\sigma_1$	8	1.998	1.578	4.293
	$\sigma'_1 = 1.10\sigma_1$	10	2.498	1.968	5.382

Table 3.2: The percentage relative errors of the fair prices of moment swaps $\Delta K^m_{\text{EBS}}(m = 2, 3, 4)$ for $\Delta P = 2, 4, 6, 8, 10\%$ of parameters r_0, r_1, σ_0 and σ_1

Moreover, since Table 3.2 shows that ΔK^m_{EBS} depends linearly on ΔP, the order of sensitivity S_p^m of each parameter is computed as the average of $\frac{\Delta K_{\text{EBS}}^m}{\Delta P}$,

$$S_P^m := \frac{1}{n}\sum_{i=1}^{n} \frac{\Delta K_{\text{EBS}}^m(P_i, P_i')}{\Delta P_i},$$

shown in Table 3.3.

Moment swaps	$S_{r_0}^m$	$S_{r_1}^m$	$S_{\sigma_0}^m$	$S_{\sigma_1}^m$
$m = 2$	1.223×10^{-3}	4.346×10^{-4}	0.749	0.250
$m = 3$	0.919	0.332	0.543	0.198
$m = 4$	2.443×10^{-3}	9.331×10^{-4}	1.501	0.535

Table 3.3: The orders of sensitivity of fair prices for $m = 2, 3, 4$ corresponding to parameters $r_0, r_1, \sigma_0, \sigma_1$

Table 3.2 shows that ΔK_{EBS}^m depends linearly on ΔP for all cases ($m = 2, 3, 4$ and all parameters). The results show that K_{EBS}^m is more sensitive to the parameter σ_0 than the others. When comparing using the orders of sensitivity, the results display that when $m = 2, 4$, K_{EBS}^m is more sensitive to the volatility $\sigma(t)$ than interest rate $r(t)$, which is not the case when $m = 3$.

Example 3.17 (Comparison fair prices)**.** In this example, we compare the fair prices K_{BS}^m to illustrate Theorem 3.9 for the BS model. The fair prices $K_{\text{BS}}^m, K_{\text{BS}}^n$ are compared based on two sets of parameters for various pairs (m, n) with $m > n$. The first set (I) of parameters is from Broadie and Jain [2], $r = 0.0319$ and $\sigma = 0.1326$. The second set (II) is from Khaled and Samai [5], $r = 0.0013$ and $\sigma = \sqrt{0.0009}$, which were used in the likelihood function for the share price of gold for the period from April 2–December 31, 2007. The evaluation is performed with $T = 1$ and $N = 252$ to find $\tau_{m,n}^*$, the smallest positive root defined in Theorem 3.12, for each pair of K_{BS}^m and K_{BS}^n, where the existing of $\tau_{m,n}^*$ implies the order $K_{\text{BS}}^m(T, N) < K_{\text{BS}}^n(T, N)$ for all $\frac{T}{N} \in (0, \tau_{m,n}^*)$. Note that the first set of parameters satisfies $r > \frac{3}{2}\sigma^2$, while the second set $\frac{1}{2}\sigma^2 < r < \frac{3}{2}\sigma^2$. The results of $\tau_{m,n}^*$ for several (m, n) pairs are shown in Table 3.4.

	(m,n)	$(3,2)$	$(4,2)$	$(4,3)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$
$\tau^*_{m,n}$	I	19.10	14.12	6.36	11.56	8.72	9.38	4.59
	II	481.71	305.35	–	245.78	156.90	196.34	–

Table 3.4: The $\tau^*_{m,n}$ of various pairs of K^m_{BS} and K^n_{BS} for the two sets of parameters

The results from Table 3.4 show that for the set I of parameters, $r > \frac{3}{2}\sigma^2$, the $\tau^*_{m,n}$ exists for all (m,n) pairs, which supports Corollary 3.14 that $\tau^*_{m,n}$ always exists in this case. However, for the set II of parameters, $\frac{1}{2}\sigma^2 < r < \frac{3}{2}\sigma^2$, the $\tau^*_{m,n}$ exists for all pairs (m,n) except for the pairs $(4,3)$ and $(6,5)$, where $n = m-1$ is odd. This illustrates that when the set of parameters does not satisfy the condition of Corollary 3.14, the existence of $\tau^*_{m,n}$ depends on (m,n) according to Theorem 3.12 and Corollary 3.13, namely, the $\tau^*_{m,n}$ exists for all (m,n) except when $n = m-1$ is odd.

3.6 Conclusion

This chapter presented a simple and easy-to-use pricing formula for discretely-sampled moment swaps when the realized higher moments defined in terms of m^{th}-moment of the log-returns of a specified underlying asset described by BS model with time-dependent parameters. The obtained analytical method is developed based on Feynman-Kac theorem, where the PDE is solved analytically, and some combinatorial techniques are used to simplify the sum of the conditional expectations. In terms of validation purposes, we have demonstrated that pricing formula has financial meaningfulness, the fair prices for moment swaps are always finite and positive in the parameter space. A comparison theorem has been proved in order to show that trading variance swaps is more expensive than trading any higher moment swaps under the BS model. The first and third numerical examples support the validity of our results. Namely, the first experiment shows that MC simulations produce the same results as that from our formula, while the third experiment illustrates the comparison results of moments for BS model. Moreover, the second example gives the sensitivity of the fair prices respect to the parameters, and the results show that the fair price is more sensitive to the volatility parameters when $m = 2, 4$ (even).

CHAPTER IV

CLOSED-FORM FORMULA FOR PRICING MOMENT SWAPS UNDER THE SCHWARTZ MODEL

This chapter derives a simple closed-form formula for pricing discretely-sampled moment swaps based on the Schwartz model (2.5) for the underlying commodity price, by improving the result from Weraprasertsakun and Rujivan [19]. Furthermore, the obtained formula of moment swaps prices is applied to extract the current convenient yield and commodity fair price.

In section 4.1, the system of recursive ordinary differential equations (ODEs) associated with the conditional moment from Weraprasertsakun and Rujivan [19] is solved analytically. In addition, a pseudocode for computing conditional moments is provided together with discussion of efficiency of the formula. The pricing formula is derived in section 4.2 and used to extract the convenience yields in the parameter space in section 4.3. Moreover, in section 4.4, we conduct Monte Carlo simulations to provide a verification of the correctness of the pricing formula, demonstrate with numerical examples for the sensitivity of the parameters. Finally, the fair price of moment swaps with initial value between the extended Black-Scholes and Schwartz model is compared in section 4.5.

4.1 Conditional Moments

This section presents the closed-form formula for conditional moment of the Schwartz model (2.5) by improving the following result [19].

Theorem 4.1 (Weraprasertsakun and Rujivan [19]). *Suppose that S_t follows the dynamics* *described in* (2.5) *and* $k \in \mathbb{N}$. *Let* $X_t = \ln S_t$ *and* $\alpha = \mu - \dfrac{\sigma^2}{2\kappa}$. *Then*

$$E^Q_{t_{i-1}}[X_t^k] = E^Q[X_t^k \mid X_{t_{i-1}} = x] = \left(\sum_{j=0}^{k} A_j^{(k)}(\tau) x^j \right) e^{-k\kappa\tau} \tag{4.1}$$

for all $t \in [t_{i-1}, t_i]$ *and* $x \in \mathbb{R}$, *where* $\tau = t - t_{i-1}$ *and* $A_j^{(k)}(\tau), j = 0, 1, 2, \ldots, k$, *can be obtained by solving the system of linear ordinary differential equations (ODEs)*

$$\frac{dA_j^{(k)}}{d\tau} = \kappa\,(k-j)\,A_j^{(k)}(\tau) + (j+1)\kappa\alpha A_{j+1}^{(k)}(\tau) + \frac{1}{2}(j+1)(j+2)\sigma^2 A_{j+2}^{(k)}(\tau) \tag{4.2}$$

subject to the initial conditions

$$A_j^{(k)}(0) = 0 \qquad \text{for all } j = k-1, k-2, \ldots, 0, \tag{4.3}$$

providing that $A_k^{(k)}(\tau) = 1$ *and* $A_{k+1}^{(k)}(\tau) = 0$ *for all* $\tau \geq 0$.

Unfortunately, they solved the recursive ODEs (4.2) only for $k = 1, 2$, and used the solutions to derive a closed-form formula for pricing variance swap on a commodity. Moreover, they did not derive explicit formulas for $A_j^{(k)}(\tau), j = k-1, k-2, \ldots, 0$ for $k \geq 3$. Therefore, we shall complete their work by deriving a closed-form formula for $A_j^{(k)}(\tau), j = k-1, k-2, \ldots, 0$ for $k \geq 3$ as following theorem.

Theorem 4.2. *The solution* (4.2) *can be written in the form*

$$A_j^{(k)}(\tau) = \left(\prod_{r=0}^{k-j-1} (k-r) \right) \sum_{\ell=0}^{\lfloor \frac{k-j}{2} \rfloor} \frac{1}{\kappa^\ell} \alpha^{k-j-2\ell} \sigma^{2\ell} \left(e^{\kappa\tau} - 1\right)^{k-j-\ell} \left(e^{\kappa\tau} + 1\right)^{\ell} c_{k,j}^{(\ell)}, \tag{4.4}$$

where $c_{k,j}^{(\ell)}$ *is defined using* $j = k - n$ *as an index in*

$$\overline{C}_{k,n} = \begin{bmatrix} c_{k,k-n}^{(0)} \\ c_{k,k-n}^{(1)} \\ \vdots \\ c_{k,k-n}^{(\lfloor \frac{n}{2} \rfloor)} \end{bmatrix} \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor + 1},$$

which is defined recursively on n *as follow;*

$$\overline{C}_{k,1} = \begin{bmatrix} c_{k,k-1}^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}, \overline{C}_{k,2} = \begin{bmatrix} c_{k,k-2}^{(0)} \\ c_{k,k-2}^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, \tag{4.5}$$

for odd $n \geq 3$*,*

$$\overline{C}_{k,n} = \frac{1}{n} \left(\overline{C}_{k,n-1} + \frac{1}{2} \begin{bmatrix} 0 \\ \overline{C}_{k,n-2} \end{bmatrix} \right), \tag{4.6}$$

and for even $n \geq 4$*,*

$$\overline{C}_{k,n} = \frac{1}{n} \left(\begin{bmatrix} \overline{C}_{k,n-1} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ \overline{C}_{k,n-2} \end{bmatrix} \right). \tag{4.7}$$

Proof. It suffices to show that the solution of

$$\begin{aligned} &\frac{dA_{k-n}^{(k)}}{d\tau} - n\kappa A_{k-n}^{(k)}(\tau) \\ &= ((k-n)+1)\kappa\alpha A_{(k-n)+1}^{(k)}(\tau) + \frac{1}{2}((k-n)+1)((k-n)+2)\sigma^2 A_{(k-n)+2}^{(k)}(\tau) \end{aligned} \tag{4.8}$$

with conditions (4.3) when $j = k - n$ is

$$A_{k-n}^{(k)}(\tau) = \left(\prod_{r=0}^{n-1} (k-r) \right) \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{\kappa^\ell} \alpha^{n-2\ell} \sigma^{2\ell} \left(e^{\kappa\tau} - 1\right)^{n-\ell} \left(e^{\kappa\tau} + 1\right)^\ell c_{k,k-n}^{(\ell)}, \tag{4.9}$$

where $c_{k,k-n}^{(\ell)}$ for $n = 1, \ldots, k$ is defined through (4.5)–(4.7).

For $n = 1$, the equation (4.8) is reduced to

$$\frac{dA_{k-1}^{(k)}}{d\tau} - \kappa A_{k-1}^{(k)}(\tau) = k\kappa\alpha,$$

with the solution subject to the initial condition (4.3) when $j = k - 1$,

$$A_{k-1}^{(k)}(\tau) = e^{\kappa\tau}\left[-k\alpha e^{-\kappa\tau} + k\alpha\right].$$

This can be written in the form of (4.9) when $n = 1$ as

$$A_{k-1}^{(k)}(\tau) = \left(\prod_{r=0}^{1-1}(k-r)\right)\sum_{\ell=0}^{\lfloor\frac{1}{2}\rfloor}\frac{1}{\kappa^{\ell}}\alpha^{1-2\ell}\sigma^{2\ell}\left(e^{\kappa\tau}-1\right)^{1-\ell}\left(e^{\kappa\tau}+1\right)^{\ell}c_{k,k-1}^{(\ell)},$$

where $c_{k,k-1}^{(0)} = 1$.

For $n = 2$, the equation (4.8) becomes

$$\frac{dA_{k-2}^{(k)}}{d\tau} - 2\kappa A_{k-2}^{(k)}(\tau) = (k-1)\kappa\alpha A_{k-1}^{(k)}(\tau) + \frac{1}{2}(k-1)k\sigma^2,$$

with the solution subject to the initial condition (4.3) when $j = k - 2$,

$$A_{k-2}^{(k)}(\tau) = k(k-1)e^{2\kappa\tau}\left[-\alpha^2 e^{-\kappa\tau} + \frac{1}{2}\alpha^2 e^{-2\kappa\tau} - \frac{1}{4\kappa}\sigma^2 e^{-2\kappa\tau} + \frac{1}{2}\alpha^2 + \frac{1}{4\kappa}\sigma^2\right].$$

By writing in the form of (4.9) when $n = 2$, we get

$$\begin{aligned} A_{k-2}^{(k)}(\tau) &= k(k-1)\left[\frac{1}{2}\alpha^2(e^{\kappa\tau}-1)^2 + \frac{1}{4\kappa}\sigma^2(e^{\kappa\tau}-1)(e^{\kappa\tau}+1)\right] \\ &= \left(\prod_{r=0}^{2-1}(k-r)\right)\sum_{\ell=0}^{\lfloor\frac{2}{2}\rfloor}\frac{1}{\kappa^{\ell}}\alpha^{2-2\ell}\sigma^{2\ell}\left(e^{\kappa\tau}-1\right)^{2-\ell}\left(e^{\kappa\tau}+1\right)^{\ell}c_{k,k-2}^{(\ell)}, \end{aligned}$$

where $c_{k,k-2}^{(0)} = \frac{1}{2}$ and $c_{k,k-2}^{(1)} = \frac{1}{4}$.

For $n = 3$, the equation (4.8) becomes

$$\frac{dA_{k-3}^{(k)}}{d\tau} - 3\kappa A_{k-3}^{(k)}(\tau) = (k-2)\kappa\alpha A_{k-2}^{(k)}(\tau) + \frac{1}{2}(k-2)(k-1)\sigma^2 A_{k-1}^{(k)}(\tau)$$

with the solution in integral form

$$\begin{aligned}
&A_{k-3}^{(k)}(\tau) \\
&= e^{3\kappa\tau}\int e^{-3\kappa\tau}(k-2)\kappa\alpha A_{k-2}^{(k)}(\tau)d\tau + e^{3\kappa\tau}\int e^{-3\kappa\tau}\frac{1}{2}(k-2)(k-1)\sigma^2 A_{k-1}^{(k)}(\tau)d\tau \\
&= e^{3\kappa\tau}\int e^{-3\kappa\tau}(k-2)\kappa\alpha\left(\left(\prod_{r=0}^{1}(k-r)\right)\sum_{\ell=0}^{1}\frac{1}{\kappa^\ell}c_{k,k-2}^{(\ell)}\alpha^{2-2\ell}\sigma^{2\ell}\left(e^{\kappa\tau}-1\right)^{2-\ell}\left(e^{\kappa\tau}+1\right)^{\ell}\right)d\tau \\
&+ e^{3\kappa\tau}\int e^{-3\kappa\tau}\frac{1}{2}(k-2)(k-1)\sigma^2\left(\left(\prod_{r=0}^{0}(k-r)\right)\sum_{\ell=0}^{0}\frac{1}{\kappa^\ell}c_{k,k-1}^{(\ell)}\alpha^{1-2\ell}\sigma^{2\ell}\left(e^{\kappa\tau}-1\right)^{1-\ell}\left(e^{\kappa\tau}+1\right)^{\ell}\right)d\tau \\
&=: R_0^{(1)}(\tau,3) + R_1^{(1)}(\tau,3) + R_0^{(2)}(\tau,3),
\end{aligned}$$

where

$$\begin{aligned}
R_0^{(1)}(\tau,3) &= \left(\prod_{r=0}^{2}(k-r)\right)\frac{1}{2}\kappa\alpha^3 e^{3\kappa\tau}\int e^{-3\kappa\tau}\left(e^{\kappa\tau}-1\right)^2 d\tau, \\
R_1^{(1)}(\tau,3) &= \left(\prod_{r=0}^{2}(k-r)\right)\frac{1}{4}\alpha\sigma^2 e^{3\kappa\tau}\int e^{-3\kappa\tau}\left(e^{\kappa\tau}-1\right)\left(e^{\kappa\tau}+1\right)d\tau, \\
R_0^{(2)}(\tau,3) &= \left(\prod_{r=0}^{2}(k-r)\right)\frac{1}{2}\sigma^2\alpha e^{3\kappa\tau}\int e^{-3\kappa\tau}\left(e^{\kappa\tau}-1\right)d\tau.
\end{aligned}$$

By integration, we obtain the solution

$$\begin{aligned}
R_0^{(1)}(\tau,3) &= \left(\prod_{r=0}^{2}(k-r)\right)\frac{1}{6}\alpha^3\left(e^{\kappa\tau}-1\right)^3 + R_0^{(1)}(0,3), \\
R_1^{(1)}(\tau,3) + R_0^{(2)}(\tau,3) &= \left(\prod_{r=0}^{2}(k-r)\right)\frac{1}{4\kappa}\alpha\sigma^2\left(e^{\kappa\tau}-1\right)^2\left(e^{\kappa\tau}+1\right) + R_1^{(1)}(0,3) + R_0^{(2)}(0,3).
\end{aligned}$$

By the initial condition (4.3) when $j = k-3$, $R_0^{(1)}(0,3) + R_1^{(1)}(0,3) + R_0^{(2)}(0,3) = 0$, and

$$\begin{aligned} A_{k-3}^{(k)}(\tau) &= R_0^{(1)}(\tau,3) + \left[R_1^{(1)}(\tau,3) + R_0^{(2)}(\tau,3)\right] \\ &= \left(\prod_{r=0}^{2}(k-r)\right) \sum_{\ell=0}^{\lfloor \frac{3}{2} \rfloor} \frac{1}{\kappa^\ell} \alpha^{3-2\ell} \sigma^{2\ell} \left(e^{\kappa\tau}-1\right)^{3-\ell} \left(e^{\kappa\tau}+1\right)^{\ell} c_{n,n-3}^{(\ell)}, \end{aligned}$$

where $c_{n,n-3}^{(0)} = \frac{1}{6}$ and $c_{n,n-3}^{(1)} = \frac{1}{4}$, which is the form of (4.9) when $n = 3$. This show that (4.6) hold for $n = 3$,

$$\begin{bmatrix} c_{k,k-3}^{(0)} \\ c_{k,k-3}^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{3}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

For $n = 4$, the equation (4.8) becomes

$$\frac{dA_{k-4}^{(k)}}{d\tau} - 4\kappa A_{k-4}^{(k)}(\tau) = (k-3)\kappa\alpha A_{k-3}^{(k)}(\tau) + \frac{1}{2}(k-3)(k-2)\sigma^2 A_{k-2}^{(k)}(\tau)$$

with the solution in integral form

$$\begin{aligned} A_{k-4}^{(k)}(\tau) &= e^{4\kappa\tau} \int e^{-4\kappa\tau}(k-3)\kappa\alpha A_{k-3}^{(k)}(\tau)d\tau + e^{4\kappa\tau} \int e^{-4\kappa\tau} \frac{1}{2}(k-3)(k-2)\sigma^2 A_{k-2}^{(k)}(\tau)d\tau \\ &=: R_0^{(1)}(\tau,4) + R_1^{(1)}(\tau,4) + R_0^{(2)}(\tau,4) + R_1^{(2)}(\tau,4), \end{aligned}$$

where

$$\begin{aligned} R_0^{(1)}(\tau,4) &= \left(\prod_{r=0}^{3}(k-r)\right) \frac{1}{6}\kappa\alpha^4 e^{4\kappa\tau} \int e^{-4\kappa\tau} \left(e^{\kappa\tau}-1\right)^3 d\tau, \\ R_1^{(1)}(\tau,4) &= \left(\prod_{r=0}^{3}(k-r)\right) \frac{1}{4}\alpha^2\sigma^2 e^{4\kappa\tau} \int e^{-4\kappa\tau} \left(e^{\kappa\tau}-1\right)^2 \left(e^{\kappa\tau}+1\right) d\tau, \\ R_0^{(2)}(\tau,4) &= \left(\prod_{r=0}^{3}(k-r)\right) \frac{1}{4}\alpha^2\sigma^2 e^{4\kappa\tau} \int e^{-4\kappa\tau} \left(e^{\kappa\tau}-1\right)^2 d\tau, \\ R_1^{(2)}(\tau,4) &= \left(\prod_{r=0}^{3}(k-r)\right) \frac{1}{8\kappa}\sigma^4 e^{4\kappa\tau} \int e^{-4\kappa\tau} \left(e^{\kappa\tau}-1\right) \left(e^{\kappa\tau}+1\right) d\tau. \end{aligned}$$

By integration, we obtain the solution

$$\begin{aligned} R_0^{(1)}(\tau,4) &= \left(\prod_{r=0}^{3}(k-r)\right)\frac{1}{24}\alpha^4\left(e^{\kappa\tau}-1\right)^4 + R_0^{(1)}(0,4), \\ R_1^{(1)}(\tau,4)+R_0^{(2)}(\tau,4) &= \left(\prod_{r=0}^{3}(k-r)\right)\frac{1}{8\kappa}\alpha^2\sigma^2\left(e^{\kappa\tau}-1\right)^3\left(e^{\kappa\tau}+1\right) + R_1^{(1)}(0,4)+R_0^{(2)}(0,4), \\ R_1^{(2)}(\tau,4) &= \left(\prod_{r=0}^{3}(k-r)\right)\frac{1}{32\kappa^2}\sigma^4\left(e^{\kappa\tau}-1\right)^2\left(e^{\kappa\tau}+1\right)^2 + R_1^{(2)}(0,4). \end{aligned}$$

By the initial condition (4.3) when $j = k-4$, $R_0^{(1)}(0,4) + R_1^{(1)}(0,4) + R_0^{(2)}(0,4) + R_1^{(2)}(0,4) = 0$, and

$$\begin{aligned} A_{k-4}^{(k)}(\tau) &= R_0^{(1)}(\tau,4) + \left[R_1^{(1)}(\tau,4)+R_0^{(2)}(\tau,4)\right] + R_1^{(2)}(\tau,4) \\ &= \left(\prod_{r=0}^{3}(k-r)\right)\sum_{\ell=0}^{\lfloor\frac{4}{2}\rfloor}\frac{1}{\kappa^\ell}\alpha^{4-2\ell}\sigma^{2\ell}\left(e^{\kappa\tau}-1\right)^{4-\ell}\left(e^{\kappa\tau}+1\right)^\ell c_{k,k-4}^{(\ell)}, \end{aligned}$$

where $c_{k,k-4}^{(0)} = \frac{1}{24}, c_{k,k-4}^{(1)} = \frac{1}{8}$ and $c_{k,k-4}^{(2)} = \frac{1}{32}$, which is the form of (4.9) when $n = 4$. This show that (4.7) hold for $n = 4$,

$$\begin{bmatrix} c_{k,k-4}^{(0)} \\ c_{k,k-4}^{(1)} \\ c_{k,k-4}^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{24} \\ \frac{1}{8} \\ \frac{1}{32} \end{bmatrix} = \frac{1}{4}\left(\begin{bmatrix} \frac{1}{6} \\ \frac{1}{4} \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}\right).$$

For other n, the solution of (4.8) in integral form is

$$A_{k-n}^{(k)}(\tau) := Q_1(\tau) + Q_2(\tau),$$

where

$$\begin{aligned} Q_1(\tau) &= e^{n\kappa\tau}\int e^{-n\kappa\tau}((k-n)+1)\kappa\alpha A_{(k-n)+1}^{(k)}(\tau)d\tau, \\ Q_2(\tau) &= e^{n\kappa\tau}\int e^{-n\kappa\tau}\frac{1}{2}((k-n)+1)((k-n)+2)\sigma^2 A_{(k-n)+2}^{(k)}(\tau)d\tau. \end{aligned}$$

Based on the same idea, we introduce $R_\ell^{(i)}$ by splitting $A_j^{(k)}(\tau)$ in $Q_1(\tau)$ and $Q_2(\tau)$ as follows. By substituting $A^{(k)}_{(k-n)+1}(\tau)$ in $Q_1(\tau)$, we have

$$
\begin{aligned}
Q_1(\tau) &= e^{n\kappa\tau} \int e^{-n\kappa\tau}((k-n)+1)\kappa\alpha \\
&\quad \left[\left(\prod_{r=0}^{(n-1)-1} (k-r) \right) \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\kappa^\ell} \alpha^{(n-1)-2\ell} \sigma^{2\ell} \left(e^{\kappa\tau}-1\right)^{(n-1)-\ell} \left(e^{\kappa\tau}+1\right)^\ell c^{(\ell)}_{k,k-(n-1)} \right] d\tau \\
&= \left(\prod_{r=0}^{n-1} (k-r) \right) \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} c^{(\ell)}_{k,k-(n-1)} \frac{1}{\kappa^{\ell-1}} \alpha^{n-2\ell} \sigma^{2\ell} e^{n\kappa\tau} \int e^{-n\kappa\tau} \left(e^{\kappa\tau}-1\right)^{(n-1)-\ell} \left(e^{\kappa\tau}+1\right)^\ell d\tau \\
&=: \left(\prod_{r=0}^{n-1} (k-r) \right) \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} R^{(1)}_\ell(\tau, n).
\end{aligned}
$$

By substituting $A^{(k)}_{(k-n)+2}(\tau)$ in $Q_2(\tau)$, we have

$$
\begin{aligned}
Q_2(\tau) &= e^{n\kappa\tau} \int e^{-n\kappa\tau} \frac{1}{2}((k-n)+1)((k-n)+2)\sigma^2 \\
&\quad \left[\left(\prod_{r=0}^{(n-2)-1} (k-r) \right) \sum_{\ell=0}^{\lfloor \frac{n-2}{2} \rfloor} \frac{1}{\kappa^\ell} \alpha^{(n-2)-2\ell} \sigma^{2\ell} \left(e^{\kappa\tau}-1\right)^{(n-2)-\ell} \left(e^{\kappa\tau}+1\right)^\ell c^{(\ell)}_{k,k-(n-2)} \right] d\tau \\
&= \left(\prod_{r=0}^{n-1} (k-r) \right) \sum_{\ell=0}^{\lfloor \frac{n-2}{2} \rfloor} c^{(\ell)}_{k,k-(n-2)} \frac{1}{2\kappa^{\ell}} \alpha^{(n-2)-2\ell} \sigma^{2\ell+2} e^{n\kappa\tau} \int e^{-n\kappa\tau} \left(e^{\kappa\tau}-1\right)^{(n-2)-\ell} \left(e^{\kappa\tau}+1\right)^\ell d\tau \\
&=: \left(\prod_{r=0}^{n-1} (k-r) \right) \sum_{\ell=0}^{\lfloor \frac{n-2}{2} \rfloor} R^{(2)}_\ell(\tau, n).
\end{aligned}
$$

For odd n, the splitting of $R^{(i)}_\ell(\tau, n)$, for $i = 1, 2$, are combined to obtain $A^{(k)}_{k-n}(\tau)$ according to the case of $n = 3$, namely, by the shifting index of $R^{(2)}_\ell(\tau, n)$,

$$
\begin{aligned}
A^{(k)}_{k-n}(\tau) &= \left(\prod_{r=0}^{n-1} (k-r) \right) \left[\sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} R^{(1)}_\ell(\tau, n) + \sum_{\ell=0}^{\lfloor \frac{n-2}{2} \rfloor} R^{(2)}_\ell(\tau, n) \right] \\
&= \left(\prod_{r=0}^{n-1} (k-r) \right) \left[R^{(1)}_0(\tau, n) + \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} \left[R^{(1)}_\ell(\tau, n) + R^{(2)}_\ell(\tau, n) \right] \right].
\end{aligned}
$$

By integration subject to initial condition (4.3) for $j = k - n$, the solution can be written in the form of (4.9) where the coefficients $c^{(\ell)}_{k,k-n}$ satisfy (4.6). Similarly, the process of even n follows the case of $k = 4$, i.e.,

$$\begin{aligned} A^{(k)}_{k-n}(\tau) &= \left(\prod_{r=0}^{n-1}(k-r)\right)\left[\sum_{\ell=0}^{\lfloor\frac{n-1}{2}\rfloor} R^{(1)}_{\ell}(\tau,n) + \sum_{\ell=0}^{\lfloor\frac{n-2}{2}\rfloor} R^{(2)}_{\ell}(\tau,n)\right] \\ &= \left(\prod_{r=0}^{n-1}(k-r)\right)\left[R^{(1)}_{0}(\tau,n) + \sum_{\ell=1}^{\lfloor\frac{n}{2}\rfloor-1}\left[R^{(1)}_{\ell}(\tau,n) + R^{(2)}_{\ell}(\tau,n)\right] + R^{(2)}_{\lfloor\frac{n-2}{2}\rfloor}(\tau,n)\right]. \end{aligned}$$

By integration subject to initial condition (4.3) for $j = k - n$, the solution can be written in the form of (4.9) where the coefficients $c^{(\ell)}_{k,k-n}$ satisfy (4.7). □

Remark 4.3. From the result, the closed-form formula for the conditional moments can be obtained from following the pseudo code.

Algorithm 1 Algorithm of the coefficient functions

Input: $k, x, \kappa, \alpha, \sigma, \tau$

Output: k^{th} condition moment

---

1. Set $\overline{C}_{k,1} = \{1\}$
2. Set $\overline{C}_{k,2} = \{1/2, 1/4\}$
3. **For** $n = 3$ to k **do**
4.   **If** n is odd **then**
5.   compute (4.6)
6.   **else**
7.   compute (4.7)
8.   **EndIf**
9. **EndFor**
10. Compute (4.4)
11. Compute (4.1)

Example 4.4. The first conditional moment is

$$E^Q[X_t|X_{t_{i-1}} = x] = \left(x + A_0^{(1)}(\tau)\right) e^{-\kappa\tau},$$

where

$$A_0^{(1)}(\tau) = \alpha\left(e^{\kappa\tau} - 1\right).$$

The second conditional moment is

$$E^Q[X_t^2|X_{t_{i-1}} = x] = \left(x^2 + A_1^{(2)}(\tau)x + A_0^{(2)}(\tau)\right) e^{-2\kappa\tau},$$

where

$$\begin{aligned} A_1^{(2)}(\tau) &= 2\alpha\left(e^{\kappa\tau} - 1\right) \\ A_0^{(2)}(\tau) &= 2\left(\frac{1}{2}\alpha^2\left(e^{\kappa\tau} - 1\right)^2 + \frac{1}{4\kappa}\sigma^2\left(e^{\kappa\tau} - 1\right)\left(e^{\kappa\tau} + 1\right)\right). \end{aligned}$$

The third conditional moment is

$$E^Q[X_t^3|X_{t_{i-1}} = x] = \left(x^3 + A_2^{(3)}(\tau)x^2 + A_1^{(3)}(\tau)x + A_0^{(3)}(\tau)\right) e^{-3\kappa\tau},$$

where

$$\begin{aligned} A_2^{(3)}(\tau) &= 3\alpha\left(e^{\kappa\tau} - 1\right), \\ A_1^{(3)}(\tau) &= 6\left(\frac{1}{2}\alpha^2\left(e^{\kappa\tau} - 1\right)^2 + \frac{1}{4\kappa}\sigma^2\left(e^{\kappa\tau} - 1\right)\left(e^{\kappa\tau} + 1\right)\right), \\ A_0^{(3)}(\tau) &= 6\left(\frac{1}{6}\alpha^3\left(e^{\kappa\tau} - 1\right)^3 + \frac{1}{4\kappa}\alpha\sigma^2\left(e^{\kappa\tau} - 1\right)^2\left(e^{\kappa\tau} + 1\right)\right). \end{aligned}$$

4.1.1 Efficiency of Closed-Form Formula

In this section, the analytical formula (4.4) and the formula (4.2) by Weraprasertsakun and Rujivan [19] are compared for efficiency in term of computational time for obtaining conditional moments (4.1), using Mathematica V9.0 program with symbolic parameters under Microsoft Windows 10 64-bit, quad-processor Intel Core i7 3.4 GHz machine with 32GB main memory. The comparison results are displayed in Figure 4.1.

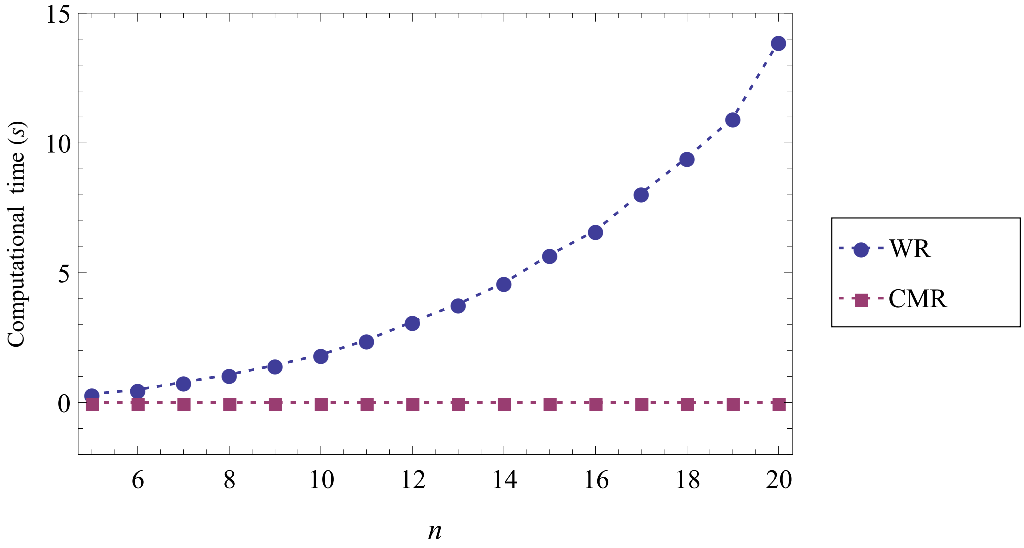

Figure 4.1: Comparison computational time between Weraprasertsakun and Rujivan (WR) and our formula (CMR)

Figure 4.1 shows that the formula from WR consumed more time when increases from 5 to 20, increase exponentially from 0.328 to 13.906 sec with the total time 74.563 sec. However, our formula only consumed 0.016 sec for the total, which is extremely fast, around 4,000 times faster.

The result concludes that our formula simplifies the result of Weraprasertsakun and Rujivan [19] for computing conditional moments, which is easier and faster to use without solving the system of recursive ordinary differential equations.

4.2 Pricing Moment Swaps

In the present section, we derive an analytical formula for pricing discretely-sampled moment swaps under the Schwartz model (2.5).

Theorem 4.5. *Suppose that S_t follows the dynamics described in (2.5). Let $X_t = \ln S_t$ and $\Delta t = t - t_{i-1}$ for all $t \in [t_{i-1}, t_i]$. Then,*

$$E_0^Q\left[\left(X_{t_i} - X_{t_{i-1}}\right)^m\right] = \sum_{j=0}^{m} \tilde{A}_{m,j}(\Delta t, t_{i-1}) X_0^j \tag{4.10}$$

for all $i = 1, 2, \ldots, N$ and $X_0 > 0$, where $\Delta t = t_i - t_{i-1}$ and

$$\tilde{A}_{m,j}(\Delta t, t_{i-1}) = \sum_{\ell=j}^{m}\sum_{k=0}^{\ell} \binom{m}{k}(-1)^k A_{\ell-k}^{(m-k)}(\Delta t) A_j^{(\ell)}(t_{i-1}) e^{-(m-k)\kappa\Delta t} e^{-\ell\kappa t_{i-1}}. \tag{4.11}$$

Proof.

$$\begin{aligned}
&E_0^Q\left[\left(X_{t_i} - X_{t_{i-1}}\right)^m\right] \\
&= E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^k X_{t_{i-1}}^k X_{t_i}^{m-k}\right] && \text{(by binomial theorem)} \\
&= E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^k X_{t_{i-1}}^k E_{t_{i-1}}^Q\left[X_{t_i}^{m-k}\right]\right] && \text{(by Tower property)} \\
&= E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^k X_{t_{i-1}}^k \left[\left(\sum_{j=0}^{m-k} A_j^{(m-k)}(\Delta t) X_{t_{i-1}}^j\right) e^{-(m-k)\kappa\Delta t}\right]\right] \\
& && \text{(by Theorem 4.1)} \\
&= E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^k \left[\left(\sum_{j=0}^{m-k} A_j^{(m-k)}(\Delta t) X_{t_{i-1}}^{j+k}\right) e^{-(m-k)\kappa\Delta t}\right]\right] \\
&= E_0^Q\left[\sum_{\ell=0}^{m}\sum_{k=0}^{\ell}\binom{m}{k}(-1)^k A_{\ell-k}^{(m-k)}(\Delta t) e^{-(m-k)\kappa\Delta t} X_{t_{i-1}}^{\ell}\right] \\
& && \text{(by rearrangement of summation)}
\end{aligned}$$

$$= \sum_{\ell=0}^{m}\sum_{k=0}^{\ell}\binom{m}{k}(-1)^k A_{\ell-k}^{(m-k)}(\Delta t)e^{-(m-k)\kappa\Delta t}E_0^Q\left[X_{t_{i-1}}^{\ell}\right]$$

$$= \sum_{\ell=0}^{m}\sum_{k=0}^{\ell}\binom{m}{k}(-1)^k A_{\ell-k}^{(m-k)}(\Delta t)e^{-(m-k)\kappa\Delta t}\left[\left(\sum_{j=0}^{\ell}A_j^{(\ell)}(t_{i-1})X_0^j\right)e^{-\ell\kappa t_{i-1}}\right]$$

(by Theorem 4.1)

$$= \sum_{j=0}^{m}\left[\sum_{\ell=j}^{m}\sum_{k=0}^{\ell}\binom{m}{k}(-1)^k A_{\ell-k}^{(m-k)}(\Delta t)A_j^{(\ell)}(t_{i-1})e^{-(m-k)\kappa\Delta t}e^{-\ell\kappa t_{i-1}}\right]X_0^j.$$

(by rearrangement of summation)

□

Theorem 4.6. *Suppose that S_t follows the Schwartz model (2.5) and $m \geq 2$ is an integer. Then, the fair delivery price of the m^{th}-moment swap can be expressed as*

$$K_{\text{S}}^m(T,\Delta t,\delta_0) = \frac{1}{T}\sum_{j=0}^{m}\sum_{i=1}^{N}\tilde{A}_{m,j}(\Delta t,t_{i-1})\left(\frac{\delta_0}{\kappa}\right)^j, \tag{4.12}$$

where $\Delta t = \frac{T}{N}, t_i = i\Delta t, i = 0,1,\ldots,N,$ and $\delta_0 = \kappa\ln S_0$.

Proof. From (2.6) and (2.7), we note that

$$K_{\text{S}}^m(T,N) = E_0^Q\left[\frac{1}{T}\sum_{i=1}^{N}\left(X_{t_i}-X_{t_{i-1}}\right)^m\right] = \frac{1}{T}\sum_{i=1}^{N}E_0^Q\left[\left(X_{t_i}-X_{t_{i-1}}\right)^m\right] \tag{4.13}$$

Insert (4.10) in Theorem 4.1 into (4.13), we immediately obtain (4.12). □

Example 4.7. We use Theorem 4.1 to derive the special case of moment swaps when $m = 2,3,4$ under the Schwartz model (2.5). The fair price of variance swaps can be expressed as

$$K_{\text{S}}^2(T,\Delta t,\delta_0) = \frac{1}{T}\sum_{j=0}^{2}\sum_{i=1}^{N}\tilde{A}_{2,j}(\Delta t,t_{i-1})\left(\frac{\delta_0}{\kappa}\right)^j, \tag{4.14}$$

where

$$\tilde{A}_{2,0}(\Delta t, t_{i-1}) = \frac{1}{2\kappa} e^{-2\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa \Delta t} - 1\right) \left(2\sigma^2 e^{\kappa(2t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right) \left(2\alpha^2\kappa - \sigma^2\right)\right),$$
$$\tilde{A}_{2,1}(\Delta t, t_{i-1}) = -2\alpha e^{-2\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^2,$$
$$\tilde{A}_{2,2}(\Delta t, t_{i-1}) = e^{-2\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^2,$$

for skewness swaps as

$$K_{\mathrm{S}}^{3}(T, \Delta t, \delta_0) = \frac{1}{T} \sum_{j=0}^{3} \sum_{i=1}^{N} \tilde{A}_{3,j}(\Delta t, t_{i-1}) \left(\frac{\delta_0}{\kappa}\right)^j, \tag{4.15}$$

where

$$\tilde{A}_{3,0}(\Delta t, t_{i-1}) = \frac{\alpha}{2\kappa} e^{-3\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^2 \left(6\sigma^2 e^{\kappa(2t_{i-1}+\Delta t)} + \left(e^{\kappa\Delta t} - 1\right) \left(2\alpha^2\kappa - 3\sigma^2\right)\right),$$
$$\tilde{A}_{3,1}(\Delta t, t_{i-1}) = \frac{3}{2\kappa} e^{-3\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^2 \left(-2\sigma^2 e^{\kappa(2t_{i-1}+\Delta t)} + \left(e^{\kappa\Delta t} - 1\right) \left(\sigma^2 - 2\alpha^2\kappa\right)\right),$$
$$\tilde{A}_{3,2}(\Delta t, t_{i-1}) = 3\alpha e^{-3\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^3,$$
$$\tilde{A}_{3,3}(\Delta t, t_{i-1}) = -e^{-3\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^3,$$

and for kurtosis swaps as

$$K_{\mathrm{S}}^{4}(T, \Delta t, \delta_0) = \frac{1}{T} \sum_{j=0}^{4} \sum_{i=1}^{N} \tilde{A}_{4,j}(\Delta t, t_{i-1}) \left(\frac{\delta_0}{\kappa}\right)^j, \tag{4.16}$$

where

$$\tilde{A}_{4,0}(\Delta t, t_{i-1}) = \frac{1}{4\kappa^2} e^{-4\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^2 \Bigg(4\alpha^4\kappa^2 \left(e^{\kappa\Delta t} - 1\right)^2 - 12\alpha^2\kappa\sigma^2 \left(e^{\kappa\Delta t} - 1\right) \left(-1 + e^{\kappa\Delta t} - 2e^{\kappa(2t_{i-1}+\Delta t)}\right) + 3\sigma^4 \left(1 + e^{\kappa\Delta t} \left(-1 + 2e^{2\kappa t_{i-1}}\right)\right)\Bigg),$$

$$\tilde{A}_{4,1}(\Delta t, t_{i-1}) = \frac{2\alpha}{\kappa} e^{-4\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^3 \left(-6\sigma^2 e^{\kappa(2t_{i-1}+\Delta t)} + \left(e^{\kappa\Delta t} - 1\right)\left(-2\alpha^2\kappa + 3\sigma^2\right)\right),$$

$$\tilde{A}_{4,2}(\Delta t, t_{i-1}) = \frac{3}{\kappa} e^{-4\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^3 \left(2\sigma^2 e^{\kappa(2t_{i-1}+\Delta t)} + \left(e^{\kappa\Delta t} - 1\right)\left(2\alpha^2\kappa - \sigma^2\right)\right),$$

$$\tilde{A}_{4,3}(\Delta t, t_{i-1}) = -4\alpha e^{-4\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^4,$$

$$\tilde{A}_{4,4}(\Delta t, t_{i-1}) = e^{-4\kappa(t_{i-1}+\Delta t)} \left(e^{\kappa\Delta t} - 1\right)^4.$$

4.3 Extraction of Convenience Yields From Fair Price of Moment swaps

In commodity markets, commodity prices depend on the convenience yield. Consequently, the current convenience yield and the spot commodity price imply the fair price of its moment swaps. However, the current convenience yield and commodity spot price are not clearly observed in markets. If we can observe the price of commodity moment swaps at current time t, then it can be used to extract the current convenience yields of commodity, given that parameters κ, μ, σ are already correctly observed. Technically, by solving the quadratic equation (4.14) based on the observed price of variance swap will gives two values of convenience yield. To obtain suitable unique value, we compare with the values extracted from observed price of the skewness swap or higher moment swaps.

We give the pseudo algorithm to extract a convenience yield as following.

Algorithm 2 Extraction of convenience yields δ_t from observed markets prices variance ($p_t^{(2)}$) and skewness ($p_t^{(3)}$) swaps.

Input: observed market prices of variance ($p_t^{(2)}$) and skewness ($p_t^{(3)}$) swaps at time t.

Output: a convenience yield δ_t.

---

1. Compute set of roots $R_2 = \{\delta_t \in \mathbb{R} \mid K^2(\delta_t) = p_t^{(2)}\}$,
2. $R_3 = \{\delta_t \in \mathbb{R} \mid K^3(\delta_t) = p_t^{(3)}\}$,
3. **For** $i = 1, \ldots, \#R_2$ and $j = 1, \ldots, \#R_3$ **do**
4. Compute $d_{ij} = |R_{2,i} - R_{3,j}|$.
5. **If** $d_{mn} = min(d_{ij})$ **do**
6. set $\delta_t = \frac{1}{2}(R_{2,n} + R_{3,m})$.
7. **EndIf**
8. **EndFor**
9. **Return** δ_t

Remark 4.8. The accuracy of the extraction of convenience yield can be improved by adding higher moment swaps into the algorithm, for comparison.

Example 4.9. Assume that formulas of fair prices (4.14) and (4.15) are used based on the parameters $N = 252, \kappa = 0.099, \mu = 2.857, \sigma = 0.129$, and $T = 1$. Suppose that the observed market price of variance and skewness swaps are 0.035 and 0.000589, respectively. Here, the observed prices are pre-assumed by using the formulas (4.14) and (4.15) based on the same parameters. Solving for quadratic (4.14) and cubic (4.15) equation gives

$$R_{2,1} \approx -2, R_{2,2} \approx 2.549 \text{ and } R_{3,1} \approx -2.$$

In this case, the suitable of convenience yield is the average of two closed solutions,

$$\delta_0 = \frac{1}{2}(R_{2,1} + R_{3,1}) \approx -2.$$

Remark 4.10. If we can observe a series of commodity moment swaps prices at anytime t, we can use the formula of fair price of moment swaps with this technique to extract a series of convenience yields in the commodity market.

4.4 Numerical Results and Discussion

For the purpose of demonstrating the correctness of our closed-form formula (4.12), we present some numerical examples in this section. We compare the results obtained from our formula and those from MC simulations and discuss the sensitivity of parameters.

Example 4.11 (Comparison to MC simulations)**.** In this example, we confirm our closed-form formula (4.12) by comparing with MC simulations when $m = 2, 3, 4$. The parameters used in the experiment are $N = 252, T = 1$ and for various $\delta_0 = -1, -0.8, -0.6, \ldots, 1$. the testing is consumed on the Schwartz model with the parameter $\mu = 2.857, \sigma = 0.129$ and $\kappa = 0.99$. The comparisons as displayed in Figure 4.2.

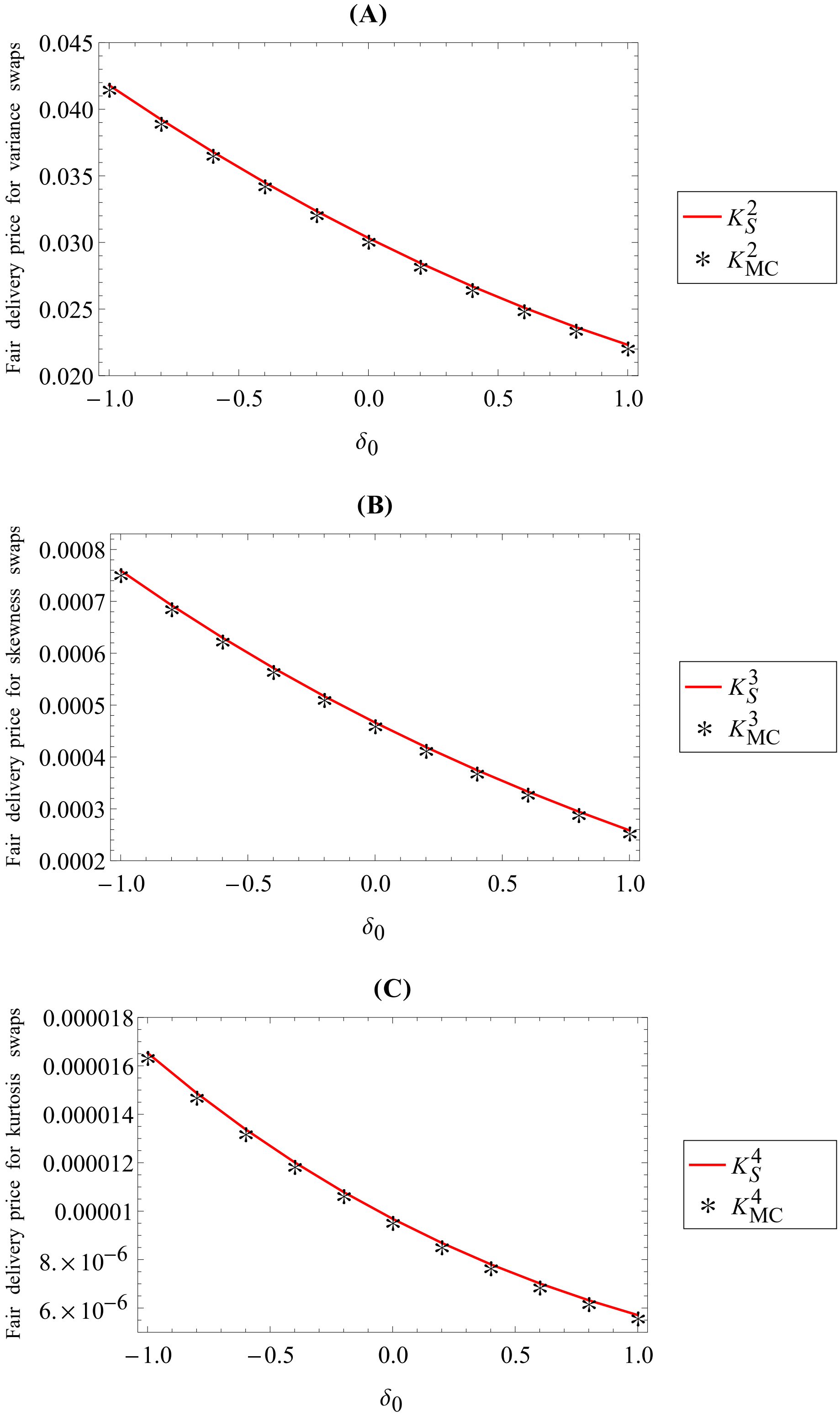

Figure 4.2: Comparisons of fair delivery prices from the closed-form solution K^m and the MC simulations for pricing K^m_{MC} of variance (A), skewness (B) and kurtosis (C) swaps

Figure 4.2 shows that the results from the closed-form solution and the MC simulations perfectly match, illustrating that the closed-form formula does not contain any algebraic errors and practitioners cam confidently use the formula for pricing moment swaps. Define the percentage relative error (ε^m) from using MC simulations by

$$\varepsilon^m(T, N, \delta_0; N_p) := \left| \frac{K^m(T, N, \delta_0) - K^m_{\text{MC}}(T, N, \delta_0; N_p)}{K^m(T, N, \delta_0)} \right| \times 100\%$$

for any δ_0. The percentage relative errors of variance, skewness and kurtosis swaps are less that 1% as displayed in Figure 4.3.

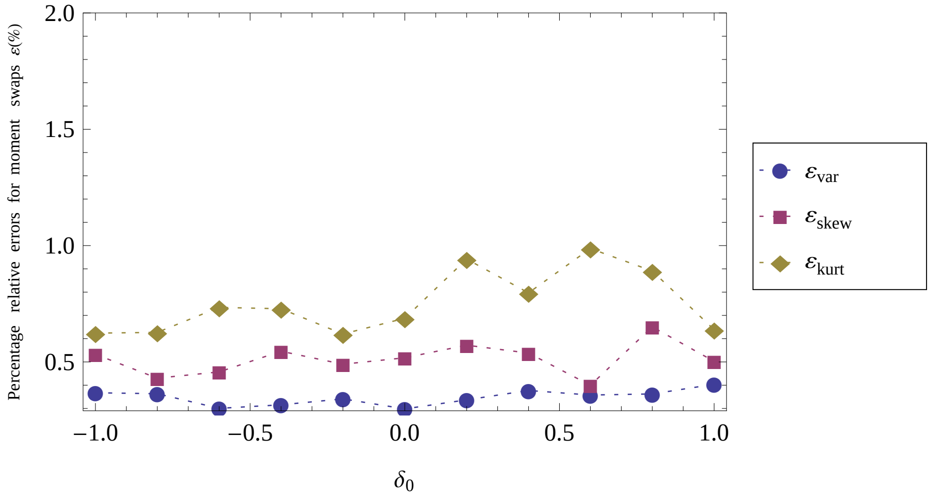

Figure 4.3: The percentage relative errors obtained by using the MC simulations for variance, skewness and kurtosis swaps

In addition to comparison in Figure 4.3, we define the average percentage relative error ($\overline{\varepsilon^m}$) from using MC simulations by

$$\overline{\varepsilon^m}(T, N, \delta_0, N_{\delta_0}; N_p) := \frac{1}{N_{\delta_0}} \sum_{i=1}^{N_{\delta_0}} \left| \frac{K^m(T, N, \delta_0^{(i)}) - K^m_{\text{MC}}(T, N, \delta_0^{(i)}; N_p)}{K^m(T, N, \delta_0^{(i)})} \right| \times 100\%$$

for $\delta_0^{(i)}, i = 1, 2, \ldots, N_{\delta_0}$ where N_{δ_0} is the number of δ_0 to measure the levels of accuracy which is shown in the Table 4.1 for $N_p = 10,000, 30,000$ and $50,000$ and $\delta_0 =$

$-1, -0.8, -0.6, \ldots, 1$ and $T = 1$.

m^{th}	K^m_{MC}			K^m_{EBS}
moment	N_p	$\overline{\varepsilon^m}(\%)$	Comp. (s)	Comp. (s)
	$10,000$	0.348	476.105	
$m = 2$	$30,000$	0.320	1501.048	0.078
	$50,000$	0.312	2447.938	
	$10,000$	0.512	480.204	
$m = 3$	$30,000$	0.507	1441.286	0.125
	$50,000$	0.477	2394.559	
	$10,000$	0.752	487.097	
$m = 4$	$30,000$	0.711	1507.094	0.172
	$50,000$	0.705	2464.436	

Table 4.1: Average percentage relative errors $\overline{\varepsilon^m}$ and computational times (Comp.) of MC simulations for pricing variance swaps ($m = 2$), skewness swaps ($m = 3$) and kurtosis swaps ($m = 4$) for $N_p = 10,000, 30,000$, and $50,000$, comparing with computational times of the closed-form formula

Table 4.1 confirms in addition that the results from the closed-form formula and the MC simulations match with high accuracy with very small $\overline{\varepsilon^m}$ for all cases of m and N_p, the highest is 0.7% when $m = 4$ and $N_p = 10,000$. Moreover, the accuracy for MC simulations is improved when N_p increases, trade-off with increasing in computational times. The experiment showed that the computational time from closed-form formula is extremely faster that that from MC simulation, around $3,000$ times faster.

Example 4.12 (Sensitivity of parameters)**.** In this study, we investigate the sensitivity of fair prices for moment swaps ($m = 2, 3, 4$) based on small changes of parameters in the Schwartz model with $\mu = 2.857, \sigma = 0.129, \kappa = 0.099$, calibrated from oil market data as proposed by Schwartz [18] and the convenience yield $\delta_0 = 2.773$. To check the sensitivity of each parameter separately, the change of fair price is computed corresponding to the change of one parameter while the other three parameter are fixed. The sensitivity is measured based on the percentage relative errors of the fair price ΔK^m and parameter ΔP, defined by

$$\Delta P := \left| \frac{P - P'}{P} \right| \times 100\% \text{ and } \Delta K^m(P, P') := \left| \frac{K^m(P) - K^m(P')}{K^m(P)} \right| \times 100\%,$$

with fixed $T = 1$ and $N = 252$. The results are shown in Table 4.2–4.3.

P	P'	$\Delta P(\%)$	$\Delta K^2(P,P')(\%)$	$\Delta K^3(P,P')(\%)$	$\Delta K^4(P,P')(\%)$
κ	$\kappa' = 1.02\kappa$	2	3.312×10^{-5}	1.335×10^{-6}	1.337×10^{-8}
	$\kappa' = 1.04\kappa$	4	6.589×10^{-5}	2.664×10^{-5}	2.706×10^{-8}
	$\kappa' = 1.06\kappa$	6	9.831×10^{-4}	3.989×10^{-6}	4.020×10^{-8}
	$\kappa' = 1.08\kappa$	8	1.304×10^{-4}	5.310×10^{-6}	5.357×10^{-8}
	$\kappa' = 1.10\kappa$	10	1.621×10^{-4}	6.625×10^{-6}	6.639×10^{-8}
μ	$\mu' = 1.02\mu$	2	2.943×10^{-5}	1.187×10^{-6}	1.209×10^{-8}
	$\mu' = 1.04\mu$	4	5.864×10^{-5}	2.373×10^{-6}	2.410×10^{-8}
	$\mu' = 1.06\mu$	6	8.761×10^{-5}	3.556×10^{-6}	3.599×10^{-8}
	$\mu' = 1.08\mu$	8	1.163×10^{-4}	4.738×10^{-6}	4.778×10^{-8}
	$\mu' = 1.10\mu$	10	1.449×10^{-4}	5.918×10^{-6}	5.948×10^{-8}
σ	$\sigma' = 1.02\sigma_0$	2	6.738×10^{-4}	5.599×10^{-6}	3.026×10^{-7}
	$\sigma' = 1.04\sigma_0$	4	1.361×10^{-3}	1.131×10^{-5}	6.224×10^{-7}
	$\sigma' = 1.06\sigma_0$	6	2.061×10^{-3}	1.715×10^{-5}	9.599×10^{-7}
	$\sigma' = 1.08\sigma_0$	8	2.775×10^{-3}	2.310×10^{-5}	1.316×10^{-6}
	$\sigma' = 1.10\sigma_0$	10	3.502×10^{-3}	2.916×10^{-5}	1.691×10^{-6}
δ_0	$\delta_0' = 1.02\delta_0$	2	1.009×10^{-3}	2.492×10^{-5}	5.839×10^{-7}
	$\delta_0' = 1.04\delta_0$	4	2.040×10^{-3}	5.049×10^{-5}	1.189×10^{-6}
	$\delta_0' = 1.06\delta_0$	6	3.093×10^{-3}	7.671×10^{-5}	1.816×10^{-6}
	$\delta_0' = 1.08\delta_0$	8	4.168×10^{-3}	1.036×10^{-4}	2.765×10^{-6}
	$\delta_0' = 1.10\delta_0$	10	5.265×10^{-3}	1.312×10^{-4}	3.137×10^{-6}

Table 4.2: The percentage relative errors of the fair prices of moment swaps $\Delta K^m (m = 2, 3, 4)$ for $\Delta P = 2, 4, 6, 8, 10\%$ of parameters κ, μ, σ and δ_0

Moreover, since Table 4.2 shows that ΔK^m depends linearly on ΔP, suggesting that the order of sensitivity S_p^m of each parameter is computed as the average of $\frac{\Delta K^m}{\Delta P}$,

$$S_P^m := \frac{1}{n}\sum_{i=1}^{n}\frac{\Delta K^m(P_i, P_i')}{\Delta P_i},$$

as shown in Table 4.3.

Moment swaps	S_κ^m	S_μ^m	S_σ^m	$S_{\delta_0}^m$
$m = 2$	9.796×10^{-5}	8.738×10^{-5}	2.075×10^{-3}	3.115×10^{-3}
$m = 3$	3.985×10^{-6}	3.554×10^{-6}	1.726×10^{-5}	7.738×10^{-5}
$m = 4$	4.012×10^{-8}	2.589×10^{-8}	9.783×10^{-7}	1.838×10^{-6}

Table 4.3: The orders of sensitivity of fair prices for $m = 2, 3, 4$ corresponding to parameters κ, μ, σ and δ_0

Table 4.3 shows that ΔK^m depends linearly on ΔP for all cases ($m = 2, 3, 4$ and all parameters). The results shows that K^m is more sensitive to the convenience yield δ_0 than the others. When comparing using the orders of sensitivity, the results display that K^m is more sensitive to $\delta_0 > \sigma > \kappa > \mu$.

4.5 Comparison of Fair Delivery Prices between Underlyings Stocks and Commodities

In this section, we compare the behavior of the fair prices of moment swaps based on the underlying assets, the extended Black-Scholes model for stocks and Schwartz model for commodities. The closed-form formula of pricing moment swaps with underlying commodities described by Schwartz model,

$$K_S^m(T, N, X_0) = \frac{1}{T}\sum_{j=0}^{m}\sum_{i=1}^{N}\tilde{A}_{m,j}(\Delta t, t_{i-1})\,(X_0)^j\,,$$

where $\tilde{A}_{m,j}(\Delta t, t_{i-1})$ defined in (4.11), depends on the initial commodity price $X_0 = \dfrac{\delta_0}{\kappa}$. The convenience yield at the current time δ_0 is associated with the storage of products or physical goods, which impacts the commodity prices. There is an inverse relationship between the commodity prices and storage levels; when the storage levels are low, the commodity prices tend to rise, and vice verse. However, The closed-form formula with underlying stocks described by extended Black-Scholes model,

$$K^m_{\mathrm{EBS}}(T, N) = \frac{1}{T} \sum_{i=1}^{N} A_m(\Delta t; t_i, m),$$

where $A_m(\Delta t; t_i, m)$ defined in (3.2)–(3.4), does not depend on X_0 because the storage is not required for stocks.

To illustrate this phenomena, the numerical experiment is carried out using parameters $N = 252$, $T = 1$, $r(T) = 0.125 + 0.05T$, $\sigma(T) = \sqrt{0.03 + 0.02T}$, $\mu = 2.857, \sigma = 0.129$ and $\kappa = 0.99$. The comparisons of fair prices K^m_{EBS} and K^m_{S} for $m = 2, 3, 4$ are displayed in Figure 4.4.

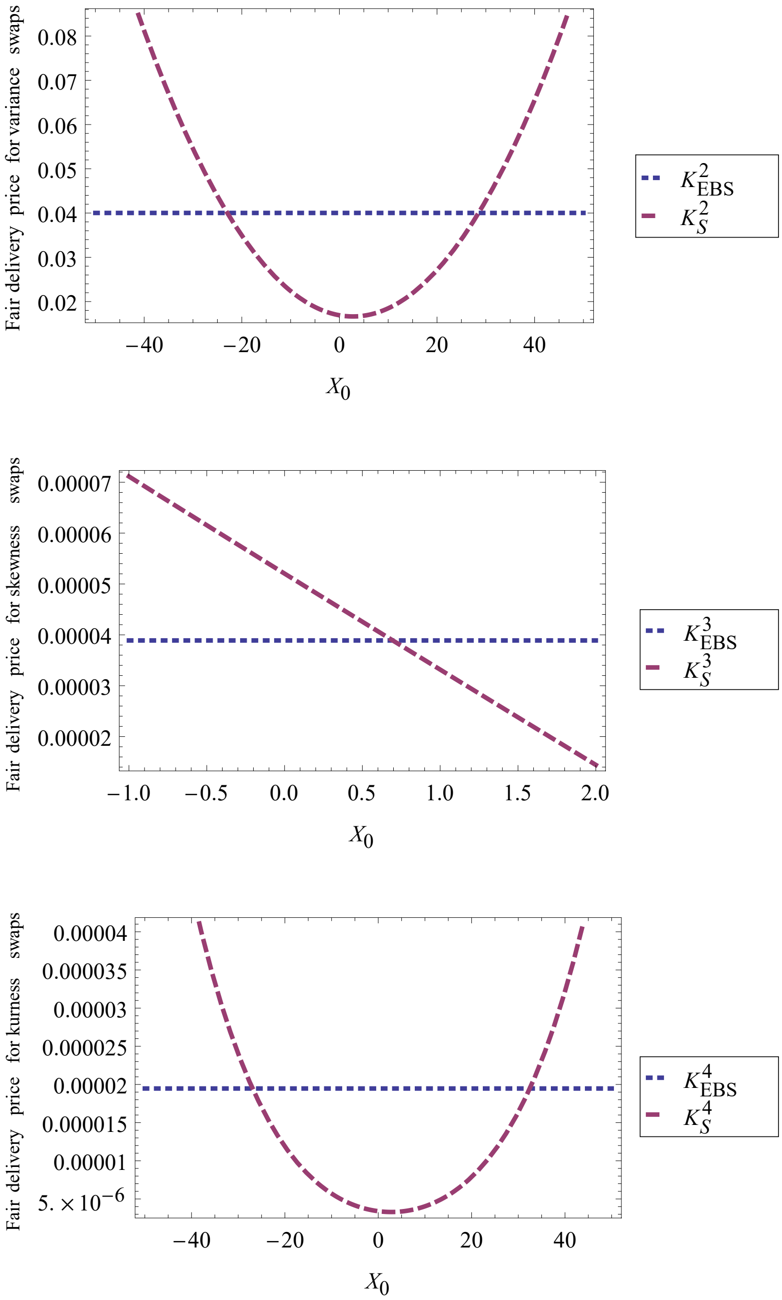

Figure 4.4: Comparisons of fair delivery prices obtained from the closed-form solutions K^m_{BS} and K^m_{S} for given initial X_0 of (A) variance, (B) skewness, and (C) kurtosis swaps

Figure 4.4 shows the fair prices of moment swaps K^m_{EBS} for $m = 2, 3, 4$ described by the the extended Black-Scholes model which do not depend on the initial value X_0. However, it is not the case for the K^m_{S} of the Schwartz model, where the fair prices depend continuously on X_0.

Remark 4.13.

1. There are many different factors that affect commodity prices and one of the main factor is the convenience yield. Our results show that the behavior of moment swaps price for commodity depends on the convenience yield as appeared in the Schwartz model but not the extended Black-Scholes model. This implies that the extended Black-Scholes model is not suitable for describing the commodity prices.
2. In this study, the results are valid for the extended Black-Scholes and Schwartz model. The results can be applied to other underlying assets that are also described by the extended Black-Scholes and Schwartz model.

4.6 Conclusion

In this chapter, we have presented a simple and easy-to-use pricing formula for discretely-sampled moment swaps when the realized higher moments defined in terms of m^{th}-moment of the log-returns of a specified underlying asset described by the Schwartz model. We have improved the conditional expectations of Weraprasertsakun and Rujivan [19] by using combinatorial technique to obtain the closed-form formula. We also applied the formula to extract the convenience yields of commodity, given that the prices of moment swaps are observed. In addition, we confirmed the result by comparing with that MC simulations. Moreover, the sensitivity of the fair prices respect to the parameters was examined numerically, showing that the fair price was more sensitive to the convenience yield when $m \geq 2$. Finally, we showed that the fair prices of moment swaps for underlying commodity depend on the initial price of the commodity X_0, which was in contrast to the moment swaps for underlying stocks, where the fair prices did not depend on the initial stock price.

CHAPTER V

CONCLUSIONS

This thesis provided simple closed-form formulas for pricing discretely-sample moment swaps with underlying assets described by Itô process, extended Black-Scholes model for stock prices and Schwartz model for commodity prices.

The obtained formulas for extended Black-Scholes model was developed based on the Feynman-Kac formula for the conditional moments, and simplified to obtained the simple-closed form formulas by combinatorial techniques.

The formula for Schwartz model was obtained by improving the result from Weraprasertsakun and Rujivan [19] for the conditional expectations by solving analytically the system of ODEs, and simplified by combinatorial techniques.

The formulas were shown numerically to have substantial advantage in terms of both accuracy and time efficiency over Monte Carlo simulations and other implicit formula. This studied can be beneficial to market practitioners to use the formulas in practice when there is obviously increasing demand of trading moment swaps in financial markets.

REFERENCES

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.

[2] M. Broadie and A. Jain, "The effect of jumps and discrete sampling on volatility and variance swaps," *International Journal of Theoretical and Applied Finance*, vol. 11, no. 8, pp. 761-797, 2008.

[3] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," *Society for Industrial and Applied Mathematics Review*, vol. 43, no. 3, pp. 525–546, 2001.

[4] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*: second edition. Springer, 1991.

[5] K. Khaled and M. Samia, "Estimation of parameters of stochastic differential equations Black-Scholes model share price of gold," *Journal of Mathematics and Statistics*, vol. 6, no. 4, pp. 421-424, 2010.

[6] M. Kijima, *Stochastic Processes with Applications to Finance.* Chapman and Hall/ CRC, 2002.

[7] R. Kozhan, A. Neuberger and P. Schneider, "The skew risk premium in the equity index market," *The Review of Financial Studies*, vol. 26, no. 9, pp. 2174-2203, 2013.

[8] A. E. Kyprianou, W. Schoutens and P. Wilmott, *Exotic Option Pricing and Advance Lévy Model.* John Wiley and Sons, 2005.

[9] G. Levy, *Computational Finance Using C and C#.* Elsevier, 2008.

[10] R. C. Merton, "Theory of rational option pricing," *The Bell Journal of Economics and Management Science*, vol. 4, no. 1, pp. 141–183, 1973.

[11] A. Neuberger, "Realized skewness," *The Review of Financial Studies*, vol. 25, no. 11, pp. 3423-3455, 2012.

[12] B. Oksendal, *Stochastic Differential Equations: An Introduction with Applications.* Springer, 2003.

[13] L. Rompolis and E. Tzavalis, "Pricing and hedging contingent claims using variance and higher-order moment swaps," *Quantitative Finance*, vol. 17, no. 4, pp. 531-550, 2017.

[14] S. Rujivan, "A novel analytical approach for pricing discretely sampled gamma swaps in the Heston model," *The ANZIAM Journal*, vol. 5, no. 3, pp. 244-268, 2016.

[15] S. Rujivan and S.-P. Zhu, "A simple closed-form formula for pricing discretely-sampled variance swaps under the Heston model," *The ANZIAM Journal*, vol. 56, no. 1, pp. 1-27, 2014.

[16] S. Rujivan and S.-P. Zhu, "A simplified analytical approach for pricing discretely-sampled variance swaps with stochastic volatility," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1644-1650, 2012.

[17] W. Schoutens, "Moment swaps," *Quantitative Finance*, vol. 5, no. 6, pp. 525-530, 2005.

[18] E.S. Schwartz, "The stochastic behavior of commodity prices: Implications for valuation and hedging," *The Journal of Finance*, vol. 52, no. 3, pp. 922-973, 1997.

[19] A. Weraprasertsakun and S. Rujivan, "A closed-form formula for pricing variance swaps on commodities," *Vietnam Journal of Mathematics*, vol. 45, no. 1-2, pp. 255-264, 2017.

[20] J. E. Zhang, F. Zhen, X. Sun and H. Zhao, "The skewness implied in the Heston model and its application," *Journal of Futures Markets*, vol. 37, no. 3, pp. 211-237, 2017.

[21] H. Zhao, J. E. Zhang and E.C. Chang, "The relation between physical and risk-neutral cumulants," *International Review of Finance*, vol. 13, no. 3, pp. 345-381, 2013.

[22] W. Zheng and Y.K. Kwok, "Closed form pricing formulas for discretely sampled generalized variance swaps," *Mathematical Finance*, vol. 24, no. 4, pp. 855-881, 2014.

[23] S.-P. Zhu and G.-H. Lian, "A closed-form exact solution for pricing variance swaps with stochastic volatility," *Mathematical Financee*, vol. 21, no. 2, pp. 233-256, 2011.

[24] S.-P. Zhu and G.-H. Lian, "On the valuation of variance swaps with stochastic volatility," *Applied Mathematics and Computation*, vol. 219, no. 4, pp. 1654-1669, 2012.

BIOGRAPHY

Name	Mr. Kittisak Chumpong
Date of Birth	29 Sebtember 1989
Place of Birth	Nakhon Si Thammarat, Thailand
Education	B.Sc. (Mathematics), Chulalongkorn University, 2011
	M.Sc. (Mathematics), Chulalongkorn University, 2014