

การครอบรูปทรงที่เกี่ยวกับสมบัติรูปเรขาคณิต

นายพงษ์บัณฑิต ต้นโพธิ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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COVERING OF OBJECTS RELATED TO RUPERT PROPERTY

Mr. Pongbunthit Tonpho

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics

Department of Mathematics and Computer Science

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By Mr. Pongbunthit Tonpho
Field of Study Mathematics
Thesis Advisor Associate Professor Wacharin Wichiramala, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial
Fulfillment of the Requirements for the Master's Degree

.....Dean of the Faculty of Science
(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

.....Chairman
(Associate Professor Nataphan Kitisin, Ph.D.)

.....Thesis Advisor
(Associate Professor Wacharin Wichiramala, Ph.D.)

.....Examiner
(Assistant Professor Pongdate Montagantirud, Ph.D.)

.....External Examiner
(Assistant Professor Wittawat Kositwattanarek, Ph.D.)

พงษ์บัณฑิต ต้นโพธิ์ : การครอบรูปทรงที่เกี่ยวกับสมบัติรูเพิร์ต. (COVERING OF OBJECTS RELATED TO RUPERT PROPERTY) อ.ที่ปรึกษาวิทยานิพนธ์
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ปัญหาหนึ่งที่เกี่ยวข้องกับปัญหาการครอบรูปทรงคือปัญหาที่ถามว่าวัตถุหนึ่งมีสมบัติรูเพิร์ตหรือไม่ กล่าวคือวัตถุ K ใน \mathbb{R}^3 มีสมบัติรูเพิร์ตก็ต่อเมื่อเราสามารถสร้างรูปในก๊อบบี้หนึ่งของ K จนกระทั่งอีกก๊อบบี้หนึ่งของ K สามารถทะลุผ่านรูนั้นได้ เนื่องจากเรามีตัวอย่างของทั้งวัตถุที่มีสมบัติรูเพิร์ตอย่างเช่นลูกบาศก์ และวัตถุที่ไม่มีสมบัติรูเพิร์ตอย่างเช่นทรงกลม สิ่งที่น่าสนใจจึงเป็นการตรวจสอบว่าวัตถุหนึ่ง ๆ ที่เราสนใจนั้นมีสมบัติรูเพิร์ตหรือไม่ ในงานนี้เราสร้างบทตั้งต่าง ๆ ที่มีประโยชน์ในการศึกษาปัญหานี้ ยิ่งไปกว่านั้นเรายังนำบทตั้งที่สร้างขึ้นและการหาค่าเหมาะที่สุดเชิงตัวเลขแสดงได้ว่าส่วนหนึ่งจากทั้งหมดของ 13 ทรงตันของอาร์คิมิดีสมีสมบัติรูเพิร์ต

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PONGBUNTHIT TONPHO : COVERING OF OBJECTS RELATED TO RU-
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ADVISOR : ASSOCIATE PROFESSOR WACHARIN WICHIRAMALA, Ph.D.

One problem related to covering problem is whether one object has Rupert property. An object K in \mathbb{R}^3 has the Rupert property if a hole could be cut through one copy of K with the same size to permit another copy to pass through it. There are objects in \mathbb{R}^3 which have Rupert property, such as a cube. But not all of objects in \mathbb{R}^3 has Rupert property, such as a sphere. It is interesting to check that a given object has Rupert property. In this work, we construct useful lemmas for studying this problem. Furthermore, we apply these lemmas to show that most of the 13 Archimedean solids have Rupert property by numerical optimization.

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CHAPTER I

INTRODUCTION

Questions about when, and whether, one set can cover another set are fundamental to study of shapes in geometry. These problems are called covering problem.

Definition 1.1. Let K , K' and H be sets in \mathbb{R}^n . We say K' is a **congruent copy** of K if there is an isometry f on \mathbb{R}^n such that $f(K) = K'$. We say a set K **covers** H if K contains a congruent copy of H . In this case, we also say H **fits in** K .

There are a lot of studies related to this question. In 1964, H. Steinhaus [5] asked for a necessary and sufficient condition on the six sides a, b, c, a', b', c' for the triangle T' with sides a', b', c' to fit into the triangle T with sides a, b, c . Separate necessary conditions and sufficient conditions are easy to give, but a condition that is both necessary and sufficient is more elusive. In 1993, K. A. Post [3] gave a list of 18 inequalities which is both necessary and sufficient in the sense that if one of the inequalities correct, the T' fits in T and if T' fits in T , then at least one of the inequalities is correct. Moreover, in this paper K. A. Post also proved the following theorem.

Theorem 1.2. *If a largest scaled copy of a triangle is a subset of another triangle, then one of its sides coincides along a side of the containing triangle.*

In 2002, J. E. Wetzel [8] found the largest square that fits into a given triangle. Later, J. M. Sullivan [6] generalized this theorem to the case of a convex polygon in a triangle.

One problem related to these questions is Prince Rupert's problem. More than three hundred years ago, Prince Rupert won a wager that a hole could be cut through one of two cubes with the same size to permit another cube to pass through

it. This property is called a Rupert property for cube. In 1968, C. J. Scriba [4] showed that the tetrahedron and the octahedron have the same property. In 2017, R. P. Jerrard, J. E. Wetzel and L. Yuan [2] found that the dodecahedron and the icosahedron also have the same property. Thus all Platonic solids, including the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron, have Rupert property. Some compact, convex subsets in \mathbb{R}^3 do not have Rupert property, such as the unit sphere and the circular cylinder with radius r and height $2r$. That is a compact, convex subset in \mathbb{R}^3 does not need to have Rupert property.

Due to lack of counterexample, J. E. Wetzel and L. Yuan conjectured in [2] that every convex polyhedron has Rupert property. Therefore, the interesting thing is to check whether a given convex polyhedron has Rupert property.

In Chapter II, we recall some basic definitions and lemmas related to this work.

In Chapter III, we study some properties about covering of objects in \mathbb{R}^2 which relate to this work.

In Chapter IV, we will mention many relations between Prince Rupert's problem and covering problem.

In the last chapter, we suggest some ideas to study the further research about Prince Rupert's problem.

CHAPTER II

PRELIMINARIES

In this chapter, we recall some definitions and lemmas used in this work. Throughout this work, we use an object to refer to a compact, convex subset of \mathbb{R}^3 . Moreover, we use a planar set to refer to a subset of \mathbb{R}^2 or an isometric image of a subset of \mathbb{R}^2 .

2.1 CONVEX SET AND AFFINE MAP

First, we introduce definitions about convex set and lemmas related to this work.

Definition 2.1. A subset $X \subseteq \mathbb{R}^n$ is called **convex** if and only if, for points x_1, x_2 in X and $\lambda \in [0, 1]$,

$$\lambda x_1 + (1 - \lambda)x_2 \in X.$$

The next definition is the definition of the convex hull of a set.

Definition 2.2. The **convex hull** of a set X of points in \mathbb{R}^n is the smallest convex set that contains X . Moreover, the convex hull C of a set of N points p_1, \dots, p_N is then given by the expression

$$C = \left\{ \sum_{i=1}^N \lambda_i p_i : \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^N \lambda_i = 1 \right\}.$$

We will use $\mathbb{H}(X)$ refers to the convex hull of a set X .

In this work, we use some mappings which are affine map.

Definition 2.3. An **affine transformation** or **affine map** $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a map of the form $x \mapsto L(x) + k$ where L is a linear map on the space \mathbb{R}^3 , x is a point in \mathbb{R}^3 , and k is a point in \mathbb{R}^3 .

The next three lemmas are about convex set and affine map related to this work.

Lemma 2.4. *Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an affine map. If a set $X \subseteq \mathbb{R}^3$ is convex, then $f(X)$ is also convex.*

Proof. Assume that a set $X \subseteq \mathbb{R}^3$ is convex. Let $x_1, x_2 \in X$ and $\lambda \in [0, 1]$. Since f is an affine map, $f(x) = L(x) + k$ for some linear map L on \mathbb{R}^3 and some point k in \mathbb{R}^3 . We have

$$\begin{aligned} \lambda f(x_1) + (1 - \lambda)f(x_2) &= \lambda(L(x_1) + k) + (1 - \lambda)(L(x_2) + k) \\ &= \lambda L(x_1) + \lambda k + (1 - \lambda)L(x_2) + (1 - \lambda)k \\ &= L(\lambda x_1 + (1 - \lambda)x_2) + k \\ &= f(\lambda x_1 + (1 - \lambda)x_2) \\ &\in f(X). \end{aligned}$$

Hence $f(X)$ is convex. □

Lemma 2.5. *Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an affine map and let $p_1, \dots, p_n \in \mathbb{R}^3$. Suppose that $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^n \lambda_i = 1$. Then $f\left(\sum_{i=1}^n \lambda_i p_i\right) = \sum_{i=1}^n \lambda_i f(p_i)$.*

Proof. Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be such that $\sum_{i=1}^n \lambda_i = 1$. Since f is an affine map, $f(x) = L(x) + k$ for some linear map L on \mathbb{R}^3 and some point k in \mathbb{R}^3 . Then

$$\begin{aligned} f\left(\sum_{i=1}^n \lambda_i p_i\right) &= L\left(\sum_{i=1}^n \lambda_i p_i\right) + k \\ &= \sum_{i=1}^n L(\lambda_i p_i) + k \\ &= \sum_{i=1}^n \lambda_i L(p_i) + \sum_{i=1}^n \lambda_i k \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \lambda_i (L(p_i) + k) \\
&= \sum_{i=1}^n \lambda_i f(p_i).
\end{aligned}$$

□

Lemma 2.6. *Let P be a set of finite points in \mathbb{R}^3 . A convex hull of an affine map f of P is the image of a convex hull of P under f .*

Proof. Since $P \subseteq \mathbb{H}(P)$, $f(P) \subseteq f(\mathbb{H}(P))$. By Lemma 2.4, $f(\mathbb{H}(P))$ is convex. Then $\mathbb{H}(f(P)) \subseteq f(\mathbb{H}(P))$.

On the contrary, let p be a point in $\mathbb{H}(p)$. Then $p = \sum_{i=1}^n \lambda_i p_i$ where $p_1, \dots, p_n \in P$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ for some $n \in \mathbb{N}$. By Lemma 2.5, we have $f(p) = f\left(\sum_{i=1}^n \lambda_i p_i\right) = \sum_{i=1}^n \lambda_i f(p_i) \in \mathbb{H}(f(P))$. Hence $f(\mathbb{H}(P)) \subseteq \mathbb{H}(f(P))$. □

2.2 OBJECTS RELATED TO THIS WORK

In this section, we will give definitions about objects which we study in this work.

Definition 2.7. A **polygon** is a compact planar set bounded by a finite number of straight line segments connected to form a closed polygonal chain.

Definition 2.8. A **polyhedron** is an object bounded by flat polygonal faces, straight edges and sharp corners.

Definition 2.9. A **Platonic solid** is a regular, convex polyhedron which is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex.

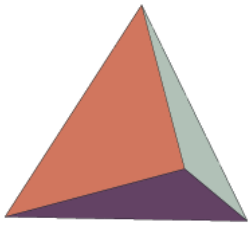
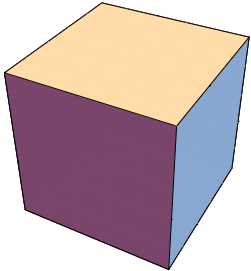
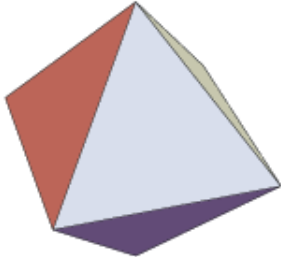
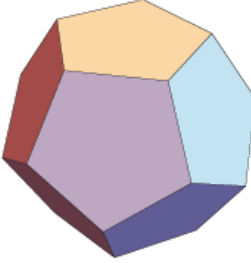
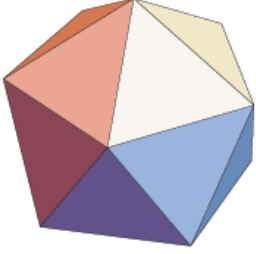
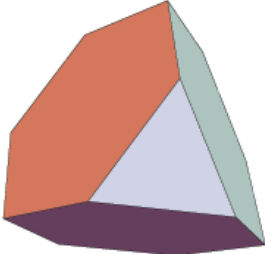
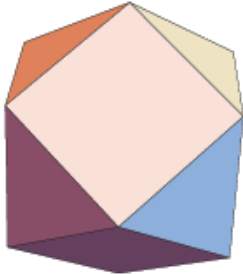

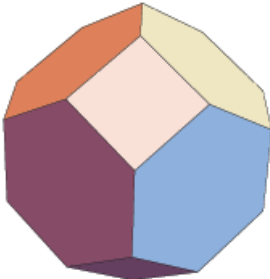
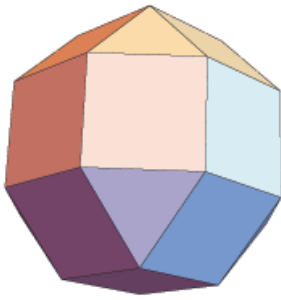
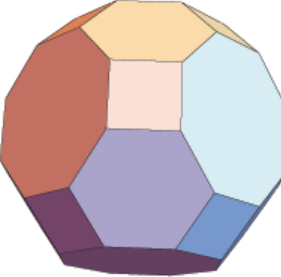

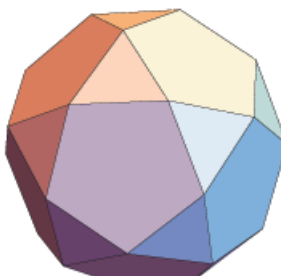
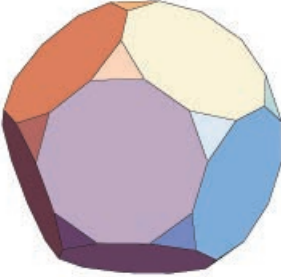
Platonic Solid	Figure
Tetrahedron	
Cube	
Octahedron	
Dodecahedron	
Icosahedron	

Table 2.2: List of all Platonic solids

Definition 2.10. An **Archimedean solid** is a convex polyhedron which has a similar arrangement of nonintersecting regular convex polygons of two or more different types arranged in the same way about each vertex with all sides the same length.

Platonic Solid	Figure
Truncated tetrahedron	
Cuboctahedron	
Truncated cube	
Truncated octahedron	

Rhombicuboctahedron	 A 3D rendering of a rhombicuboctahedron, a polyhedron with 14 faces: 6 squares and 8 hexagons. The faces are colored in a variety of shades including orange, light blue, purple, and dark blue.
Truncated cuboctahedron	 A 3D rendering of a truncated cuboctahedron, a polyhedron with 14 faces: 6 squares and 8 hexagons. The faces are colored in a variety of shades including orange, light blue, purple, and dark blue.
Snub cube	 A 3D rendering of a snub cube, a polyhedron with 14 faces: 6 squares and 8 hexagons. The faces are colored in a variety of shades including orange, light blue, purple, and dark blue.
Icosidodecahedron	 A 3D rendering of an icosidodecahedron, a polyhedron with 14 faces: 6 squares and 8 hexagons. The faces are colored in a variety of shades including orange, light blue, purple, and dark blue.
Truncated dodecahedron	 A 3D rendering of a truncated dodecahedron, a polyhedron with 14 faces: 6 squares and 8 hexagons. The faces are colored in a variety of shades including orange, light blue, purple, and dark blue.

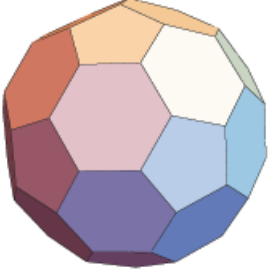

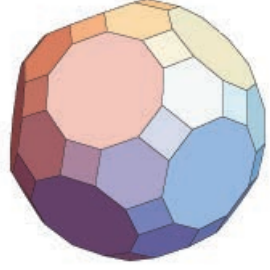

Truncated icosahedron	
Rhombicosidodecahedron	
Truncated icosidodecahedron	
Snub dodecahedron	

Table 2.4: List of all Archimedean solids

The following is a definition of width of a nonempty compact subset of \mathbb{R}^2 .

Definition 2.11. Let K be a nonempty compact subset of \mathbb{R}^2 and l be a line in \mathbb{R}^2 . The *width* of K in direction of l is the minimum distance between two lines perpendicular to l with K in between them. The *minimum width* of K

is the minimum of width among every direction. Similarly, for an object K in \mathbb{R}^3 , the *width* of K in direction of l is the minimum distance between two planes perpendicular to l with K in between them.

The following is a definition of an interesting type of object in \mathbb{R}^3 .

Definition 2.12. An object K is said to have *constant width* w if the width is w in every direction.

A sphere and Meissner tetrahedra are the examples of objects with constant width.

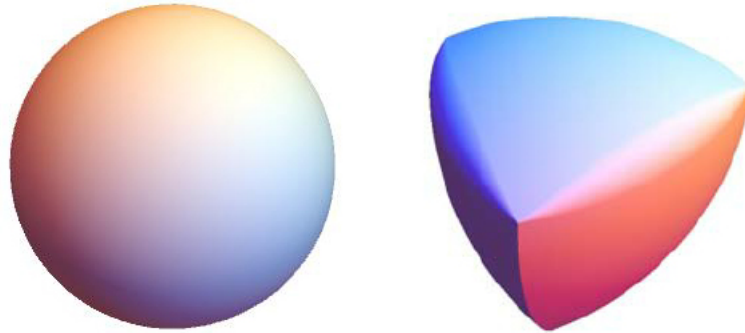


Figure 2.1: the sphere (left) and the Meissner tetrahedra (right)

2.3 RUPERT PROPERTY AND NIEUWLAND CONSTANT

From [2], R. P. Jerrard, J. E. Wetzel and L. Yuan gave a definition of Rupert property stating as follows.

Definition 2.13. Let K be an object in \mathbb{R}^3 . K has **Rupert property** if and only if there are planes M and N so that the (orthogonally) projected images of K to M fits in the interior of the (orthogonally) projected images of K to N .

From its definition, we may simply regard Rupert property as follows.

Lemma 2.14. *Let K be an object in \mathbb{R}^3 . K has **Rupert property** if and only if there is a plane M and two congruent copies of K whose (orthogonally) projected images to M has one image in the interior of the other.*

Remark 2.15. It is clear that Prince Rupert's problem relates to a covering problem of the projections of 2 congruent copies of any given object.

In 1816, Pieter Nieuwland published [7] the solution of the question of what is the largest cube that can be passed through a cube of unit size, i.e., how large a positive scalar ν that a scaled congruent copy νC of cube C can pass through in C . This question for any object is called Nieuwland's question. For an object K in \mathbb{R}^3 , we define the Nieuwland's constant $\nu(K)$ by

$$\nu(K) = \sup\{\nu > 0 \mid \text{there is a hole in } K \text{ that an object } \nu K \text{ can pass through it}\}.$$

Note that for any convex object K , we have $\nu(K) \geq 1$. By direct observation, we have the following obvious lemma.

Lemma 2.16. *An object K in \mathbb{R}^3 has Rupert property if and only if $\nu(K) > 1$.*

Table 2.5 from [2] shows the known estimates of the Nieuwland constants for all Platonic solids.

Platonic Solid	Nieuwland Estimate
Tetrahedron \mathfrak{T}	$\nu(\mathfrak{T}) \geq \frac{2}{5}\sqrt{3}(\sqrt{6} - 1) > 1.004235$
Cube \mathfrak{C}	$\nu(\mathfrak{C}) = \frac{3}{4}\sqrt{2} \geq 1.060660$
Octahedron \mathfrak{O}	$\nu(\mathfrak{O}) \geq \frac{3}{4}\sqrt{2} \geq 1.060660$
Dodecahedron \mathfrak{D}	$\nu(\mathfrak{D}) \geq \frac{171}{170} > 1.005882$
Icosahedron \mathfrak{I}	$\nu(\mathfrak{I}) \geq \frac{1108}{1098} > 1.009107$

Table 2.5: Nieuwland constant estimates for all Platonic solids [2]

Besides 5 Platonic solids, Y. Chai, L. Yuan and T. Zamfirescu studied 8 of the 13 Archimedean solids to find the estimates of their Nieuwland constants [1]. Table 2.6 from [1] shows the estimates of the Nieuwland constants for 8 of the 13 Archimedean solids.

Solid	Nieuwland Estimate
Cuboctahedron \mathcal{C}	$\nu(\mathcal{C}) > 1.01461$
Truncated cube \mathfrak{C}_t	$\nu(\mathfrak{C}_t) > 1.02036$
Truncated octahedron \mathcal{O}_t	$\nu(\mathcal{O}_t) > 1.00815$
Rhombicuboctahedron \mathcal{R}	$\nu(\mathcal{R}) > 1.00609$
Truncated cuboctahedron \mathcal{C}_t	$\nu(\mathcal{C}_t) > 1.00370$
Icosidodecahedron \mathcal{I}	$\nu(\mathcal{I}) > 1.00015$
Truncated dodecahedron \mathcal{D}_t	$\nu(\mathcal{D}_t) > 1.00014$
Truncated icosahedron I_t	$\nu(I_t) > 1.00004$

Table 2.6: Nieuwland constant estimates for 8 of the 13 Archimedean solids [1]

In chapter IV, we will improve Nieuwland constant estimates for the tetrahedron, the dodecahedron and the icosahedron. For the 13 Archimedean solids, the truncated tetrahedron and the truncated icosidodecahedron have not been studied yet. We show that their Nieuwland constant is greater than 1. Consequently, they have Rupert property.

CHAPTER III

COVERING OF OBJECTS IN 2D

In this chapter, we give some properties about covering of objects in \mathbb{R}^2 which relate to this work.

Recall that Prince Rupert's problem can be converted to a covering problem of the projections of 2 congruent copies of any given object. We will construct lemmas and corollaries related to covering problem that may be helpful to study Prince Rupert's problem.

First, we have an obvious lemma by the property of width as follows.

Lemma 3.1. *Let K and H be compact planar sets. If K is in the interior of H , then the width of K is less than width of H for every direction.*

The following corollary follows from Lemma 3.1.

Corollary 3.2. *Let K and H be compact, convex planar sets. If there exists a direction such that the width of K is more than or equal to the width of H , then K is not in the interior of H .*

We will use Corollary 3.2 to construct a lemma to check that a given object does not have Rupert property in chapter IV.

If we use the property of convex hull, we will have a lemma about covering of a set of points in a convex planar set as follows.

Lemma 3.3. *A planar set H fits in the interior of a convex planar set K if and only if a convex hull of H fits in the interior of K .*

Proof. Assume that H fits in the interior of a convex planar set K . There is an isometry f on \mathbb{R}^2 such that $f(H) \subseteq \text{int}(K)$. Then $H \subseteq f^{-1}(\text{int}(K))$. Since $f^{-1}(\text{int}(K))$ is convex, $\mathbb{H}(H) \subseteq \mathbb{H}(f^{-1}(\text{int}(K))) \subseteq f^{-1}(\text{int}(K))$. Thus $f(\mathbb{H}(H)) \subseteq$

$\text{int}(K)$. On the contrary, assume that a convex hull of H fits in $\text{int}(K)$. There is an isometry f on \mathbb{R}^2 such that $f(\mathbb{H}(H)) \subseteq \text{int}(K)$. Then $H \subseteq \mathbb{H}(H) \subseteq f^{-1}(\text{int}(K))$. Therefore $f(H) \subseteq \text{int}(K)$. \square

In the case that we know the order of vertices of a convex polygon, we construct lemmas and corollaries which are useful for numerical computation. By basic property of a convex polygon, we have the following remark.

Remark 3.4. The interior of a convex polygon K is represented as the intersection of every left side of counterclockwise edge around the polygon K .

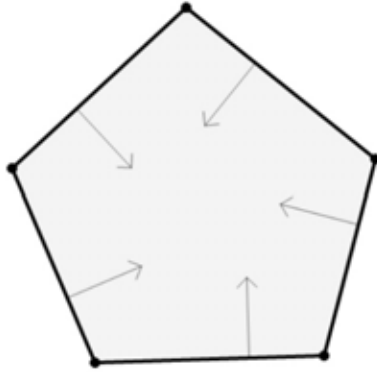


Figure 3.1: the interior of a pentagon is the intersection of every left side of counterclockwise edge around the pentagon

This remark leads to the following lemma.

Lemma 3.5. *Let a_1, a_2, \dots, a_n be vertices of a convex polygon K in counterclockwise order. A point p is in the interior of K if and only if p lies to the left side of the ray $a_i a_{i+1}$ for all $1 \leq i \leq n$ where $a_{n+1} = a_1$.*

In this work, we use two concepts to check whether a point lies to the left of a ray. The first one is a concept using dot product. For another concept, we use a mapping similar to a cross product in \mathbb{R}^3 which is defined as follows.

Definition 3.6. A map $\otimes : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by $(x_1, y_1) \otimes (x_2, y_2) = x_1 y_2 - x_2 y_1$ for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

For convenience, we will define the following definition.

Definition 3.7. Let \vec{v} be a vector in \mathbb{R}^2 . A vector \vec{v}^* is the counterclockwise rotation of the vector \vec{v} around $(0, 0)$ for angle $\frac{\pi}{2}$.

Remark 3.8. For points $p, q, r \in \mathbb{R}^2$, we notice as follows.

1. $\vec{qr}^* \cdot \vec{qp}$ is equal to $\vec{qr} \otimes \vec{qp}$.
2. A point p lies to the left side of ray qr if and only if $\vec{qr}^* \cdot \vec{qp} > 0$ if and only if $\vec{qr} \otimes \vec{qp} > 0$.
3. A point p lies to the right side of ray qr if and only if $\vec{qr}^* \cdot \vec{qp} < 0$ if and only if $\vec{qr} \otimes \vec{qp} < 0$.
4. Points p, q, r are collinear if and only if $\vec{qr}^* \cdot \vec{qp} = 0$ if and only if $\vec{qr} \otimes \vec{qp} = 0$.

By Lemma 3.5 and Remark 3.8, we have two following lemmas.

Lemma 3.9. Let a_1, a_2, \dots, a_n be vertices of a convex polygon K in counterclockwise order. A point p is in the interior of K if and only if $\overrightarrow{a_i a_{i+1}}^* \cdot \overrightarrow{a_i p} > 0$ for all $1 \leq i \leq n$ where $a_{n+1} = a_1$.

Lemma 3.10. Let a_1, a_2, \dots, a_n be vertices of a convex polygon K in counterclockwise order. A point p is in the interior of K if and only if $\overrightarrow{a_i a_{i+1}} \otimes \overrightarrow{a_i p} > 0$ for all $1 \leq i \leq n$ where $a_{n+1} = a_1$.

By basic property of convex sets, we have the following lemma.

Lemma 3.11. Let K and H be two convex polygons. Then K is in the interior of H if and only if every vertex of K lies in the interior of H .

By Lemma 3.9 and Lemma 3.11, we obtain the following corollary.

Corollary 3.12. Let a_1, a_2, \dots, a_n be vertices of a convex polygon H in counterclockwise order. A convex polygon K is in the interior of H if and only if every vertex p of K satisfies conditions that $\overrightarrow{a_i a_{i+1}}^* \cdot \overrightarrow{a_i p} > 0$ for all $1 \leq i \leq n$ where $a_{n+1} = a_1$.

By Lemma 3.10 and Lemma 3.11, we obtain the following corollary.

Corollary 3.13. *Let a_1, a_2, \dots, a_n be vertices of a convex polygon H in counter-clockwise order. A convex polygon K is in the interior of H if and only if every vertex p of K satisfies conditions that $\overrightarrow{a_i a_{i+1}} \otimes \overrightarrow{a_i p} > 0$ for all $1 \leq i \leq n$ where $a_{n+1} = a_1$.*

Remark 3.14. If we know about the order of vertices of a convex polygon in clockwise order, then we will have similar properties by flipping the inequality sign.

Next, we will construct lemmas to verify whether a point fits in a convex hull of arbitrary finite points. First, we define a definition of tangent strip as follows.

Definition 3.15. The **tangent strip** containing a set H in direction of \vec{v} is the region between the two lines parallel to \vec{v} and tangent to H .

We obtain the following lemma from a property of convex polygon.

Lemma 3.16. *A convex polygon H is the intersection the tangent strips containing H in every direction parallel to an edge of H . Moreover, let V be a set of vectors, if vectors parallel to each edge of H is in V , then H is also the intersection the tangent strips containing H in every direction \vec{v} in V .*

The next two lemmas follow from the geometric interpretations of the dot product and the map \otimes .

Lemma 3.17. *Let S be a tangent strip between lines l_1 and l_2 parallel to v with $p_1 \in l_1$ and $p_2 \in l_2$. A point p lies on the interior of S if and only if $\vec{v}^* \cdot \vec{p}$ is between $\vec{v}^* \cdot \vec{p}_1$ and $\vec{v}^* \cdot \vec{p}_2$.*

Lemma 3.18. *Let S be a tangent strip between lines l_1 and l_2 parallel to v with $p_1 \in l_1$ and $p_2 \in l_2$. A point p lies on the interior of S if and only if $\vec{v} \otimes \vec{p}$ is between $\vec{v} \otimes \vec{p}_1$ and $\vec{v} \otimes \vec{p}_2$.*

By Lemma 3.16 and Lemma 3.17, we have the following lemma.

Lemma 3.19. *Let H be a set of points a_1, a_2, \dots, a_n . A point p is in the interior of $\mathbb{H}(H)$ if and only if $\min_{1 \leq k \leq n} \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i a_k} < \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i p} < \max_{1 \leq k \leq n} \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i a_k}$ for all $1 \leq i, j \leq n$ and $i \neq j$.*

By Lemma 3.16 and Lemma 3.18, we have the following lemma.

Lemma 3.20. *Let H be a set of points a_1, a_2, \dots, a_n . A point p is in the interior of $\mathbb{H}(H)$ if and only if $\min_{1 \leq k \leq n} \overrightarrow{a_i a_j} \otimes \overrightarrow{a_i a_k} < \overrightarrow{a_i a_j} \otimes \overrightarrow{a_i p} < \max_{1 \leq k \leq n} \overrightarrow{a_i a_j} \otimes \overrightarrow{a_i a_k}$ for all $1 \leq i, j \leq n$ and $i \neq j$.*

By Lemma 3.11 and Lemma 3.19, we obtain the following corollary.

Corollary 3.21. *Let H be a set of points a_1, a_2, \dots, a_n . A convex polygon K is in the interior of $\mathbb{H}(H)$ if and only if every vertex p of K satisfies conditions that $\min_{1 \leq k \leq n} \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i a_k} < \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i p} < \max_{1 \leq k \leq n} \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i a_k}$ for all $1 \leq i, j \leq n$ and $i \neq j$.*

By Lemma 3.11 and Lemma 3.20, we obtain the following corollary.

Corollary 3.22. *Let H be a set of points a_1, a_2, \dots, a_n . A convex polygon K is in the interior of $\mathbb{H}(H)$ if and only if every vertex p of K satisfies conditions that $\min_{1 \leq k \leq n} \overrightarrow{a_i a_j} \otimes \overrightarrow{a_i a_k} < \overrightarrow{a_i a_j} \otimes \overrightarrow{a_i p} < \max_{1 \leq k \leq n} \overrightarrow{a_i a_j} \otimes \overrightarrow{a_i a_k}$ for all $1 \leq i, j \leq n$ and $i \neq j$.*

Remark 3.23. In Lemma 3.19, Lemma 3.20, Corollary 3.21 and Corollary 3.22, it is enough to consider a set of all vectors $\overrightarrow{a_i a_j}$ which are not parallel to each other in this set instead of a set of vectors $\overrightarrow{a_i a_j}$ for all $1 \leq i, j \leq n$ and $i \neq j$.

CHAPTER IV

COVERING OF OBJECTS RELATED TO RUPERT PROPERTY

In this chapter, we will construct lemmas related to covering problem that may be helpful to study Prince Rupert's problem.

In this work, we will use XY -plane in Lemma 2.14 for convenience to numerical optimization. Let $\pi_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a projection to XY -plane such that $\pi_0((x, y, z)) = (x, y)$ for every point (x, y, z) in \mathbb{R}^3 . By Lemma 2.14 and basic property of rotation, without loss of generality we may use the projection π_0 as in the following lemma.

Lemma 4.1. *An object K in \mathbb{R}^3 has Rupert property if and only if there are congruent copies K_1 and K_2 of K that $\pi_0(K_1)$ is a subset of the interior of $\pi_0(K_2)$.*

4.1 OBJECTS WHICH DO NOT HAVE RUPERT PROPERTY

Lemma 4.1 is suitable to check whether a given object K has Rupert property but it is hard to check that a given object K does not have Rupert property because there are uncountably many congruent copies of K that we need to consider. Thus we will construct a lemma that can help to check that a given object does not have Rupert property. The property related to this lemma is the width of compact subsets of \mathbb{R}^2 . From Corollary 3.2 and Lemma 4.1, we then have the following criteria for an object that does not have Rupert property.

Lemma 4.2. *Let K be an object in \mathbb{R}^3 . If every congruent copy of K has the same minimum width under the projection to the XY -plane, then K does not have Rupert property.*

Proof. Assume that every congruent copy of K has the same minimum width w under the projection to the XY -plane. Let K_1 and K_2 be two congruent copies of K . There is a line l in \mathbb{R}^2 such that the width of $\pi_0(K_2)$ in direction of l is w . Then the width of $\pi_0(K_1)$ is greater than or equal to width of $\pi_0(K_2)$ in direction of l . By Corollary 3.2, $\pi_0(K_1)$ is not in the interior of $\pi_0(K_2)$. By Lemma 4.1, K does not have Rupert property. \square

Remark 4.3. By Lemma 4.2, we then have the following observations.

1. We notice that an object with constant width w has every projection with minimum width w . Then all objects of constant width do not have Rupert property. Hence, as it is a constant width, every sphere does not have Rupert property.
2. For the circular cylinder with its radius is r and its height is greater than or equal to $2r$, the minimum width of its projection to the XY -plane is $2r$. Hence the circular cylinder does not have Rupert property.
3. For the prism which the base is a planar set of constant width l and its height is greater than or equal to l , the minimum width of its projection to the XY -plane is l . Hence it does not have Rupert property.

Due to lack of counterexample, J. E. Wetzel and L. Yuan conjectured in [2] that every convex polyhedron has Rupert property. Hence we will study about Rupert property for two famous kinds of convex polyhedrons which are Platonic solids and Archimedean solids in the next section.

4.2 OBJECTS WHICH HAVE RUPERT PROPERTY

In this section, we apply lemmas and corollaries to approximate the Nieuwland constants for some convex polyhedrons.

By Lemma 4.1, we can regard one congruent copy of polyhedron as its vertices.

Lemma 4.4. *A polyhedron K has Rupert property if and only if there are congruent copies K_1 and K_2 of K that every point which is a projected vertex from K_1 by π_0 fits in the interior of $\pi_0(K_2)$.*

Proof. Let A be a set of points which are projected vertices from K_1 by π_0 . By Lemma 2.6 and Lemma 3.3, for congruent copies K_1 and K_2 of K , we have that $\pi_0(K_1)$ fits in the interior of $\pi_0(K_2)$ if and only if $\pi_0(\mathbb{H}(A))$ fits in the interior of $\pi_0(K_2)$ if and only if $\mathbb{H}(\pi_0(A))$ fits in the interior of $\pi_0(K_2)$ if and only if $\pi_0(A)$ fits in the interior of $\pi_0(K_2)$. By Lemma 4.1, we are done. \square

Let R_β^X be the rotation around X -axis for angle β (from Y -axis to Z -axis). Similarly, let R_ϕ^Y and R_θ^Z be the rotations around Y -axis for angle ϕ (from X -axis to Z -axis) and around Z -axis for angle θ (from X -axis to Y -axis), respectively. Regardless of translation, a congruent copy of an object K can be written as $R_\theta^Z(R_\phi^Y(R_\beta^X(K)))$ for some β , ϕ and θ . To check Rupert property for a given object K in \mathbb{R}^3 we can consider a projection π_0 to XY -plane of all two possible congruent copies of K as $\pi_0(R_{\theta_1}^Z(R_{\phi_1}^Y(R_{\beta_1}^X(K))))$ and $\pi_0(R_{\theta_2}^Z(R_{\phi_2}^Y(R_{\beta_2}^X(K))))$ for some β_1 , ϕ_1 , θ_1 , β_2 , ϕ_2 and θ_2 . By elementary property of compact, convex sets with nonempty interiors and Lemma 4.1, Nieuwland constant can be written as follows.

Lemma 4.5. *Let $R_\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation around $(0, 0)$ for angle ω . For an object K ,*

$$\begin{aligned} \nu(K) &= \max\{\alpha > 0 \mid \alpha R_\omega(\pi_0(R_{\theta_1}^Z(R_{\phi_1}^Y(R_{\beta_1}^X(K)))) + (x_0, y_0) \subseteq \pi_0(R_{\theta_2}^Z(R_{\phi_2}^Y(R_{\beta_2}^X(K))))\} \\ &\quad \text{for } \omega, \beta_1, \phi_1, \theta_1, \beta_2, \phi_2, \theta_2, x_0, y_0 \in \mathbb{R}\} \\ &= \max\{\alpha > 0 \mid \alpha R_\omega(\pi_0(R_{\phi_1}^Y(R_{\beta_1}^X(K)))) + (x_0, y_0) \subseteq \pi_0(R_{\phi_2}^Y(R_{\beta_2}^X(K)))\} \\ &\quad \text{for } \omega, \beta_1, \phi_1, \beta_2, \phi_2, x_0, y_0 \in \mathbb{R}\}. \end{aligned}$$

Proof. The first equality follows from the definition of Nieuwland constant and elementary property of compact, convex set. We note that $R_\omega \circ \pi_0 \circ R_{\theta_1}^Z = R_{\omega+\theta_1} \circ \pi_0$. For $A, B \subseteq \mathbb{R}^2$, we have that $R_{\theta_1}(A) \subseteq R_{\theta_2}(B)$ if and only if $R_{\theta_1-\theta_2}(A) \subseteq R_0(B) = B$. Hence, for $K, H \subseteq \mathbb{R}^3$, we have that $R_\omega(\pi_0(R_{\theta_1}^Z(K))) \subseteq \pi_0(R_{\theta_2}^Z(H))$ if and only

if $R_{\omega+\theta_1}(\pi_0(K)) \subseteq R_{\theta_2}(\pi_0(H))$ if and only if $R_{\omega+\theta_1-\theta_2}(\pi_0(K)) \subseteq \pi_0(H)$. Since ω is arbitrary, we may omit θ_1 and θ_2 as in the last expression. \square

For each object, we estimate its Nieuwland constant by using numerical optimization according to Lemma 4.5 to find $\alpha > 1$ under variables $\omega, \beta_1, \phi_1, \beta_2, \phi_2, x_0$ and y_0 . In particular, to verify whether a planar image fits in another image, we use Lemmas and corollaries in chapter 3 as the following corollary.

Corollary 4.6. *These statements are equivalent.*

1. *A polyhedron K has Rupert property.*
2. *There are congruent copies K_1 and K_2 of K that $\pi_0(K_1)$ is a subset of interior of $\pi_0(K_2)$.*
3. *There are congruent copies K_1 and K_2 of K that every point which is a projected vertex from K_1 by π_0 fits in the interior of $\pi_0(K_2)$.*
4. *There are congruent copies K_1 and K_2 of K that every point p which is a projected vertex from K_1 by π_0 satisfies conditions that, for all $1 \leq i \leq n$, $\overrightarrow{a_i a_{i+1}}^* \cdot \overrightarrow{a_i p} > 0$ which a_1, a_2, \dots, a_n be vertices of $\pi_0(K_2)$ in counterclockwise order and $a_{n+1} = a_1$.*
5. *There are congruent copies K_1 and K_2 of K that every point p which is a projected vertex from K_1 by π_0 satisfies conditions that, for all $1 \leq i, j \leq n$ and $i \neq j$, $\min_{1 \leq k \leq n} \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i a_k} < \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i p} < \max_{1 \leq k \leq n} \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i a_k}$ which points a_1, a_2, \dots, a_n are projected vertices from K_2 by π_0 .*
6. *There are $\beta_1, \phi_1, \theta_1, \beta_2, \phi_2, \theta_2, x_0$ and y_0 that every point p which is a projected vertex from $R_{\theta_1}^Z(R_{\phi_1}^Y(R_{\beta_1}^X(K))) + (x_0, y_0)$ by π_0 satisfies conditions that, for all $1 \leq i, j \leq n$ and $i \neq j$, $\min_{1 \leq k \leq n} \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i a_k} < \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i p} < \max_{1 \leq k \leq n} \overrightarrow{a_i a_j}^* \cdot \overrightarrow{a_i a_k}$ which points a_1, a_2, \dots, a_n are projected vertices from $R_{\theta_2}^Z(R_{\phi_2}^Y(R_{\beta_2}^X(K)))$ by π_0 .*

Remark 4.7. By Remark 3.8 and Remark ??, we will obtain a similar corollary as Corollary 4.6 by using concept of a map \otimes .

We notice that we can find the Nieuwland constant as Lemma 4.5 by replacing a congruent copy K_1 of K in Lemma 4.6 by αK_1 and then constructing processes to find the maximum of α . These processes give a lower bound of Nieuwland constant for each given object. In this work, we improve Nieuwland constant estimates for 3 of 5 Platonic solids as in Table 4.1. We also have result for 7 of 13 Archimedean solids about estimates of its Nieuwland constant as in Table 4.2.

Platonic solid	Previous result	Our result
Tetrahedron \mathfrak{T}	$\nu(\mathfrak{T}) \geq \frac{2}{5}\sqrt{3}(\sqrt{6}-1) > 1.004235$	$\nu(\mathfrak{T}) > \mathbf{1.014611}$
Cube \mathfrak{C}	$\nu(\mathfrak{C}) = \frac{3}{4}\sqrt{2} \geq 1.060660$	$\nu(\mathfrak{C}) \geq 1.060660$
Octahedron \mathfrak{O}	$\nu(\mathfrak{O}) \geq \frac{3}{4}\sqrt{2} \geq 1.060660$	$\nu(\mathfrak{O}) \geq 1.060660$
Dodecahedron \mathfrak{D}	$\nu(\mathfrak{D}) \geq \frac{171}{170} > 1.005882$	$\nu(\mathfrak{D}) > \mathbf{1.010788}$
Icosahedron \mathfrak{I}	$\nu(\mathfrak{I}) \geq \frac{1108}{1098} > 1.009107$	$\nu(\mathfrak{I}) > \mathbf{1.010823}$

Table 4.1: Our Nieuwland constant estimates for all Platonic solids compare with the previous result in [2]

Archimedean solid	Previous result	Our result
Truncated tetrahedron \mathcal{T}_t	-	$\nu(\mathcal{T}_t) > \mathbf{1.014049}$
Cuboctahedron \mathcal{C}	$\nu(\mathfrak{C}) > 1.01461$	$\nu(\mathcal{C}) > 1.014611$
Truncated cube \mathfrak{C}_t	$\nu(\mathfrak{T}) > \mathbf{1.02036}$	$\nu(\mathfrak{C}_t) > 1.013669$
Truncated octahedron \mathcal{O}_t	$\nu(\mathcal{O}_t) > \mathbf{1.00815}$	-
Rhombicuboctahedron \mathcal{R}	$\nu(\mathfrak{T}) > 1.00609$	$\nu(\mathcal{R}) > \mathbf{1.012819}$
Truncated cuboctahedron \mathcal{C}_t	$\nu(\mathfrak{T}) > \mathbf{1.00370}$	$\nu(\mathcal{C}_t) > 1.003005$
Icosidodecahedron \mathcal{I}	$\nu(\mathfrak{T}) > 1.00015$	$\nu(\mathcal{I}) > \mathbf{1.000604}$
Truncated dodecahedron \mathcal{D}_t	$\nu(\mathcal{D}_t) > \mathbf{1.00014}$	-
Truncated icosahedron \mathcal{I}_t	$\nu(\mathcal{I}_t) > \mathbf{1.00004}$	-
Truncated icosidodecahedron \mathcal{I}_t	-	$\nu(\mathcal{I}_t) > \mathbf{1.001915}$

Table 4.2: Our Nieuwland constant estimates for 7 of 13 Archimedean solids

Therefore these 7 solids have Rupert property. Among these 7 solids, the truncated tetrahedron and the truncated icosidodecahedron have not yet been studied before.

CHAPTER V

FURTHER RESEARCH

In this dissertation, we study some properties to check whether some objects have Rupert property. Moreover, we get a lower bound of Nieuwland constant of some polyhedrons. We suggest some ideas to extend the research as follows.

1. In this work, we study Rupert property for 7 of the 13 Archimedean solids. So there are the remaining 6 of 13 Archimedean solids that we did not study yet. Hence we may find a new method to check whether the remaining object has Rupert property.
2. We may find a new method to improve a lower bound of Nieuwland constant of some polyhedrons.
3. We can use this method to check Rupert property of some kinds of polyhedron, such as Catalan solids and Johnson solids.
4. We may construct some lemmas to check that some objects do not have Rupert property.

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CHAPTER VI

APPENDIX

We will give the algorithm that we use to find a lower bound of Nieuwland constant for a given convex polyhedron K using concept of dot product as follows.

1. Find a set of vertices of a given convex polyhedron K .
2. Find set of all vector edges of K which are not parallel to each other in set.
3. Find the maximum value of scalar α such that every point p which is a projected vertex from $R_{\theta_1}^Z(R_{\phi_1}^Y(R_{\beta_1}^X(K))) + (x_0, y_0)$ by π_0 satisfies conditions that $\min_{1 \leq k \leq n} \overrightarrow{a_i a_j^*} \cdot \overrightarrow{a_k} < \overrightarrow{a_i a_j^*} \cdot \overrightarrow{\alpha p} < \max_{1 \leq k \leq n} \overrightarrow{a_i a_j^*} \cdot \overrightarrow{a_k}$ which vectors $\overrightarrow{a_i a_j^*}$ are projected edges from $R_{\phi_2}^Y(R_{\beta_2}^X(K))$ by π_0 from step 2.
4. Show the result, including the maximum value of α with values $\beta_1, \phi_1, \theta_1, \beta_2, \phi_2, x_0, y_0$ and figure of two projected images which one image in the interior of the other.

Using concept of a map \otimes , we have the algorithm as follows.

1. Find a set of vertices of a given convex polyhedron K .
2. Find set of all vector edges of K which are not parallel to each other in set.
3. Find the maximum value of scalar α such that every point p which is a projected vertex from $R_{\theta_1}^Z(R_{\phi_1}^Y(R_{\beta_1}^X(K))) + (x_0, y_0)$ by π_0 satisfies conditions that $\min_{1 \leq k \leq n} \overrightarrow{a_i a_j} \otimes \overrightarrow{a_k} < \overrightarrow{a_i a_j} \otimes \overrightarrow{\alpha p} < \max_{1 \leq k \leq n} \overrightarrow{a_i a_j} \otimes \overrightarrow{a_k}$ which vectors $\overrightarrow{a_i a_j}$ are projected edges from $R_{\phi_2}^Y(R_{\beta_2}^X(K))$ by π_0 from step 2.
4. Show the result, including the maximum value of α with values $\beta_1, \phi_1, \theta_1, \beta_2, \phi_2, x_0, y_0$ and figure of two projected images which one image in the interior of the other.

```

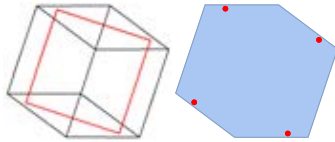
In[1]:= name = "Cube";
proj[{x_, y_, z_}] = {x, y};
rot[{x_, y_}] = {-y, x};
scale[α_, p1_, p0_] := α p1 + p0;
p0 = {x0, y0};

(*Find a set of vertices of a given convex polyhedron*)
vertices = N[PolyhedronData[name, "VertexCoordinates"]];
(*Find set of all vector edges which are not parallel to each other*)
allvec = {};
allvecid = {};
Do[
  v = vertices[[vi[[1]]]] - vertices[[vi[[2]]]];
  Do[If[Cross[u, v] == {0, 0, 0}, Goto[nextv]], {u, allvec}];
  AppendTo[allvec, v];
  AppendTo[allvecid, vi];
  Label[nextv]
, {vi, PolyhedronData[name, "EdgeIndices"]]];
(*Find the maximum value of scalar alpha*)
nm = FindMaximum[Join[{α, α < 1.5},
  pls = RotationTransform[θ1] /@proj /@RotationTransform[φ1, {0, -1, 0}] /@
  RotationTransform[β1, {1, 0, 0}] /@vertices;
  p2s = proj /@RotationTransform[φ2, {0, -1, 0}] /@
  RotationTransform[β2, {1, 0, 0}] /@vertices;
  n2 = Table[rot[p[[1]] - p[[2]]], {p, Table[p2s[[i]], {i, allvecid}}]];
  Table[{min, max} = MinMax[Table[p.n, {p, p2s}]];
  min ≤ scale[α, p1, p0].n ≤ max, {p1, pls}, {n, n2}]],
  {α, {x0, 0}, {y0, 0}, θ1, φ1, β1, φ2, β2}, MaxIterations → 5000];
(*Show Result*)
pls = (RotationTransform[θ1] /@proj /@RotationTransform[φ1, {0, -1, 0}] /@
  RotationTransform[β1, {1, 0, 0}] /@vertices) /. nm[[2]];
p2s = (proj /@RotationTransform[φ2, {0, -1, 0}] /@
  RotationTransform[β2, {1, 0, 0}] /@vertices) /. nm[[2]];
p0 = p0 /. nm[[2]];
Print[Column[{nm[[2]]}]];
If[nm[[1]] > 1, Print["***** Rupert *****"]];
Print[
Graphics3D[{{Red, Translate[Rotate[Rotate[Rotate[PolyhedronData[name, "Edges"],
  β1, {1, 0, 0}], φ1, {0, -1, 0}], θ1, {0, 0, 1}], {x0, y0, 0}],
  Rotate[Rotate[PolyhedronData[name, "Edges"], β2, {1, 0, 0}], φ2, {0, -1, 0}]} /.
  nm[[2]], Boxed → False, ViewPoint → {0, 0, 100}],
Show[ConvexHullMesh[p2s], Graphics[{Red, Translate[Point[pls], p0]}]]]

```

$\{\alpha \rightarrow 1.06066, x_0 \rightarrow -1.50327 \times 10^{-14}, y_0 \rightarrow 3.63224 \times 10^{-15}, \theta_1 \rightarrow -0.321751,$
 $\phi_1 \rightarrow 4.12443 \times 10^{-11}, \beta_1 \rightarrow -2.34299 \times 10^{-12}, \phi_2 \rightarrow 0.729728, \beta_2 \rightarrow 1.10715\}$

***** Rupert *****



```

In[17]:= name = "Cube";
proj[{x_, y_, z_}] = {x, y};
rot[{x_, y_}] = {-y, x};
cross[{x1_, y1_}, {x2_, y2_}] := x1 y2 - x2 y1;
scale[α_, p1_, p0_] := α p1 + p0;
p0 = {x0, y0};

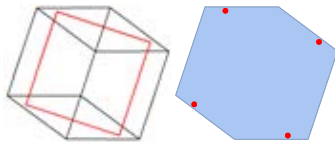
(*Find a set of vertices of a given convex polyhedron*)
vertices = N[PolyhedronData[name, "VertexCoordinates"]];
(*Find set of all vector edges which are not parallel to each other*)
allvec = {};
allvecid = {};
Do[
  v = vertices[[vi[[1]]]] - vertices[[vi[[2]]]];
  Do[If[Cross[u, v] == {0, 0, 0}, Goto[nextv]], {u, allvec}];
  AppendTo[allvec, v];
  AppendTo[allvecid, vi];
  Label[nextv]
, {vi, PolyhedronData[name, "EdgeIndices"]]];
(*Find the maximum value of scalar alpha*)
nm = FindMaximum[Join[{α, α < 1.5},
  pls = RotationTransform[θ1] /@proj /@RotationTransform[φ1, {0, -1, 0}] /@
  RotationTransform[β1, {1, 0, 0}] /@vertices;
  p2s = proj /@RotationTransform[φ2, {0, -1, 0}] /@
  RotationTransform[β2, {1, 0, 0}] /@vertices;
  E2 = Table[p[[1]] - p[[2]], {p, Table[p2s[[i]], {i, allvecid}]}];
  Table[{min, max} = MinMax[Table[cross[e, p], {p, p2s}]];
  min ≤ cross[e, scale[α, p1, p0]] ≤ max, {p1, pls}, {e, E2}]],
{α, {x0, 0}, {y0, 0}, θ1, φ1, β1, φ2, β2}, MaxIterations → 5000];
(*Show Result*)
pls = (RotationTransform[θ1] /@proj /@RotationTransform[φ1, {0, -1, 0}] /@
  RotationTransform[β1, {1, 0, 0}] /@vertices) /. nm[[2]];
p2s = (proj /@RotationTransform[φ2, {0, -1, 0}] /@

```

```

RotationTransform[ $\beta_2$ , {1, 0, 0}] /@vertices) /. nm[[2]];
p0 = p0 /. nm[[2]];
Print[Column[{{nm[[2]]}}]];
If[nm[[1]] > 1, Print["***** Rupert *****"]];
Print[
Graphics3D[{{Red, Translate[Rotate[Rotate[Rotate[PolyhedronData[name, "Edges"],
 $\beta_1$ , {1, 0, 0}],  $\phi_1$ , {0, -1, 0}],  $\theta_1$ , {0, 0, 1}], {x0, y0, 0}],
Rotate[Rotate[PolyhedronData[name, "Edges"],  $\beta_2$ , {1, 0, 0}],  $\phi_2$ , {0, -1, 0}]} /.
nm[[2]], Boxed -> False, ViewPoint -> {0, 0, 100}},
Show[ConvexHullMesh[p2s], Graphics[{{Red, Translate[Point[p1s], p0]}]}]]
{ $\alpha$  -> 1.06066, x0 ->  $-1.50327 \times 10^{-14}$ , y0 ->  $3.63224 \times 10^{-15}$ ,  $\theta_1$  -> -0.321751,
 $\phi_1$  ->  $4.12443 \times 10^{-11}$ ,  $\beta_1$  ->  $-2.34299 \times 10^{-12}$ ,  $\phi_2$  -> 0.729728,  $\beta_2$  -> 1.10715}
***** Rupert *****

```



VITA

Name	Mr. Pongbunthit Tonpho
Date of Birth	July 3, 1993
Place of Birth	Khon Kaen, Thailand
Education	B.Sc. (Mathematics)(First Class Honours), Khon Kaen University, 2015
Scholarship	Development and Promotion of Science and Technology Talents Project (DPST)