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VARIANT OF D'ALEMBERT FUNCTIONAL EQUATION ON COMPACT HOMOGENEOUS SPACE

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In this work, we introduce a new functional equation on compact homogeneous spaces based on the d'Alembert functional equation on compact groups. We solve the functional equation by using tools from harmonic analysis.

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CONTENTS

CHAPTER I
INTRODUCTION

Let G be a locally compact group. A **nonzero continuous** function $f : G \to \mathbb{C}$ is said to be a solution of the *d'Alembert functional equation* if for any $x, y \in G$, we have

$$f(xy) + f(xy^{-1}) = 2f(x)f(y).$$

This equation can be solved by an algebraic method, see [3]. However, in 2013, Yang provided a solution of this functional equation on compact groups in his paper [4] using methods from abstract harmonic analysis.

In 2015, Chahbi et al. generalized d'Alembert functional equation to the equation

$$f(xy) + \sum_{k=1}^{n-1} f(\sigma^k(y)x) = nf(x)f(y). =, \tag{1.1}$$

where σ is an automorphism on G such that $\sigma^n = \mathrm{id}_G$. They solved equation (1.1) on compact groups by applying Fourier transform to the functional equation. The solution they gave in their paper [1] is

$$f(x) = \frac{1}{n} \sum_{k=0}^{n-1} \chi \circ \sigma^k(x)$$

where $\chi : G \to \mathbb{C} \setminus \{0\}$ is a continuous group homomorphism.

In this work, we generalized the domain of solutions of the functional equation. We define the functional equation on a compact homogeneous space based on equation (1.1) and give the solution to this functional equation by using Fourier analysis on compact homogeneous spaces.

CHAPTER II
PRELIMINARIES

In this chapter, we introduce a definition of a topological group and a homogeneous space. we also give a reference to important results about Haar measure on a topological group and G-invariant Radon measure on a homogeneous space. The material in this chapter can be found in [2].

2.1 Locally Compact Groups

Definition 2.1. A group G is called a *topological group* if it is equipped with Hausdorff topology such that the group operations are continuous; that is the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.
If G is locally compact Hausdorff, then we call G a *locally compact group.*

Definition 2.2. Let f be a function on a group G. For each $y \in G$, we define the *left translation of f through y* by

$$L_y f(x) = f(y^{-1}x), \qquad \text{for } x \in G,$$

and the *right translation of f through y* by

$$R_y f(x) = f(xy), \qquad \text{for } x \in G.$$

Note that $L_{yz}f = L_y L_z f$ and $R_{yz}f = R_y R_z f$.

Definition 2.3. Let G be a locally compact group. A *left (resp. right) Haar measure* on G is a nonzero Radon measure μ on G that satisfies $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$) for every $x \in G$ and every Borel set $E \subset G$.
A *Haar measure* on G is a measure μ that is both left and right Haar measure.

Observe that if μ is a left Haar measure, then the measure $\tilde{\mu}$ defined by $\tilde{\mu}(E) = \mu(E^{-1})$ is a right Haar measure, and vice versa. Thus it suffices to consider only left Haar measures on G. Let $C_c(G)$ denote the space of continuous functions on G with compact support.

Proposition 2.4. *Let μ be a Radon measure on a locally compact group G. Then*

i) μ is a left Haar measure if and only if $\int L_y f\, d\mu = \int f\, d\mu$ for all $y \in G$ and for all $f \in C_c(G)$.

ii) μ is a right Haar measure if and only if $\int R_y f\, d\mu = \int f\, d\mu$ for all $y \in G$ and for all $f \in C_c(G)$.

Theorem 2.5. *Every locally compact group has a left Haar measure and it is unique up to scaling.*

Let μ be a left Haar measure on G. For each $x \in G$, we define a measure μ_x by $\mu_x(E) = \mu(Ex)$ for every Borel set $E \subset G$. Notice that μ_x is also a left Haar measure. By the uniqueness of left Haar measure, there is a positive real number c_x such that $\mu_x = c_x\mu$.

Definition 2.6. A function $\Delta : G \to \mathbb{R}^+$ defined by $\Delta(x) = c_x$ is called a *modular function* of G.

Note that $\Delta \equiv 1$ if and only if every left Haar measure on G is a right Haar measure. We call a group G with such property a *unimodular group.*

Proposition 2.7. *Let G be a locally compact group and μ be the left Haar measure on G. If $f \in C_c(G)$, then*

$$\int_G f(x)\, d\mu(x) = \int_G f(x^{-1})\Delta(x^{-1})\, d\mu(x).$$

Proposition 2.8. *Δ is a continuous group homomorphism from G to $(\mathbb{R}^+, \times)$.*

Theorem 2.9. *If G is a compact group, then G is unimodular.*

Proof. Since Δ is continuous, $\Delta(G)$ is a compact subgroup of $(\mathbb{R}^+, \times)$, and the only compact subgroup of $(\mathbb{R}^+, \times)$ is $\{1\}$. $\square$

Let G be a compact group. Then G possesses a Haar measure μ. Since Haar measure is a Radon measure, $\mu(G) < \infty$. We say that a Haar measure μ on G is *normalized* if $\mu(G) = 1$.

2.2 Homogeneous Spaces

Let G be a locally compact group and S be a locally compact Hausdorff space. An *action* of G on S is a continuous map $(x, s) \mapsto xs$ from $G \times S$ to S such that

1. For each $x \in G$, the map $s \mapsto xs$ is a homeomorphism on S.
2. $(xy)s = x(ys)$ for all $x, y \in G$ and $s \in S$.

A space S equipped with an action of G is called a *G-space*.

Definition 2.10. A locally compact Hausdorff space S is called a *homogeneous space* if S is a G-space such that the action of G is transitive and S is homeomorphic to a quotient space G/H for some closed subgroup H of G.

Here we regard G/H as the space of left cosets of H in G. Let dx and $d\xi$ be the left Haar measure on G and its closed subgroup H respectively, Δ_G and Δ_H be the modular functions of G and H, and let $q : G \to G/H$ be the canonical quotient map $q(x) = xH$. We define the map $P : C_c(G) \to C_c(G/H)$ by

$$Pf(xH) = \int_H f(x\xi)\, d\xi.$$

Lemma 2.11. *If $K \subseteq G/H$ is compact, then there exists $\phi \in C_c(G)$ such that $\phi \geq 0$ and $P\phi = 1$ on K.*

Proposition 2.12. *If $F \in C_c(G/H)$, then there exists $f \in C_c(G)$ such that $Pf = F$, $q(\operatorname{supp} f) = \operatorname{supp} F$ and $f \geq 0$ if $F \geq 0$.*

The following theorem gives a condition for the existence of a G-invariant Radon measure on G/H, and its relation with the Haar measure on the underlying group G, which will be used in section 3.3. To cover this aspect, we present a proof for this theorem, which can also be found in [2].

Theorem 2.13. *There is a left G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$. In this case, μ is unique up to a constant factor, and if this factor is suitably chosen we have*

$$\int_G f(x)\,dx = \int_{G/H} Pf\,d\mu = \int_{G/H}\int_H f(x\xi)\,d\xi\,d\mu(xH) \tag{2.1}$$

for all $f \in C_c(G)$.

Proof. Assume that $\Delta_G|_H = \Delta_H$. Let $f \in C_c(G)$. Firstly, we show that $\int_G f\,dx = 0$ whenever $Pf = 0$. Indeed, suppose $Pf = 0$. By Lemma 2.11, We can choose $\phi \in C_c(G)$ such that $\phi \geq 0$ and $P\phi = 1$ on $q(\operatorname{supp} f)$. Then we have

$$\begin{aligned}
0 &= \int_H f(x\xi)\,d\xi \\
&= \int_H f(x\xi^{-1})\ \Delta_H(\xi^{-1})\,d\xi \\
&= \int_G \phi(x) \int_H f(x\xi^{-1})\Delta_H(\xi^{-1})\,d\xi\,dx \\
&= \int_G \int_H \phi(x) f(x\xi^{-1})\Delta_G(\xi^{-1})\,d\xi\,dx.
\end{aligned}$$

Since the map $(x,\xi) \mapsto \phi(x)f(x\xi^{-1})$ has support contained in $\operatorname{supp}\phi \times \big((\operatorname{supp} f)^{-1}\operatorname{supp}\phi\big)$, which is compact in $G \times H$, we can apply Fubini's Theorem to get

$$\begin{aligned}
0 &= \int_G \int_H \phi(x) f(x\xi^{-1})\Delta_G(\xi^{-1})\,d\xi\,dx \\
&= \int_H \int_G \phi(x) f(x\xi^{-1})\Delta_G(\xi^{-1})\,dx\,d\xi. \\
&= \int_H \int_G \phi(x\xi) f(x)\,dx\,d\xi.
\end{aligned}$$

Since $\operatorname{supp}\phi$ is compact, we can apply Fubini's Theorem again to get

$$\begin{aligned}
0 &= \int_H \int_G \phi(x\xi) f(x)\, dx\, d\xi \\
&= \int_G \left(\int_H \phi(x\xi)\, d\xi \right) f(x)\, dx \\
&= \int_G P\phi(xH) f(x)\, dx \\
&= \int_G f(x)\, dx.
\end{aligned}$$

Then we have $\displaystyle\int_G f\, dx = \int_G g\, dx$ whenever $Pf = Pg$. From Lemma 2.12, the map $Pf \mapsto \displaystyle\int_G f\, dx$ is a well-defined left G-invariant positive linear functional on $C_c(G/H)$. By Riesz representation theorem, there is a unique Radon measure μ on G/H that satisfies the equation (2.1).

Conversely, suppose that the G-invariant Radon measure μ on G/H exists. Then the map $f \mapsto \displaystyle\int_{G/H} Pf\, d\mu$ is a left G-invariant positive linear functional on $C_c(G)$. By Theorem 2.5, $\displaystyle\int_{G/H} Pf\, d\mu = c \int_G f\, dx$ for some $c > 0$ and we can assume that $c = 1$ so that the equation (2.1) holds. Let $f \in C_c(G)$ and $h \in H$. Then we have

$$\begin{aligned}
\Delta_G(h) \int_G f(x)\, dx &= \int_G f(xh^{-1})\, dx \\
&= \int_{G/H} \int_H f(xh^{-1}\xi)\, d\xi\, d\mu(xH) \\
&= \Delta_H(h) \int_{G/H} \int_H f(x\xi)\, d\xi\, d\mu(xH) \\
&= \Delta_H(h) \int_G f(x)\, dx.
\end{aligned}$$

Thus $\Delta_G(h) = \Delta_H(H)$ for all $h \in H$. $\square$

Corollary 2.14. *If G is compact, then G/H admits a G-invariant Radon measure.*

Proof. Assume that G is compact. Since H is a closed subgroup of G, H is also compact. By Theorem 2.9, we have $\Delta_G|_H = 1 = \Delta_H$. $\square$

Theorem 2.15. *If G is compact and dx, $d\xi$ are normalized, then the G-invariant Radon measure μ in* (2.1) *is the pushforward measure of the measure dx under the quotient map $q : G \to G/H$.*

Proof. Assume that G is compact. Then $C_c(G) = C(G)$ and $C_c(G/H) = C(G/H)$. Let $F \in C(G/H)$ and let $d(xH)$ denote the pushforward measure on G/H. Then we have

$$\int_{G/H} F\, d(xH) = \int_G F \circ q\, dx.$$

Observe that $F \circ q \in C(G)$ and $F \circ q(x\xi) = F \circ q(x)$ for all $\xi \in H$. Thus

$$\begin{aligned}
\int_{G/H} F\, d(xH) &= \int_G F \circ q\, dx \\
&= \int_{G/H} \int_H F \circ q(x\xi)\, d\xi\, d\mu(xH) \\
&= \int_{G/H} \int_H F \circ q(x)\, d\xi\, d\mu(xH) \\
&= \int_{G/H} F(xH) \left(\int_H d\xi \right) d\mu(xH) \\
&= \int_{G/H} F(xH)\, d\mu(xH).
\end{aligned}$$

□

CHAPTER III
ABSTRACT HARMONIC ANALYSIS

In this chapter, we introduce a tool that is used for solving the functional equation. The basic definitions about unitary representations are provided in Section 3.1. In Section 3.2, we give a well-known result about Fourier analysis on compact group. Our main tool for solving the functional equation on a compact homogeneous space is developed in Section 3.3. The detailed proof in Section 3.1 and 3.2 can be found in [2].

3.1 Unitary Representations

Definition 3.1. A *unitary representation* of G is a continuous group homomorphism $\pi : G \to U(V_\pi)$ where V_π is a nonzero Hilbert space and $U(V_\pi)$ is the group of unitary operators on V_π equipped with the strong operator topology. The dimension of π is defined by $d_\pi := \dim V_\pi$.

If V_π is one-dimensional, then for each $x \in G$, $\pi(x)$ is a scalar multiplication. In this case, we identify $U(V_\pi)$ with the circle group $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ and call π a *character* of G.

The most natural unitary representation of G arises from the translation action of G on itself. Suppose G is unimodular. Let $L^2(G)$ denote the space of square-integrable functions on G with respect to the Haar measure dx. The *left regular representation* of G is a representation $\pi_L : G \to U(L^2(G))$ given by the left translation

$$(\pi_L(y)f)(x) = L_y f(x) = f(y^{-1}x), \qquad x, y \in G.$$

Similarly, the *right regular representation* $\pi_R : G \to U(L^2(G))$ is defined by

$$(\pi_R(y)f)(x) = R_y f(x) = f(xy), \qquad x, y \in G.$$

Proposition 2.4 implies that π_L and π_R are unitary representations.

Definition 3.2. Let π_1, π_2 be representations of G. An *intertwining operator* for π_1 and π_2 is a bounded linear map $T : V_{\pi_1} \to V_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. The set of all intertwining operator for π_1 and π_2 is denoted by $\mathcal{C}(\pi_1, \pi_2)$.

Note that $\mathcal{C}(\pi_1, \pi_2)$ is a vector space over $\mathbb{C}$. For convenience, we write $\mathcal{C}(\pi) := \mathcal{C}(\pi, \pi)$. We say that π_1 and π_2 are *equivalent* if $\mathcal{C}(\pi_1, \pi_2)$ contains a unitary isomorphism. In this case, we write $\pi_1 \sim \pi_2$. It is easy to see that $\sim$ is an equivalent relation.

Definition 3.3. Let $\pi : G \to U(V_\pi)$ be a representation of G. A **closed** subspace W of V_π is called an *invariant subspace* for π if $\pi(x)[W] \subseteq W$ for all $x \in G$. The representation $\pi^W : G \to U(W)$ defined by

$$\pi^W(x) = \pi(x)|_W$$

is called a *subrepresentation* of π.

A representation π is said to be *irreducible* if π has no invariant subspace other than $\{0\}$ and V_π. Otherwise π is *reducible.*

Theorem 3.4 (Schur's Lemma)**.**

i) A unitary representation π of G is irreducible if and only if $\mathcal{C}(\pi)$ contains only scalar multiple of identity map.

ii) Let π_1, π_2 be irreducible unitary representations of G. If $\pi_1 \sim \pi_2$, then $\mathcal{C}(\pi_1, \pi_2)$ is one-dimensional. Otherwise, $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.

3.2 Fourier Analysis on Compact Groups

For this section, let G be a compact group with the normalized Haar measure dx, $\widehat{G}$ denote the set of all equivalence classes of irreducible unitary representations of G. For each $C \in \widehat{G}$, we fix the representative $\pi \in C = [\pi]$, so $\widehat{G} = \{[\pi] \mid \pi \text{ are those fixed representatives}\}$.

One of the most important result in harmonic analysis on compact group is the following theorem.

Theorem 3.5. *If G is compact, then every irreducible representation of G is finite dimensional, and every unitary representation of G is a direct sum of irreducible representations.*

Let G be a compact group. From Theorem 3.5, we know that for each $[\pi] \in \widehat{G}$, V_π is finite dimensional. Let $\{e_1, \ldots, e_{d_\pi}\}$ be an ordered orthonormal basis for V_π. For each $i, j \in \{1, \ldots, d_\pi\}$, the *matrix coefficient* π_{ij} is a continuous function on G defined by

$$\pi_{ij}(x) = \langle \pi(x)e_j, e_i \rangle \qquad x \in G.$$

Notice that if we identify V_π with $\mathbb{C}^{d_\pi}$ with this ordered orthonormal basis, the matrix representation of a linear map $\pi(x)$ is precisely the matrix $[\pi_{ij}(x)]$. Let $\mathcal{M}_\pi$ be the closure of subspace of $L^2(G)$ spanned by the set

$$\{\langle \pi(x)u, v \rangle \mid u, v \in V_\pi\}.$$

Theorem 3.6 (Schur Orthogonality Relations)**.** *Let π and π' be irreducible representations of G.*

i) If $\pi \nsim \pi'$, then $\mathcal{M}_\pi \perp \mathcal{M}_{\pi'}$.

ii) The set

$$\{\sqrt{d_\pi}\pi_{ij} \mid i, j = 1, \ldots, d_\pi\}$$

is an orthonormal basis for $\mathcal{M}_\pi$.

Theorem 3.7 (Peter-Weyl Theorem)**.** *Let* $\mathcal{M}_\pi^{(i)}$ *be the subspace of* $L^2(G)$ *spanned by* i^{th} *row of the matrix* $[\pi_{ij}]$*. Then* $\mathcal{M}_\pi^{(i)}$ *is invariant under right regular representation, and the restriction of the right regular representation to* $\mathcal{M}_\pi^{(i)}$ *is equivalent to* π*. Moreover,* $L^2(G)$ *can be decomposed into the Hilbert space direct sum:*

$$L^2(G) = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{M}_\pi$$

and the set

$$\{\sqrt{d_\pi}\pi_{ij} \mid i, j = 1, \ldots, d_\pi, [\pi] \in \widehat{G}\}$$

is an orthonormal basis of $L^2(G)$*.*

Definition 3.8. Let $f \in L^1(G)$ and $[\pi] \in \widehat{G}$. The *Fourier transform* of f at π is the operator on V_π defined by

$$\widehat{f}(\pi)u = \int_G f(x)\pi(x)u\,dx, \qquad u \in V_\pi,$$

which we interprete in the *weak sense*, that is

$$\langle \widehat{f}(\pi)u, v\rangle = \int_G f(x)\langle \pi(x)u, v\rangle\,dx$$

for all $u, v \in V_\pi$.

Note that our definition of the Fourier transform is different from the usual one (we regard G/H as a space of left cosets). The reason for using this definition will become clear in the next section. For usual definition of Fourier transform on compact groups, see [2].

Theorem 3.9 (Fourier Inversion Formula)**.** *Let* $f \in L^2(G)$*. Then*

$$f(x) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\widehat{f}(\pi)\right)$$

where the sum converges in L^2 *norm.*

Proof. Let $f \in L^2(G)$. From Peter-Weyl Theorem, $\{\sqrt{d_\pi}\pi_{ij} \mid i,j = 1,\dots,d_\pi, [\pi] \in \widehat{G}\}$, is an orthonormal basis for $L^2(G)$. Thus, in $L^2(G)$,

$$\begin{aligned}
\overline{f(x)} &= \sum_{[\pi]\in\widehat{G}} \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} \left(\int_G \overline{f(y)\sqrt{d_\pi}\pi_{ij}(y)}\, dy\right) \sqrt{d_\pi}\pi_{ij}(x) \\
&= \sum_{[\pi]\in\widehat{G}} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} \left(\int_G \overline{f(y)\pi_{ij}(y)}\, dy\right) \pi_{ij}(x).
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} \left(\int_G \overline{f(y)\pi_{ij}(y)}\, dy\right) \pi_{ij}(x) &= \int_G \overline{f(y)} \left(\sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} \pi_{ij}(x)\overline{\pi_{ij}(y)}\right) dy \\
&= \int_G \overline{f(y)} \left(\sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} \langle\pi(x)e_j, e_i\rangle\overline{\langle\pi(y)e_j, e_i\rangle}\right) dy \\
&= \int_G \overline{f(y)} \left(\sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \langle\pi(x)e_j, e_i\rangle\langle e_i, \pi(y)e_j\rangle\right) dy \\
&= \int_G \overline{f(y)} \left(\sum_{j=1}^{d_\pi} \langle\pi(x)e_j, \pi(y)e_j\rangle\right) dy \\
&= \sum_{j=1}^{d_\pi} \int_G \overline{f(y)}\langle\pi(x)e_j, \pi(y)e_j\rangle\, dy \\
&= \sum_{j=1}^{d_\pi} \overline{\int_G f(y)\langle\pi(y)e_j, \pi(x)e_j\rangle\, dy} \\
&= \sum_{j=1}^{d_\pi} \overline{\langle\widehat{f}(\pi)e_j, \pi(x)e_j\rangle} \\
&= \overline{\sum_{j=1}^{d_\pi} \langle\pi(x)^*\widehat{f}(\pi)e_j, e_j\rangle} \\
&= \overline{\operatorname{tr}\left(\pi(x)^*\widehat{f}(\pi)\right)}.
\end{aligned}$$

Hence, in $L^2(G)$,

$$f(x) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}\left(\pi(x)^*\widehat{f}(\pi)\right) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\widehat{f}(\pi)\right). \qquad \square$$

The following lemmas is frequently used to solve functional equations.

Lemma 3.10. *Let* $f \in L^2(G)$*. Then*

$$\widehat{L_y f}(\pi) = \pi(y)\widehat{f}(\pi) \qquad \text{and} \qquad \widehat{R_y f}(\pi) = \widehat{f}(\pi)\pi(y)^{-1} \tag{3.1}$$

for all $x, y \in G$ *and* $[\pi] \in \widehat{G}$*.*

Proof. Let $f \in L^2(G)$, $[\pi] \in \widehat{G}$ and $y \in G$. Since dx is a Haar measure on G, we have

$$\begin{aligned}
\widehat{L_y f}(\pi) &= \int_G L_y f(x)\pi(x)\,dx \\
&= \int_G f(y^{-1}x)\pi(x)\,dx \\
&= \int_G f(x)\pi(yx)\,dx \\
&= \pi(y)\int_G f(x)\pi(x)\,dx \\
&= \pi(y)\widehat{f}(\pi).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widehat{R_y f}(\pi) &= \int_G R_y f(x)\pi(x)\,dx \\
&= \int_G f(xy)\pi(x)\,dx \\
&= \int_G f(x)\pi(xy^{-1})\,dx \\
&= \left(\int_G f(x)\pi(x)\,dx\right)\pi(y^{-1}) \\
&= \widehat{f}(\pi)\pi(y)^{-1}.
\end{aligned}$$

□

Lemma 3.11. *If* $f \in L^2(G)$ *is nonzero, then there exists* $[\pi] \in \widehat{G}$ *such that* $\widehat{f}(\pi) \neq 0$*.*

Proof. Suppose $\widehat{f}(\pi) = 0$ for all $[\pi] \in \widehat{G}$. By Fourier inversion formula, we have

$$\begin{aligned} f(x) &= \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\widehat{f}(\pi)\right) \\ &= \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}(0) \\ &= 0. \end{aligned}$$

□

3.3 Fourier Analysis on Compact Homogeneous Spaces

Let G be a compact group and H be a closed subgroup of G with normalized Haar measure dx and $d\xi$ respectively. Let $q : G \to G/H$ be the canonical quotient map, let $d(xH)$ be the G-invariant Radon measure on G/H as in Theorem 2.13 and $L^2(G/H)$ be the space of square-integrable functions on G/H with respect to this measure. The following theorem gives a relation between $L^2(G)$ and $L^2(G/H)$.

Definition 3.12. The space of right H-invariant square-integrable functions is denoted by

$$L^2(G)^H = \{f \in L^2(G) \mid \forall x \in G\ \forall \xi \in H, f(x\xi) = f(x)\}.$$

Equivalently, $f \in L^2(G)^H$ if and only if $R_\xi f = f$ for all $\xi \in H$.

Theorem 3.13. *$L^2(G)^H$ is a closed subspace of $L^2(G)$, and the map $q_* : L^2(G/H) \to L^2(G)^H$ given by*

$$q_*(F) = F \circ q, \qquad F \in L^2(G/H),$$

is a unitary isomorphism.

Proof. Let $\{f_n\}$ be a sequence in $L^2(G)^H$ that converges to $f \in L^2(G)$. Then

$\int_G |f_n - f|^2\,dx$ converges to 0. For each $\xi \in H$, we have

$$\begin{aligned}\int_G |R_\xi f - f|^2\,dx &= \int_G |R_\xi f - R_\xi f_n + R_\xi f_n - f|^2\,dx \\ &\le \int_G |R_\xi f - R_\xi f_n|^2\,dx + \int_G |R_\xi f_n - f|^2\,dx \\ &= \int_G R_\xi(|f - f_n|^2)\,dx + \int_G |f_n - f|^2\,dx \\ &= 2\int_G |f_n - f|^2\,dx \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus $R_\xi f = f$, so $f \in L^2(G)^H$.

Clearly q_* is linear, also, by universal mapping property of G/H, we can see that q_* is onto. Let $F \in L^2(G/H)$. Then for any $\xi \in H$, $F \circ q(x\xi) = F(x\xi H) = F(xH) = F \circ q(x)$, so $F \circ q \in L^2(G)^H$. Since $d(xH)$ is the pushforward measure, we have

$$\int_{G/H} |F|^2\,d(xH) = \int_G |F \circ q|^2\,dx.$$

Hence q_* is a unitary isomorphism. □

For each unitary representation $\pi : G \to U(V)$, we define the space of H-fixed vectors by

$$V^H = \{v \in V \mid \forall h \in H, \pi(h)v = v\}.$$

Since V is finite dimensional, V^H is a closed subspace of V.

Proposition 3.14. *For $[\pi] \in \widehat{G}$, the map $P_\pi : V \to V^H$ given by*

$$P_\pi v = \int_H \pi(\xi)v\,d\xi$$

is an orthogonal projection.

Proof. Let $v \in V$ and $h \in H$. Then

$$\pi(h)P_\pi v = \int_H \pi(h)\pi(\xi)v\,d\xi = \int_H \pi(h\xi)v\,d\xi = \int_H \pi(\xi)v\,d\xi = P_\pi v,$$

and thus, $P_\pi v \in V^H$. Note that if $v \in V^H$, then $\int_H \pi(\xi)v\, d\xi = \int_H v\, d\xi = v$. Thus P_π is a projection. To show that P_π is self-adjoint, let $u, v \in V$. Then we have

$$\begin{aligned}
\langle P_\pi u, v\rangle &= \int_H \langle \pi(\xi)u, v\rangle\, d\xi \\
&= \int_H \langle u, \pi(\xi)^* v\rangle\, d\xi \\
&= \int_H \langle u, \pi(\xi^{-1})v\rangle\, d\xi \\
&= \int_H \langle u, \pi(\xi)v\rangle\, d\xi \\
&= \langle u, P_\pi v\rangle.
\end{aligned}$$

Thus $P_\pi^* = P_\pi$, and so P_π is a self-adjoint projection. Therefore, it is an orthogonal projection. □

Theorem 3.15. *If $f \in L^2(G)^H$, we have the identity*

$$\widehat{f}(\pi)v = \widehat{f}(\pi)P_\pi v, \quad \textit{for } v \in V,\ [\pi] \in \widehat{G} \tag{3.2}$$

Proof. Let $f \in L^2(G)^H$. Then we have for $v \in V$ and $[\pi] \in \widehat{G}$,

$$\begin{aligned}
\widehat{f}(\pi)P_\pi v &= \widehat{f}(\pi)\left(\int_H \pi(\xi)v\ d\xi\right) \\
&= \int_G \int_H f(x)\pi(x)\pi(\xi)v\, d\xi dx \\
&= \int_G \int_H f(x)\pi(x\xi)v\, d\xi\, dx \\
&= \int_H \int_G f(x)\pi(x\xi)v\, dx\, d\xi \\
&= \int_H \int_G f(x\xi^{-1})\pi(x)v\, dx\, d\xi \\
&= \int_H \int_G f(x)\pi(x)v\, dx\, d\xi \\
&= \int_H \widehat{f}(\pi)v\, d\xi \\
&= \widehat{f}(\pi)v.
\end{aligned}$$

□

Definition 3.16. An H*-spherical representation* of G is a unitary representation $\pi : G \to U(V)$ such that $V^H \neq \{0\}$. Moreover, we define the set

$$\widehat{G/H} = \{[\pi] \in \widehat{G} \mid \pi \text{ is a } H\text{-spherical representation}\}.$$

Lemma 3.17. *Let $f \in L^2(G)^H$. If $[\pi] \notin \widehat{G/H}$, then $\widehat{f}(\pi) = 0$.*

Proof. Suppose $[\pi] \notin \widehat{G/H}$. Then $V^H = \{0\}$, so that $P_\pi v = 0$ for any $v \in V$. Thus

$$\widehat{f}(\pi)v = \widehat{f}(\pi)P_\pi v = \widehat{f}(\pi)0 = 0$$

for all $v \in V$. □

Combining Theorem 3.9 and Lemma 3.17, we get the following theorem.

Theorem 3.18. *Let $f \in L^2(G)^H$. Then*

$$f(x) = \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}\left(\pi(x^{-1})\widehat{f}(\pi)\right)$$

where the sum converges in L^2 norm.

CHAPTER IV
FUNCTIONAL EQUATION ON COMPACT HOMOGENEOUS SPACES

In this chapter, we define the functional equation on a **compact homogeneous space** G/H based on the following functional equation on G:

$$f(xy) + \sum_{k=1}^{n-1} f(\sigma^k(y)x) = nf(x)f(y), \quad \text{for } x, y \in G, \tag{4.1}$$

where σ is a continuous automorphism on G such that $\sigma^n(x) = x$ for all $x \in G$. We give a solution to the functional equation in Theorem 4.4.

4.1 The Functional Equation on G/H

Let G be a compact group and H a closed subgroup of G with normalized Haar measure dx and $d\xi$ respectively. Let σ be a continuous automorphism such that σ^n is the identity map on G for some $n \geq 2$. Our main focus is the functional equation on G/H given by

$$F(xyH) + \sum_{k=1}^{n-1} F(\sigma^k(y)xH) = nF(xH)F(yH), \quad \text{for } x, y \in G, \tag{4.2}$$

where $F : G/H \to \mathbb{C}$ is a nonzero continuous function.

4.2 Solution of the Functional Equation

In this section, we prove lemmas used in solving the functional equation (4.2). The proofs of these lemmas are similar to the ones given in [1] with a slight modification since the definition of Fourier transform in [1] is different from what we use here.

Lemma 4.1. *If $f : G \to \mathbb{C}$ is a nonzero continuous solution of* (4.1)*, then* $f(e) = 1$ *and* $f \circ \sigma = f$ *where* e *denotes the identity of* G*.*

Proof. Take $y = e$ in the equation (4.1), we get for any $x \in G$,

$$nf(x) = nf(x)f(e).$$

Since f is nonzero, we can choose $x \in G$ such that $f(x) \neq 0$. Then we have $f(e) = 1$.
Next, We take $x = e$ in (4.1). Then we get the equation

$$\sum_{k=0}^{n-1} f(\sigma^k(y)) = nf(y).$$

Since $\sigma^n(y) = y$, we have

$$f(\sigma(y)) = \frac{1}{n}\sum_{k=1}^{n} f(\sigma^k(y)) = \frac{1}{n}\sum_{k=0}^{n-1} f(\sigma^k(y)) = f(y)$$

for all $y \in G$. □

Lemma 4.2. *Let* $f : G \to \mathbb{C}$ *be a nonzero continuous solution of* (4.1) *Then for* $[\pi] \in \widehat{G}$*, either* $\widehat{f}(\pi) = 0$ *or* $\widehat{f}(\pi)$ *is invertible.*

Proof. Let $f \in C(G)$ be a nonzero continuous solution of the functional equation (4.1). Rewrite the equation as

$$R_y f(x) + \sum_{k=1}^{n-1} L_{\sigma^k(y)^{-1}} f(x) = nf(y)f(x).$$

Let $[\pi] \in \widehat{G}$. By Lemma 3.10, taking Fourier transform at π with respect to x gives

$$\widehat{f}(\pi)\pi(y^{-1}) + \sum_{k=1}^{n-1} \pi(\sigma^k(y^{-1}))\widehat{f}(\pi) = nf(y)\widehat{f}(\pi) \tag{4.3}$$

Let $v \in \ker \widehat{f}(\pi)$. Then

$$\widehat{f}(\pi)\pi(y^{-1})v = nf(y)\widehat{f}(\pi)v - \sum_{k=1}^{n-1} \pi(\sigma^k(y^{-1}))\widehat{f}(\pi)v = 0,$$

so that $\pi(y^{-1})v \in \ker \widehat{f}(\pi)$ for all $y \in G$. Thus $\ker \widehat{f}(\pi)$ is an invariant subspace of π. Since π is irreducible, $\ker \widehat{f}(\pi)$ is $\{0\}$ or V_π, which means that $\widehat{f}(\pi)$ is invertible or $\widehat{f}(\pi) = 0$. □

Lemma 4.3. *Let $f \in L^2(G)^H$ be a solution of* (4.1)*. For each $[\pi] \in \widehat{G}$, if $\widehat{f}(\pi) \neq 0$, then $[\pi] \in \widehat{G/H}$.*

Proof. Let $f \in L^2(G)^H$ satisfies the equation (4.1) and $[\pi] \in \widehat{G}$. Assume that $\widehat{f}(\pi) \neq 0$. By Lemma 4.2, $\widehat{f}(\pi)$ is invertible. Since $f \in L^2(G)^H$, $\widehat{f}(\pi)$ satisfies the identity (3.2). Then

$$P_\pi v = \widehat{f}(\pi)^{-1}\widehat{f}(\pi)v = v$$

for all $v \in V_\pi$, which implies that P_π is the identity map and hence $V_\pi^H = V_\pi \neq \{0\}$. Thus $[\pi] \in \widehat{G/H}$. □

Our main result is the following theorem.

Theorem 4.4. *Let $F : G/H \to \mathbb{C}$ be a nonzero continuous solution of the functional equation* (4.2)*, i.e.,*

$$F(xyH) + \sum_{k=1}^{n-1} F(\sigma^k(y)xH) = nF(xH)F(yH).$$

Then there exists a one-dimensional H-spherical representation $\chi : G \to \mathbb{T}$ such that $H \leq \ker(\chi \circ \sigma^k)$ for all $k \in \{0, 1, \ldots, n-1\}$ and

$$F(xH) = \frac{1}{n}\sum_{k=0}^{n-1} \chi \circ \sigma^k(x) \tag{$\star$}$$

for all $x \in G$. Moreover, the function F defined in ($\star$) is a solution of the functional equation (4.2)*.*

Proof. The idea of the proof is the same as the one given in [1]. Let $F \in C(G/H)$ be a nonzero solution of (4.2). Let $f = F \circ q$. Then $f \in L^2(G)^H$ and f satisfies the equation (4.1). Since F is nonzero, f is also nonzero. Then there exists $[\pi] \in \widehat{G}$ such that $\widehat{f}(\pi) \neq 0$. By Lemmas 4.2 and 4.3, $\widehat{f}(\pi)$ is invertible and $[\pi] \in \widehat{G/H}$. From equation (4.3), we have

$$\pi(x^{-1}) + \sum_{k=1}^{n-1} \widehat{f}(\pi)^{-1}\pi(\sigma^k(x^{-1}))\widehat{f}(\pi) = nf(x)I$$

where I denotes the identity map on V_π. Taking trace both sides gives

$$\operatorname{tr}(\pi(x^{-1})) + \sum_{k=1}^{n-1} \operatorname{tr}(\pi(\sigma^k(x^{-1}))) = nd_\pi f(x).$$

Thus

$$f(x) = \frac{1}{nd_\pi} \sum_{k=0}^{n-1} \operatorname{tr}(\pi(\sigma^k(x)^{-1})). \tag{4.4}$$

Observe that $f(xy) = f(yx)$ for all $x, y \in G$. Then $\widehat{f}(\pi)\pi(y) = \pi(y)\widehat{f}(\pi)$ for all $y \in G$, so that $\widehat{f}(\pi) \in \mathcal{C}(\pi)$. Since π is irreducible, by Schur's lemma, $\widehat{f}(\pi)$ is a nonzero scalar multiple of I. Thus equation (4.3) becomes

$$\sum_{k=0}^{n-1} \pi \circ \sigma^k(x) = nf(x^{-1})I. \tag{4.5}$$

Note that $\pi \circ \sigma^k$ is also an irreducible representation of G on V_π. Let $i, j \in \{1, \ldots, d_\pi\}$. Consider the matrix coefficients in (4.5). Since I is the identity map, we have

$$\sum_{k=0}^{n-1} (\pi \circ \sigma^k)_{ij}(x) = \begin{cases} nf(x^{-1}), & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases} \tag{4.6}$$

To show that $d_\pi = 1$, assume that $d_\pi \geq 2$. Let $S = \{k \in \mathbb{N} \mid k < n \text{ and } \pi \sim \pi \circ \sigma^k\}$. If $S = \varnothing$, by Schur's orthogonality relation, $\mathcal{M}_\pi \perp \mathcal{M}_{\pi\circ\sigma^k}$ for all $k < n$, so $\{(\pi \circ \sigma^k)_{ij} \mid k = 1, \ldots, n-1\}$ is a linearly independent set. But (4.6) implies that

$$\sum_{k=0}^{n-1}(\pi \circ \sigma^k)_{12}(x) = 0,$$

which is a contradiction.

On the other hand, if $S \neq \varnothing$, let s be the smallest element in S. We see that $\pi \sim \pi \circ \sigma^{ms}$ for all $m \in \mathbb{N}$. Let $q = \gcd(n, s)$. Then there exist $a, b \in \mathbb{Z}$ such that $an + bs = q$. Hence $\pi \sim \pi \circ \sigma^{bs} = \pi \circ \sigma^{an+bs} = \pi \circ \sigma^q$, so we must have $q = s$. Thus $s|n$ and $S = \{s, 2s, \dots, Ns = n - s\}$. Since $\pi \sim \pi \circ \sigma^s$, there is a unitary operator T on V_π such that $\pi \circ \sigma^s(x) = T^*\pi(x)T$ for all $x \in G$. Then we have

$$\pi \circ \sigma^{as+b}(x) = (T^a)^* \pi \circ \sigma^b(x) T^a$$

for all $x \in G$ and $a, b \in \mathbb{N}$. Since T is unitary, there is an orthonormal eigenbasis $\{e_1, \dots, e_{d_\pi}\}$ for T. Let λ_i be the eigenvalue of T associated with e_i. Since T is unitary, $|\lambda_i| = 1$. If we compute matrix coefficients on the diagonal line with respect to this orthonormal basis, we get

$$\begin{aligned}
(\pi \circ \sigma^{as+b})_{ii}(x) &= \langle \pi \circ \sigma^{as+b}(x) e_i, e_i \rangle \\
&= \langle (T^a)^* \pi \circ \sigma^b(x) T^a e_i, e_i \rangle \\
&= \langle \pi \circ \sigma^b(x) T^a e_i, T^a e_i \rangle \\
&= \langle \pi \circ \sigma^b(x) \lambda_i^a e_i, \lambda_i^a e_i \rangle \\
&= |\lambda_i^a|^2 \langle \pi \circ \sigma^b(x) e_i, e_i \rangle \\
&= (\pi \circ \sigma^b)_{ii}(x).
\end{aligned}$$

From (4.6), we have

$$(N+1)\sum_{k=0}^{s-1}(\pi \circ \sigma^k)_{ii}(x) = nf(x^{-1}).$$

for all $i = 1, \ldots, d_\pi$, which implies that

$$\sum_{k=0}^{s-1} \big((\pi \circ \sigma^k)_{11}(x) - (\pi \circ \sigma^k)_{22}(x)\big) = 0. \tag{4.7}$$

Since $\pi \circ \sigma^k \nsim \pi \circ \sigma^l$ for $0 \leq k < l \leq s-1$, Schur's orthogonality relation implies that $\{(\pi \circ \sigma^k)_{ii} \mid 0 \leq k \leq s-1, 1 \leq i \leq d_\pi\}$ is an orthogonal set. Thus the equation (4.7) is impossible, so we must have $d_\pi = 1$. Hence π is a character of G.

Define $\chi : G \to \mathbb{T}$ by $\chi(x) = \mathrm{tr}(\pi(x^{-1})) = \pi(x^{-1})$. Then χ is a character and also an H-spherical representation of G. Then (4.4) becomes

$$F \circ q(x) = f(x) = \frac{1}{n} \sum_{k=0}^{n-1} \chi \circ \sigma^k(x). \tag{4.8}$$

Let $\xi \in H$. Since $f(e) = 1$ and $f \in L^2(G)^H$, we have $f(\xi) = f(e\xi) = f(e) = 1$. Then

$$n = nf(\xi) = \sum_{k=0}^{n-1} \chi \circ \sigma^k(\xi) \tag{4.9}$$

Since χ is unitary, $|\chi \circ \sigma^k(\xi)| = 1$ for all $k \in \{0, 1, \ldots, n-1\}$. Using the triangle inequality, we can deduce that $\chi \circ \sigma^k(\xi) = 1$ for all k. Thus $H \leq \ker(\chi \circ \sigma^k)$ for all $k \in \{0, 1, \ldots, n-1\}$, so that $\frac{1}{n} \sum_{k=0}^{n-1} \chi \circ \sigma^k(x)$ is a well-defined function on G/H.

Finally, we check that the function F given by (4.8) is indeed a solution of the functional equation (4.2). Since χ and σ are continuous, the right-hand side of (4.8) is a continuous function. Moreover,

$$\frac{1}{n} \sum_{k=0}^{n-1} \chi \circ \sigma^k(xy) = \frac{1}{n} \sum_{k=0}^{n-1} \chi \circ \sigma^k(x) \chi \circ \sigma^k(y)$$

and

$$\frac{1}{n} \sum_{k=0}^{n-1} \chi \circ \sigma^k(\sigma^l(y)x) = \frac{1}{n} \sum_{k=0}^{n-1} \chi \circ \sigma^k(x) \chi \circ \sigma^{k+l}(y)$$

for all $l \in \{1, \ldots, n-1\}$. Thus

$$\begin{aligned}
&\frac{1}{n}\sum_{k=0}^{n-1}\chi\circ\sigma^k(xy)+\sum_{l=1}^{n-1}\left(\frac{1}{n}\sum_{k=0}^{n-1}\chi\circ\sigma^k(\sigma^l(y)x)\right)\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\chi\circ\sigma^k(x)\chi\circ\sigma^k(y)+\frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=1}^{n-1}\chi\circ\sigma^k(x)\chi\circ\sigma^{k+l}(y)\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\chi\circ\sigma^k(x)\left(\sum_{l=0}^{n-1}\chi\circ\sigma^{k+l}(y)\right)\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\chi\circ\sigma^k(x)\left(\sum_{l=0}^{n-1}\chi\circ\sigma^{l}(y)\right)\\
&= \frac{1}{n}\left(\sum_{k=0}^{n-1}\chi\circ\sigma^k(x)\right)\left(\sum_{l=0}^{n-1}\chi\circ\sigma^{l}(y)\right)\\
&= n\left(\frac{1}{n}\sum_{k=0}^{n-1}\chi\circ\sigma^k(x)\right)\left(\frac{1}{n}\sum_{l=0}^{n-1}\chi\circ\sigma^{l}(y)\right).
\end{aligned}$$

Hence the function

$$F(xH)=\frac{1}{n}\sum_{k=0}^{n-1}\chi\circ\sigma^k(x)$$

is a nonzero continuous solution of (4.2). □

4.3 An Example: The Circle Group

Let $G := \mathbb{T}$ the circle group, $\zeta_n := e^{\frac{2\pi i}{n}}$ where n is a positive integer, $H := \langle\, \zeta_n \,\rangle$ the group of n^{th} roots of unity in $\mathbb{T}$ and $\sigma(x) := x^{-1}$ for $x \in G$. Note that H is closed because it is the kernel of the continuous group homomorphism $z \mapsto z^n$. Then the functional equation (4.2) on G/H is

$$F(xyH) + F(y^{-1}xH) = 2F(xH)F(yH), \qquad x, y \in G.$$

It is well-known that every irreducible representation of G is one-dimensional and $\widehat{G} = \{\chi_m \mid m \in \mathbb{Z}\}$ where $\chi_m(x) := x^m$. Note that $\chi_m \in \widehat{G/H}$ if and only if there exists $z \in \mathbb{C} \setminus \{0\}$ such that $\chi_m(x)z = z$ for all $x \in H$. Since H is a cyclic group generated by ζ_n, we have for $m \in \mathbb{Z}$,

$$\chi_m \in \widehat{G/H} \text{ if and only if } \chi_m(\zeta_n) = \zeta_n^m = 1$$
$$\text{if and only if } n \text{ divides } m.$$

Hence

$$\widehat{G/H} = \{\chi_m \mid m \in n\mathbb{Z}\}.$$

Let $\chi_m \in \widehat{G/H}$. To show that $H \leq \ker \chi_m \cap \ker(\chi_m \circ \sigma)$, it suffices to prove that $\zeta_n \in \ker \chi_m \cap \ker(\chi_m \circ \sigma)$. Since $\chi_m \in \widehat{G/H}$, m is divisible by n. Thus $\chi_m(\zeta_n) = \zeta_n^m = 1$ and $\chi_m \circ \sigma(\zeta_n) = \zeta_n^{-m} = 1$. By Theorem 4.4, every nonzero continuous solution of

$$F(xyH) + F(y^{-1}xH) = 2F(xH)F(yH), \qquad x, y \in G,$$

is of the form

$$F(xH) = \frac{x^{kn} + x^{-kn}}{2}$$

where $k \in \mathbb{Z}$.

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