CHAPTER 4

TYPE II SEMIRINGS

4.1.1 Proposition : Let S be a type II semiring of order greater than 2. Then no element in S is AC.

Proof : ∞ is not AC since $1 + \infty = \infty + \infty$, but $1 \neq \infty$. Now apply Theorem 2.1.6. #

If $||S|| = 2$ then $S = \{1, \infty\}$ and by setting $1 + 1 = 1$ we see that 1 is AC. Thus the restriction on the order of S is necessary.

4.1.2 Proposition : Let S be a type II semiring and $x \in S$. Then $\langle x \rangle + S = S$ or $\langle x \rangle + S = \{\infty\}$

Proof : $\langle x \rangle + S$ is a double ideal in S. Since S is congruence free S has two possible double ideals : $\{\infty\}$ and S. #

4.1.1 Definition : Let S be a semiring with ∞. We say that S has the <u>trivial structure</u> iff $x + y = \infty \ \forall\ x,y \in S$. If $x + y = \infty \ \forall x \neq y \in S$ and $x + x = x \ \forall\ x \in S$ then we say that S has the <u>almost trivial structure</u>.

4.1.3 Proposition : Let S be an ∞ -semifield which has the trivial structure. Then S is congruence-free iff $S = \{1, \infty\}$

Proof : First we prove sufficiency. Suppose S is congruence free.

Let $\sim = (S \setminus \{\infty\} \times S \setminus \{\infty\}) \bigcup (\infty, \infty)$. $\sim$ is an equivalence relation on S. But since S is a semifield $S \setminus \{\infty\}$ is closed with respect to multiplication. Now suppose $x \sim y$ and $x \neq \infty$. Then $y \neq \infty$. $\infty x = \infty$ and $\infty y = \infty$. Thus $\infty x \sim \infty y$. For $s \in S \setminus \{\infty\}$ $sx \neq \infty$ and $sy \neq \infty$. Thus $sx \sim sy$. Thus for all $s \in S$, and for all $x, y \in S$, $x \sim y$ implies $sx \sim sy$. Suppose $x, y \in S$ and $x \sim y$. Then for all $s \in S$. $x + s = \infty = y + s$. Therefore $x + s \sim y + s$ so $\sim$ is a congruence on S. But since S is congruence-free $\sim = \Delta$. Thus $||S \setminus \{\infty\}|| = 1$ and so $S \setminus \{\infty\} = \{1\}$. Thus $S = \{1, \infty\}$. Next we prove necessity. Suppose $S = \{1, \infty\}$. Well there are only two possible equivalence relations on S i.e. $S \times S$ and Δ. Thus S is congruence-free. #

The next proposition concerns ∞ -semifields with the almost trivial structure.

<u>4.1.4 Proposition</u> : Let S be an ∞ -semifield which has the almost trivial structure. Then S is congruence free.

Proof : Let S be an ∞ -semifield with the almost trivial structure. Let $\sim$ be a congruence on S such that $\sim \neq \Delta$. Then there exists $x \neq \infty$ and $y \neq x \in S$ such that $x \sim y$ Thus $x + x \sim x + y$, so $x \sim \infty$. Therefore $x^{-1}x \sim x^{-1}\infty$ so $1 \sim \infty$. Now choose $y \in S$. $y.1 \sim y.\infty$. Thus $y \sim \infty$. Thus $\sim = S \times S$. #

At this stage it might seem that the two trivial results above are not very interesting. However, as will become apparent later, there is a close relationship between ∞ -semifields with the trivial and almost trivial structures and type II semirings. The following proposition is an indication of this.

4.1.5 Proposition : Let S be a type II semiring with the trivial or almost trivial structures. Then S is an ∞ -semifield.

Proof : We distinguish two cases :

Case A : S has the trivial structure. Choose $x \neq \infty \in S$. Define $\sim \; = (\langle x \rangle \times \langle x \rangle) \cup \Delta$. $\sim$ is clearly an equivalence relation on S. Now suppose $a \sim b$ and $a \neq b$. Then there exist $k_1, k_2 \in S$ such that $k_1 x = a$ and $k_2 x = b$. Thus for all $m \in S$, $ma \sim mb$ since $ma = mk_1 x \in \langle x \rangle$ and $mb = mk_2 x \in \langle x \rangle$. Clearly $ma \sim ma$ for all $a \in S$. Therefore for all $m, a, b \in S$ $a \sim b \Rightarrow ma \sim mb$. Suppose $a, b \in S$ and $a \sim b$. Then $\forall m \in S$, $m + a = \infty = m + b$ $m + a \sim m + b$. Thus $\sim$ is a congruence relation on S. Thus $\sim = \Delta$ or $S \times S$. But $\sim = \Delta$ is impossible since $1.x = x$ and $\infty.x = \infty$ $\forall x \in S$. Thus $\sim = S \times S$, so $1 \in \langle x \rangle$. Therefore there exists a $k \in S$ such that $kx = 1$ i.e. $x^{-1} \in S$. Thus since x was an arbitrary nonzero element in S, S is an ∞ -semifield.

Case B : S has the almost trivial structure. Again claim that for $x \neq \infty \in S$, $\sim \; = (\langle x \rangle \times \langle x \rangle) \cup \Delta$ is a congruence on S. By the argument in Case A. $\sim$ is an equivalence relation and $\sim$ preserves multiplication. Suppose $a \sim b$ and $a \neq b$. Choose $y \in S$. If $y \neq a$ and $y \neq b$ then $y + a = \infty = y + b$ so $y + a \sim y + b$. If $y = a$ then $y + a = a \in \langle x \rangle$ and $y + b = \infty \in \langle x \rangle$. Thus $y + a \sim y + b$. Similarly if $y = b$ we get that $y + a \sim y + b$. Thus $\sim$ preserves addition. Thus by the argument in Case A, S is an ∞ -semifield. #

To simplify the following proofs we introduce the following notation. Let S be a commutative semiring with ∞. Then for $x \in S$, the core of x denoted by $\mathrm{cor}(x) = \{a \in S \mid a + x = \infty\}$ If there is no confusion about the universal set we will sometimes shorten $\mathrm{cor}_S(x)$ to $\mathrm{cor}(x)$. The anticore of x, denoted by $\mathrm{acor}_S(x) = S \setminus \mathrm{cor}_S(x)$.

Clearly $\{\infty\} \in \mathrm{cor}(x)$, $\forall x \in S$. It can be shown that if S is an ∞-semifield then for each $x,y \in S \setminus \{\infty\}$, there exists $b \in S$ such that $\mathrm{cor}(x) = b\,\mathrm{cor}(y)$. Suppose $x + a = \infty$. Then $1 + \frac{a}{x} = \infty$. Thus $\frac{a}{x} \in \mathrm{cor}(1)$, so $\mathrm{cor}(1) \supseteq \frac{\mathrm{cor}(x)}{x}$. Similarly $\mathrm{cor}(x) \supseteq x\,\mathrm{cor}(1)$ since if $a \in \mathrm{cor}(1)$, $x(a + 1) = \infty$ and so $xa + x = \infty$. Therefore $\mathrm{cor}(x) = x\,\mathrm{cor}(1)$. By the same argument $\mathrm{cor}(y) = y\,\mathrm{cor}(1)$ Thus $\frac{\mathrm{cor}(x)}{x} = \frac{\mathrm{cor}(y)}{y}$ so $\frac{x}{y}\mathrm{cor}(y) = \mathrm{cor}(x)$ As the next few results show, the concept of the "cor " of an element is critical in the study of congruences on ∞-semifields.

<u>4.1.6 Theorem</u> : Let S be an ∞-semifield. Then say $x \sim y$ if $\mathrm{cor}(x) = \mathrm{cor}(y)$. $\sim$ is a congruence on S. Moreover if S doesn't have the trivial structure then $\sim$ is a maximum proper congruence on S.

Proof : Clearly $\sim$ is an equivalence relation on S. Suppose $x \sim y$ and $z \in S$. Let $w \in S$, and suppose $a \in \mathrm{cor}(x + w)$. Then $a + x + w = \infty$, so $x + (a + w) = \infty$. Thus $a + w \in \mathrm{cor}(x)$ so $a + w \in \mathrm{cor}(y)$. Therefore $y + a + w = \infty$ so we get $(y + w) + a = \infty$. Thus $a \in \mathrm{cor}(y + w)$ so $\mathrm{cor}(y + w) \supseteq \mathrm{cor}(x + w)$. By the same argument $\mathrm{cor}(y + w) \subseteq \mathrm{cor}(x + w)$. Thus $\mathrm{cor}(y + w) = \mathrm{cor}(x + w)$. Thus $y \sim x \Rightarrow y + w \sim x + w$ for all $w \in S$. Again suppose that $x \sim y$ and $w \in S$. Suppose $w = \infty$. Then $wx = wy$ so $wx \sim wy$. Suppose $w \neq \infty$. and suppose that $a + wx = \infty$. Then $\frac{a}{w} + x = \infty$, so $\frac{a}{w} \in \mathrm{cor}(x) = \mathrm{cor}(y)$. Therefore $\frac{a}{w} + y = \infty$. Thus $a + wy = \infty$. Therefore $a \in \mathrm{cor}(wy)$, so $\mathrm{cor}(wx) \supseteq \mathrm{cor}(wy)$. By reversing the roles of x and y above we get that $\mathrm{cor}(wx) = \mathrm{cor}(wy)$ and thus that $wx \sim wy$. Thus $\sim$ is indeed a congruence on S. Now suppose S doesn't have the trivial structure. Since $\mathrm{cor}(\infty) = S$. $\sim \neq S \times S$. Now let p be a proper congruence on S. We must show that $p \subseteq \sim$. Let xpy. We must show that $x \sim y$. Suppose that $x \not\sim y$. Then $\mathrm{cor}(x) \neq \mathrm{cor}(y)$. Without loss of generality assume there exists $z \in S$ such that $x + z = \infty$ but $y + z \neq \infty$. Then $(x + z)p(y + z)$, so $\infty\, p\; y + z$. Thus $\infty \cdot \frac{1}{y+z}\; p\; (y + z) \frac{1}{y+z}$ Therefore $\infty\, p\, 1$. Thus for all $s \in S$ $s \cdot \infty\, p\, s.1$ Thus $\infty\, p\, s$ for all $s \in S$. Thus p is $S \times S$ which contradicts the fact that p is proper. Thus $p \subseteq \sim$ and since p was arbitrary, $\sim$ is a maximum proper congruence on S. #

4.1.2 Definition : We call $\sim$ in the theorem above the fundamental congruence on an ∞-semifield. The following corollary is then immediate.

4.1.7 Corollary : Let S be an ∞-semifield which does not have the trivial structure. Then S is congruence-free iff the fundamental congruence on S is Δ.

4.1.8 Lemma. Let S be an ∞-semifield which does not have the trivial structure. Then $\mathrm{acor}(1)$ is a division semiring.

Proof : Since S doesn't have the trivial structure $\mathrm{acor}(1) \neq \phi$ Choose $x_1 \in \mathrm{acor}(1)$. Then $1 + x_1 \neq \infty$, so $\frac{1}{x_1}(1 + x_1) \neq \infty$. Thus $\frac{1}{x_1} + 1 \neq \infty$. Thus $\frac{1}{x_1} \in \mathrm{acor}(1)$. Suppose $x, y \in \mathrm{acor}(1)$. Then $1 + x \neq \infty$ and $1 + y \neq \infty$ so $(1 + x)(1 + y) \neq \infty$.

Thus $1 + xy + x + y \neq \infty$. Thus $xy \in \mathrm{acor}(1)$ and $(x + y) \in \mathrm{acor}(1)$. Therefore $\mathrm{acor}(1)$ is a division semiring. #

By the lemma above if S is an ∞-semifield without the trivial structure then $||acor(1)|| = 1$ (and thus $acor(1) = \{1\}$) or $||acor(1)|| = \infty$ since the only finite division semiring is $\{1\}$. If S has the trivial structure, then $acor(1) = \phi$.

4.1.9. Theorem.

Let S be a finite ∞-semifield. Then S has the trivial or almost trivial structure.

Proof : First suppose that $1 + 1 = \infty$. Choose $x \in S$. Claim that $x + 1 = \infty$. Suppose not. Then clearly $cor(x + 1) \supseteq cor(1)$. But $cor(x + 1) = (x + 1)\, cor(1)$. Thus $||cor(x + 1)|| = ||cor(1)||$. Thus $cor(x + 1) = cor(1)$. Now $x \in cor(x)$ since $x + x = x(1 + 1) = x \cdot \infty = \infty$. Thus $x \in cor(x + 1)$. Therefore $x \in cor(1)$ i.e. $x + 1 = \infty$. Contradiction. Therefore for all $x \in S$. $x + 1 = \infty$. Suppose $y \neq \infty$. Then $\frac{x}{y} + 1 = \infty$ for all $x \in S$. Thus $x + y = \infty$ for all $x \in S$ and for all $y \in S \setminus \{\infty\}$. But surely $x + y = \infty$ if $y = \infty$. Thus $x + y = \infty$ for all $x, y \in S$. Therefore S has the trivial structure.

Now suppose that $1 + 1 \neq \infty$. Suppose that $||acor(1)|| = 1$ Claim that $1 + 1 = 1$. To prove this suppose not. Let $1 + 1 = \alpha \neq 1$. If $\alpha + \alpha = \infty$ then $\alpha^2 = \alpha(1 + 1) = \infty$ so $\alpha = \infty$ which is a contradiction. Thus $\alpha + \alpha \neq \infty$.
Thus $(1 + 1) + (1 + 1) \neq \infty$ so $1 + (1 + 1) \neq \infty$ i.e. $\alpha \in acor(1)$. which contradicts the fact that $||acor(1)|| = 1$ (since $\{1, \alpha\} \subseteq acor(1)$). Thus $1 + 1 = 1$ and we have the claim.
Therefore $x + x = x$ $\forall$ $x \in S$. And by assumption above $||acor(1)|| = 1$ so $||cor(1)|| = ||S|| - 1$. Thus for all $x \in S \setminus \{\infty\}$

$||cor(x)|| = ||S|| - 1$. Thus S has the almost trivial structure.

But since $||S||$ is finite $||acor(1)|| \geq 2$ is impossible by the remark following the previous lemma. This completes the proof. #

We must assume in the general case that $||S||$ is finite in the theorem above. For example let $S = \mathbb{Q}^+_\infty$ with the usual multiplication, but for $x,y \in S$ define $x + y = \max \{x,y\}$. Then S is an ∞ -semifield without the trivial or almost trivial structure.

If S is a finite ∞ -semifield then by the theorem above S has the almost trivial or the trivial structure. S is congruence-free if S has the almost trivial structure. Thus the following result.

<u>4.1.10Corollary</u> : If S is a finite ∞ -semifield, then S is congruence-free if S does not have the trivial structure.

<u>4.1.11 Corollary</u> : The only finite MC commutative semirings with 1 and ∞ are trivial and almost trivial finite semifields.

Proof : Let S be a finite MC commutative semiring with 1 and ∞ . Then S is an -semifield. Thus S has the trivial or almost trivial structure. #

The next corollary is also an immediate result of previous results.

<u>4.1.12 Corollary</u> : Every finite type II semiring S is an ∞- semifield with the trivial or almost trivial structure.

Proof: By an earlier result S is MC. Now by the result above

S has the trivial or almost trivial structure and S is an ∞-semifield. #

The previous results depend on the finiteness of S. The next two theorems remove this condition.

<u>4.1.13 Theorem</u> : Let S be a type II semiring. Then $1 + 1 = 1$ or $1 + 1 = \infty$.

Proof : Suppose not. Then $1 + 1 = \alpha$ where $\alpha \neq 1$ and $\alpha \neq \infty$. Let K be the quotient semifield of S. Let $\sim$ be the fundamental congruence on K. Since $1 + 1 \neq \infty$ by Thm. 4.1.6. $\sim \neq S \times S$ (i.e. $\sim$ is proper). Thus since K is congruence-free, $\sim = \Delta$. Thus $cor(\alpha) \neq cor(1)$. But clearly $cor(\alpha) \supseteq cor(1)$. Thus $cor(\alpha) \supset cor(1)$. Therefore $acor(\alpha) \subset acor(1)$. Now suppose $x \in acor(1)$ i.e. $1 + x \neq \infty$. Since $1 + 1 \neq \infty$, $\alpha + \alpha \neq \infty$. Thus $\alpha \in acor(\alpha)$ so $\alpha \in acor(1)$. Thus $1 + \alpha \neq \infty$ so $(1 + \alpha)(1 + x) \neq \infty$. Thus $\alpha + x + \alpha x + 1 \neq \infty$. so $\alpha + x \neq \infty$ i.e. $x \in acor(\alpha)$. Thus $acor(\alpha) \supseteq acor(1)$ which is a contradiction. Thus $1 + 1 = \infty$ or $1 + 1 = 1$. #

The next result is perhaps the most interesting result of this thesis.

<u>4.1.14 Theorem</u> : Let S be a type II semiring. Then S is an ∞-semifield. In particular, every noninfinity element has a multiplicative inverse. Furthermore, either S has the almost trivial structure or S has the trivial structure and is of order 2.

Proof : By the preceeding theorem $1 + 1 = 1$ or $1 + 1 = \infty$. We consider these cases separately.

Case A ; $1 + 1 = \infty$.

Suppose that $K = QS$ doesn't have the trivial structure. Then $acor_K(1) \neq \phi$. Thus $acor_K(1)$ is a division semiring. Therefore $1 \in acor_K(1)$, so $1 + 1 \neq \infty$ which is a contradiction. By Theorem 4.1.5 S is an ∞-semifield, since S must also have the trivial structure.

Case B ; $1 + 1 = 1$

Let K be the quotient semifield of S. Suppose that $acor_K(1) \supsetneq \{1\}$. Choose $x \neq 1 \in acor_K(1)$. Then $(1 + x) \neq \infty$. Suppose that $1 + y \neq \infty$. Then $1 + x + y + xy = (1 + x)(1 + y) \neq \infty$. Thus $x + y \neq \infty$. Therefore $acor_K(x) \supseteq acor_K(1)$. so $cor_K(x) \subseteq cor_K(1)$ Since K does not have the trivial structure the fundamental congruence on K is Δ. Thus $cor_K(x) \subset cor_K(1)$.

Claim that $x + 1 = 1$. To prove this suppose $x + 1 \neq 1$. Suppose that $a \in cor_K(x + 1)$. Then $(1 + x) + a = \infty$ so $x + (1 + a) = \infty$. Thus $1 + a \in cor_K(x)$ so $1 + a + 1 = \infty$. Now $1 + a = \infty$ since $1 + 1 = 1$. Thus $a \in cor_K(1)$. Thus $cor_K(x + 1) \subseteq cor_K(1)$. Since K is congruence free $cor_K(x + 1) = cor_K(1)$, which is obviously untrue so we have the claim. Thus $x + 1 = 1$ for all $x \in acor_K(1)$. But $acor_K(1)$ is a division semiring, by a previous result. For all $x \neq y \in acor_K(1)$ $(x + y) = x(1 + \frac{y}{x}) = x.1 = x$ since $\frac{y}{x} \in acor_K(1)$. Similarly $x + y = y$ Thus $acor_K(1) \supset \{1\}$ is impossible. Thus $acor_K(1) = 1$ so S has the almost trivial structure which can be shown as follows. Since $1 + 1 = 1$, $\alpha + \alpha = \alpha$ for all $\alpha \in K$. Suppose that $\alpha \neq \beta \in S$ and $\beta \neq \infty$. Then in K $\frac{\alpha}{\beta} \neq 1$. Thus $\frac{\alpha}{\beta} + 1 = \infty$. Thus $\alpha + \beta = \infty$, i.e S has the almost trivial structure. Therefore S is an ∞-semifield. This completes the proof. #

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<u>4.1.15 Corollary</u> : Let S be an ∞-semifield such that $1 + 1 = \infty$. Then S has the trivial structure.

Proof : As in Case A of the theorem above if a cor(1) $\neq \phi$ then a cor(1) is a division semiring so $1 \in$ a cor(1). Thus $1 + 1 \neq \infty$ which is a contradiction, so acor(1) $= \phi$ i.e. S has the trivial structure. #

There exist ∞-semifields such that $1 + 1 = 1$ but which do not have the almost trivial structure. For example let $X = \mathbb{Q}^+$ for $x,y \in S$ define $x + y = \max(x,y)$ and give multiplication the usual definition. Then $1 + 1 = 1$ but X does not have the almost trivial structure.

From the corollary above we see that every MC com. semiring S with ∞ and 1 such that $1 + 1 = \infty$ has the trivial structure (just embed it in its quotient semifield and apply the corollary). If S is not MC then S may not have the trivial structure. For example let $S = \{0,1,\infty\}$ with addition defined as $0+0=0, 1+0=0+1=1$, $1+1=\infty$ and $a+\infty=\infty\ \forall a \in S$. Define multiplication by $1.a=a.1=a\ \forall a \in S$, $\infty.a=a.\infty=\infty$ $\forall a \in S$ and $1.0=0.1=0$. S doesn't have the trivial structure but $1+1=\infty$.

Using the results above we are able to obtain a sharp characterization of type II semirings in terms of double ideals. The following theorem which is a partial converse to Proposition 2.2.1 is useful in the study of semirings of the form $\frac{S}{M}$ where M is a maximum proper double ideal.

<u>4.1.15 Theorem.</u> Let S be an MC commutative semiring with 1 and with ∞. Then S is congruence-free iff S is double ideal free and the

quotient ∞-semifield of S is congruence-free (and hence S $\cong$ its quotient ∞-semifield).

Proof : Necessity. Let S be an MC commutative semiring with 1 and with ∞ such that S is double ideal free and the quotient semifield QS of S is congruence-free. By Theorem 4.1.13 QS = $\{1,\infty\}$ with the trivial structure or QS has the almost trivial structure. Now if QS = $\{1, \infty\}$ above then S = $\{1,\infty\}$ with the trivial structure so S is congruence free. So suppose that QS has the almost trivial structure, Then S has the almost trivial structure. Now choose $x \neq \infty \in S$. Claim that $\langle x\rangle$ is a double ideal in S. Clearly $ks \in \langle x\rangle$ for all $s \in \langle x\rangle$ since if $s \in \langle x\rangle$ $s = k_1 x$ for some $k_1 \in S$. Thus $ks = kk_1 x \in \langle x\rangle$. Choose $s \in \langle x\rangle$ and $1 \in S$.

$$s + 1 = \begin{cases} \infty & \text{if } 1 \neq s \\ s & \text{if } 1 = s \end{cases}$$

Then . Thus $s + 1 \in \langle x\rangle$. Therefore $\langle x\rangle$ is a double ideal in S. But $x \neq \infty \in \langle x\rangle$. so $\langle x\rangle \neq \{\infty\}$. Thus $\langle x\rangle = S$ since S is double ideal free. Therefore $1 \in \langle x\rangle$ so $x^{-1} \in S$. Since $x \neq \infty$ was arbitrary in S, every non ∞ element in S has a multiplicative inverse. Thus S is a semifield with the almost trivial structure. By Proposition 4.1.4 S is congruence-free. This finishes the proof for necessity.

Now to prove sufficiency suppose that S is a type II semiring. Then by Proposition 1.2.1 S is double ideal free. Also by Theorem 1.1.3 QS is congruence-free. #

Now let S be a commutative MC semiring with 1 which admits a proper double ideal. In Section 1.2 we proved that S has a maximum proper double ideal M and we showed that S/M is MC. In fact S/M is an MC commutative semiring with 1 and with ∞. As shown before S is not ingeneral an ∞-semifield. But the results of this chapter enable us to give the following characterization :

4.1.17 Theorem. Let S be as described above

Then : 1) If S/M is finite then S/M is an ∞-semifield with the trivial or almost trivial structure.

2) If the quotient ∞-semifield of S/M is congruence-free then S/M is congruence free and thus S/M is a ∞-semifield with the trivial or almost trivial structure.

3) If S/M has the almost trivial structure then S/M is an ∞-semifield. and S/M is congruence-free.

4) If S/M has the trivial structure then S/M is a ∞-semifield.

5) If $1 + 1 = \infty$ in S/M then S/M is an ∞-semifield.

Proof : 1) Suppose S/M is finite By Corollary 4.1.10 S/M is an ∞-semifield with the trivial or almost trivial structure.

2) Suppose the quotient ∞-semifield Q of S/M is congruence-free. Obviously S/M is double ideal free. Therefore by Theorem 4.1.15 S/M is congruence free. Thus by Theorem 4.1.12 S/M = $\{1, \infty\}$ with the trivial structure or S/M is an ∞-semifield with the almost trivial structure.

3) Suppose that S/M has the almost trivial structure. Choose $x \neq \infty \in S/M$. By the proof of Theorem 4.1.16 $\langle x \rangle$ is a double ideal in S/M which is double ideal free. $\langle x \rangle \neq \{\infty\}$ since $x \in \langle x \rangle$. Thus $\langle x \rangle = S/M$ so $x^{-1} \in S/M$ As x was an arbitrary non-infinity element in S ,S/M is an ∞-semifield. Thus by Proposition 4.1.4 S/M is

congruence free.

4) Suppose that S/M has the trivial structure. Again for all $x \neq \infty \in S/M$, $\langle x \rangle$ is a double ideal so $1 \in \langle x \rangle$. Thus $x^{-1} \in S/M$. Thus S/M is a semifield.

5) Suppose that $1 + 1 = \infty$ in S/M. Then the quotient - semifield Q of S/M has the property that $1+1=\infty$ Thus Q and hence S/M has the trivial structure. Thus S/M is an ∞-semifield by 4) above. #

Thus it can be seen that the theory of type II semirings and the theory of double ideals are closely related. It should be noted that Theorem 4.1.17 gives some fascinating clues into the structure of commutative MC semirings with 1, S, which admit proper double ideals. For example. Suppose that $||S \setminus M||$ is finite. Then by part 1) $a \in M$ iff $a^{-1} \notin S$, and for all $x, y \in S$ such that $x \neq y$, $x + y \in M$. Thus $\frac{1}{x + y}$ does not exist.

As a final application we prove the analogue of Theorem 3.1.6.

<u>4.1.18 Theorem</u> : Let S be a commutative semiring with 1 and ∞, and $\sim$ a maximal proper congruence on S. Then $\frac{S}{\sim}$ is an ∞-semifield with the trivial or almost trivial structure.

Proof : By Theorem 3.1.5 $\frac{S}{\sim}$ is congruence-free. Now apply Theorem 4.1.13. #