CHAPTER II

PROPERTIES OF DIFFERENCE RINGS

In this chapter we consider semirings of order greater than one and we shall study the relationship between properties of an A.C. semiring and properties of its difference ring.

The first problem that we shall study is this chapter is to find necessary and sufficient conditions on an A.C. semiring S so that D(S) has a multiplicative identity.

It is possible that an A.C. semiring S has no multiplicative identity but D(S) has a multiplicative identity. We now give an example.

Example 2.1. Let $S = \{n \in \mathbb{Z} \mid n \geq 2\}$ with the usual addition and multiplication. Then S is an A.C. semiring with no multiplicative identity. Claim that $D(S) \cong \mathbb{Z}$. Define $i: D(S) \longrightarrow \mathbb{Z}$ as follows: for $\alpha \in D(S)$, choose $(x,y) \in \alpha$ and define $i(\alpha) = x-y$, Suppose that $(x',y') \in \alpha$. Then $x+y' = x'+y$ so in $\mathbb{Z}$, $x-y = x'-y'$. Hence i is well-define. To show that i is onto. Let $z \in \mathbb{Z}$. Then $|z|+z+2 \in S$ and $i(z+|z|+2, |z|+2) = z+|z|+2-|z|-2 = z$. Hence i is onto. To show that i is 1-1, suppose $i(\alpha) = i(\beta)$. Choose $(x,y) \in \alpha$, $(x',y') \in \beta$. Then $x-y = x'-y'$ so $x+y' = x'+y$. Hence i is 1-1. Lastly, we must show that i is a homomorphism. Let $\alpha, \beta \in D(S)$. Choose $(x,y) \in \alpha$, $(x',y') \in \beta$. Then $i(\alpha+\beta) = i([(x+x',y+y')]) = x+x'-y-y' = x-y+x'-y' = i(\alpha)+i(\beta)$.

$i(\alpha\beta) = i([(xx'+yy', xy'+x'y)]) = xx'+yy'-xy'-x'y = (x-y)(x'-y') = i(\alpha)i(\beta)$. Therefore $D(S) \cong \mathbb{Z}$. Hence $D(S)$ has a multiplicative identity.

From Proposition 1.19, we get that if S is an A.C. semiring with multiplicative identity 1, then 1 is the multiplicative identity of D(S).

We shall now give a property on an A.C. semiring S which is a necessary and sufficient condition that D(S) has a multiplicative identity.

Definition 2.2. Let S be a semiring. Then S is called unitive if and only if there exist $a,b \in S$ such that for all $x,y \in S$, $ax+by+y = ay+bx+x$.

Proposition2.3. Let S be a semiring with multiplicative identity 1. Then S is unitive.

Proof. Let $a = 1+1$, $b = 1$. Then for all $x,y \in S$, $ax+by+y = (1+1)x+1y+y = x+x+y+y = (1+1)y+x+x = ay+bx+x$. Hence S is unitive. #

As a result, $(\mathbb{Q}^+, \min \text{ or } \max, \cdot)$ is a unitive semiring which is not A.C. whereas $\mathbb{Q}^+$ with the usual addition and multiplication is an A.C. unitive semiring.

Theorem 2.4. Let S be an A.C. semiring. Then D(S) has a multiplicative identity if and only if S is unitive.

Proof. Assume D(S) has a multiplicative identity 1. Let $a,b \in S$ be such that $1 = [(a,b)] = a-b$. For all $x,y \in S$, $x-y = (x-y)(a-b) = ax+by-ay-bx$ so $ax+by+y = ay+bx+x$.

Conversely, suppose that S is unitive. Then there exist $a,b \in S$ such that $ax+by+y = ay+bx+x$ for all $x,y \in S$. Hence in D(S) $ax+by-ay-bx = x-y$. Thus $(a-b)(x-y) = x-y$ for all $x,y \in S$. Therefore $a-b$ is a multiplicative identity in D(S). #

Remark 2.5. Let S be an A.C. unitive semiring and let $a,b \in S$ satisfy the property that for all $x,y \in S$, $ax+by+y = ay+bx+x$. If $c,d \in S$ also satisfy this property, then $a+d = b+c$.

Proof. By Theorem 2.4, $a-b$ and $c-d$ are multiplicative identities in D(S). Then $a-b = c-d$. Hence $a+d = b+c$. #

Next we shall consider the problem of finding necessary and sufficient condition on an A.C. semiring S so that D(S) is an integral domain.

Theorem 2.6. Let S be an A.C. semiring. Then D(S) is 0-M.C. if and only if S is S.M.C..

See [1], page 42.

Note that D(S) may be 0-M.C. but not an integral domain since it may not contain a multiplicative identity. We shall give an example.

Example 2.7. Let $S = \{2n \mid n \in Z^+\}$ with the usual addition and multiplication. Then $D(S) \cong \{2n \mid n \in Z\}$ (using the same proof as in Example 2.1) which is 0-M.C. but D(S) has no multiplicative identity, so it is not an integral domain.

Theorem 2.8. Let S be an A.C. semiring. Then D(S) is an integral domain if and only if S is S.M.C. and unitive.

Proof. It follows directly from Theorem 2.4 and Theorem 2.6. #

Corollary 2.9. Let S be an A.C. semiring with 1. Then D(S) is an integral domain if and only if S is S.M.C..

Now we shall study the problem of finding necessary and sufficient condition on an A.C. semiring S so that D(S) is a field.

Using the same proof as in Example 2.1, we can show that $D(\mathbb{Q}^+) \cong D(\mathbb{Q}_0^+) \cong \mathbb{Q}$ so we have examples of A.C. semirings with the property that D(S) is a field. $\mathbb{Q}^+$ and $\mathbb{Q}_0^+$ are very special semirings, they are a ratio semiring and a 0-semifields respectively. It is possible for an A.C. semiring S which is neither a ratio semiring nor a 0-semifield to have the property that D(S) is a field. In fact, S need not even have a multiplicative identity and still D(S) can be a field.

Example 2.10. Let $S = [2, \infty)$ with the usual addition and multiplication. Then S is A.C. semiring without a multiplicative identity. Using the same proof as in Example 2.1, we can show that $D(S) \cong \mathbb{R}$ which is a field.

Note. Since a field contains a multiplicative identity, If an A.C. semiring S has the property that D(S) is a field then S must be unitive

We shall now give a property on a unitive semiring S which is a necessary and sufficient condition that D(S) is a field when S is A.C..

<u>Definition 2.11</u>. Let S be a unitive semiring. S is called <u>exact</u> if and only if there exist $c,d \in S$ such that

i) for all $x,y \in S$, $cx+dy+y = cy+dx+x$ and

ii) for all distinct $x,y \in S$ there exist $u,v \in S$ such that $d+xu+yv = c+xv+yu$.

<u>Theorem 2.12</u>. Let S be an A.C. semiring. Then D(S) is a field if and only if S is exact.

<u>Proof</u>. Assume D(S) is a field. Then S is unitive so there exist $a,b \in S$ such that for all $x,y \in S$, $ax+by+y = ay+bx+x$. Furthermore, $a-b = 1$ in D(S). Let $x,y \in S$ be distinct. Then $x-y \in D(S)-\{0\}$, so there exist $u,v \in S$ such that $(x-y)(u-v) = (a-b)$ so $xu+yv-xv-yu = a-b$. Therefore $xu+yv+b = xv+yu+a$. Hence S is exact.

Conversely, assume S is exact. Then there exist $a,b \in S$ such that for all $x,y \in S$, $ax+by+y = ay+bx+x$ and for all distinct $x,y \in S$ there exist $u,v \in S$ such that $b+xu+yv = a+xv+yu$. Furthermore, D(S) is a ring with $1 = [(a,b)]$. Let $\alpha \in D(S)-\{0\}$. Choose $(x,y) \in \alpha$. Hence $x \neq y$. Then there exist $u,v \in S$ such that $b+xu+yv = a+xv+yu$. Therefore $1 = a-b = xu+yv-xv-yu = (x-y)(u-v)$. Hence D(S) is a field. #

From the above theorem, we see that $\mathbb{Q}^+$, $\mathbb{Q}_0^+$, $[2, \infty)$ with the usual addition and multiplication are A.C. exact semirings.

Now we shall give an example of an exact semiring which is not A.C..

<u>Example 2.13</u>. Let $K = \{0,1\}$. Define + and $\cdot$ on K as follows: $1+1 = 0+1 = 1+0 = 1$, $0+0 = 0$ and $0\cdot0 = 0\cdot1 = 0$, $1\cdot1 = 1$. Then $(K,+,\cdot)$ is a semiring with multiplicative identity 1.

Let $b = 1$, $a = 1+1 = 1$. Then by the same proof as in Proposition 2.3, $ax+by+y = ay+bx+x$. Hence K is unitive.
Claim that K is exact, Let $x,y \in K$. Choose $u = v = 0$ then $b+xu+yv = b = a = a+xv+yu$. Therefore K is exact.
Clearly K is not A.C. since $0+1 = 1+1$ but $1 \neq 0$. #

If an A.C. semiring S has a multiplicative identity, then we can give a simple property on S which is a necessary and sufficient condition that D(S) is a field.

<u>Definition 2.14.</u> Let S be a semiring with multiplicative identity 1. Then S is called <u>total</u> if and only if for all distinct $x,y \in S$ there exist $a,b \in S$ such that $1+ay+bx = ax+by$.

Now we shall give two example of semirings one of which is total but not A.C. and the other is both total and A.C..

<u>Example 2.15.</u> In Example 2.13, K is a semiring with 1.
Claim that K is total. Let $x,y \in K$ be distinct. Without loss of generality assume that $x = 1$, $y = 0$. Choose $a = b = 1$.
Then $1+ay+bx = 1 = ax+by$. Therefore K is total.

<u>Example 2.16.</u> Let Q^+ with the usual addition and multiplication.
Claim that Q^+ is total. Let $x,y \in Q^+$ be such that $x \neq y$.
Without loss of generality assume that $x > y$. Choose $a = \frac{1}{x-y}+1$ and $b = 1$. Then $1+ay+bx = 1+\frac{y}{x-y}+y+x = \frac{x}{x-y}+x+y = ax+by$.
Hence Q^+ is total.

<u>Theorem 2.17.</u> Let S be an A.C. semiring with multiplicative identity. Then D(S) is a field if and only if S is total.

Proof. Assume that S is total. We must show that D(S) is a field. Let $\alpha \in D(S)-\{0\}$. Choose $(x,y) \in \alpha$, so $x \neq y$. Since S is total, there exist $a,b \in S$ such that $1+ay+bx = ax+by$. Hence $1 = ax+by-ay-ax = (a-b)(x-y)$. Therefore D(S) is a field.

Conversely, assume that D(S) is a field. Let $x,y \in S$ be distinct. Hence $x-y \in D(S)-\{0\}$, so there exist $a,b \in S$ such that $(x-y)(a-b) = 1$. Then $ax+by-ay-bx = 1$. Thus $ax+by = 1+ay+bx$. #

Corollary 2.18. Let S be an A.C. semiring with multiplicative identity. Then S is exact if and only if S is total.

Proposition 2.19. Let S be a semiring with multiplicative identity. If S is total, then S is exact.

Proof. Let 1 be the multiplicative identity of S. Let $a = 1+1$ and $b = 1$. Then $ax+by+y = ay+bx+x$. Let $x,y \in S$ be distinct. Since S is total, there exist $z,w \in S$ such that $1+zy+wx = zx+wy$. Also $1+1+zy+wx = 1+zx+wy$. Hence S is exact. #

The converse of this proposition is not always true as the following example shows.

Example 2.20. Let $S = [1,\infty)$. Define $\oplus$ and $\odot$ on S as follows: for $x,y \in S$, let $x \oplus y = \min\{x,y\}$ and $x \odot y = \max\{x,y\}$. It is easy to show that $(S,\oplus,\odot)$ is a semiring with multiplicative identity 1. Claim that S is exact. Let $a = b = 1$. Clearly, for all $x,y \in S$, $a\odot x \oplus b \odot y \oplus y = a \odot y \oplus b \odot x \oplus x$. For all $x,y,u,v \in S$, $a \oplus x \odot u \oplus y \odot v = 1 = b \oplus x \odot v \oplus y \odot u$. Hence we have the claim. To show that S is not total, note that there do not exist $z,w \in S$ such that $1 \oplus 2 \odot z \oplus 3 \odot w = 2 \odot w \oplus 3 \odot z$.

Proposition 2.21. Let E be a ratio semiring. Then E is exact if and only if E is total.

Proof. It suffices to show that E is exact implies E is total. Let 1 be the multiplicative identity of E and let $a,b \in E$ be such that $ax+by+y = ay+bx+x$ and for all distincts $x,y \in E$ there exist $u,v \in E$ such that $b+xu+yv = a+xv+yu$. Since $1,1+1 \in E$. Then $a1+b(1+1)+(1+1) = a(1+1)+b1+1$. Hence $a+b+b+1+1 = a+a+b+1$ (*). Let $x,y \in E$ be distinct. Then there exist $u,v \in E$ such that $b+xu+yv = a+xv+yu$. Let $z = (\frac{1+b+a}{y}+\frac{a}{x}+u)$ and $w = (\frac{1+b+a}{y}+\frac{a}{x}+v)$. Therefore $1+zy+wx = 1+1+b+a+\frac{ay}{x}+uy+\frac{x}{y}(1+b+a)+a+xv =$ $1+1+b+a+b+xu+yv+\frac{ay}{x}+\frac{x}{y}(1+b+a)$ since $a+xv+yu = b+xu+yv$. $zx+wy = (\frac{1+b+a}{y}+\frac{a}{x}+u)x+(\frac{1+b+a}{y}+\frac{a}{x}+v)y = \frac{x}{y}(1+b+a)+a+ux+1+b+a+\frac{ay}{x}+vy =$ $1+1+b+a+b+xu+yv+\frac{ay}{x}+\frac{x}{y}(1+b+a)$ by (*). Hence $1+zy+wx = zx+wy$. #

Theorem 2.22. Let E be a ratio semiring. Then we can embed E in a field if and only if E is precise and A.C..

See [1], pages 46-47.

We see from this theorem that the properties A.C. and precise are necessary and sufficient conditions for a ratio semiring to be embeddable in a field. However, if E is an A.C. and precise ratio semiring, then $D(E)$ may not be a field i.e. E may not be total. We shall now give two examples of this.

Remark 2.23. Let $Q_0^+[x] = \{$ nonzero polynomials $|$ coefficients belong to $Q_0^+\}$ and let $Q^+[x] = \{$polynomials $|$ coefficients belong to $Q^+\}$. Then $Q_0^+[x]$ and $Q^+[x]$ with the usual addition and multiplication are A.C. and M.C. semiring. Let $Q_0^+(x)$ denote $QR(Q_0^+[x])$ and $Q^+(x)$ denote $QR(Q^+[x])$.

Example 2.24. $Q^+(x)$ and $Q_0^+(x)$ are A.C. and precise ratio semirings whose difference rings are not fields.

Proof. We shall prove this for $Q_0^+(x)$, the same proof works for $Q^+(x)$. Define $i: Q_0^+(x) \longrightarrow Q(x)$ as follows: for $\alpha \in Q_0^+(x)$ choose $(p,q) \in \alpha$ and define $i(\alpha) = \frac{p}{q}$, cleary i is well-defined monomorphism. By Theorem 2.22, $Q_0^+(x)$ is precise and A.C.. Claim that $D(Q_0^+(x))$ is not a field. Define $f_2 : D(Q_0^+(x)) \longrightarrow Q$ as follows: for $\alpha \in D(Q_0^+(x))$, choose $(\alpha_1, \alpha_2) \in \alpha$ and choose $(p(x),q(x)) \in \alpha_1$ $(r(x),s(x)) \in \alpha_2$. Define $f_2(\alpha) = \frac{p(2)}{q(2)} - \frac{r(2)}{s(2)}$. To show that f_2 is well-define, first let $(p'(x),q'(x)) \in \alpha_1$ also. Then $p(x)q'(x) = p'(x)q(x)$. Thus $p(2)q'(2) = p'(2)q(2)$. Since $q(x), q'(x) \in Q_0^+[x]$. Therefore $\frac{p(2)}{q(2)} = \frac{p'(2)}{q'(2)}$. If $(\alpha_1', \alpha_2') \in \alpha$, choose $(p'(x),q'(x)) \in \alpha_1'$, and $(r'(x),s'(x)) \in \alpha_2'$ so $\alpha_1 - \alpha_2 = \alpha_1' - \alpha_2'$. Hence $\frac{p(x)}{q(x)} - \frac{r(x)}{s(x)} = \frac{p'(x)}{q'(x)} - \frac{r'(x)}{s'(x)}$. Therefore $\frac{p(2)}{q(2)} - \frac{r(2)}{s(2)} = \frac{p'(2)}{q'(2)} - \frac{r'(2)}{s'(2)}$

Thus f_2 is well-defined. To show f_2 is a homomorphism. Let $\eta_1, \eta_2 \in D(Q_0^+(x))$. Choose $(\alpha_1, \alpha_2) \in \eta_1$, $(\beta_1, \beta_2) \in \eta_2$, $(p(x),q(x)) \in \alpha_1$, $(r(x),s(x)) \in \alpha_2$, $(p'(x),q'(x)) \in \beta_1$ and $(r'(x),s'(x)) \in \beta_2$. Then $f_2(\eta_1 + \eta_2) = f_2(\alpha_1 - \alpha_2 + \beta_1 - \beta_2) =$

$$f_2\left(\left(\frac{p(x)}{q(x)} + \frac{p'(x)}{q'(x)}\right) - \left(\frac{r(x)}{s(x)} + \frac{r'(x)}{s'(x)}\right)\right) = \left(\frac{p(2)}{q(2)} + \frac{p'(2)}{q'(2)}\right) - \left(\frac{r(2)}{s(2)} + \frac{r'(2)}{s'(2)}\right) =$$

$$\frac{p(2)}{q(2)} - \frac{r(2)}{s(2)} + \frac{p'(2)}{q'(2)} - \frac{r'(2)}{s'(2)} = f_2(\eta_1) + f_2(\eta_2).$$

$$f_2(\eta_1\eta_2) = f_2((\alpha_1 - \alpha_2)(\beta_1 - \beta_2)) = f_2((\alpha_1\beta_1 + \alpha_2\beta_2) - (\alpha_1\beta_2 + \alpha_2\beta_1))$$

$$= f_2\left(\left(\frac{p(x)}{q(x)}\frac{p'(x)}{q'(x)} + \frac{r(x)}{s(x)}\frac{r'(x)}{s'(x)}\right) - \left(\frac{p(x)}{q(x)}\frac{r'(x)}{q'(x)} + \frac{r(x)}{s(x)}\frac{p'(x)}{q'(x)}\right)\right)$$

$$\frac{p(2)}{q(2)}\frac{p'(2)}{q'(2)} + \frac{r(2)}{s(2)}\frac{r'(2)}{s'(2)} - \frac{p(2)}{q(2)}\frac{r(2)}{s(2)} - \frac{r(2)}{s(2)}\frac{p'(2)}{q'(2)} =$$

$$\left(\frac{p(2)}{q(2)} - \frac{r(2)}{s(2)}\right)\left(\frac{p'(2)}{q'(2)} - \frac{r'(2)}{s'(2)}\right) = f_2(\eta_1)f_2(\eta_2).$$

$x-2 \in D(Q_0^+(x))-\{0\}$. If $D(Q^+(x))$ is a field, then $\frac{1}{x-2} \in D(Q_0^+(x))$ so there exist $p(x), q(x), r(x), s(x) \in Q_0^+(x)$ such that

$\frac{1}{x-2} = \frac{p(x)}{q(x)} - \frac{r(x)}{s(x)}$. Hence $(x-2)\left(\frac{p(x)}{q(x)} - \frac{r(x)}{s(x)}\right) = 1$ so $1 = f_2(1) =$

$f_2\left((x-2)\frac{p(x)}{q(x)} - \frac{r(x)}{s(x)}\right) = f_2((x-2))f_2\left(\frac{p(x)}{q(x)} - \frac{r(x)}{s(x)}\right) = 0$.

Thus $1 = 0$, a contradiction. Hence $D(Q_0^+(x))$ is not a field.

We would now like to give a sufficient condition on a ratio semiring E which guarantees that E is total. To do this, we shall need the concept of a compatible partial order on a semiring.

Definition 2.25. A partial order on a semiring S is said to be compatible if and only if for all $x,y,a \in S$, $x \geqslant y$ implies $ax \geqslant ay$ and $a+x \geqslant a+y$.

Remark 2.26. Let S be an A.C. semiring satisfy the properties that if S has an additive identity 0, $x+y = 0$ if and only if $x = y = 0$ for all $x,y \in S$. Then S has a natural partial order denoted by $\leq_+$ given as follows: For $x,y \in S$, $x \leq_+ y$ if and only if either $x = y$ or there exists a $z \in S$ such that $y = x+z$.

1) $x \leq_+ x$

2) Assume that $x \leq_+ y$ and $y \leq_+ z$. If $x = y$ or $y = z$, then $x \leq_+ z$. Suppose $x \neq y$ and $y \neq z$. Then there exist $a,b \in S$ such that $y = x+a$ and $z = y+b$, so $z = x+a+b$. Hence $x \leq_+ z$.

3) Assume that $x \leq_+ y$ and $y \leq_+ x$. If $x = y$ then done. Suppose $x \neq y$. Then there exist $a,b \in S$ such that $y = x+a$, $x = y+b$. Thus $y = y+a+b$. Hence $a+b$ is an additive identity. We get that $a = b = 0$ then $x = y$, a contradiction since $x \neq y$.

Hence $\leq_+$ is a partial order. Claim that $\leq_+$ is a compatible.

Suppose $x \leq_+ y$. Let $a \in S$. If $x = y$ then $x+a = y+a$ and $xa = ya$. Hence $x+a \leq_+ y+a$ and $xa \leq_+ ya$. Suppose there exists a $z \in S$ such that $y = x+z$. Hence $y+a = x+a+z$ and $ya = xa+za$. Thus $x+a \leq_+ y+a$ and $xa \leq_+ ya$.

Note that an A.C. O-semifield which is not a field is an example of a semiring satisfying property above.
$\mathbb{Z}_o^+$ satisfies property above.

Theorem 2.27. Let S be a semiring satisfying Remark 2.26. Then the natural partial order is a total order if and only if $D(S) = S \cup \{0\} \cup (-S)$ where by $-S$ we mean $-f(S)$ ($f: S \longrightarrow D(S)$ is the natural embedding).

Proof. Assume that the natural partial order is a total order. We must show that $D(S) = S \cup \{0\} \cup (-S)$.
Clearly $S \cup \{0\} \cup (-S) \subseteq D(S)$. We must show that $D(S) \subseteq S \cup \{0\} \cup (-S)$.
Let $\alpha \in D(S)$ and choose $(x,y) \in \alpha$. Since $\leq_+$ is a total order, either $x = y$ or $x = y+a$ or $y = x+a$ for some $a \in S$.

Case 1. $x = y$, so $\alpha = 0$.

Case 2. $x = y+a$ for some $a \in S$, so $x-y = a \in S$.

Case 3. $y = x+a$ for some $a \in S$, so $x-y = -a \in (-S)$.

Hence $D(S) = S \cup \{0\} \cup (-S)$.

Conversely, suppose that $D(S) = S \cup \{0\} \cup (-S)$.
We must show that $\leq_+$ is a total order. Let $x,y \in S$. If $x = y$ then $x \leq_+ y$. Suppose $x \neq y$, so $x-y \in D(S)-\{0\}$.

Case 1. $x-y \in S$ then there exists an $a \in S$ such that $x-y = a$. Hence $x = y+a$ so $y \leq_+ x$.

Case 2. $x-y \in (-S)$ then there exists an $a \in S$ such that $x-y = -a$. Hence $y = x+a$ so $x \leq_+ y$. #

Theorem 2.28. Let E be an A.C. ratio semiring. If $\leq_+$ on E is a total order then E is total.

Proof. It suffices to show that if E is totally ordered by $\leq_+$, then D(E) is a field. Let $1 \in E$ be the multiplicative identity, so 1 is the multiplicative identity of D(E).
Let $\alpha \in D(E)-\{0\}$. We must show that the multiplicative inverse of α exists. Choose $(x,y) \in \alpha$, so $x-y \in E$ or $x-y \in (-E)$ (by Theorem 2.27).
Case 1. $x-y \in E$, then the multiplicative inverse exists.
Case 2. $x-y \in (-E)$, then there exists an $a \in E$ such that $(x-y) = -a$. Hence $-(x-y) = a \in E$. Therefore there exists a $b \in E$ such that $1 = ab = -(x-y)b = (x-y)(-b)$. Hence there exists an $-b \in (-E)$ such that $(-b)(x-y) = 1$. #

We see that if the natural partial order on an A.C. ratio semiring E is a total order then E is total. However, if an A.C. ratio semiring has a compatible total order which is not the natural partial order then E may not be total. We now give an example of this.

Example 2.29. Let $S = \mathbb{Q}^+[x]$ with the usual addition and multiplication. Then $(S,+,\cdot)$ is an M.C. and A.C. semiring. Let $f,g \in \mathbb{Q}^+[x]$, then $f(x) = a_m x^m + \ldots + a_o$, $g(x) = b_n x^n + \ldots + b_o$ for some $a_o, a_1, \ldots, a_m$, $b_o, b_1, \ldots, b_n \in \mathbb{Q}^+$. We say that $f \leq g$ if and only if either

1) $f = g$, or
2) $m < n$, or
3) $m = n$ and there exists a $j \leq n$ such that $a_i = b_i$ for all $i > j$ and $a_j < b_j$.

Claim that $\leq$ is a partial order. Let $f,g,h \in Q^{+}[x]$ where $f(x) = a_m x^m + \ldots + a_o$, $g(x) = b_n x^n + \ldots + b_o$ and $h(x) = c_l x^l + \ldots + c_o$.

1) Clearly $f \leq f$,

2) if $f \leq g$ and $g \leq f$, then $m \leq n$ and $n \leq m$ so $m = n$. If $f = g$ then done. Suppose $f \neq g$. Since $f < g$, let $j \in \{0,1,\ldots,m\}$ be such that $a_i = b_i$ for all $i > j$ and $a_j < b_j$. Since $g < f$, let $k \in \{0,1,\ldots,m\}$ be such that $a_l = b_l$ for all $l > k$ and $b_k < a_k$. If $k < j$ then $a_j = b_j$, a contradiction. If $j < k$ then $a_k = b_k$, a contradiction. Hence $j = k$, a contradiction. Thus $f = g$.

3) Suppose $f \leq g$ and $g \leq h$.

<u>Case 1</u>. $m < n$ and $n < l$, so $m < l$. Hence $f \leq h$.

<u>Case 2</u>. $m = n$ and $n < l$, or $m < n$ and $n = l$, so $m < l$. Hence $f \leq h$.

<u>Case 3</u>. $m = n$ and $n = l$ then $m = n = l$. If $f = g$ or $g = h$ then $f = h$. Suppose $f \neq g$ and $g \neq h$. Since $f \leq g$, let $j \in \{0,1,\ldots,m\}$ be such that $a_i = b_i$ for all $i > j$ and $a_j < b_j$. Since $g \leq h$,, let $k \in \{0,1,\ldots,m\}$ be such that $b_l = c_l$ for all $l > k$ and $b_k < c_k$. If $j \geq k$ then $a_j < b_j = c_j$ so $f \leq h$. If $j < k$ then $a_k = b_k < c_k$ so $f \leq h$.

Therefore $\leq$ is a partial order. Claim that $\leq$ is compatible. Let $f,g \in Q^{+}[x]$ and $h \in Q^{+}[x]$ be arbitrary.

Let $f(x) = a_m x^m + \ldots + a_o$, $g(x) = b_n x^n + \ldots + b_o$ and $h(x) = c_l x^l + \ldots + c_o$. Suppose $f \leq g$. Then $m \leq n$.

<u>Case 1</u>. $m < n$ then $m+l < n+l$ also $fh = gh$, and clearly $f+h \leq g+h$.

<u>Case 2</u>. $m = n$. If $f = g$ then $fh = gh$ and $f+h = g+h$. Suppose $f \neq g$. Since $f \leq g$ let $j \in \{0,\ldots,m\}$ be such that $a_i = b_i$ for all $i > j$ and $a_j < b_j$. Clearly $f+h \leq g+h$. Let u be the coefficient of x^{j+l} in fh, v be the coefficient of x^{j+l} in gh. Then $u < v$. Hence $fh \leq gh$.

Therefore $\leq$ is a compatible. To show $\leq$ is a total order, let $f, g \in \mathbb{Q}^+[x]$ be such that $f(x) = a_m x^m + \ldots + a_o$ and $g(x) = b_n x^n + \ldots + b_o$. If $f \neq g$, we must show that $f < g$ or $g < f$. Suppose $g \nless f$, so $m \leq n$.

Case 1. $m < n$ then clearly $f < g$.

Case 2. $m = n$ so $b_n = a_m$ or $a_m < b_m$. If $a_m < b_n$ then $f < g$. Suppose $a_m = b_n$. Since $g \nless f$ then $a_{m-1} \leq b_{n-1}$. If $a_{m-1} < b_{n-1}$ then $f < g$. Assume $a_{m-1} = b_{n-1}$. Continue this process. We get that there exists a $j \in \{0, 1, \ldots, m\}$ such that $a_j < b_j$. Therefore $f < g$. Hence $\leq$ is total partial order in $\mathbb{Q}^+[x]$.

Define $\leq$ on $\mathbb{Q}^+(x)$ as follows: Let $\alpha, \beta \in \mathbb{Q}^+(x)$ choose $(f,g) \in \alpha$ and $(h,k) \in \beta$. We say that $\alpha \leq \beta$ if and only if $fk \leq gh$. Claim that $\leq$ is a total order. First we shall show that for $f, g, h \in \mathbb{Q}^+[x]$ if $fh \leq gh$ then $f \leq g$. To prove this, let $f(x) = a_m x^m + \ldots + a_o$, $g(x) = b_n x^n + \ldots + b_o$, and $h(x) = c_l x^l + \ldots + c_o$. If $fh = gh$ then $f = g$. Hence $f \leq g$. Assume $fh \neq gh$.

Case 1. $m + l < n + l$ then $m < n$. Hence $f \leq g$.

Case 2. $m + l = n + l$ then $m = n$. If $a_m c_l < b_n c_l$ then $a_m < b_n$ so $f \leq g$. Assume $a_m c_l = b_n c_l$. If $a_m c_{l-1} + a_{m-1} c_l < b_n c_{l-1} + b_{n-1} c_l$ then $a_{m-1} c_l < b_{n-1} c_l$ (since $a_m c_l = b_n c_l$ implies $a_m = b_n$) hence $a_{m-1} < b_{n-1}$, so $f \leq g$.

Assume $a_m c_{l-1} + a_{m-1} c_l = b_n c_{l-1} + b_{n-1} c_l$. Then $a_{m-1} = b_{n-1}$. If $a_m c_{l-2} + a_{m-1} c_{l-1} + a_{m-2} c_l < b_n c_{l-2} + b_{n-1} c_{l-1} + b_{n-2} c_l$ then $a_{m-2} < b_{n-2}$, so $f \leq g$. Continue this process. We get that there exists a $j \in \{0, 1, \ldots, m\}$ such that $a_j < b_j$. Then $f \leq g$.

Next to show that $\leq$ is well-defined. If $(f_1,g_1)\in\alpha$ and $(h_1,k_1)\in\beta$ also then $fg_1 = gf_1$ and $hk_1 = kh_1$. Since $fk \leq gh$ so $fkf_1k_1 \leq ghf_1k_1 = fg_1kh_1 = fkg_1h_1$, so $f_1k_1 \leq g_1h_1$. Thus $\leq$ is well-defined. Let $\alpha, \beta, \gamma \in \mathbb{Q}^+(x)$. Choose $(f,g)\in\alpha$, $(h,k)\in\beta$ and $(i,j)\in\gamma$.

1) Since $fg = fg$ then $\alpha \leq \alpha$.

2) If $\alpha \leq \beta$ and $\beta \leq \alpha$ then $fk \leq gh$ and $hg \leq kf$, so $fk = gh$. Hence $\alpha = \beta$.

3) If $\alpha \leq \beta$ and $\beta \leq \gamma$ then $fk \leq gh$ and $hj \leq ki$, so $fkj \leq ghj \leq gki$, hence $fjk \leq gik$, so $fj \leq gi$, therefore $\alpha \leq \gamma$.

4) Suppose $\alpha \neq \beta$. Then $fk \neq gh$. Since $\leq$ is a total order on $\mathbb{Q}^+[x]$, either $fk < gh$ or $gh < fk$. If $fk < gh$ then $\alpha < \beta$. If $gh < fk$ then $\beta < \alpha$.

5) Suppose $\alpha \leq \beta$ then $fk \leq gh$. Hence $fkij \leq ghij$. Therefore $\alpha\gamma = \beta\gamma$ and $fkj^2 + gkij \leq hgj^2 + gkij$. Hence $\alpha + \gamma = \beta + \gamma$.

Hence $\leq$ is a total order and $\leq$ is compatible.

By Example 2.24 $D(\mathbb{Q}^+(x))$ is not a field. #

Proposition 2.30. Let S be an A.C. semiring. If S has no multiplicative zero then S has no additive identity.

Proof. Let $0 \in S$ be an additive identity. Hence $0+xx = xx = x(0+x) = x0+xx$ for all $x \in S$ which implies that $0 = x0$ for all $x \in S$. Therefore 0 is a multiplicative zero. #

Proposition 2.31. Let S be an A.C. and M.C. semiring. If S is totally ordered with respect to $\leq_+$ then QR(S) is totally ordered with respect to $\leq_+$.

Proof. Assume that S is totally ordered with respect to $\leq_+$. Must show that QR(S) is totally ordered with respect to $\leq_+$. Let $\alpha, \beta \in$ QR(S) be such that $\alpha \neq \beta$. Choose $(a,b) \in \alpha$ and $(x,y) \in \beta$ then $ay \neq bx$. Since S is totally ordered with respect to $\leq_+$, $ay = bx+z$ or $bx = ay+z$ for some $z \in S$. Without loss of generality, suppose $ay = bx+z$ for some $z \in S$. Hence $\frac{a}{b} = \frac{x}{y} + \frac{z}{yb}$ in QR(S). Therefore $\beta \leq_+ \alpha$. #

We now give an example to show that the converse of Proposition 2.31 is not always true.

Example 2.32. Let $S = [1, \infty)$ with the usual addition and multiplication. Then S is an A.C. and M.C. semiring with no multiplicative zero. Note that S is not totally ordered w.r.t.$\leq_+$ since $1, 1.5 \in S$ but 1 and 1.5 are not comparable with respect to $\leq_+$. Claim QR(S) $\cong \mathbb{R}^+$. Define $i: QR(S) \longrightarrow \mathbb{R}^+$ as follows: for $\alpha \in$ QR(S). Choose $(x,y) \in \alpha$ and define $i(\alpha) = \frac{x}{y}$. To show that i is well-defined, let $(x',y') \in \alpha$ also. Then $xy' = x'y$ so $\frac{x}{y} = \frac{x'}{y'}$. To show that i is 1-1, let $\alpha, \beta \in$ QR(S) be such that $i(\alpha) = i(\beta)$. Choose $(x,y) \in \alpha$, $(x',y') \in \beta$. Then $\frac{x}{y} = \frac{x'}{y'}$ so $xy' = x'y$. Therefore $\alpha = \beta$. To show that i is onto, let $r \in \mathbb{R}^+$. If $r \geq 1$ then $i([(r,1)]) = \frac{r}{1} = r$. If $r < 1$ then $\frac{1}{r} > 1$. Hence $i([(1,\frac{1}{r})]) = \frac{1}{\frac{1}{r}} = r$. Thus i is onto. To show that i is a homomorphism, let $\alpha, \beta \in$ QR(S). Choose $(a,b) \in \alpha$ and $(x,y) \in \beta$.
$i(\alpha + \beta) = i([(ay+bx, by)]) = \frac{ay+bx}{by} = \frac{a}{b} + \frac{x}{y} = i(\alpha) + i(\beta)$ and $i(\alpha\beta) = i([(ax,by)]) = \frac{ax}{by} = \frac{a}{b} \cdot \frac{x}{y} = i(\alpha)\, i(\beta)$. Hence we have the claim. Since $\mathbb{R}^+$ is a totally ordered with respect to $\leq_+$, we get that QR(S) is totally ordered with respect to $\leq_+$.

Corollary 2.33. Let S be an A.C. and M.C. semiring. If S is totally ordered with respect to $\leq_+$, then D(QR(S)) is a field.

Proof. Use Proposition 2.31 and Theorem 2.28. #

We now give another sufficient condition on a ratio semiring E which guarantees that E is total.

Proposition 2.34. Let E be a ratio semiring. If 1+1 = 1 then E is total.

Proof. Let $x, y \in E$. Then $1+x\cdot\frac{1}{x}+y\frac{1}{y} = 1+1+\frac{y}{x} = 1+\frac{y}{x} = x\frac{1}{x}+y\frac{1}{x}$. Hence E is total. #

Example 2.35. $(\mathbb{Q}^+,\min,\cdot)$, $(\mathbb{Q}^+,\max,\cdot)$, $(\mathbb{R}^+,\min,\cdot)$ and $(\mathbb{R}^+,\max,\cdot)$ are all total ratio semirings.

We see that on a ratio semiring, there are three important properties relating to the difference ring, A.C., precise and total. We would now like to study the relationship between these three properties on a ratio semiring.

Let S be an A.C. and total ratio semiring. Then D(S) is a field. Hence S is precise.

There exist ratio semirings which are not A.C. and not precise but are total as the following example shows.

Example 2.36. Consider $\mathbb{Q}^+\times\mathbb{Q}^+$ with the usual multiplication. Define $(a,b)+(c,d) = (\min\{a,c\},\min\{b,d\})$. Then $(\mathbb{Q}^+\times\mathbb{Q}^+,+,\cdot)$ is a ratio semiring. Since $(1,1)+(2,2) = (1,1) = (1,1)+(3,3)$ and $(2,2)\neq(3,3)$, we get that $\mathbb{Q}^+\times\mathbb{Q}^+$ is not A.C..

Since $(1,1)+(1,2)(2,1) = (1,2)+(2,1)$ and $(1,2) \neq (1,1) \neq (2,1)$, we get that $Q^+ \times Q^+$ is not precise. **Since** $(1,1)+(1,1) = (1,1)$, we get that $Q^+ \times Q^+$ is total.

There exist ratio semirings which are neither A.C., no precise nor total as the following example shows.

Example 2.37. Consider $Q^+ \times Q^+$ with the usual multiplication define $(a,b)+(c,d) = (a+c, \min\{b,d\})$. Then $(Q^+ \times Q^+, +, \cdot)$ is a ratio semiring. Since $(1,1)+(1,3) = (2,1) = (1,1)+(1,2)$ and $(1,3) \neq (1,2)$, $Q^+ \times Q^+$ is not A.C..
Since $(1,1)+(1,2)(2,1) = (3,1) = (1,2)+(2,1)$ and $(1,2) \neq (1,1) \neq (2,1)$, $Q^+ \times Q^+$ is not precise. If there exist (y_1,y_2) and $(x_1,x_2) \in Q^+ \times Q^+$ such that
$(1,1)+(1,2)(y_1,y_2)+(1,3)(x_1,x_2) = (1,2)(x_1,x_2)+(1,3)(y_1,y_2)$.
Then $(1,1)+(y,2y_2)+(x_1,3x_2) = (x_1 2x_2)+(y_1,3y_2)$ so
$(x_1+y_1, \min\{2x_2,3y_2\}) = (1+y_1+x_1, \min\{1,2y_2,3x_2\})$, a contradiction.
Thus $Q^+ \times Q^+$ is not total.

Q^+ with the usual addition and multiplication is a ratio semiring and Q^+ is A.C., precise and total.

$Q^+(x)$ with the usual addition and multiplication is a ratio semiring and $Q^+(x)$ is A.C., precise but not total.

$(Q^+, \min, \cdot)$ is a ratio semiring which is precise, total but not A.C..

Every ratio semiring which is A.C. and total must be precise.

$Q^+ \times Q^+$ with the usual addition and multiplication is A.C. but not precise. (Since $(1,1)+(1,2)(2,1) = (1,2)+(2,1)$ and $(1,2) \not\leq (1,1)$, $(2,1) \not\leq (1,1)$, we see that $Q^+ \times Q^+$ is not precise). Hence it is not total since A.C. and total imply precise.

$Q^+ \times Q^+$ in Example 2.36 is not A.C. and not precise but total.

$Q^+ \times Q^+$ in Example 2.37 is neither A.C. nor precise nor total.

An open problem is the following: Does there exist a ratio semiring which is precise but neither A.C. nor total

We shall now give a necessary and sufficient condition which guarantees that a ratio semiring is precise. Again we shall need a natural partial order, but this time it will be a natural partial order on a ratio semiring with the poperty that $1+1 = 1$.

Remark 2.38. Let E be a ratio semiring such that $1+1 = 1$. Then $(E, +)$ is a band so E has a natural partial order given as follows: if $x, y \in S$ say that $x \leq y$ if and only if $x+y = y$.

Proposition 2.39. Let E be a ratio semiring such that $1+1 = 1$. Then the natural partial order is a total order if and only if E is precise.

Proof. Assume the natural partial order is a total order. We must show that E is precise. Let $u, v \in E$ be such that $1+uv = u+v$. We must show that $u = 1$ or $v = 1$. Since $u, v \in E$ Then either $u \leq v$ or $v \leq u$. Without loss of generallity, assume $u \leq v$. Then $u+v = v$.

Case 1. $uv \leq 1$. Then $1+uv = 1$. Hence $v = 1$.

Case 2. $1 \leq uv$. Then $1+uv = uv$. Hence $uv = v$. Thus $u = 1$.

Conversely, assume that E is precise. Let $x,y \in E$.
Then $1+(1+\frac{y}{x})(1+\frac{x}{y}) = 1+\frac{x}{y}+\frac{y}{x}$ and $1+\frac{y}{x}+1+\frac{x}{y} = 1+\frac{x}{y}+\frac{y}{x}$, so $1+(1+\frac{y}{x})(1+\frac{x}{y}) = (1+\frac{y}{x})+(1+\frac{x}{y})$. Since E is precise, $1+\frac{y}{x} = 1$ or $1+\frac{x}{y} = 1$. If $1+\frac{y}{x} = 1$, then $x+y = x$ so $y \leq x$. If $1+\frac{x}{y} = 1$, then $y+x = y$, so $x \leq y$. #

The proposition is not always true if E has a compatible total order which is not the natural partial order as the following example shows.

Example 2.40. Let $E = (\mathbb{Q}^+, \min, \cdot)$. Then $E \times E$ is not precise.
Let $(x,y) \in E \times E$ and $(u,v) \in E \times E$. Define $(x,y) \leq (u,v)$ if and only if either i) $x < u$ or ii) $x = u$ and $y \leq v$.
Claim that $\leq$ is a total order.

1) $(x,x) \leq (x,x)$

2) Suppose that $(x,y) \leq (u,v)$ and $(u,v) \leq (x,y)$.

Case 1. $x < u$ so $(u,v) \nleq (x,y)$, a contradiction.

Case 2. $x = u$, so $y \leq v$ and $v \leq y$ so $y = v$. Hence $(x,y) = (u,v)$.

3) If $(x,y) \leq (u,v)$ and $(u,v) \leq (a,b)$ then $x \leq u$ and $u \leq a$ so $x \leq a$.

Case 1. $x < a$, so $(x,y) \leq (a,b)$.

Case 2. $x = a$, so $x = u = a$ so $y \leq v$ and $v \leq b$ so $y \leq b$.
Hence $(x,y) \leq (a,b)$ so $\leq$ is a partial order.

4) Let $(a,b) \in E \times E$. Claim that if $(x,y) \leq (u,v)$, then $(x,y)(a,b) \leq (u,v)(a,b)$ and $(x,y)+(a,b) \leq (u,v)+(a,b)$.
To prove this, suppose $(x,y) \leq (u,v)$.

Case 1. $x < u$ then $x+a < u+a$ and $xa < ua$ so $(x,y)+(a,b) \leq (u,v)+(a,b)$ and $(x,y)(a,b) \leq (u,v)(a,b)$.

<u>Case 2.</u> $x = u$ then $y \leq v$ so $y+b \leq v+b$ and $yb \leq vb$. Therefore $(x,y)+(a,b) \leq (u,v)+(a,b)$ and $(x,y)(a,b) \leq (u,v)(a,b)$.

5) Let (x,y) and $(u,v) \in E \times E$.

<u>Case 1.</u> $x < u$ then $(x,y) \leq (u,v)$.

<u>Case 2.</u> $u < x$ then $(u,v) \leq (x,y)$.

<u>Case 3.</u> $u = x$ then either $y < v$ or $v < y$ or $v = y$. If $y = v$, then $(x,y) = (u,v)$. If $y < v$, then $(x,y) \leq (u,v)$. If $v < y$, then $(u,v) \leq (x,y)$. Thus $E \times E$ has a compatible total order but $E \times E$ is not precise. #