CHAPTER IV

ALMOST MULTIPLICATIVELY CANCELLATIVE SEMIRING

In [2] a generalization of the concept of semifield was given. In [1] it was shown that a semiring S with zero element is embeddable in a O-semifield if and only if S is O-M.C. In this chapter we generalize the property of a O-M.C. to a new concept, almost multiplicative cancellatively, so that semiring with this property can be embedded in generalize semifields. Before we give the definition of almost multiplicative cancellativity we shall give two short propositions of independent interest.

Proposition 4.1. Let S be a semiring with a multiplicative zero a which is O-M.C.. Then a is either the additive identity or the additive zero.

Proof. We can embeded S in a O-semifield or an ∞-semifield. If S can be embedded in a O-semifield then a is the additive identity. If S can be embedded in an ∞-semifield then ais the additive zero. #

Proposition 4.2. Let S be a semiring. If there exists an $a \in S$ such that every $x \in S - \{a\}$ is M.C. then either S is M.C. or S is O-M.C..

Proof. We must show that a is a multiplicative zero or that a is an M.C. element. Suppose that a is not a multiplicative zero. Then there exists a $b \in S$ such that $ab \neq a$. Let $x,y \in S$ be such that $ax = ay$ then $abx = aby$. Hence $x = y$. #

Definition 4.3. Let $(S,+,\cdot)$ be a semiring. S is called an almost multiplicatively cancellative semiring (A.M.C. semiring) if and only if there exists an $a \in S$ such that $(S-\{a\},\cdot)$ is a cancellative semigroup. If $a \in S$ has the property that $(S-\{a\},\cdot)$ is a cancellative semigroup then S is called A.M.C. semiring w.r.t. a.

Proposition 4.4. Let S be an A.M.C. semiring. Then the number of multiplicative idempotents in S is ≤ 2.

Proof. Let $a \in S$ be such that $(S-\{a\},\cdot)$ is a cancellative semigroup. If $e \in S-\{a\}$ is an idempotent then for all $x \in S-\{a\}$, $e^2x = ex$ so $ex = x$. Hence e is the identity of $(S-\{a\},\cdot)$. Thus e is the only idempotent in $(S-\{a\},\cdot)$. Hence e,a are the only possible idempotents in $(S,\cdot)$. #

Proposition 4.5. Let S be an A.M.C. semiring. Let $A = \{a \in S \mid (S-\{a\},\cdot)$ is a cancellative semigroup$\}$. If there exists an $a \in A$ such that a is not M.C. in S then $|A| \leq 2$.

Proof. Since $a \in A$ is not M.C. then there exist $x,y \in S$ such that $x \neq y$ and $ax = ay$. Suppose that $|A| > 2$. Then there exist $b,c \in A-\{a\}$ such that $b \neq c$. Clearly $x,y \in S-\{a\}$ or $x,y \in S-\{b\}$ or $x,y \in S-\{c\}$. If $x,y \in S-\{b\}$ or $x,y \in S-\{c\}$ then $x = y$, a contradiction. Hence $x,y \in S-\{a\}$.

Case 1. a is a multiplicative zero. Then $aa = ab$ which implies that $a = b$, a contradiction.

Case 2. a is not a multiplicative zero. Then there exists a $u \in S$ such that $au \neq a$. Since $ax = ay$ so $aux = auy$ which implies that $x = y$, a contradiction. #

Theorem 4.6. Let S be an A.M.C. semiring w.r.t. a. Then exactly one of the following holds:

1) $ax = a$ for all $x \in S$, or

2) $a^2 = a$ and there exists a $1 \in S-\{a\}$ such that $1x = x$ for all $x \in S$ and there exists a $b \in S-\{a\}$ such that $ab \neq a$, or

3) $ax = x$ for all $x \in S$, or

4) there exists a $1 \in S-\{a\}$ such that $1x = x$ for all $x \in S$ and $a^2 = a$, or

5) $xy \neq a$ for all $x, y \in S$.

Proof. Consider a^2.

Case 1. $a^2 = a$.

Subcase 1.1. a is a multiplicative zero.

Subcase 1.2. a is not a multiplicative zero so there exists a $b \in S-\{a\}$ such that $ab \neq a$.

Subcase 1.2.1. there exists a $1 \in S-\{a\}$ such that $1a = a$. Claim that $1x = x$ for all $x \in S$. Since $a^2 = a$ and $ab \neq a$. Hence $abab = abb$ so $ab = b$. Hence $1b = 1(ab) = ab = b$. Let $y \in S-\{a\}$. We have that $(1y)b = (y1)b = y(1b) = yb$ so $1y = y$. Since $1a = a$, we have $1x = x$ for all $x \in S$.

Subcase 1.2.2. Assume that for all $x \in S-\{a\}$, $ax \neq a$. Claim that $ax = x$ for all $x \in S$. Since $a^2 = a$, we get that for all $y \in S-\{a\}$, $ayay = ayy$ so $ay = y$. Thus $ax = x$ for all $x \in S$.

Case 2. $a^2 \neq a$.

Subcase 2.1. there exists a $1 \in S-\{a\}$ such that $1a = a$. Claim that $1x = x$ for all $x \in S$. Since $1a = a$ we get that for all $x \in S-\{a\}$, $1aax = aax$ so $1x = x$. Hence $1x = x$ for all $x \in S$.

Subcase 2.2. for all $x \in S-\{a\}$, $ax \neq a$. Since $a^2 \neq a$ therefore $ax \neq a$ for all $x \in S$ and for all $x,y \in S-\{a\}$, $xy \neq a$. Hence $xy \neq a$ for all $x,y \in S$. #

From Theorem 4.6. We see that if S is an A.M.C. semiring w.r.t. a. Then there are exactly five mutually exclusive possibilities:

In 1) we say that S is a Classification I semiring w.r.t. a.

In 2) we say that S is a Classification II semiring w.r.t. a.

In 3) we say that S is a Classification III semiring w.r.t. a.

In 4) we say that S is a Classification IV semiring w.r.t. a.

In 5) we say that S is a Classification V semiring w.r.t. a.

Corollary 4.7. i) Let S be a Classification II semiring w.r.t. a. Then $ax = x$ for all $x \neq 1$.

ii) Let S be a Classification IV semiring w.r.t. a. Then either $ax \neq x$ for all $x \in S$ or $a^2 = 1$.

iii) Let S be a Classification V semiring w.r.t. a. Then either $ax \neq x$ for all $x \in S$ or $a^2 = a^n$ for all $n \in Z^+-\{1\}$.

Proof. i) Since S is a Classification II semiring w.r.t. a. there exists a $b \in S-\{a\}$ such that $ab \neq a$. Hence $ab = b$. Let $x \in S-\{1\}$ be arbitrary. Then $axb = xb$ therefore $ax = x$ (since $ax \neq a$, if $ax = a$ then $1b = b = ab = axb = xb$ so $x = 1$, a contradiction).

ii) If $ax = x$ for some $x \in S$ then $a^2x = ax = x = 1x$. Therefore $a^2 = 1$.

iii) If $ax = x$ for some $x \in S$ then $a^2x = ax$, so $a^3x = a^2x$. Hence $a^3 = a^2$. Thus $a^2 = a^n$ for all $n \in Z^+-\{1\}$. #

Theorem 4.8. Let $S = \{1,a\}$ be an A.M.C. semiring. Then S must be isomorphic to one of the structures given below:

(1)

.	a	1
a	a	a
1	a	1

and

+	a	1
a	a	a
1	a	a

(2)

.	a	1
a	a	a
1	a	1

and

+	a	1
a	a	a
1	a	1

(3)

.	a	1
a	a	a
1	a	1

and

+	a	1
a	a	1
1	1	1

(4)

.	a	1
a	a	a
1	a	1

and

+	a	1
a	a	1
1	1	a

(5)

·	a	1
a	1	1
1	1	1

and

+	a	1
a	1	1
1	1	1

(6)

·	a	1
a	1	1
1	1	1

and

+	a	1
a	a	a
1	a	1

(7)

·	a	1
a	1	1
1	1	1

and

+	a	1
a	a	1
1	1	1

(8)

·	a	1
a	1	1
1	1	1

and

+	a	1
a	1	a
1	a	1

In (1)-(4) S is a Classification I semiring w.r.t. a. Furthermore S is a type I semifield w.r.t. a and S is a Classification III semiring w.r.t. 1. Furthermore S is a type II semifield w.r.t. 1. In (5)-(8) S is a Classification V semiring w.r.t. a. Furthermore S is a type III semifield w.r.t. a.

From now on we assume that $|S| > 2$ for all semiring S.

Theorem 4.9. There does not exist a Classification II semifield.

Proof. Let S be a Classification II semiring w.r.t. a. Suppose that S is a semifield w.r.t. b. By Theorem 1.23 either $bx = b$ for all $x \in S$ or $bx = x$ for all $x \in S$ or $b^2 \neq b$ and $be \neq b$ where e is the identity of $(S-\{a\}, \cdot)$.

Case 1. $bx = b$ for all $x \in S$. Clearly $b \neq a$. Hence $bb = b1$ so $b = 1$ therefore $1a = 1$, a contradiction.

Case 2. $bx = x$ for all $x \in S$. If $b = a$ then $a1 = 1$, a contradiction so $b \neq a$. Since $b^2 = b$, $bb = b1$ so $b = 1$. Since $(S-\{b\},\cdot)$ is a group, let e be the identity of $(S-\{b\},\cdot)$. Then $ea = a$. By Corollary 4.7, $ae = e$ so $a = e$. Let $c \in S-\{a,1\}$. Then there exists a $d \in S-\{a,1\}$ such that $cd = e = a$, a contradiction.

Case 3. $b^2 \neq b$ and $be \neq b$ where e is the identity of $(S-\{b\},\cdot)$. Clearly $b \neq a$ and $b \neq 1$, so $1, a \in S-\{b\}$ but $1^2 = 1$ and $a^2 = a$, a contradiction since a group has only one idempotent. #

Theorem 4.10. There does not exist a Classification IV semifield.

Proof. Let S be a Classefication IV semiring w.r.t. a. Then there exists a $1 \in S-\{a\}$ such that $1x = x$ for all $x \in S$. Suppose that S is a semifield w.r.t. b. By Theorem 1.23, either $bx = b$ for all $x \in S$ or $bx = x$ for all $x \in S$ or $b^2 \neq b$ and $be \neq b$ where e is the identity of $(S-\{b\},\cdot)$.

Case 1. $bx = b$ for all $x \in S$. Clearly $b \neq a$ and $b \neq 1$. Consider $1, b \in S-\{a\}$. Then $1b = b = bb$, hence $b = 1$, a contradiction.

Case 2. $bx = x$ for all $x \in S$. Clearly $b \neq a$, so $1, b \in S-\{a\}$ hence $1b = b = bb$ therefore $1 = b$. Let e be the identity of $(S-\{b\},\cdot)$. Since $a \in S-\{b\}$, $ea = a$. Hence $eaa = aa = aa1$ which implyies that $e = 1$, a contradiction.

Case 3. $b^2 \neq b$ and $be \neq b$ where e is the identity of $(S-\{b\},\cdot)$. Clearly $b \neq 1$ so $1 \in S-\{b\}$. Hence $1 = 1e = e$. But $1b = b$, a contradiction. #

Proposition 4.11. Let S be a Classification I semiring w.r.t. a. If there is an element b in S such that $(S-\{b\},\cdot)$ is a cancellative semigroup then a = b.

Proof. Suppose not. Let $c \in S-\{a,b\}$. Then $aa = ac = a$ so a = c, a contradiction. #

Propsition 4.12. Let S be a Classification II semiring w.r.t. a. If S is a Classification II semiring w.r.t. b then a = b.

Proof. Let $1 \in S-\{a\}$ be such that $1x = x$ for all $x \in S$ and let $1' \in S-\{b\}$ be such that $1'x = x$ for all $x \in S$. Then $1 = 1'$. By Corollary 4.7 (i), a = ab = b. #

Proposition 4.13. Let S be a Classification III semiring w.r.t. a. If S is a Classification III semiring w.r.t. b then a = b.

Proof. Clearly the identity of any semigroup is unique. #

We shall now show that if S is Classification IV or V semiring w.r.t. a then a may be not unique.

Example 4.14. $\mathbb{Z}^+$ with the usual addition and multiplication is a Classification IV semiring w.r.t. 2 and a Classification IV semiring w.r.t. 3.

$\mathbb{Z}^+ -\{1\}$ with the usual addition and multiplication is a Classification V semiring w.r.t. 2 and a Classification V semiring w.r.t. 3.

Example 4.15. Let $K = \{x \in \mathbb{Q}^+ \mid x \geq 1\}$ with the usual addition and multiplication. Then K is an M.C. semiring. Let a be a symbol not representing any element of K. Extend + and $\cdot$

from K to $S = K \cup \{a\}$ by $aa = 1$, $a1 = 1a = a$, $ax = xa = x$ for all $x \neq 1$ and $a+x = x+a = 1+x$ for all $x \in S$. Claim that $(S,+,\cdot)$ is a semiring. We must show that (a) $(x+y)+z = x+(y+z)$ for all $x,y,z \in S$, (b) $(xy)z = x(yz)$ for all $x,y,z \in S$, and (c) $(x+y)z = xz+yz$ for all $x,y,z \in S$.

To show (a), we shall consider the following cases:

Case 1. $x = y = z = a$.

$(x+y)+z = (a+a)+a = a+(a+a) = x+(y+z)$.

Case 2. $x = y = a$, $z \neq a$.

$(x+y)+z = (a+a)+z = (1+1)+z = 1+(1+z) = 1+(a+z) = a+(a+z) = x+(y+z)$.

Case 3. $x = a$, $y \neq a$, $z = a$.

$(x+y)+z = (a+y)+a = a+(a+y) = a+(y+a) = x+(y+z)$.

Case 4. $x = a$, $y \neq a$, $z \neq a$.

$(x+y)+z = (a+y)+z = (1+y)+z = 1+(y+z) = a+(y+z) = x+(y+z)$.

Using Case 2 - Case 4, the commutativity of + and the fact that $(K,+)$ is a semigroup we can prove the remaining cases of (a).

To show (b), we shall consider the following cases:

Case 1. $x = y = z = a$.

$(xy)z = (aa)a = a(aa) = x(yz)$.

Case 2. $x = y = a$, $z \neq a$.

Subcase 2.1. $z = 1$. $(xy)z = (aa)1 = 11 = 1 = aa = a(a1) = x(yz)$.

Subcase 2.2. $z \neq 1$. $(xy)z = (aa)z = 1z = z = az = a(az) = x(yz)$.

<u>Case 3.</u> $x = a,\ y \neq a,\ z = a.$

$(xy)z = (ay)a = a(ay) = a(ya) = x(yz).$

<u>Case 4.</u> $x = a,\ y \neq a,\ z \neq a.$

<u>Subcase 4.1.</u> $y = z = 1.$ $(xy)z = (a1)1 = a1 = a = a1 = a(11) = x(yz).$

<u>Subcase 4.2.</u> $y = 1,\ z \neq 1.$ $(xy)z = (ay)z = az = z = 1z = x(yz).$

<u>Subcase 4.3.</u> $y \neq 1, z = 1.$ $(xy)z = (ay)1 = y1 = ay = a(y1) = x(yz).$

<u>Subcase 4.4.</u> $y = 1,\ z = 1.$ $(xy)z = (ay)z = yz = a(yz) = x(yz).$

Using Case2 - Case 4, the commutativity of $\cdot$ and the fact that $(K, \cdot)$ is a semigroup we can prove the remaining case of (b).

To show (c), note that if $x = 1$ then $x(y+z) = xy+xz$. Assume $x \neq 1$. We shall consider the following cases:

<u>Case 1.</u> $x = y = z = a.$

$a(a+a) = a(1+1) = 1+1 = aa+aa.$

<u>Case 2.</u> $x = y = a,\ z \neq a.$

<u>Subcase 2.1.</u> $z = 1.$ $x(y+z) = a(a+1) = a(1+1) = 1+1 = 1+a = aa+a1 = xy+xz.$

<u>Subcase 2.2.</u> $z \neq 1.$ $x(y+z) = a(a+z) = a(1+z) = 1+z = aa+az = xy+xz.$

<u>Case 3.</u> $x = a,\ y \neq a,\ z = a.$

$x(y+z) = a(y+a) = a(a+y) = aa+ay = ay+aa = xy+xz.$

Case 4. $x = a$, $y \neq a$, $z \neq a$.

Subcase 4.1. $y = z = 1$. $x(y+z) = a(1+1) = 1+1 = a+a = a1+a1 = xy+xz$.

Subcase 4.2. $y = 1$, $z \neq 1$. $x(y+z) = a(1+z) = 1+z = a+z = a1+az = xy+xz$.

Subcase 4.3. $y \neq 1$, $z = 1$. $x(y+z) = a(y+1) = a(1+y) = a1+ay = ay+a1 = xy+xz$.

Subcase 4.4. $y \neq 1$, $z \neq 1$. $x(y+z) = a(y+z) = y+z = ay+az = xy+xz$.

Case 5. $x \neq a$, $y = z = a$.

$x(y+z) = x(a+a) = x(1+1) = x1+x1 = x+x = xa+xa = xy+xz$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

$x(y+z) = x(a+z) = x(1+z) = x1+xz = x+xz = xa+xz = xy+xz$.

Case 7. $x \neq a$, $y \neq a$, $z = a$. Done by Case 6.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$. Done.

Hence $(S,+,\cdot)$ is a semiring. Furthermore S is a Classification IV semiring w.r.t. a and a is the unique element in S such thar $(S-\{a\},\cdot)$ is a cancellative semigroup. #

Example 4.16. Let $K = \{x \in Q^+ \mid x > 1\}$ with the usual addition and multiplication. Then K is an M.C. semiring. Let a be a symbol not representing any element of K. Exten + and $\cdot$ from K to $S = K \cup \{a\}$ by $ax = xa = 2x$ for all $x \in S$ and $a+x = x+a = 2+x$ for all $x \in S$. Claim that $(S,+,\cdot)$ is a semiring. We must show that

(a) $(x+y)+z = x+(y+z)$ for all $x,y,z \in S$,

(b) $(xy)z = x(yz)$ for all $x,y,z \in S$ and

(c) $x(y+z) = xy+xz$ for all $x,y,z \in S$.

To show (a), we shall consider the following cases:

<u>Case 1</u>. $x = y = z = a$.

$(x+y)+z = x+(y+z)$

<u>Case 2</u>. $x = y = z,\ z \neq a$.

$(x+y)+z = (a+a)+z = (2+2)+z = 2+(2+z) = 2+(2+z) = a+(a+z) = x+(y+z)$.

<u>Case 3</u>. $x = a,\ y \neq a,\ z = a$. Done.

<u>Case 4</u>. $x = a,\ y \neq a,\ z \neq a$.

$(x+y)+z = (a+y)+z = (2+y)+z = 2+(y+z) = a+(y+z) = x+(y+z)$.

Using Case 2 - Case 4, the commutativity of + and the fact $(K,+)$ is a semigroup, we can prove the remaining case of (a).

To show (b), we shall consider the following cases:

<u>Case 1</u>. $x = y = z = a$.

$(xy)z = x(yz)$.

<u>Case 2</u>. $x = y = a,\ z \neq a$.

$(xy)z = (aa)z = (22)z = 2(2z) = 2(az) = a(az) = x(yz)$.

<u>Case 3</u>. $x = a,\ y \neq a,\ z = a$. Done.

<u>Case 4</u>. $x = a,\ y \neq a,\ z \neq a$.

$(xy)z = (ay)z = (2y)z = 2(yz) = a(yz) = x(yz)$.

Using Case 2 - Case 4, the commutative of $\cdot$ and the fact $(K,\cdot)$ is a semigroup, we can prove the remaining cases of (b).

To show (c), we shall consider the following cases:

Case 1. $x = y = z = a$.

$x(y+z) = a(a+a) = a(2+2) = 2(2+2) = 22+22 = aa+aa = xy+xz$.

Case 2. $x = y = a$, $z \neq a$.

$x(y+z) = a(a+z) = a(2+z) = 2(2+z) = 22+2z = aa+az = xy+xz$.

Case 3. $x = a$, $y \neq a$, $z = a$. Done by Case 2.

Case 4. $x = a$, $y \neq a$, $z \neq a$.

$x(y+z) = a(y+z) = 2(y+z) = 2y+2z = ay+az = xy+xz$.

Case 5. $x \neq a$, $y = z = a$.

$x(y+z) = x(a+a) = x(2+2) = x2+x2 = xa+xa = xy+xz$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

$x(y+z) = x(a+z) = x(2+z) = x2+xz = xa+xz = xy+xz$.

Case 7. $x \neq a$, $y \neq a$, $z = a$. Done by Case 6.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$. Done.

Hence $(S,+,\cdot)$ is a semiring. Furthermore, S is a Classification V semiring w.r.t. a and a is the unique element such that $(S-\{a\},\cdot)$ is a cancellative semigroup. #

Proposition 4.17. Let S be a Classification I semiring w.r.t. a. Then either a is the additive zero or a is the additive identity.

Proof. Since a is a multiplicative zero and $S-\{a\}$ is M.C. , S is 0-M.C. semiring. By Proposition 4.1 a is the additive zero or a is the additive identity. #

Proposition 4.18. Let S be aClassification II semiring w.r.t.a. Then $S-\{1\}$ is an M.C. semiring where 1 is the multiplicative identity of S.

Proof. To show that $S-\{1\}$ is a semiring, we shall show that for all $x,y \in S-\{1\}$, $x+y \neq 1$ and $xy \neq 1$. To prove this, suppose not. Let $x,y \in S- 1$ be such that $x+y = 1$.
By Corollary 4.7 i), $1 = x+y = xa+ya = (x+y)a = 1a = a$, a contradiction. Suppose there exist $x,y \in S-\{1\}$ such that $1 = xy$. Again, by Corollary 4.7 i), $1 = xy = x(ya) = (xy)a = a$, a contradiction. Hence $(S-\{1\},+,\cdot)$ is a semiring.
Next, we must show that $S-\{1\}$ is M.C.. Let $x,y,z \in S-\{1\}$ be such that $xy = xz$. We must show that $y = z$. If $x = a$ then done. Assume that $x \neq a$.

Case 1. $y = a$. Then $xz = xa = x$. If $z \neq a$ then $xz = x1$ which implies that $z = 1$, a contradiction. Hence $z = a$ so $y = z$. Similarly, if $z = a$ then $y = a$.

Case 2. $y \neq a$. Then $x,y,z \in S-\{a\}$ so $y = z$.

Hence $(S-\{1\},+,\cdot)$ is an M.C. semiring. #

Corollary 4.19. For all Classification II semiring w.r.t. a is always a Classification III semiring w.r.t. 1 where 1 is the multiplicative identity of S.

Proof. Follows directly from Proposition 4.18. #

Proposition 4.20. Let S be a Classification III semiring w.r.t. a. If there is a $b \in S-\{a\}$ such that b is M.C. in S then S is M.C..

Proof. Let $x,y,z \in S$ be such that $xy = xz$. We must show that $y = z$. If $x = a$ then done. Assume that $x \neq a$.

Case 1. $y = a$. Then $x = xz$ so $xb = xzb$ which implies that $b = bz$. Therefore $ab = bz$ which implies that $a = z$. Similarly, if $z = a$ then $y = a$.

Case 2. $y \neq a$. Then $x,y,z \in S-\{a\}$. Hence $y = z$.

Therefore S is M.C.. #

$\mathbb{Z}^+$ with the usual addition and multiplication is a Classification III semiring w.r.t. 1 satisfying Proposition 4.20.

Let S be a Classification III semiring w.r.t. a. If a is M.C. in S then S may be not M.C.. Every type II semifield w.r.t. a is an example of this.

In Example 4.15, S is a Classification IV semiring w.r.t. a such that a and 1 are M.C. in S but S is not M.C..

Proposition 4.21. Let S be a Classification V semiring w.r.t. a. If there is a $b \in S$ such that b is M.C. in S then S is M.C..

Proof. Let $x,y,z \in S$ be such that $xy = xz$. We must show that $y = z$. Since $xy = xz$, $xbxy = xbxz$ so $xxby = xxbz$ which implies that $by = bz$. Hence $y = z$. #

$\mathbb{Z}^+-\{1\}$ with the usual addition and multiplication is a Classification V semiring w.r.t. 2 satisfying Proposition 4.21.

<u>Theorem 4.22</u>. Let S be a Classification II semiring w.r.t. a.

Then i) $1+x = 1$ or $1+x = a+x$ for all $x \neq 1$.

ii) exactly one of the following holds:

1) $(S,+)$ is a band i.e. $x+x = x$ for all $x \in S$.

2) $1+1 = a$ and $(S-\{1\},+)$ is a band.

3) $1+1 = a+a$ and $x+x = y+y$ if and only if $x = y$ for all $x,y \in S-\{1\}$.

<u>Proof</u>. i) Let $x \in S-\{1\}$. If $1+x \neq 1$ then

$1+x = a(1+x) = a1+ax = a+x$.

ii) Consider $1+1$.

<u>Case 1</u>. $1+1 = 1$. Then $x+x = x$ for all $x \in S$ so $(S,+)$ is a band.

<u>Case 2</u>. $1+1 = a$. Then $a+a = a1+a1 = a(1+1) = a+a = a$ and for all $x \in S-\{1\}$, $x+x = ax+ax = x(a+a) = xa = x$.

<u>Case 3</u>. $1+1 \in S-\{1,a\}$. Then $1+1 = a(1+1) = a1+a1 = a+a$.

Let $x,y \in S-\{1\}$. Clearly, if $x = y$ then $x+x = y+y$. If $x+x = y+y$ then $x(1+1) = x1+x1 = x+x = y+y = y(1+1)$, so $x = y$. #

<u>Note</u>. If S is a Classification II semiring w.r.t. a then $1+1 = 1$ or $1+1 = a+a$.

Now we shall give some example of Theorem 4.22.

<u>Example 4.23</u>. $(\mathbb{Z}^+,\min,\cdot)$ is a semiring. Let a be a symbol not representing any element of $\mathbb{Z}^+$. Extend + and $\cdot$ from $\mathbb{Z}^+$ to $S = \mathbb{Z}^+ \cup \{a\}$ by $ax = xa = x$ for all $x \neq 1$, $a1 = 1a = aa = a$. For all $x \neq 1$, $a+x = x+a = a = a+a$. $1+a = a+1 = 1$.

Claim that $(S,+,\cdot)$ is a semiring. We must show that

(a) $(x+y)+z = x+(y+z)$ for all $x,y,z \in S$,

(b) $(xy)z = x(yz)$ for all $x,y,z \in S$, and

(c) $x(y+z) = xy+xz$ for all $x,y,z \in S$.

First we shall show (a).

Case 1. $x = 1$ or $y = 1$ or $z = 1$. Then $(x+y)+z = x+(y+z) = 1$.

Assume that $x \neq 1$, $y \neq 1$ and $z \neq 1$.

Case 2. $x = a$ or $y = a$ or $z = a$. Then $(x+y)+z = x+(y+z) = a$.

Case 3. x,y,z $S- 1,a$. Then done.

Next, we shall show (b).

Case 1. $x = 1$ or $y = 1$ or $z = 1$. Then $(xy)z = x(yz)$.

Assume that $x,y,z \in S-\{1\}$.

Case 2. $x = a$ or $y = a$or $z = a$. Then $(xy)z = x(yz)$.

Case 3. $x,y,z \in S-\{1,a\}$. Then done.

Lastly, we shall show (c).

Case 1. $x = 1$. Then done.

Assume that $x \neq 1$.

Case 2. $y = 1$. Then $x(y+z) = x$ and $x+xz = \begin{cases} x+x = x \text{ if } z = 1, \\ x+x = x \text{ if } z = a, \\ x+xz=x \text{ if } z \in S-\{1,a\}. \end{cases}$

Case 3. $z = 1$. Done by Case 2.

Assume that $x,y,z \in S-\{1\}$.

Case 4. $x = a$. Then $x(y+z) = xy+xz$.

Assume that $x \neq a$.

Case 5. $y = a$. $x(a+z) = xa = x$ and $xy+xz = x+xz = x$.

Case 6. $z = a$. Done by Case 5.

Suppose that $x,y,z \in S-\{1,a\}$. Then done.

Hence S is a Classification II semiring w.r.t. a such that $(S,+)$ is a band. Furthermore 1 is the additive zero. #

Example 4.24. $\mathbb{Z}^+$ with the usual addition and multiplication is a semiring. Let a be a symbol not representing any element of $\mathbb{Z}^+$ Extend + and $\cdot$ from $\mathbb{Z}^+$ to $S = \mathbb{Z}^+ \cup \{a\}$ by $a+x = 1+x$ for all $x \in S$ and $ax = xa = x$ for all $x \in S-\{1\}$, $a1 = 1a = a$.

Claim that $(S,+,\cdot)$ is a semiring. We must show that

(a) $(x+y)+z = x+(y+z)$ for all $x,y,z \in S$,

(b) $(xy)z = x(yz)$ for all $x,y,z \in S$, and

(c) $x(y+z) = xy+xz$ for all $x,y,z \in S$.

The proof of (a) is similar to the proof in Example 4.15 used to show (a).

We shall first to show (b).

Case 1. $x = 1$ or $y = 1$ or $z = 1$. Then $(xy)z = x(yz)$.

Suppose that $x,y,z \in S-\{1\}$.

Case 2. $x = a$ or $y = a$ or $z = a$. Then $(xy)z = x(yz)$.

Suppose that $x,y,z \in S-\{1,a\}$. Then $(xy)z = x(yz)$.

Lastly, we shall show (c).

Case 1. $x = 1$. Then $x(y+z) = xy+xz$.

Suppose that $x \neq 1$.

Case 2. $x = y = z = a$.

$x(y+z) = a(a+a) = a(1+1) = 1+1 = a+a = aa+aa = xy+xz$.

Case 3. $x = y = a$, $z \neq a$.

Subcase 3.1. $z = 1$. Then $x(y+z) = a(a+1) = a(1+1) = 1+1 = a+a = aa+a1 = xy+xz$.

Subcase 3.2. $z \neq 1$. Then $x(y+z) = a(a+z) = a(1+z) = 1+z = a+z = aa+az = xy+xz$.

Case 4. $x = a$, $y \neq a$, $z = a$. Done by Case 3.

Case 5. $x = a$, $y \neq a$, $z \neq a$.

Subcase 5.1. $y = z = 1$. Then $x(y+z) = a(1+1) = 1+1 = a+a = a1+a1 = xy+xz$.

Subcase 5.2. $y = 1$, $z \neq 1$. Then $x(y+z) = a(1+z) = 1+z = a+z = a1+az = xy+xz$.

Subcase 5.3. $y \neq 1$, $z = 1$. Done by Subcase 5.2.

Subcase 5.4. $y \neq 1$, $z \neq 1$. Then $x(y+z) = a(y+z) = y+z = ay+az = xy+xz$.

Case 6. $x \neq a$, $y = a$, $z = a$.

$x(y+z) = x(a+a) = x(1+1) = x+x = xa+xa = xy+xz$.

Case 7. $x \neq a$, $y = a$, $z \neq a$.

$x(y+z) = x(a+z) = x(1+z) = x+xz = xa+xz = xy+xz$.

Case 8. $x \neq a$, $y \neq a$, $z = a$. Done by Case 7.

Case 9. $x \neq a$, $y \neq a$, $z \neq a$. Done.

Hence S is a Classification II semiring w.r.t. a such that $1+1 \in S-\{1,a\}$. #

Example 4.25. $(\mathbb{Z}^+-\{1\},\min,\cdot)$ is a semiring. Let $a, 1'$ be symbols not representing any element of $\mathbb{Z}^+-\{1\}$.

Extend + and $\cdot$ from $\mathbb{Z}^+-\{1\}$ to $S = (\mathbb{Z}^+-\{1\})\cup\{1',a\}$ by $1'1' = 1'$
$1'a = a1' = aa = a$, $1'x = x1' = ax = xa = x$ for all $x \in \mathbb{Z}^+-\{1\}$,
$1'+1' = 1'+a = a+1' = a+a = a$, $1'+x = x+1' = 1'$ for all $x \in \mathbb{Z}^+-\{1\}$,
$a+x = x+a = a$ for all $x \in \mathbb{Z}^+-\{1\}$.

Claim that $(S,+,\cdot)$ is a semiring. We must show that

(a)	$(x+y)+z$	=	$x+(y+z)$	for all $x,y,z \in S$,
(b)	$(xy)z$	=	$x(yz)$	for all $x,y,z \in S$, and
(c)	$x(y+z)$	=	$xy+xz$	for all $x,y,z \in S$.

First, we shall show (a).

Case 1. $x = a$ or $y = a$ or $z = a$. Then $(x+y)+z = a = x+(y+z)$.

Suppose that $x,y,z \in S-\{a\}$.

Case 2. $x = y = z = 1'$. Then $(x+y)+z = a = x+(y+z)$.

Case 3. $x = y = 1'$ or $x = z = 1'$ or $y = z = 1'$. Then $(x+y)+z = a = x+(y+z)$.

Case 4. exactly one of the following holds: $x = 1'$, $y = 1'$, $z=1'$.

Then $(x+y)+z = 1' = x+(y+z)$.

Suppose that $x,y,z \in S-\{1',a\}$. Then done.

Next, we shall show (b).

Case 1. $x = 1'$ or $y = 1'$ or $z = 1'$. Then $(xy)z = x(yz)$.

Suppose that $x,y,z \in S-\{1'\}$.

Case 2. $x = a$ or $y = a$ or $z = a$. Then $(xy)z = x(yz)$.

Suppose That $x,y,z \in S-\{1',a\}$. Then $(xy)z = x(yz)$.

Lastly, we shall show (c).

Case 1. $x = 1'$. Then $x(y+z) = xy+xz$.

Suppose that $x \neq 1'$.

Case 2. $y = z = 1'$ or $y = 1'$, $z = a$ or $y = a$, $z = 1'$ or $y = z = a$.
$x(y+z) = xa = x = x+x = xy+xz$.

Suppose that $y,z \in S-\{1',a\}$.

Case 3. $x = a$. Then $x(y+z) = xy+xz$.

Suppose that $x,y,z \in S-\{1',a\}$. Then $x(y+z) = xy+xz$.

Hence S is a Classification II semiring w.r.t. a such that $1'+1' = a$. Furthermore a is the additive zero. #

Theorem 4.26. Let S be a Classification II semiring w.r.t. a. Define $D = S-\{1\}$. Then

1) $I_D(1) = \emptyset$ or $I_D(1)$ is an additive subsemigroup of $I_D(a)$.

2) If $a \in I_D(1)$ then $I_D(1) = I_D(a)$.

3) $D-I_D(1) = \emptyset$ or $D-I_D(1)$ is an ideal of $(D,+)$.

Proof. 1) Suppose $I_D(1) \neq \emptyset$. Let $x \in I_D(1)$. Then $1+x = 1$. $a = a1 = a(1+x) = a1+ax = a+x$, so $x \in I_D(a)$. Hence $I_D(1) \subseteq I_D(a)$. For all $x,y \in I_D(1)$, $1+(x+y) = (1+x)+y = 1+y = 1$, so $x+y \in I_D(1)$. Hence $I_D(1)$ is an additive subsemigroup of $I_D(a)$.

2) Suppose $a \in I_D(1)$. Then $1+a = 1$. We must show that $I_D(a) = I_D(1)$. Let $x \in I_D(a)$. Then $a+x = a$ so $1 = 1+a = 1+(a+x) = 1+x$. Hence $x \in I_D(1)$. Thus $I_D(1) = I_D(a)$.

3) Suppose $D-I_D(1) \neq \emptyset$. Let $u \in D-I_D(1)$ and $v \in D$. To show that $u+v \in D-I_D(1)$. Since $u,v \in D$, $u+v \in D$ and $1+(u+v) = (1+u)+v \neq 1$ (by Proposition 4.18). #

Theorem 4.27. Let S be a Classification II semiring. Then S is not finite.

Proof. Let $a \in S$ be such that $(S-\{a\},\cdot)$ is a cancellative semigroup. Let $1 \in S$ be such that $1x = x$ for all $x \in S$. Clearly $|S| > 2$. Suppose that the order of S is finite. By Proposition 4.18, $S-\{1\}$ is a finite M.C. semiring of order greater than one, a contradiction with Corollary 1.10. #

Proposition 4.28. Let S be a Classification II semiring w.r.t. a. Then S is not A.C..

Proof. Let $x,y \in S-\{1,a\}$ be such that $x \neq y$. By Theorem 4.22 (i), $1+x = 1$ or $1+x = a+x$. If $1+x = a+x$ then x is not A.C. so done. Suppose $1+x = 1$. Again, $1+y = 1$ or $1+y = a+y$. Hence y is not A.C. or 1 is not A.C.. #

Note. $\mathbb{Z}_o^+$ with the usual addition and multiplication is a Classification I semiring w.r.t. 0 such that $\mathbb{Z}_o^+$ is A.C..

$\mathbb{Z}^+$ with the usual addition and multiplication is a Classification III semiring w.r.t. 1 and a Classification IV semiring w.r.t. 2 such that $\mathbb{Z}^+$ is A.C..

$\mathbb{Z}^+-\{1\}$ with the usual addition and multiplication is a Classification V semiring w.r.t. 2 such that $\mathbb{Z}^+-\{1\}$ is A.C..

Proposition 4.29. Let S be a Classification II semiring w.r.t. a. with either an additive zero or an additive identity. Then 1 or a(1 is the identity of $(S,\cdot)$) is the additive zero or the additive identity.

Proof. Suppose that S has an additive zero c and $c \neq 1$. Since $ac = c = c+cc = c(a+c) = cc$. Therefore $a = c$.

Suppose that S has an additive identity 0 and $0 \neq 1$. Since $0a = 0(a+0) = 0a+00 = 0+00 = 00$. Therefore $a = 0$. #

Proposition 4.30. Let S be a Classification III semiring w.r.t. a. Then either $(S-\{a\},\cdot)$ has an identity or S is M.C..

Proof. Case 1. There exists an $e \in S-\{a\}$ such that $e^2 = e$. Claim that $ex = x$ for all $x \in S-\{a\}$. Let $x \in S-\{a\}$, then $e^2x = ex$ which implies that $ex = x$.

Case 2. Suppose that $x^2 \neq x$ for all $x \in S-\{a\}$. Claim that S is M.C.. Let $x,y,z \in S$ be such that $xy = xz$.

Subcase 2.1. $x = a$. Then $y = z$.

Subcase 2.2. $x \neq a$. If $y = a$ then $x = xz$ which implies that $xz = xz^2$. Then $z = a$(if $z \neq a$ then $z = z^2$, a contradiction). Similarly if $z = a$ then $y = a$. Now assume that $y,z \in S-\{a\}$. Then $y = z$. Hence S is M.C.. #

If S is a Classification III semiring w.r.t. a such that $(S-\{a\},\cdot)$ has an identity then we say that S is a Classification III semiring w.r.t. a of form 1.

If S is a Classification III M.C. semiring then we say that S is a Classification III semiring of form 2.

Proposition 4.31. Let S be a Classification III semiring w.r.t. a of form 1. Then

i) $a+x = a$ or $a+x = e+x$ for all $x \neq a$ (e is the identity of $(S-\{a\},\cdot)$.

ii) exactly one of the following holds:

1) $(S,+)$ is a band.

2) $a+a = e$ and $(S-\{a\},+)$ is a band.

3) $a+a = e+e$ and $x+x = y+y$ if and only if $x = y$ for all $x,y \in S-\{a\}$.

Proof. The proof is similar to the proof of Theorem 4.22. #

Note that if S is a Classification III semiring w.r.t. a of form 1 then $a+a = a$ or $a+a = e+e$.

Proposition 4.32. Let S be a Classification III semiring w.r.t.a. of form 1. Define $D = S-\{a\}$. Then

1) $I_D(a) = \emptyset$ or $I_D(a)$ is an additive subsemigroup of $I_D(e)$ (e is the identity of $(S-\{a\},\cdot)$).

2) if $e \in I_D(a)$ then $I_D(a) = I_D(e)$.

3) $D-I_D(a) = \emptyset$ or $D-I_D(a)$ is an ideal of $(D,+)$.

Proof. The proof is similar to the proof of Theorem 4.26. #

Proposition 4.33. Let S be a finite Classification III semiring. Then $|S| = 2$.

Proof. Since $(S-\{a\},\cdot)$ is a finite cancellative semigroup then $(S-\{a\},\cdot)$ is a group. Let e be the identity of $(S-\{a\},\cdot)$. Claim that $x+y \neq a$ for all $x,y \in S-\{a\}$. To prove this, suppose not. Let $x,y \in S-\{a\}$ be such that $x+y = a$. Then $a = x+y = xe+ye = (x+y)e = ae = e$, a contradiction. Hence we have the claim. Thus $(S-\{a\},+,\cdot)$ is a ratio semiring. Therefore $|S-\{a\}| = 1$, so $|S| = 2$. #

Remark 4.34. i) Let S be a Classification III semiring w.r.t. a of form 1 with either an additive zero or an additive identity. Then a or e (e is the identity of $(S-\{a\},\cdot)$) is the additive zero or the additive identity.

ii) Let S be a Classification III semiring w.r.t. a of form 2 with either an additive zero or an additive identity. Then a is the additive zero or the additive identity.

Proof. i) Similar to the proof of Proposition 4.29.

ii) Suppose that S has an additive zero c. Then $c = c+c^2 = c(a+c) = cc$ which implies that $c = a$.

Suppose that S has an additive identity 0. Then $0a = 0(a+0) = 0a+00 = 0+00 = 00$ which implies that $a = 0$. #

Proposition 4.35. Let S be a Classification IV semiring w.r.t. a. Then 1) $ax = a$ if and only if $x = 1$.

2) if $1+1 = 1$ then $(S,+)$ is a band.

3) if $1+1 = a$ then $a+a = a^2$.

4) if $1+1 \in S-\{1,a\}$ then $x+x = y+y$ if and only if $x = y$ for all $x,y \in S-\{a\}$.

Proof. 1) If $x = 1$ then $ax = a$. Assume that $ax = a$. Clearly $x \neq a$, so $a^2x = a^2 = a^21$ which implies that $x = 1$.

2) Clearly if $1+1 = 1$ then $(S,+)$ is a band.

3) If $1+1 = a$ then $a^2 = a(1+1) = a+a$.

4) Suppose that $1+1 \in S-\{1,a\}$. Let $x,y \in S-\{a\}$. If $x+x = y+y$ then $x(1+1) = x+x = y+y = y(1+1)$ which implies that $x = y$. Clearly $x = y$ so $x+x = y+y$. #

Note that in a Classification II semiring w.r.t. a, $1+x = 1$ or $1+x = a+x$ for all $x \neq 1$, $1+1 = 1$ or $1+1 = a+a$. But in a Classification IV semiring this properties may not hold:

$\mathbb{Z}^+$ with the usual addition and multiplication is a Classification IV semiring w.r.t. 3 which is an example of this i.e. $1+x \neq 1$ and $1+x \neq 3+x$ for all $x \in \mathbb{Z}^+$, $1+1 \neq 1$ and $1+1 \neq 3+3$.

Now we shall give some example of Theorem 4.35.

Example 4.36. $(\mathbb{Z}^+, \min, \cdot)$ $((\mathbb{Z}^+, \max, \cdot))$ is a Classification IV semiring w.r.t. 2. Furthermore, 1 is the additive zero (1 is the additive identity).

Example 4.37. $\mathbb{Z}^+$ with the usual addition and multiplication is a Classification IV semiring w.r.t. 2 such that $1+1 = 2$ and $a+a = a^2$.

Example 4.38. $\mathbb{Z}^+$ with the usual addition and multiplication is a semiring. Let a be asymbol not representing any element of $\mathbb{Z}^+$. Extend + and $\cdot$ from $\mathbb{Z}^+$ to $S = \mathbb{Z}^+ \cup \{a\}$ by

$ax = xa = 2x$ for all $x \in S-\{1\}$, $1a = a1 = a$, and

$a+x = x+a = 2+x$ for all $x \in S$.

Claim that $(S,+,\cdot)$ is a semiring. We must show that

(a) $(x+y)+z = x+(y+z)$ for all $x,y,z \in S$,

(b) $(xy)z = x(yz)$ for all $x,y,z \in S$, and

(c) $x(y+z) = xy+xz$ for all $x,y,z \in S$.

To show (a), use a proof similar to the proof of Example 4.16 (a).

We shall show (b). If $x = 1$ or $y = 1$ or $z = 1$ then $(xy)z = x(yz)$. Suppose that $x,y,z \in S-\{1\}$. The proof is similar to the proof of (b) in Example 4.16.

Lastly, we shall show (c).

Case 1. $x = 1$. Then $x(y+z) = xy+xz$.

Suppose $x \neq 1$.

Case 2. $x = y = z = a$.

$x(y+z) = a(a+a) = a(2+2) = 2(2+2) = 22+22 = aa+aa = xy+xz$.

Case 3. $x = y = a$, $z \neq a$.

Subcase 3.1. $z = 1$. $x(y+z) = a(a+1) = a(2+1) = 2(2+1) = 22+21 = 22+2 = 22+a = aa+a = aa+a1 = xy+xz$.

Subcase 3.2. $z \neq 1$. $x(y+z) = a(a+z) = a(2+z) = 2(2+z) = 22+2z = aa+az = xy+xz$.

Case 4. $x = a$, $y \neq a$, $z \neq a$. Done by Case 3.

Case 5. $x = a$, $y \neq a$, $z \neq a$.

Subcase 5.1. $y = z = 1$. Then $x(y+z) = a(1+1) = 2(1+1) = 2+2 = a+a = a1+a1 = xy+xz$.

Subcase 5.2 $y = 1$, $z \neq 1$. Then $x(y+z) = a(1+z) = 2(1+z) = 2+2z = a+2z = a+az = a1+az = xy+xz$.

Subcase 5.3. $y \neq 1$, $z + 1$. Done by Subcase 5.2.

Subcase 5.4. $y \neq 1$, $z \neq 1$. Then $x(y+z) = a(y+z) = 2(y+z) = 2y+2z = ay+az = xy+xz$.

The proof of the other cases is clear from Case 2 - Case 5 and the fact that + and $\cdot$ are commutative.

Thus S is a Classification IV semiring w.r.t. a such that $a+a = a^2$, $1+1 \in S-\{1,a\}$ and a is not M.C. in S. #

Proposition 4.39. Let S be a Classification IV semiring w.r.t. a with either an additive zero or an additive identity. Then 1 (1 is the multiplicative identity of S) is the additive zero or the additive identity.

Proof. Suppose that S has an additive zero c. Then $1c = c = c+c^2 = c(1+c) = cc$ which implies that $c = 1$.

Suppose that S has an additive identity 0. Then $0 = 01 = 0(1+0) = 01+00 = 0+00 = 00$ which implies that $0 = 1$. #

Proposition 4.40. Let S be a Classification IV semiring. Then S is not finite.

Proof. Let $a \in S$ be such that $(S-\{a\},\cdot)$ is a cancellative semigroup. Then $a^2 \neq a$. Let $1 \in S$ be the multiplicative identity of S. By Theorem 4.8, it is clear that $|S| > 2$. Suppose that S is finite. Then $(S-\{a\},\cdot)$ is a finite cancellative semigroup, so $(S-\{a\},\cdot)$ is a group. Let e be the identity of $(S-\{a\},\cdot)$, then $1 = e$. Since $a^2 \neq a$, there exists an $x \in S-\{a\}$ such that $a^2x = 1$, so a has a multiplicative inverse. Hence $(S,\cdot)$ is a group. Thus $(S,+,\cdot)$ is a finite ratio semiring. Hence $|S| = 1$, a contradiction. #

Proposition 4.41. Let S be a Classification V semiring w.r.t. a. If there exists a $d \in S-\{a\}$ such that $a+a = d+d$ then $ax = dx$ for all $x \in S$.

Proof. $ad^2(a+a) = ad(da+da)$, so $ad = a^2 = d^2$. Let $y \in S-\{a\}$. Then $ayd = ady = d^2y = dyd$ which implies that $ay = dy$. Hence $ax = dx$ for all $x \in S$. #

Proposition 4.42. Let S be a Classification V semiring w.r.t. a. If a is not M.C. in S then there exists a $d \in S-\{a\}$ such that $ax = dx$ for all $x \in S$.

Proof. Since a is not M.C. in S, there exist $d, y \in S$ such that $d \neq y$ and $ay = ad$. Then $d = a$ or $y = a$ since if $d, y \in S-\{a\}$ then $a^2y = a^2d$ which implies that $y = d$, a contradiction. Without loss of generality, we may assume that $y = a$. Then $a^2 = ad$ where $d \neq a$, so $a^2d^2 = ad^3$ which implies that $ad = d^2$, so $ad = a^2 = d^2$. Let $x \in S-\{a\}$
Then $axd = adx = d^2x = dxd$, so $ax = dx$. Therefore $ax = dx$ for all $x \in S$. #

Proposition 4.43. Let S be a Classification V semiring w.r.t. a. Assume that there exists a $d \in S-\{a\}$ such that $ax = dx$ for all $x \in S$. Then 1) $a+x = a$ or $a+x = d$ or $a+x = d+x$ for all $x \in S$.

2) If $a+a = a$ then $x+x = x$ for all $x \in S-\{d\}$.

3) If $a+a \neq a$ then $a+a = d$ or $a+a = d+d$ and for all $x, y \in S-\{a\}$, $x+x = y+y$ if and only if $x = y$.

Proof. 1) Assume that $a+x \neq a$. Then $d(a+x) = da+dx = dd+dx = d(d+x)$. If $d+x = a$. then $d(a+x) = ad = dd$ which implies that $a+x = d$. Suppose that $d+x \neq a$, so $d(a+x) = d(d+x)$ which implies that $a+x = d+x$.

2) Assume that $a+a = a$. Let $x \in S-\{d,a\}$. Then $ax(x+x) = ax^2+ax^2 = (a+a)x^2 = ax^2$ which implies that $x+x = x$ since $x+x \neq a$ if $x+x = a$ then $x+x = a+a$, so $xy = ay$ for all $y \in S$ therefore $xd = ad = dd$ which implies that $x = d$ or a, contradiction.

3) Suppose that $a+a \neq a$. Then $d(a+a) = da+da = dd+dd = (d+d)d$. If $d+d = a$, then $d(a+a) = ad = dd$ which implies that $a+a = d$. Suppose that $d+d \neq a$. Then $d(a+a) = d(d+d)$ which implies that $a+a = d+d$. #

Proposition 4.44. Let S be a Classification V semiring w.r.t. a. Assume that there exists a $d \in S-\{a\}$ such that $ax = dx$ for all $x \in S$ and for all $x,y \in S-\{a\}$, $x+y \neq a$. Then

1) $a+x = a$ or $a+x = d+x$ for all $x \in S$.

2) if $a+a = a$ then $(S,+)$ is a band.

Define $D = S-\{a\}$. Then

3) $I_D(a)$ is an additive subsemigroup of $I_D(d)$.

4) if $d \in I_D(a)$ then $I_D(a) = I_D(d)$.

5) $(D-I_D(a),+)$ is an ideal of $(D,+)$.

6) if $x,y \in I_D(d)$ then $xy \in I_D(d^2)$.

Proof. 1) Let $x \in S-\{a\}$. Suppose that $a+x \neq a$. Then $d(a+x) = da+dx = dd+dx = d(d+x)$ which implies that $a+x = d+x$. If $a+a \neq a$ then $d(a+a) = da+da = dd+dd = d(d+d)$ which implies that $a+a = d+d$.

2) Let $x \in S-\{a\}$. Then $ax(x+x) = ax^2+ax^2 = (a+a)x^2 = ax^2$ which implies that $x+x = x$. Thus $(S,+)$ is a band.

3) Let $x \in I_D(a)$. We must show that $x \in I_D(d)$. $d^2 = ad = (x+a)d = xd+ad = xd+dd = d(x+d)$ which implies that $d = x+d$. Thus $I_D(a) \subseteq I_D(d)$. Since $(I_D(d),+)$ is a semigroup and for all $x,y \in I_D(a)$, $a+(x+y) = (a+x)+y = a+y = a$ we get that $x+y \in I_D(a)$. Hence $I_D(a)$ is an additive subsemigroup of $I_D(d)$.

4) If $d \in I_D(a)$, let $x \in I_D(d)$. Then $a = a+d = a+(d+x) = (a+d)+x = a+x$, so $x \in I_D(a)$. Thus $I_D(a) = I_D(d)$.

5) Let $x \in D-I_D(a)$ and $y \in D$. If $y \in I_D(a)$ then $(x+y)+a = x+(y+a) = x+a \neq a$. Suppose that $y \notin I_D(a)$. Then $(x+y)+a = x+(y+a) \neq a$. Hence $x+y \in D-I_D(a)$.

6) Let $x,y \in I_D(d)$. Then $x+d = y+d = d$. Hence $d^2 = (x+d)(y+d) = xy+xd+yd+d^2 = xy+xd+d(y+d) = xy+xd+d^2 = xy+(x+d)d = xy+d^2$. Thus $xy \in I_D(d^2)$. #

Now we shall give some example of Theorem 4.44.

Example 4.45. $(\mathbb{Q}^+,\min,\cdot)$ is a ratio semiring. Let a be a symbol not representing any element in $\mathbb{Q}^+$. Then $I_{\mathbb{Q}^+}(3) = \{x \in \mathbb{Q}^+ \mid x \geq 3\}$. Let $S = \{x \in \mathbb{Q}^+ \mid x > 6\}$. Clearly S is an additive subsemigroup of $I_{\mathbb{Q}^+}(3)$ and $\mathbb{Q}^+-S$ is an ideal of $(\mathbb{Q}^+,+)$. Then by Theorem 1.42 we can extend the binary operations of $\mathbb{Q}^+$ to $K = \mathbb{Q}^+ \cup \{a\}$ making K into a semifield of type III and also a Classification V semiring w.r.t. a by

1) $ax = xa = 3x$ for all $x \in \mathbb{Q}^+$ and $a^2 = 9$,
2) $a+x = x+a = a$ for all $x \in S$ and
   $a+x = x+a = 3+x$ for all $x \in \mathbb{Q}^+-S$,
3) $a+a = 3$.

Proposition 4.46. Let S be a finite Classification V semiring. Then $|S| = 2$.

Proof. Let $a \in S$ be such that $(S-\{a\},\cdot)$ is a cancellative semigroup. Then $(S-\{a\},\cdot)$ is a finite cancellative semigroup so $(S-\{a\},\cdot)$ is a group. Let e be the identity of $(S-\{a\},\cdot)$. Claim that $x+y \neq a$ for all $x,y \in S-\{a\}$. To prove this, suppose not. Let $x,y \in S-\{a\}$ be such that $x+y = a$.

Then $a = x+y = xe+ye = (x+y)e = ae$, a contradiction. Hence we have the claim. Thus $(S-\{a\},+,\cdot)$ is a finite ratio semiring. Therefore $|S-\{a\}| = 1$ so $|S| = 2$. #

Now we shall give some examples of a Classification V semirings S w.r.t. a such that a is not M.C. in S. Then there exists a $d \in S-\{a\}$ such that $ax = dx$ for all $x \in S$ and S has the following properties:

1) there exist $x,y \in S-\{a\}$ such that $x+y = a$ and there exist $u,v \in S-\{d\}$ such that $u+v = d$.

2) there exist $x,y \in S-\{a\}$ such that $x+y = a$ and for all $u,v \in S$, $u+v \neq d$ but there exist $z,w \in S$ such that $zw = d$.

3) there exist $x,y \in S-\{a\}$ such that $x+y = a$ and for all $u,v \in S$, $u+v \neq d$ and $uv \neq d$.

Example 4.47. $\mathbb{Z}^+-\{1,3\}$ with the usual addition and multiplication is an M.C. semiring. Let a, b be symbols not representing any element of $\mathbb{Z}^+-\{1,3\}$. We can extend the binary operation + and $\cdot$ of $\mathbb{Z}^+-\{1,3\}$ to $S = (\mathbb{Z}^+-\{1,3\})\cup\{a,b\}$ by defining

(1) $aa = 36$, $ab = ba = 18$ and $ax = 6x$ for all $x \in \mathbb{Z}^+-\{1,3\}$.
$bb = 9$ and $bx = xb = 3x$ for all $x \in \mathbb{Z}^+-\{1,3\}$.

(2) $a+a = 12$, $a+b = b+a = 9$ and $a+x = x+a = 6+x$ for all $x \in \mathbb{Z}^+-\{1,3\}$.
$b+b = a$ and $b+x = x+b = 3+x$ for all $x \in \mathbb{Z}^+-\{1,3\}$.

Claim that $(S,+,\cdot)$ is a semiring. We must show that

(a) $(x+y)+z = x+(y+z)$ for all $x,y,z \in S$,

(b) $(xy)z = x(yz)$ for all $x,y,z \in S$, and

(c) $x(y+z) = xy+xz$ for all $x,y,z \in S$.

To show (a), we shall consider the following cases:

Case 1. $x = y = z = a$. Then $(x+y)+z = x+(y+z)$.

Case 2. $x = y = a$, $z \neq a$.

Subcase 2.1. $z = b$. Then $(x+y)+z = (a+a)+b = 12+b = 15$ and $x+(y+z) = a+(a+b) = a+9 = 6+9 = 15$.

Subcase 2.2. $z \neq b$. Then $(x+y)+z = (a+a)+z = 12+z$ and $a+(a+z) = a+(6+z) = 6+(6+z) = 12+z$.

Case 3. $x = a$, $y \neq a$, $z = a$. Done.

Case 4. $x = a$, $y \neq a$, $z \neq a$.

Subcase 4.1. $y = z = b$. Then $(x+y)+z = (a+b)+b = 9+b = 12$ and $x+(y+z) = a+(b+b) = a+a = 12$.

Subcase 4.2. $y = b$, $z \neq b$. Then $(x+y)+z = (a+b)+z = 9+z$ and $x+(y+z) = a+(b+z) = a+(3+z) = 6+(3+z) = 9+z$.

Subcase 4.3. $y \neq b$, $z = b$. Then $(x+y)+z = (a+y)+b = (6+y)+3 = 6+(y+3) = a+(y+3) = a+(y+b) = x+(y+z)$.

Subcase 4.4. $y \neq b$, $z \neq b$. Then $(x+y)+z = (a+y)+z = (6+y)+z = 6+(y+z) = a+(y+z) = x+(y+z)$.

Case 5. $x \neq a$, $y = a$, $z = a$.

$(x+y)+z = (x+a)+a = a+(a+x) = (a+a)+x = x+(a+a) = x+(y+z)$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

$(x+y)+z = (x+a)+z = (a+x)+z = a+(x+z) = a+(z+x) = (a+z)+x =$
$x+(a+z) = x+(y+z)$.

Case 7. $x \neq a$, $y \neq a$, $z = a$.

$(x+y)+a = a+(y+x) = (a+y)+x = x+(y+z)$.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

Subcase 8.1. $x = y = z = b$. Then $(x+y)+z = x+(y+z)$.

Subcase 8.2. $x = y = b$, $z \neq b$. Then $(x+y)+z = (b+b)+z =$
$a+z = 6+z = 3+(3+z) = b+(3+z) = b+(b+z) = x+(y+z)$.

Subcase 8.3. $x = b$, $y \neq b$, $z = b$. Then $(x+y)+z =$
$(b+y)+b = b+(y+b) = x+(y+z)$.

Subcase 8.4. $x = b$, $y \neq b$, $z \neq b$. Then $(x+y)+z =$
$(b+y)+z = (3+y)+z = 3+(y+z) = b+(y+z) = x+(y+z)$.

All other subcase can be proven in the same way that Subcase 8.2, Subcase 8.3 and Subcase 8.4 were proven.

To show (b) and (c), we shall consider the following cases:

Case 1. $x = y = z = a$.

$(xy)z = x(yz)$.

$x(y+z) = a(a+a) = a(12) = 72 = 36+36 = aa+aa = xy+xz$.

Case 2. $x = y = a$, $z \neq a$.

Subcase 2.1. $z = b$. Then $(xy)z = (aa)b = (36)b = 108 =$
$6(18) = 6(ab) = a(ab) = x(yz)$.

$x(y+z) = a(a+b) = a(9) = 54 = 36+18 = aa+ab = xy+xz$.

Subcase 2.2. $z \neq b$. Then $(xy)z = (aa)z = 36z = 6(6z) =$
$a(az) = x(yz)$.

$x(y+z) = a(a+z) = a(6+z) = 6(6+z) = 36+6z = aa+az = xy+xz$.

Case 3. $x = a,\ y \neq a,\ z = a$.

$(xy)z = (ay)a = a(ya) = x(yz)$.

$x(y+z) = a(y+a) = a(a+y) = aa+ay = ay+aa = xy+xz$.

Case 4. $x = a,\ y \neq a,\ z \neq a$.

Subcase 4.1. $y = z = b$. Then $(xy)z = (ab)b = 18b = 54 =$
$(6)(9) = a(9) = a(bb) = x(yz)$.

$x(y+z) = a(b+b) = aa = 36 = 18+18 = ab+ab = xy+xz$.

Subcase 4.2. $y = b,\ z \neq b$. Then $(xy)z = (ab)z = 18z =$
$6(3z) = a(3z) = a(bz) = x(yz)$.

$x(y+z) = a(b+z) = a(3+z) = 6(3+z) = 18+\ 6z = ab+az = xy+xz$.

Subcase 4.3. $y \neq b,\ z = b$. Then $(xy)z = (ay)b = (6y)b =$
$(6y)3 = 6(y3) = a(y3) = a(yb) = x(yz)$.

$x(y+z) = a(y+b) = a(b+y) = ab+ay = ay+ab = ab+ay = xy+xz$.

Subcase 4.4. $y \neq b,\ z \neq b$. Then $(xy)z = (ay)z = (6y)z =$
$6(yz) = a(yz) = x(yz)$.

$x(y+z) = a(y+z) = 6(y+z) = 6y+6z = ay+az = xy+xz$.

Case 5. $x \neq a,\ y = a,\ z = a$.

$(xy)z = (xa)a = a(ax) = (aa)x = x(aa) = x(yz)$.

Subcase 5.1. $x = b$. Then $x(y+z) = b(a+a) = b(12) = 36 =$
$18+18 = ba+ba = xy+xz$.

Subcase 5.2. $x \neq b$. Then $x(y+z) = x(12) = x6+x6 =$
$xa+xa = xy+xz$.

Case 6. $x \neq a,\ y = a,\ z \neq a.$

$(xy)z = (xa)z = (ax)z = a(xz) = a(zx) = (az)x = x(az) = x(yz)$

Subcase 6.1. $x = b,\ z = b.$ Then $x(y+z) = b(a+b) =$ $b(9) = 27 = 18+9 = ba+bb = xy+xz.$

Subcase 6.2. $x = b,\ z \neq b.$ Then $x(y+z) = b(a+z) =$ $b(6+z) = 3(6+z) = 18+3z = ba+bz = xy+xz.$

Subcase 6.3. $x \neq b,\ z = b.$ Then $x(y+z) = x(a+b) =$ $x(9) = x6+x3 = xa+xb = xy+xz.$

Subcase 6.4. $x \neq b,\ z \neq b.$ Then $x(y+z) = x(a+z) =$ $x(6+z) = x6+xz = xa+xz = xy+xz.$

Case 7. $x \neq a,\ y \neq a,\ z = a.$

$(xy)z = (xy)a = a(yx) = (ay)x = x(ya) = x(yz).$

$x(y+z) = x(y+a) = x(a+y) = xa+xy = xy+xa = xy+xz.$

Case 8. $x \neq a, y \neq a,\ z \neq a.$

Subcase 8.1. $x = y = z = b$. Then $(xy)z = x(yz).$

$x(y+z) = b(b+b) = ba = 18 = 9+9 = bb+bb = xy+xz.$

Subcase 8.2. $x = y = b,\ z \neq b.$ Then $(xy)z = (bb)z =$

$9z = 3(3z) = 3(bz) = b(bz) = x(yz).$

$x(y+z) = b(b+z) = b(3+z) = 3(3+z) = 9+3z = bb+bz = xy+xz.$

Subcase 8.3. $x = b,\ y \neq b,\ z = b.$ Then

$(xy)z = (by)b = b(yb) = x(yz).$

$x(y+z) = b(y+b) = b(b+y) = bb+by = by+bb = xy+xz.$

Subcase 8.4. $x = b$, $y \neq b$, $z \neq b$. Then

$(xy)z = (by)z = (3y)z = 3(yz) = b(yz) = x(yz)$.

$x(y+z) = b(y+z) = 3(y+z) = 3y+3z = by+bz = xy+xz$.

All other subcase can be proven in the same way that Subcase 8.2, Subcase 8.3, and Subcase 8.4 were proven.

Hence S is a semiring. Clearly $(S-\{a\},\cdot)$ is a semigroup, $a^2 \neq a$, a is not M.C. in S, $ax = 6x$ for all $x \in S$, $b \in S-\{a\}$ has the property that $b+b = a$, and $2,4 \in S-\{6\}$ have the property that $2+4 = 6$.

Claim that $(S-\{a\},\cdot)$ is a cancellative semigroup. Let $x,y,z \in S-\{a\}$ be such that $xy = xz$. We must show that $y = z$.

Case 1. $x = b$.

Subcase 1.1. $y = b$. Then $9 = bz = 3z$. Hence $z = b = y$. Similarly if $z = b$ then $y = b$.

Subcase 1.2. $y \neq b$. Then $3y = 3z$, $y = z$.

Case 2. $x \neq b$.

Subcase 2.1. $y = b$. Then $x3 = xz$ so $z = b$. Similarly if $z = b$ then $y = b$.

Subcase 2.2. $y \neq b$. Then $x,y,z \in S-\{a,b\}$ so $y = z$.

Hence S is a Classification V semiring w.r.t. a. #

Example 4.48. $\mathbb{Z}^+-\{1,2\}$ with the usual addition and multiplication is an M.C. semiring. Let a,b be symbols not representing any element of $\mathbb{Z}^+-\{1,2\}$. We can extend the binary operation of $\mathbb{Z}^+-\{1,2\}$ to $S = (\mathbb{Z}^+-\{1,2\})\cup\{a,b\}$ by defining

(1) $aa = 16$, $ab = ba = 8$ and $ax = 4x$ for all $x \in \mathbb{Z}^+ - \{1,2\}$

$bb = 4$, $bx = xb = 2x$ for all $x \in \mathbb{Z}^+ - \{1,2\}$.

(2) $a+a = 8$, $a+b = b+a = 6$ and $a+x = x+a = 4+x$

for all $x \in \mathbb{Z}^+ - \{1,2\}$.

$b+b = a$ and $b+x = x+b = 2+x$ for all $x \in \mathbb{Z}^+ - \{1,2\}$.

Using a proof similar to the proof in Example 4.47, we can show that $(S,+,\cdot)$ is a semiring. Furthermore, S is a Classification V semiring w.r.t. a such that $b+b = a$, $bb = 4$, $ax = 4x$ for all $x \in S$ and for all $u,v \in S$ $u+v \neq 4$. #

Example 4.49. $\mathbb{Z}^+ - \{1\}$ with the usual addition and multiplication is an M.C. semiring. Let a be a symbol not representing any element of $\mathbb{Z}^+ - \{1\}$. We can extend the binary operations of $\mathbb{Z}^+ - \{1\}$ to $S = (\mathbb{Z}^+ - \{1\}) \cup \{a\}$ by defining

(1) $ax = xa = 5x$ for all $x \in S$,

(2) $a+x = x+a = 5+x$ for all $x \in S$.

Using a proof similar to the proof of Example 4.16, we can show that S is a Classification V semiring w.r.t. 5, such that $5x = ax$ for all $x \in S$, $2+3 = 5$, $xy \neq a$, and $x+y \neq a$ for all $x,y \in S$. #