CHAPTER V

GENERALIZED QUOTIENT SEMIFIELDS

In [1] it was show that if S is a O-M.C. semiring then S can be embedded in a O-semifield and, in fact there exists a smallest O-semifield K (up to isomorphism) containing S. K is called the quotient O-semifield of S.

In [2] the concept of semifield was generalized and in Chapter IV we generalized the concept of O-M.C. to A.M.C. We now study the problem of whether or not an A.M.C. semiring can be embedded in a generalized semifield and if so we would like to know whether or not a smallest such semifield exists up to isomorphism. In some cases, in order to find a smallest generalized semifield containing a certain A.M.C. semiring, we shall have to generalized the concept of a quotient semifield by restricting the category of semirings under consideration.

In this Chapter every semiring is assumed to have order greater than two.

Theorem 5.1. Let S be a Classification I semiring. Then we can embed S in a type I semifield and not in any other type of semifield.

Proof. Let $a \in S$ be such that $(S-\{a\},\cdot)$ is a cancellative semigroup. Then $ax = a$ for all $x \in S$. Claim that S is O-M.C. Let $x,y,z \in S$ be such that $xy = xz$ and $x \neq a$. To show $y = z$.

If $y = a$, then $z = a$. Similarly if $z = a$ then $y = a$. Suppose $y, z \in S-\{a\}$. Hence $y = z$, so we have the claim. By Theorem 1.27, S can be embed in a type I semifield.

To show that S **cannot** be embedded in any other type of semifield, suppose not. Let $f:S \longrightarrow K$ be a monomorphism where K is a type II semifield w.r.t. a' or K is a type III semifield w.r.t. a'.

Case 1. K is a type II semifield w.r.t. a'. Let $x \in S-\{a\}$. Then $aa = ax = a$. Hence $f(a) \neq a'$. If $f(x) \neq a'$ then $f(a)f(a) = f(a)f(x)$ which implies that $f(a) = f(x)$. Thus $x = a$, contradiction. Suppose that $f(x) = a'$. Let $y \in S-\{a,x\}$. Then $aa = ay$. Hence $f(a)f(a) = f(a)f(y)$ which implies that $f(a) = f(y)$. Hence $a = y$, a contradiction.

Case 2. K is a type III semifield w.r.t. a'. Let $x \in S-\{a\}$ be such that $f(x) \neq a'$. Since $f(a) \neq a'$ and $aa = ax$, $f(a) = f(x)$ which implies that $a = x$, a contradiction. #

Theorem 5.2. Let S be a classification II semiring w.r.t. a. Assume that $1+x \neq 1$ for all $x \neq 1$. Then we can embed S in a type II semifield and not in any other type of semifield.

Proof. First we shall show that S cannot be embedded in a type I semifield or a type III semifield. To prove this, suppose not. Then there is a monomorphism $f:S \longrightarrow K$ where K is a type I semifield w.r.t. a' or K is a type III semifield w.r.t. a'.

Case 1. K is a type I semifield w.r.t. a'. Let $x \in S-\{1,a\}$, so $1x = ax = x$. Hence $f(a) \neq a'$ and $f(1) \neq a'$. Since $1a = aa$, $f(1)f(a) = f(a)f(a)$ which implies that $f(1) = f(a)$. Hence $1 = a$, a contradiction.

Case 2. K is a type III semifield w.r.t. a'. Since $11 = 1$ and $aa = a$, clearly $f(1) \neq a'$ and $f(a) \neq a'$. Since $1a = aa$, $f(1)f(a) = f(a)f(a)$ which implies that $1 = a$, a contradiction.

Next to show that S can be embedded in a type II semifield. By Proposition 4.18, $(S-\{1\},+,\cdot)$ is an M.C. semiring. By Theorem 1.12. $S-\{1\}$ has a quotient ratio semiring say $(D,+,\cdot)$ where $D = \underline{(S-\{1\})\times(S-\{1\})}_{\sim}$. Let $f:S-\{1\}\longrightarrow D$ be the natural embedding. Let $e' = [(a,a)] \in D$ be the multiplicative identity. Let a' be a symbol not representing any element of D. Then we can extend the binary operations of D to $K = D \cup \{a'\}$ by defining $a'\alpha = \alpha a' = \alpha$ for all $\alpha \in K$ and $a'+\alpha = \alpha+a' = e'+\alpha$ for all α D. In order to define $a'+a$ we consider two cases. If $1+1 = 1$ then $a+a = a$. Hence $e'+e' = e'$ so define $a'+a' = a'$. If $1+1 \neq 1$ then $a+a = a$ or $a+a \neq a$. Hence $e'+e' = e'$ or $e'+e' \neq e'$, so define $a'+a' = e'+e'$. Then K is a type II semifield w.r.t. a' (by Theorem 1.39). Extend $f:S-\{1\}\longrightarrow D$ to $f:S\longrightarrow K$ by defining $f(1) = a'$. Clearly f is 1-1. To show that f is a homomorphism, let $x,y \in S$. We shall show that

(a) $f(x+y) = f(x)+f(y)$ and

(b) $f(xy) = f(x)f(y)$.

To show (a) we shall consider the following cases:

Case 1. $x = y = 1$.

$$f(x+y) = f(1+1) = \begin{cases} a' & \text{if } 1+1 = 1, \\ f(a+a) & \text{if } 1+1 \neq 1 \text{ by Theorem 4.22 (3).} \end{cases}$$

$$\text{Hence } f(1+1) = \begin{cases} a' & \text{if } 1+1 = 1, \\ f(a)+f(a) = e'+e' & \text{if } 1+1 \neq 1. \end{cases}$$

$$f(1)+f(1) = a'+a' = \begin{cases} a' & \text{if } 1+1 = 1, \\ e'+e' & \text{if } 1+1 \neq 1. \end{cases}$$ Thus $f(x+y) = f(x)+f(y)$.

Case 2. $x = 1$, $y \neq 1$.

$f(x+y) = f(1+y) = f(a+y) = f(a)+f(y) = e'+f(y) = a'+f(y) = f(1)+f(y) = f(x)+f(y)$.

Case 3. $x \neq 1$, $y = 1$ (same proof as Case 2).

Case 4. $x \neq 1$, $y \neq 1$. Done.

To show (b), if $x = 1$ or $y = 1$ then $f(xy) = f(x)f(y)$. Suppose that $x \neq 1$ and $y \neq 1$ then clearly $f(xy) = f(x)f(y)$. #

Theorem 5.3. Let S be a classification II semiring w.r.t. a. Assume that $1+a = 1$ (hence $I_{S-\{1\}}(1) = I_{S-\{1\}}(a)$). Then we can embed S in a type II semifield and not in any other type of semifield.

Proof. Clearly, by Theorem 5.2, S cannot be embedded in a type I or type III semifield. Since $1+a = 1$, $x+x = x$ for all $x \in S-\{1\}$. By Proposition 4.18, $(S-\{1\},+,\cdot)$ is an M.C. semiring so $QR(S-\{1\})$ exists. Let $D = QR(S-\{1\})$. Let $e' = [(a,a)] \in D$. Then e' is the multiplicative identity of D. Let $f: S-\{1\} \longrightarrow D$ be the natural embedding. By Proposition 1.21, $I_D(e') = \emptyset$ or $I_D(e')$ is additive subsemigroup of D. Claim that $D-I_D(e')$ is an ideal of $(D,+)$. Let $\alpha \in D-I_D(e')$ and $\beta \in D$. Choose $(x,y) \in \alpha$, $(z,w) \in \beta$.

Case 1. $\beta \in I_D(e')$. Then $(\alpha+\beta)+e' = \alpha+(\beta+e') = \alpha+e' \neq e'$. Hence $\alpha+\beta \in D-I_D(e')$.

Case 2. $\beta \notin I_D(e')$. Then $e' \neq \beta+e' = [(z,w)]+[(a,a)] = [(z+w,w)]$. Hence $z+w \neq w$. Similarly $x+y \neq y$. Claim that $yz+xw+yw \neq yw$. To prove this, suppose not. Then $yz+yw = yz+yz+xw+yw = yz+xw+yw = yw$. Hence $y(z+w) = yw$ which implies that $z+w = w$, a contradiction. Therefore $([(z,w)]+[(x,y)])+[(a,a)] = [(yz+xw,yw)]+[(a,a)] = [(yz+xw+yw,yw)] \neq [(a,a)]$.

Hence $D-I_D(e')$ is an ideal of $(D,+)$ so we have the claim. Let a' be a symbol not representing any element of D. We can extend the binary operation of D to $K = D\cup\{a'\}$ by defining

(1) $a'\alpha = \alpha a' = \alpha$ for all $\alpha \in K$,

(2) $a'+\alpha = \alpha+a' = a'$ for all $\alpha \in I_D(e')$

$a'+\alpha = \alpha+a' = e'+\alpha$ for all $\alpha \in D-I_D(e')$,

(3) $$a'+a' = \begin{cases} a' & \text{if } 1+1 = 1, \\ e' & \text{if } 1+1 \neq 1. \end{cases}$$

Then K is a type II semifield w.r.t. a'(by Theorem 1.39). Extend $f:S-\{1\}\longrightarrow D$ to $f:S\longrightarrow K$ by defining $f(1) = a'$. Clearly f is 1-1. We must show that f is a homomorphism. Let $x,y \in S$. We must show that

(a) $f(x+y) = f(x)+f(y)$,

(b) $f(xy) = f(x)f(y)$.

The proof of (b) is similar to the proof of (b) in Theorem 5.2.

To show (a), we shall consider the following cases:

Case 1. $x = y = 1$.

Subcase 1.1. 1+1 = 1. Then f(1+1) = f(1) = a' = a'+a' = f(1)+f(1).

Subcase 1.2. 1+1 $\neq$ 1. Then f(1+1) = f(a+a) = f(a) = e' = a'+a' = f(1)+f(1).

Case 2. x = 1, y $\neq$ 1.

Subcase 2.1. 1+y = 1. Then f(1+y) = f(1) = a' and f(1)+f(y) = a'+ [(y,a)]. Since 1+y = 1, a+y = a. Hence [(y,a)] + [(a,a)] = [(y+a,a)] = [(a,a)] therefore a'+ [(y,a)] = a'.

Subcase 2.2. 1+y $\neq$ 1. Then f(1+y) = f(a+y) = f(a)+f(y) = e'+ [(y,a)]. Since 1+y $\neq$ 1, a+y $\neq$ a. Hence [(y,a)] + [(a,a)] $\neq$ [(a,a)] therefore f(1)+f(y) = a'+f(y) = a'+ [(y,a)] = e'+ [(y,a)].

Case 3. x $\neq$ 1, y = 1. The proof is the same as Case 2.

Case 4. x $\neq$ 1, y $\neq$ 1. Done. #

Theorem 5.4. Let S be a Classification III semiring w.r.t. a of form 1. Assume that a+x $\neq$ a for all x $\neq$ a. (Hence $I_{S-\{a\}}(a) = \emptyset$) or a+e = a (Hence $I_{S-\{a\}}(a) = I_{S-\{a\}}(e)$). Then we can embed S in a type II semifield and not in any other type of a semifield.

Proof. Suppose that a+x = a for all x $\neq$ a i.e. $I_{S-\{a\}}(a) = \emptyset$. Using a proof similar to the proof of Theorem 5.2 (substitute a for 1 and e for a) we can show that we can embed S in a type II semifield and not in any other type of semifield.

Suppose a+e = a. By Proposition 4.32 (2) $I_{S-\{a\}}(a) = I_{S-\{a\}}(e)$. Using a proof similar to the proof of Theorem 5.3 (substitute a for 1 and e for a) we can show that we can embed S in a type II semifield and not inany other type of semifield. #

<u>Theorem 5.5</u>. Let S be a Classification III semiring of form 2. Then S can be embedded in any type of semifield.

<u>Proof</u>. Since S is a Classification III semiring of form 2 S is an M.C. semiring. By Corollary 1.45, Propostion 1.46 and Proposition 1.47. S can be embedded in all type of semifield. #

<u>Theorem 5.6</u>. Let S be a Classification IV semiring w.r.t. a. If a is not M.C. in S then we cannot embbed S in any semifield.

<u>Proof</u>. Suppose not. Then there exists a monomorphism $f:S \longrightarrow K$ where K is a semifield. Let $a' \in K$ be such that $(K-\{a'\},\cdot)$ is a group. Let e' be the identity of $(K-\{a'\},\cdot)$. Since a is not M.C. in S, there exist $x,y \in S$ such that $x \neq y$ and ax = ay. Therefore x = a or y = a (since if $x \neq a$ and $y \neq a$ then $a^2x = a^2y$ which implies that x = y, a contradiction). Assume that x = a. Hence $y \in S-\{a\}$ and we have that aa = ay. Claim that $f(a) \neq a'$. To prove this, suppose not.

<u>Case 1</u>. K is a type I ro type II semifield. Then $f(aa) = f(a)f(a) = a'a' = a' = f(a)$. Hence $a^2 = a$, a contradiction.

<u>Case 2</u>. K is a type III semifield. Then $a' = f(a) = f(1a) = f(1)f(a) = f(1)a'$ contradiction Proposition 1.36 ($xy \neq a'$ for all $x,y \in K$).

Hence we have the claim. Since $aa = ay$, $aaay = ayay = ayya = aayy$ which implies that $ay = yy$ (since $y \neq 1$). Thus $aa = ay = yy$. Clearly $yy \neq y$. Thus $f(y) \neq a'$ (the proof is similar to the above proof). Hence $f(a)f(a) = f(a)f(y)$ which implies that $f(a) = f(y)$ so $a = y$, a contradiction. #

Theorem 5.7. Let S be a Classification IV semiring w.r.t. a. Assume a is M.C. in S but S is not M.C. Then we cannot embed S in any type of semifield.

Proof. Since S is not M.C. there exists an $x \in S-\{a\}$ such that x is not M.C.. Then there exist distinct $y, d \in S$ such that $xy = xd$. Clearly $y = a$ or $d = a$. Assume that $y = a$. Hence $d \in S-\{a\}$ and we have $xa = xd$. Claim that $d = 1$. To prove this, suppose not. Then $ad \neq a$ (if $ad = a$ then $a^2d = a^2 1$ which implies that $d = 1$, a contradiction). Hence $xaa = xda$ which implies that $aa = da$. Therefore $a = d$ (since a is M.C. in S), a contradiction. Hence we have the claim. Therefore $xa = x$ and $a^2x = ax = x = 1x$ so we have $a^2 = 1$. Since for all $y \in S-\{1,a\}$, $ay \neq a$, we get that $ayx = axy = xy = yx$, which implies that $ay = y$. Suppose that S can be embedded in a semifield. Then there exists a monomorphism $f: S \longrightarrow K$ where K is a semifield. Let $a' \in K$ be such that $(K-\{a'\}, \cdot)$ is a group. Let e' be the identity of $(K-\{a'\}, \cdot)$. Claim that $f(1) \neq a'$ and $f(a) \neq a'$.

Case 1. K is a type I semifield. Clearly $f(a) \neq a'$. If $f(1) = a'$ then $f(a) = f(1a) = f(1)f(a) = a'f(a) = a' = f(1)$. Hence $a = 1$, a contradiction.

Case 2. K is a type II semifield. Clearly $f(a) \neq a'$.
If $f(1) = a'$ then $f(a)f(a) = f(aa) = f(1) = a'$, contradiction the fact that $(K-\{a'\}, \cdot)$ is a group.

Case 3. K is a type III semifield. Clearly $f(1) \neq a'$.
If $f(a) = a'$ then $a' = f(a) = f(1a) = f(1)f(a) = f(1)a'$, a contradiction.

Hence we have the Claim. Let $z \in S-\{1,a\}$ be such that $f(z) \neq a'$. Then $az = 1z = z$ so $f(a)f(z) = f(1)f(z)$. Therefore $f(a) = f(1)$. Thus $1 = a$, a contradiction. #

Theorem 5.8. Let S be a Classification IV semiring w.r.t. a. If S is M.C. then S can be embedded in all type of semifield.

Proof. Use Corollary 1.45, Proposition 1.46 and Proposition 1.47. #

Theorem 5.9. Let S be a Classification V semiring w.r.t. a. If a is M.C. in S then S can be embedded in all type of semifield.

Proof. Since a is M.C. in S, S is M.C.. Using Corollary 1,45, Proposition 1.46 and Proposition 1.47 we see that S can be embedded in all type of semifield. #

Theorem 5.10. Let S be a ClassificationV semiring w.r.t. a. If a is not M.C. in S then S cannot be embedded in a type I semifield and S cannot embedded in a type II semifield.

Proof. Since a is not M.C. in S, there exists $d \in S-\{a\}$ such that $ax = dx$ for all $x \in S$. Suppose that S can be embedded in a type I semifield or a type II semifield. Then there exists a monomorphism $f:S\longrightarrow K$ where K is a type I or type II semifield. Let $a'\in K$ be such that $(K-\{a'\},\cdot)$ is a group. Clearly $f(a)\neq a'$. Claim that $f(d)\neq a'$. If $f(d)=a'$ then $a'=a'a'=f(d)f(d)=f(dd)=f(aa)=f(a)f(a)$ contradicting the fact that $(K-\{a'\},\cdot)$ is a group. Hence we have the claim. Since $aa=ad$, $f(a)f(a)=f(a)f(d)$. Therefore $f(a)=f(d)$. Thus $a=d$, a contradiction. #

<u>Theorem 5.11.</u> Let S be a Classification V semiring w.r.t. a. If a is not M.C. in S and $x+y\neq a$ for all $x,y\in S$. Then S can be embedded in a type III semifield.

<u>Proof</u>. Since a is not M.C. in S, there exists $d\in S-\{a\}$ such that $ax=dx$ for all $x\in S$. Since for all $x,y\in S$, $x+y\neq a$, $(S-\{a\},+,\cdot)$ is an M.C. semiring. Hence $QR(S-\{a\})$ exists. Let $D=QR(S-\{a\})$. Let $f:S-\{a\}\longrightarrow D$ be the natural embedding i.e. $f(x)=[\![(xd,d)]\!]$ for all $x\in S-\{a\}$. Let a' be a symbol not representing any element of D. We can extend the binary operation of D to $K=D\cup\{a'\}$ by defining $a'\alpha=\alpha a'=f(d)\alpha$ for all $\alpha\in K$ and $a'+\alpha=\alpha+a'=f(d)+\alpha$ for all $\alpha\in K$. Then $(K,+,\cdot)$ is a type III semifield w.r.t. a' (by Theorem 1.42). Extend $f:S-\{a\}\longrightarrow D$ to $f:S\longrightarrow K$ by defining $f(a)=a'$. Clearly f is 1-1. To show that f is a homomorphism. Since $x+y\neq a$ for all $x,y\in S$, $a+x=d+x$ for all $x\in S$ (by Theorem 4.44 (1)). Let $x,y\in S$. To show that $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$.

Case 1. $x = y = a$.

$f(x+y) = f(a+a) = f(d+d) = f(d)+f(d) = a'+a' = f(a)+f(a) = f(x)+f(y)$.

$f(xy) = f(aa) = f(dd) = f(d)f(d) = a'a' = f(a)f(a) = f(x)f(y)$.

Case 2. $x = a$, $y \neq a$.

$f(x+y) = f(a+y) = f(d+y) = f(d)+f(y) = a'+f(y) = f(a)+f(y) = f(x)+f(y)$.

$f(xy) = f(ay) = f(dy) = f(d)f(y) = a'f(y) = f(a)f(y) = f(x)f(y)$.

Case 3. $x \neq a$, $y = a$. Same proof as Case 2.

Case 4. $x \neq a$, $y \neq a$. Done. #

Theorem 5.12. Let S be a Classification V semiring w.r.t. a. Assume that a is not M.C. in S. Let $d \in S-\{a\}$ be such that $ax = dx$ for all $x \in S$. If there exist $x,y \in S-\{a\}$ such that $x+y = a$ and there exist $u,v \in S-\{d\}$ such that $u+v = d$ then S cannot be embedded in a type III semifield.

Proof. Suppose not. Then there exists a monomorphism $f:S \longrightarrow K$ where K is a type III semifield. Let $a' \in K$ be such that $(K-\{a'\},\cdot)$ is a group. Let e be the identity of $(K-\{a'\},\cdot)$. Claim that $f(a) \neq a'$ and $f(d) \neq a'$. Suppose $f(a) = a'$. Since there exist $x,y \in S-\{a\}$ such that $x+y = a$, so $a' = f(a) = f(x+y) = f(x)+f(y) = f(x)e+f(y)e = (f(x)+f(y))e = a'e$, a contradiction. Similary $f(d) \neq a'$. Hence we have the claim. But $f(a)f(d) = f(ad) = f(dd) = f(d)f(d)$. Therefore $f(a) = f(d)$ so $a = d$, a contradiction. #

Theorem 5.13. Let S be a Classification V semiring w.r.t. a. Assume that a is not M.C. in S. Let $d \in S-\{a\}$ be such that

$ax = dx$ for all $x \in S$. If there exist $x,y \in S-\{a\}$ such that $x+y = a$ and for all $u,v \in S$ $u+v \neq d$ but there exist $z,w \in S$ such that $zw = d$ then S **cannot** be embedded in a type III semifield.

<u>Proof</u>. Suppose not. Then there exists a monomorphism $f:S \longrightarrow K$ where K is a type III semifield w.r.t. a'. Using a similar proof to the proof of Theorem 5.12 we can show that $f(a) \neq a'$. Since there exist $z,w \in S$ such that $zw = d$, $f(d) \neq a'$. Again, using a similar proof to the one in Theorem 5.12, $f(a) = f(d)$ which implies that $a = d$, a contradiction. #

<u>Theorem 5.14</u>. Let S be a Classification V semiring w.r.t. a. Assume that a is not M.C. in S. Let $d \in S-\{a\}$ be such that $ax = dx$ for all $x \in S$. If there exist $x,y \in S-\{a\}$ such that $x+y = a$ and for all $u,v \in S$, $u+v \neq d$ and $uv \neq d$ then S can be embedded in a type III semifield.

<u>Proof.</u> Claim that $(S-\{d\},+,\cdot)$ is an M.C. semiring. Clearly $(S-\{d\},+)$ and $(S-\{d\},\cdot)$ are commutative semigroups and $S-\{d\}$ is distributive. To show that $S-\{d\}$ is M.C.. Let $x,y,z \in S-\{d\}$ be such that $xy = xz$. To show that $y = z$. We shall consider the following cases:

<u>Case 1</u>. $x = a$ then $ay = az$.

<u>Subcase 1.1</u>. $y = a$. Claim that $z = a$. If $z \neq a$ then $dd = aa = az = dz$ which implies that $d = z$, a contradiction. Hence we have the claim. Similarly if $z = a$ then $y = a$. Hence $y = z$.

Subcase 1.2. $y, z \in S-\{d, a\}$. Then $dy = ay = az = dz$ which implies that $y = z$.

Case 2. $x \neq a$.

Subcase 2.1. $y = a$. Claim that $z = a$. If $z \neq a$, $xd = xz$ which implies that $d = z$, a contradiction. Hence we have the claim. Similarly if $z = a$ then $y = a$. Hence $y = z$.

Subcase 2.2. $y, z \in S-\{a, d\}$. Clearly $y = z$.

Thus we have $(S-\{d\}, +, \cdot)$ is an M.C. semiring. Then $QR(S-\{d\})$ exists. Let $D = QR(S-\{d\})$. Let $f: S-\{d\} \longrightarrow D$ be the natural embedding $f(x) = [(xa, a)]$ for all $x \in D-\{d\}$. Let a' be a symbol not representing any element of D. We can extend the binary operation of D to $K = D \cup \{a'\}$ by defining $a'\alpha = \alpha a' = f(a)\alpha$ for all $\alpha \in K$ and $a'+\alpha = \alpha+a' = f(a)+\alpha$ for all $\alpha \in K$. Then $(K, +, \cdot)$ is a type III semifield. Extend $f: S-\{d\} \longrightarrow D$ to $f: S \longrightarrow K$ by defining $f(d) = a'$. Clearly f is 1-1. Note that $a+x = d+x$ for all $x \in S$ since $d^2(a+x) = d^2a+d^2x = d^2d+d^2x = d^2(d+x)$ which implies that $a+x = d+x$ (since d^2, $a+x$, $d+x \in S-\{d\}$). Using a proof similar to the one in Theorem 5.11 (substitute a for d and d for a) we get that f is a homonorphism. #

Definition 5.15. Let K be a semifield w.r.t. a. If for all $x \in K-\{a\}$, $a+x \neq a$ then K is called almost full. If for all $x \in K$, $a+x \neq a$ thwn K is called full.

Definition 5.16. Let S be a semiring and $a \in S$. Then (S, a) is called a pointed semiring.

Definition 5.17. Let (S,a) and (T,b) be pointed semirings. Then $f:(S,a)\longrightarrow(T,b)$ is called a pointed homomorphism if and only if

1) $f(x+y)=f(x)+f(y)$, $f(xy)=f(x)f(y)$ for all $x,y\in S$,

2) $f(a) = b$.

Definition 5.18. Let $\mathcal{C}$ be a category whose objects are semirings and whose morphisms are semiring homomorphisms. Let S be an object of $\mathcal{C}$. A quotient semifield of S w.r.t. the category $\mathcal{C}$ is a triple (S,f,K) where K is a semifield in $\mathcal{C}$ and $f\in \mathrm{Mor}(S,K)$ is 1-1 such that for each semifield object K' in $\mathcal{C}$ and for each $i\in \mathrm{Mor}(S,K')$ there exists a unique $g\in \mathrm{Mor}(K,K')$ such that $g\circ f = i$.

Theorem 5.19. Let S be a Classification I semiring. Let K be the semifield of type I given by the construction and $f:S\longrightarrow K$ the embedding given by the construction. Let K' be any type I semifield and $i:S\longrightarrow K'$ a monomorphism. Then there exists a unique monomorphism $g:K\longrightarrow K'$ such that $g\circ f = i$.

Proof. Define $g:K\longrightarrow K'$ as follows: for $\alpha\in K$, choose $(x,y)\in\alpha$ define $g(\alpha)=\frac{i(x)}{i(y)}$. By Theorem 1.29 we have that g is a well defined monomorphism. We must show that $g\circ f = i$. Let $x\in S$. Then $(g\circ f)(x)=g(f(x))=g([(xu,u)])$ (where $u\in S$ is not a multiplicative zero) $=\frac{i(xu)}{i(u)}=\frac{i(x)i(u)}{i(u)}=i(x)$. Hence $g\circ f = i$. To show uniquness, suppose that there exists a monomorphism $h:K\longrightarrow K'$ such that $h\circ f = i$ must show that $h=g$. Let $\alpha\in K$. Choose $(x,y)\in\alpha$. Then $g(\alpha)=\frac{i(x)}{i(y)}=\frac{(h\circ f)(x)}{(h\circ f)(y)}=$ $\frac{h(f(x))}{h(f(y))}=h([(xu,u)])\,h([(yu,u)])^{-1}=h([(xu,u)]\cdot[(u,yu)])=$ $h([(xuu,yuu)])=h([(x,y)])=h(\alpha)$. Thus $g = h$. #

Corollary 5.20. Let S be a Classification I semiring, K the type I semifield given by the construction and $f:S\longrightarrow K$ the embedding given by the construction. Let $\mathcal{C}_1$ be the category whose objects are either Classification I semirings or type I semifields and whose morphisms are semiring homomorphisms. Then (S,f,K) is a quotient semifield w.r.t. $\mathcal{C}_1$.

We shall give an example to show that there exists a Classification II semiring S such that $1+x \neq 1$ for all $x \neq 1$ and the type II semifield K given by the construction in Theorem 5.2 is not the smallest type II semifield containing S.

Example 5.21. Consider $(\mathbb{Z}^+,\max,\cdot)$. Then $(\mathbb{Z}^+,\max,\cdot)$ is an M.C. semiring. Let a be a symbol not representing any element of $\mathbb{Z}^+$. Let $S = \mathbb{Z}^+ \cup \{a\}$ define $ax = xa = x$ for all $x \in \mathbb{Z}^+-\{1\}$, $1a = a1 = aa = a$, $a+x = x+a = x$ for all $x \in \mathbb{Z}^+-\{1\}$ and $1+a = a+1 = a+a = a$. Then S is a Classification II semiring w.r.t. a and $1+x \neq 1$ for all $x \neq 1$. By Theorem 5.2 we can embed S in a type II semifield K w.r.t. a'. Let $f:S\longrightarrow K$ be the natural embedding. $(\mathbb{Q}^+,\max,\cdot)$ is a ratio semiring. $I_{\mathbb{Q}^+}(1) = \{x \in \mathbb{Q}^+ | x \leq 1\}$. Let $T = \{x \in \mathbb{Q}^+ | x < \frac{1}{2}\}$ Let $\bar{a}$ be a symbol not representing any element of $\mathbb{Q}^+$. Define (1) $\bar{a}x = x\bar{a} = x$ for all $x \in \bar{K} = \mathbb{Q}^+ \cup \{\bar{a}\}$, (2) $\bar{a}+x = x+\bar{a} = \bar{a}$ for all $x \in T$, $\bar{a}+x = x+\bar{a} = 1+x$ for all $x \in \mathbb{Q}^+-T$ and $\bar{a}+\bar{a} = \bar{a}$. Hence $(\bar{K},+,\cdot)$ is a type II semifield w.r.t. $\bar{a}$ (by Theorem 1.39). Define $i:S\longrightarrow \bar{K}$ by $i(1) = \bar{a}$, $i(a) = 1$ and $i(x) = x$ for all $x \in \mathbb{Z}^+$. Clearly i is 1-1. To show that i is a homomorphism we shall consider the following cases:

Let $x,y \in S$.

<u>Case 1</u>. $x = y = 1$.

$i(x+y) = i(1+1) = i(1) = \bar{a} = \bar{a}+\bar{a} = i(1)+i(1) = i(x)+i(y)$.

$i(xy) = i(1) = \bar{a} = \bar{a}\bar{a} = i(1)i(1) = i(x)i(y)$.

<u>Case 2</u>. $x = 1$, $y \neq 1$.

$i(x+y) = i(1+y) = i(y) = 1+i(y) = \bar{a}+i(y) = i(1)+i(y)$

$i(xy) = i(y) = \bar{a}i(y) = i(1)i(y) = i(x)i(y)$.

<u>Case 3</u>. $x \neq 1$, $y = 1$. (the proof similar to Case 2).

<u>Case 4</u>. $x \neq 1$, $y \neq 1$.

$$i(x+y) = \begin{cases} 1 & \text{if } x = a,\ y = a, \\ y & \text{if } x = a,\ y \neq a, \\ x & \text{if } x \neq a,\ y = a, \\ x+y & \text{if } x \neq a,\ y \neq a, \end{cases} \quad \text{and } i(x)+i(y) = \begin{cases} 1 & \text{if } x = a,\ y = a, \\ 1+y = y & \text{if } x = a,\ y \neq a, \\ x+1 = x & \text{if } x \neq a,\ y = a, \\ x+y & \text{if } x \neq a,\ y \neq a. \end{cases}$$

$$i(xy) = \begin{cases} 1 & \text{if } x = a,\ y = a, \\ y & \text{if } x = a,\ y \neq a, \\ x & \text{if } x \neq a,\ y = a, \\ xy & \text{if } x \neq a,\ y \neq a, \end{cases} \quad \text{and } i(x)i(y) = \begin{cases} 1 & \text{if } x = a,\ y = a, \\ y & \text{if } x = a,\ y \neq a, \\ x & \text{if } x \neq a,\ y = a, \\ xy & \text{if } x \neq a,\ y \neq a. \end{cases}$$

Claim that there is not a monomorphism $h:K \longrightarrow \bar{K}$ such that $hof = i$. To prove this, suppose not. Since $(hof)(1) = i(1) = \bar{a}$, $h(a') = \bar{a}$ and $1 = i(a) = hof(a) = h([(a,a)])$. Since $2 = i(2) = hof\ (2) = h([(2,a)])$ and $5 = h([(5,a)])$, $\frac{2}{5} = \frac{h([(2,a)])}{h([(5,a)])} = h([(2,5)])$. Since h is homomorphism, $h(a'+[(2,5)]) = h([(a,a)]+[(2,5)]) = h([(5,5)]) = 1$. But $h(a')+h(|(2,5)|) = \bar{a}+\frac{2}{5} = \bar{a}$. Hence $\bar{a} = 1$, a contradiction. #

<u>Theorem 5.22</u>. Let S be a Classification II semiring w.r.t. a. such that $1+x \neq 1$ for all $x \neq 1$. Let K be the type II semifield w.r.t. a' given by the construction and let $f:S \longrightarrow K$ the natural

embedding given by the construction. Let $\bar{K}$ be any type II semifield w.r.t. $\bar{a}$ and $i:S \longrightarrow \bar{K}$ a monomorphism.

Then the following hold:

(1) if there exist $x,y \in S-\{1\}$ such that $\bar{a}+\frac{i(x)}{i(y)} = \bar{a}$ then there is no monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

(2) if $\bar{K}$ is almost full then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

Proof. (1) Suppose not. Then there exists a monomorphism. $g:K \longrightarrow \bar{K}$ such that $gof = i$. Claim that $i(1) = \bar{a}$. To prove this, suppose not. Then $i(1) \neq \bar{a}$. If $i(a) = \bar{a}$ then $i(a) = i(1a) = i(1)i(a) = i(1)\bar{a} = i(1)$ so $1 = a$, a contradiction. Hence $i(a) \neq \bar{a}$. Thus $i(1)i(a) = i(a) = i(aa) = i(a)i(a)$, so $i(1) = i(a)$ which implicies that $1 = a$, a contradiction. Thus $i(1) = \bar{a}$, so we have the claim. Since $i(a) = i(aa) = i(a)i(a)$, $i(a) = \bar{e}$ where $\bar{e}$ is the identity of $(\bar{K}-\{\bar{a}\},\cdot)$. Since $gof = i$, $g(a') = \bar{a}$ and $g([(a,a)]) = \bar{e}$. Since there exist $x,y \in S-\{1\}$ such that $\bar{a}+\frac{i(x)}{i(y)} = \bar{a}$, $i(y)+i(x) = i(y)$ which implies that $x+y = y$. Since $g([(x,y)]) = g([(x,a)][(a,y)]) = \frac{g([(x,a)])}{g([(y,a)])} = \frac{g(f(x))}{g(f(y))} = \frac{i(x)}{i(y)}$, we get that $\bar{a} = \bar{a}+\frac{i(x)}{i(y)} = g(a')+g([(x,y)]) = g(a'+[(x,y)]) = g([(a,a)]+[(x,y)]) = g([(x+y,y)]) = g([(a,a)]) = \bar{e}$. Hence $\bar{a} = \bar{e}$, a contradiction.

(2) Using a proof similar to the proof of (1) we get that $i(1) = \bar{a}$ and $i(a) = \bar{e}$. Let $\alpha \in K-\{a'\}$. Choose $(x,y) \in \alpha$. Define $g(\alpha) = \frac{i(x)}{i(y)}$ and $g(a') = \bar{a}$. To show that g is well-defined, suppose that $(x',y') \in \alpha$ also. Then $xy' = x'y$ which implies that $i(x)i(y') = i(x')i(y)$. Hence $\frac{i(x)}{i(y)} = \frac{i(x')}{i(y')}$. Hence g is

well-defined. To show that g is 1-1, let $\alpha, \beta \in K$ be such that $g(\alpha) = g(\beta)$. If $\alpha = a'$ then $\beta = a'$ suppose $\alpha \neq a'$ and $\beta \neq a'$. Choose $(x,y) \in \alpha$ and $(z,w) \in \beta$. Then $\frac{i(x)}{i(y)} = \frac{i(z)}{i(w)}$ so $i(xw) = i(yz)$ which implies that $xw = yz$ i.e. $\alpha = \beta$. Thus g is 1-1. To show that g is a homomorphism, we shall show

(a) $g(\alpha\beta) = g(\alpha)g(\beta)$ and

(b) $g(\alpha+\beta) = g(\alpha)+g(\beta)$ for all $\alpha, \beta \in K$.

To show (a) we shall consider the following cases:

<u>Case 1</u>. $\alpha = \beta = a'$.

$g(\alpha\beta) = g(a'a') = g(a') = \bar{a} = \bar{a}\bar{a} = g(a')g(a') = g(\alpha)g(\beta)$.

<u>Case 2</u>. $\alpha = a'$, $\beta \neq a'$.

$g(\alpha\beta) = g(a'\beta) = g(\beta) = \bar{a}\, g(\beta) = g(a')\, g(\beta) = g(\alpha)g(\beta)$.

<u>Case 3</u>. $\alpha \neq a'$, $\beta = a'$. Using a proof similar to the proof in Case 2 we get that $g(\alpha\beta) = g(\alpha)g(\beta)$.

<u>Case 4</u>. $\alpha \neq a'$, $\beta \neq a'$. Choose $(x,y) \in \alpha$, $(z,w) \in \beta$.

$g(\alpha\beta) = g([(xz,yw)]) = \frac{i(xz)}{i(yw)} = \frac{i(x)i(z)}{i(y)i(w)} = g(\alpha)g(\beta)$.

To show (b) we shall consider the following cases.

<u>Case 1</u>. $\alpha = \beta = a'$.

<u>Subcase 1.1</u>. $1+1 = 1$. Then $\bar{a}+\bar{a} = \bar{a}$ so $g(\alpha+\beta) = g(a'+a') = g(a') = \bar{a} = \bar{a}+\bar{a} = g(a')+g(a') = g(\alpha)+g(\beta)$.

<u>Subcase 1.2</u>. $1+1 \neq 1$. Then $\bar{a}+\bar{a} \neq \bar{a}$ so by Theorem 1.37, $\bar{a}+\bar{a} = \bar{e}+\bar{e}$. $g(\alpha+\beta) = g(a'+a') = g([(a,a)]+[(a,a)]) = g([(a+a,a)]) = \frac{i(a+a)}{i(a)} = \frac{i(a)}{i(a)}+\frac{i(a)}{i(a)} = \bar{e}+\bar{e} = \bar{a}+\bar{a} = g(a')+g(a') = g(\alpha)+g(\beta)$.

Case 2. $\alpha = a'$, $\beta \neq a'$. Choose $(z,w)\in\beta$.

$g(\alpha+\beta) = g(a'+\beta) = g([(a,a)]+[(z,w)]) = g([(w+z,w)]) = \frac{i(w+z)}{i(w)} =$

$\bar{e} + \frac{i(z)}{i(w)} = \bar{a} + \frac{i(z)}{i(w)} = g(a')+g(\beta) = g(\alpha) + g(\beta)$ (since $\bar{K}$ is an almost full, $\bar{a}+u = \bar{e}+u$ for all $u \in \bar{K}-\{\bar{a}\}$ by Theorem 1.37).

Case 3. $\alpha \neq a'$, $\beta = a'$. The proof is similar to the proof of Case 2.

Case 4. $\alpha \neq a'$, $\beta \neq a'$. Choose $(x,y)\in\alpha$ and $(z,w)\in\beta$

$g(\alpha+\beta) = g([(xw+yz,yw)]) = \frac{i(xw+yz)}{i(yw)} = \frac{i(x)}{i(y)} + \frac{i(z)}{i(w)} = g(\alpha) + g(\beta)$.

Hence g is a homomorphism. Let $x \in S$ be arbitrary. If $x = 1$ then $(gof)(1) = g(f(1)) = g(a') = \bar{a} = i(1)$ and for all $x \in S-\{1\}$ $(gof)(x) = g(f(x)) = g([(x,a)]) = \frac{i(x)}{i(a)} = i(x)$. Hence $gof = i$. To show uniquness, let $h:K\longrightarrow\bar{K}$ be a monomorphism such that $hof = i$. Let $\alpha \in K$ be arbitrary. If $\alpha = a'$ then $g(a') = \bar{a} = i(1) = hof(1) = h(a')$ and for $\alpha \in K-\{a'\}$ choose $(x,y)\in\alpha$. Then $g(\alpha) = \frac{i(x)}{i(y)} = \frac{(hof)(x)}{(hof)(y)} = \frac{h([(x,a)])}{h([(y,a)])} = h([(x,y)]) = h(\alpha)$. Hence g is unique. #

Corollary 5.23. Let S be a Classification II semiring w.r.t. a such that $1+x \neq 1$ for all $x\neq 1$. Let K be the type II semifield given by the construction and $f:S\longrightarrow K$ the embedding given by the construction. Let $\mathcal{C}_2$ be the category whose objects are either Classification II semirings S^+. w.r.t. a^+ such that $1^+ + x \neq 1^+$ for all $x \neq 1^+$ (where $1^+ \in S^+ - \{a^+\}$ is such that $1^+x = x$ for all $x \in S^+$) or type II almost full semifields and whose morphisms are semiring homomorphisms. Then (S,f,K) is a quotient semifield w.r.t. $\mathcal{C}_2$.

Theorem 5.24. Let S be a Classification II semiring w.r.t. a such that $1+a = 1$. Let K be the type II semifield given by the construction and let $f: S \longrightarrow K$ be the embedding given by the construction. Let $\bar{K}$ be any type II semifield and $i: S \longrightarrow \bar{K}$ a monomorphism. Then there exists a unique monomorphism $g: K \longrightarrow \bar{K}$ such that $g \circ f = i$.

Proof. Let $a' \in K$, be such that $(K-\{a'\}, \cdot)$ is a group and let e' be the identity of $(K-\{a'\}, \cdot)$. Let $\bar{a} \in \bar{K}$ be such that $(\bar{K}-\{\bar{a}\}, \cdot)$ is a group and let $\bar{e}$ be the identity of $(\bar{K}-\{\bar{a}\}, \cdot)$. Using the same proof as in Theorem 5.22 we can show that $i(1) = \bar{a}$ and $i(a) = \bar{e}$. Since $1+a = 1$, $\bar{a}+\bar{e} = \bar{a}$. Hence $\bar{e} \in I_{\bar{K}-\{\bar{a}\}}(\bar{a})$. Thus $I_{\bar{K}-\{\bar{a}\}}(\bar{a}) = I_{\bar{K}-\{\bar{a}\}}(\bar{e})$ by Theorem 1.38 (2). Let $g: K \longrightarrow \bar{K}$ be defined as follows: for $\alpha \in K-\{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = \frac{i(x)}{i(y)}$ and $g(a') = \bar{a}$. Then the same proof used in Theorem 5.22 shows that g is well-defined, 1-1 and $g(\alpha\beta) = g(\alpha)g(\beta)$ for all $\alpha, \beta \in K$.

To show that $g(\alpha+\beta) = g(\alpha)+g(\beta)$ for all $\alpha, \beta \in K$, we shall consider the following cases:

Case 1. $\alpha = a'$, $\beta = a'$.

Subcase 1.1. $1+1 = 1$. Then $\bar{a}+\bar{a} = \bar{a}$. $g(\alpha+\beta) = g(a'+a') = g(a') = \bar{a} = \bar{a}+\bar{a} = g(a')+g(a') = g(\alpha)+g(\beta)$.

Subcase 1.2. $1+1 \neq 1$. Then $\bar{a}+\bar{a} = \bar{e}+\bar{e} = i(a)+(a) = i(a+a) = i(a) = \bar{e}$. $g(\alpha+\beta) = g(a'+a') = g(e') = g([(a,a)]) = \frac{i(a)}{i(a)} = \bar{e} = \bar{a}+\bar{a} = g(\alpha)+g(\beta)$.

Case 2. $\alpha \neq a'$, $\beta = a'$. Choose $(x,y) \in \alpha$.

Subcase 2.1. $\alpha \in I_D(e')$. Then $\alpha + a' = a'$. Since $\alpha \in I_D(e')$, $[(x,y)] + [(a,a)] = [(a,a)]$ hence $x+y = y$. Therefore $i(x)+i(y) = i(y)$. Therefore $\frac{i(x)}{i(y)} + \bar{e} = \bar{e}$. Thus $\frac{i(x)}{i(y)} + \bar{a} = \bar{a}$. $g(\alpha+\beta) = g(\alpha + a') = g(a') = \bar{a} = \frac{i(x)}{i(y)} + \bar{a} = g(\alpha) + g(a') = g(\alpha) + g(\beta)$.

Subcase 2.2. $\alpha \notin I_D(e')$. Then $a' + \alpha = e' + \alpha$. Claim that $\frac{i(x)}{i(y)} + \bar{a} \neq \bar{a}$. To prove this, suppose not. Then $i(x)+i(y) = i(y)$ which implies that $x+y = y$ so $[(x,y)] + [(a,a)] = [(x+y,y)] = [(a,a)]$. Hence $\alpha \in I_D(e')$, a contradiction. Thus $\frac{i(x)}{i(y)} + \bar{a} = \frac{i(x)}{i(y)} + \bar{e}$. $g(\alpha+\beta) = g(\alpha + a') = g(\alpha + e') =$ $g([(x,y)] + [(a,a)]) = g([(x+y,y)]) = \frac{i(x+y)}{i(y)} = \frac{i(x)}{i(y)} + \bar{e} = \frac{i(x)}{i(y)} + \bar{a} =$ $g(\alpha) + g(a') = g(\alpha) + g(\beta)$.

Case 3. $\alpha = a'$, $\beta \neq a'$. Using the same proof as used in Case 2 we get tgat $g(\alpha+\beta) = g(\alpha) + g(\beta)$.

Case 4. $\alpha \neq a'$, $\beta \neq a'$. Choose $(x,y) \in \alpha$ and $(z,w) \in \beta$. $g(\alpha+\beta) = g([(xw, yz, yw)]) = \frac{i(xw+yz)}{i(yw)} = \frac{i(x)}{i(y)} + \frac{i(z)}{i(w)} = g(\alpha) + g(\beta)$.

Hence g is a monomorphism. Using the same proof as in Theorem 5.22 we can show that g is the unique monomorphism from K to $\bar{K}$ such that $g \circ f = i$. #

Corollary 5.25. Let S be a Classification II semiring w.r.t. a such that $1+a = 1$. Let K be the type II semifield given by the construction and let $f: S \longrightarrow K$ be the embedding given by the construction. Let $\mathcal{C}_2^*$ be the category whose objects are either Classification II semirings S^* w.r.t. a^* such that $1^* + a^* = 1^*$ (where $1^* \in S^* - \{a^*\}$ is such that $1^* x = x$ for all $x \in S^*$) or type II semifields and whose morphisms are semiring homomorphisms. Then (S,f,K) is a quotient semifield w.r.t. $\mathcal{C}_2^*$.

Theorem 5.26. Let S be a Classification III semiring w.r.t. a of form 1 such that $a+x \neq a$ for all $x \neq a$. Let K be the type II semifield given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction. Let $\bar{K}$ be any type II semifield w.r.t. $\bar{a}$ and $i:S \longrightarrow \bar{K}$ a monomorphism. Then the following hold:

(1) If there exist $x,y \in S-\{a\}$ such that $\frac{i(x)}{i(y)} + \bar{a} = \bar{a}$ then there is not monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

(2) If K is almost full then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

Proof. Let e be the identity of $(S-\{a\},\cdot)$ use a proof similar to the one used in Theorem 5.22 (substitute a for 1 and e for a). #

Corollary 5.27. Let S be a Classification III semiring w.r.t. a of form 1 such that $a+x \neq a$ for all $x \neq a$. Let K be the type II semifield given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction. Let $\mathcal{C}_3$ be the category whose objects are either Classification III semirings S^* w.r.t. a^* of form 1 such that $a^*+x \neq a^*$ for all $x \in S^*-\{a^*\}$ or type II almost full semifields and whose morphisms are semiring homomorphisms. Then (S,f,K) is a quotient semifield w.r.t. $\mathcal{C}_3$.

Theorem 5.28. Let S be a Classification III semiring w.r.t. a of form 1. Let e be the identity of $(S-\{a\},\cdot)$. Assume that $a+e = a$. Let K be the type II semifield given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction . Let $\bar{K}$ be any type II semifield and $i:S \longrightarrow \bar{K}$ a monomorphism. Then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

Proof. Similar to the proof of Theorem 5.24. #

Corollary 5.29. Let S be a Classification III semiring w.r.t. a of form 1. Let e be the identity of $(S-\{a\},\cdot)$ and assume that $e+a = a$. Let K be the type II semifield given by the construction and let $f:S\longrightarrow K$ be the embedding given by the construction. Let $\mathcal{C}_3^*$ be the category whose objects are either Classification III semirings S^* w.r.t. a^* such that $a^*+e^* = a^*$. (e^* is the identity of $(S^*-\{a^*\},\cdot)$) or type II semifields and whose morphisms are semiring homomorphisms. Then (S,f,K) is a quotient semifield w.r.t. $\mathcal{C}_3^*$.

Theorem 5.30. Let S be a Classification III (IV,V) semiring w.r.t. a such that S is M.C.. Let K be the O-semifield [∞ -semifield] given by the construction and let $f:S\longrightarrow K$ be the embedding given by the construction. Let $\bar{K}$ be any O-semifield [∞ -semifield] and $i:S\longrightarrow\bar{K}$ a monomorphism. Then there exists a unique monomorphism $g:K\longrightarrow\bar{K}$ such that $g\circ f = i$.

Proof. We shall prove the O-semifield case (the ∞ -semifield is proven similarly). By the construction of K, $K = QR(S)\cup\{a'\}$ where a' is a zero element of K and the natrual embedding $f:S\longrightarrow K$ is given by $f:S\longrightarrow QR(S)$. Let $\bar{a}$ be a zero element of $\bar{K}$. Claim $i(x)\neq\bar{a}$ for all $x\in S$. To prove this, suppose not. Let $x\in S$ be such that $i(x) = \bar{a}$. Let $y\in S-\{x\}$. Then $i(xx) = i(x)i(x) = \bar{a}\bar{a} = \bar{a} = \bar{a}i(y) = i(xy)$ which implies that $xx = xy$. Thus $x = y$, a contradiction. Hence we have the claim. Define $g:K\longrightarrow\bar{K}$ as follows: for $\alpha\in K- a'$, choose $(x,y)\in\alpha$. Define $g(\alpha) = \frac{i(x)}{i(y)}$ and $g(a') = \bar{a}$. To show that g is 1-1, let $\alpha,\beta\in K$ be such that $g(\alpha) = g(\beta)$.

If $\alpha = a'$ then $\beta = a'$. Suppose that $\alpha, \beta \neq a'$. Choose $(x,y) \in \alpha$ and $(z,w) \in \beta$. Then $\frac{i(x)}{i(y)} = \frac{i(z)}{i(w)}$. Hence $i(xw) = i(yz)$ which implies that $xw = yz$. Hence $\alpha = \beta$. Let $\alpha, \beta \in K$. We shall show that $g(\alpha + \beta) = g(\alpha) + g(\beta)$ and $g(\alpha\beta) = g(\alpha)g(\beta)$.

<u>Case 1</u>. $\alpha = \beta = a'$.

$g(\alpha + \beta) = g(a' + a') = g(a') = \bar{a} = \bar{a} + \bar{a} = g(a') + g(a') = g(\alpha) + g(\beta)$.

$g(\alpha\beta) = g(a'a') = g(a') = \bar{a} = \bar{a}\bar{a} = g(a')g(a') = g(\alpha)g(\beta)$.

<u>Case 2</u>. $\alpha = a'$, $\beta \neq a'$.

$g(\alpha + \beta) = g(a' + \beta) = g(\beta) = \bar{a} + g(\beta) = g(a') + g(\beta) = g(\alpha) + g(\beta)$.

$g(\alpha\beta) = g(a'\beta) = g(a') = \bar{a} = \bar{a}g(\beta) = g(a')g(\beta) = g(\alpha)g(\beta)$.

<u>Case 3</u>. $\alpha \neq a'$, $\beta = a'$. The proof is similar to the proof of Case 2.

<u>Case 4</u>. $\alpha \neq a'$, $\beta \neq a'$. Choose $(x,y) \in \alpha$ and $(z,w) \in \beta$.

$g(\alpha + \beta) = g([(xw+yz, yw)]) = \frac{i(xw+yz)}{i(yw)} = \frac{i(x)}{i(y)} + \frac{i(z)}{i(w)} = g(\alpha) + g(\beta)$.

$g(\alpha\beta) = g([(xz, yw)]) = \frac{i(xz)}{i(yw)} \quad \frac{i(x)i(z)}{i(y)i(w)} = g(\alpha)g(\beta)$.

We must show that $g \circ f = i$. Let $x \in S$ be arbitrary.

$g \circ f(x) = g(f(x)) = g([(xa, a)]) = \frac{i(xa)}{i(a)} = \frac{i(x)i(a)}{i(a)} = i(x)$.

Hence $g \circ f = i$. To show the uniqueness, suppose that there exists a monomorphism $h: K \longrightarrow \bar{K}$ such that $h \circ f = i$.

Then $h(a') = \bar{a} = g(a')$. Let $\alpha \in K - \{a'\}$ and choose $(x,y) \in \alpha$.

Then $g(\alpha) = \frac{i(x)}{i(y)} = \frac{h \circ f(x)}{h \circ f(y)} = \frac{h([(xa, a)])}{h([(ya, a)])} = h([(xa, a)][(a, ya)]) = h([(x,y)]) = h(\alpha)$. Thus $g = h$. #

Corollary 5.31. Let S be a Classification III (IV, V) semiring such that S is M.C., let K be the O-semifield (∞-semifield) given by the construction and let $f:S\longrightarrow K$ be the embedding given by the construction. Let $\mathcal{C}_0$ ($\mathcal{C}_\infty$) be the category whose objects are either M.C. Classification III semiring or M.C. Classification IV semirings or M.C.Classification V semirings or O-semifields (∞-semifields) and whose morphisms are semiring homomorphisms.
Then (S,f,K) is a quotient semifield w.r.t. $\mathcal{C}_0$ ($\mathcal{C}_\infty$)

We shall give some example that there exists a Classification III (IV) semiring S such that S is M.C. and the type II semifield given by the construction in Theorem 5.5 (Theorem 5.8) is not the smallest type II semifield containing S. (i.e. there exists a Classification III (IV) semiring S such that S is M.C. and there exists a monomorphism $i:S\longrightarrow K'$ where K' is a type II semifield but there does not a monomorphism $g:K\longrightarrow K'$ such that $g \circ f = i$ where $f:S\longrightarrow K$ is the embedding given by the construction and K is the type II semifield given by the constrution.

Example 5.32. $\mathbb{Z}^+$ with the usual addition and multiplication is an M.C. Classification III semiring w.r.t. 1 and also an M.C. Classification IV semiring w.r.t. 2. Let a' be a symbol not representing any element of $QR(\mathbb{Z}^+)$.
Define $a'+a' = [(2,1)]$, $a'+\alpha = \alpha+a' = [(1,1)]+\alpha$ for all $\alpha \in QR(\mathbb{Z}^+)$

$a'a' = a'$ and $a'\alpha = \alpha a' = \alpha$ for all $\alpha \in QR(\mathbb{Z}^+)$.

Then $K = QR(\mathbb{Z}^+)\cup\{a'\}$ is a type II semifield given by the construction and $f:\mathbb{Z}^+\longrightarrow K$ define by $f(x) = [(x,1)]$ for all $x \in \mathbb{Z}^+$ is the embedding given by the construction.

$\mathbb{Q}^+$ with the usual addition and multiplication is a ratio semiring
Let a be a symbol not representing any element of $\mathbb{Q}^+$.
Define $ax = xa = x$ for all $x \in \mathbb{Q}^+ \cup \{a\}$ and

$$a+x = x+a = 1+x \text{ for all } x \in \mathbb{Q}^+ \cup \{a\}.$$

Then $\bar{K} = \mathbb{Q}^+ \cup \{a\}$ is a type II semifield.
Define $i:\mathbb{Z}^+ \longrightarrow \bar{K}$ by $i(x) = x$ for all $x \in \mathbb{Z}^+ - \{1\}$ and $i(1) = a$.
Clearly i is 1-1 and $i(xy) = i(x)i(y)$ for all $x,y \in \mathbb{Z}^+$ and

$$i(x+y) = x+y = \begin{cases} 1+1 = a+a = i(1)+i(1) & \text{if } x = y = 1, \\ 1+y = a+y = i(1)+i(y) & \text{if } x = 1,\ y \neq 1, \\ x+1 = x+a = i(x)+i(1) & \text{if } x \neq 1,\ y = 1, \\ x+y = i(x)+i(y) & \text{if } x = 1,\ y = 1. \end{cases}$$

Hence i is a homomorphism. Claim that there does not exist a monomorphism $g:K \longrightarrow \bar{K}$ such that $g \circ f = i$. To prove this, suppose not. Hence $a = i(1) = g\circ f(1) = g(f(1)) = g([(1,1)])$.
Thus $g(a') = g(a')a = g(a')g([(1,1)]) = g(a'[(1,1)]) = g([(1,1)])$.
Therefore $a' = [(1,1)]$, a contradiction. #

Example 5.33. $(\mathbb{Z}^+,\max,\cdot)$ is an M.C. Classification III semiring w.r.t. 1 and an M.C. Classification IV semiring w.r.t. 2.
Let a' be a symbol not representing any element of $QR(\mathbb{Z}^+)$.
Define $a'+a' = [(1,1)]$, $a'+\alpha = \alpha+a' = [(1,1)]+\alpha$ for all $\alpha \in QR(\mathbb{Z}^+)$

$$a'a' = a' \text{ and } a'\alpha = \alpha a' = \alpha \text{ for all } \alpha \in QR(\mathbb{Z}^+).$$

Then $K = QR(\mathbb{Z}^+) \cup \{a'\}$ is a type II semifield given by the construction and $f:\mathbb{Z}^+ \longrightarrow K$ define by $f(x) = [(x,1)]$ for all $x \in \mathbb{Z}^+$ is the embedding given by the construction.
$(\mathbb{Q}^+,\max,\cdot)$ is a ratio semiring. Let $S = \{x \in \mathbb{Q}^+ \mid x < ½\}$.
Clearly S is an additive subsemigroup of $I_{\mathbb{Q}^+}(1)$ and $\mathbb{Q}^+ - S$ is an ideal of $(\mathbb{Q}^+,+)$. Let a be a symbol not representing any element in $\mathbb{Q}^+$. Extend + and $\cdot$ from $\mathbb{Q}^+$ to $\bar{K} = \mathbb{Q}^+ \cup \{a\}$ by defining

(1) $ax = xa = x$ for all $x \in K$,

(2) $a+x = x+a = a$ for all $x \in S$,

$a+x = x+a = 1+x$ for all $x \in Q^+ - S$

(3) $a+a = 1$.

By Theorem 1.39, $\bar{K}$ is a type II semifield.

Define $i:\mathbb{Z}^+ \longrightarrow \bar{K}$ by $i(x) = x$ for all $x \in \mathbb{Z}^+$. Clearly i is a monomorphism and $i(1) \neq a$. But there does not exist a monomorphism $g:K \longrightarrow \bar{K}$ such that $g \circ f = i$. To prove this, suppose not. Claim that $g(a') = a$. If $g(a') \neq a$, then $g(a') = g(a')g(a')$. Hence $g(a') = 1 = i(1) = g(f(1)) = g([(1,1)])$ Thus $a' = [(1,1)]$, a contradiction. Hence we have the claim. Since $1 = i(1) = g(f(1)) = g([(1,1)])$ and $4 = i(4) = g([(1,4)])$, $\frac{1}{4} = \frac{g([(1,1)])}{g([(4,1)])} = g([(1,1)][(1,4)]) = g([(1,4)])$.
Therefore $a = a+\frac{1}{4} = g(a)+g([(1,4)]) = g(a'+[(1,4)]) = g([(1,1)]+[(1,4)]) = g([(4,4)])$. Thus $a = [(4,4)]$, a contradiction. #

Theorem 5.34. Let S be a Classification III semiring w.r.t. a such that S is M.C., K be the type II semifield given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction. Let $\bar{K}$ be any type II semifield w.r.t. $\bar{a}$ and $i:S \longrightarrow \bar{K}$ a monomorphism. Then the following hold:

1) if $i(a) = \bar{a}$ then there is no monomorphism $g:K \longrightarrow \bar{K}$ such that $g \circ f = i$.

2) if $i(a) \neq \bar{a}$ and $\bar{K}$ is a full then there is a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $g \circ f = i$.

Proof. Let $a' \in K$ be such that $(K-\{a'\},\cdot)$ is a group.

1) Suppose not.

Hence $\bar{a} = i(a) = gof(a) = g(f(a)) = g([(a,a)])$.

Thus $g(a') = g(a')\bar{a} = g(a')g([(a,a)]) = g(a'[(a,a)]) = g([(a,a)])$.

Therefore $a' = [(a,a)]$, a contradiction.

2) If $i(x) = \bar{a}$ for some $x \in S-\{a\}$ then

$\bar{a} = i(x) = i(xa) = i(x)i(a) = \bar{a}i(a) = i(a)$. Hence $x = a$, a contradiction. Then $i(x) \neq a$ for all $x \in S$.

Define $g:K \longrightarrow \bar{K}$ as follows: for $\alpha \in K-\{a'\}$, choose $(x,y) \in \alpha$.

Define $g(\alpha) = \dfrac{i(x)}{i(y)}$ and $g(a') = \bar{a}$. Using a similar proof to the one used in the proof of Theorem 5.22, we get that g is well-define, 1-1 and $g(\alpha\beta) = g(\alpha)g(\beta)$ for all $\alpha,\beta \in K$.

We must show that $g(\alpha+\beta) = g(\alpha)+g(\beta)$ for all $\alpha,\beta \in K$,

Let $\bar{e}$ be the identity of $(\bar{K}-\{\bar{a}\},\cdot)$.

Case 1. $\alpha = \beta = a'$.

$g(\alpha+\beta) = g(a'+a') = g([(a,a)]+[(a,a)]) = g([(a+a,a)]) = \bar{e}+\bar{e}$
$= \bar{a}+\bar{a}$ (since K is full , $a+a = e+e$) $= g(\alpha)+g(\beta)$.

Case 2. $\alpha = a'$, $\beta \neq a'$. Choose $(z,w) \in \beta$.

$g(\alpha+\beta) = g(a'+\beta) = g([(a,a)]+[(z,w)]) = g([(w+z,w)]) =$

$\dfrac{i(w)}{i(w)} + \dfrac{i(z)}{i(w)} = \bar{e}+\dfrac{i(z)}{i(w)} = \bar{a}+\dfrac{i(z)}{i(w)} = g(\alpha)+g(\beta)$.

Case 3. $\alpha \neq a'$, $\beta = a'$. Use the same proof as in Case 2.

Case 4. $\alpha \neq a'$, $\beta = a'$. Choose $(x,y) \in \alpha$, $(z,w) \in \beta$.

$g(\alpha+\beta) = g([(xw+yz,yw)]) = \dfrac{i(x)}{i(y)} + \dfrac{i(z)}{i(w)} = g(\alpha)+g(\beta)$.

Using a proof similar to the one used in Theorem 5.22, we get that g is the unique monomorphism such that $gof = i$. #

Corollary 5.35. Let S be an M.C. Classification III semiring w.r.t. a. Then (S,a) is a pointed semiring.
Let K be the type II semifield given by the construction.
Let $a' \in K$ be such that $(K-\{a'\},\cdot)$ is a group.
Let $e' \in K$ be the identity of $(K-\{a'\},\cdot)$. Then (K,e') is a pointed semifield. Let $f:S \longrightarrow K$ be the embedding given by the construction.
Let $\mathcal{C}_{3,2}$ be the category whose objects are either pointed semirings (S^*,a^*) where S^* is an M.C. Classification III semiring w.r.t. a^* or a pointed semifields $(\bar{K},\bar{e})$ where $\bar{K}$ is a type II full semifields w.r.t. $\bar{a}$ and $\bar{e}$ is the identity of $(\bar{K}-\{\bar{a}\},\cdot)$ and whose morphism are pointed semiring homomorphisms.
Then $((S,a),f,(K,e'))$ is a quotient semifield w.r.t. $\mathcal{C}_{3,2}$.

Theorem 5.36. Let S be an M.C. Classification IV semiring w.r.t. a and let 1 be the identity of $(S,\cdot)$.
Let K be the type II semifield given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction.
Let $\bar{K}$ be any type II semifield w.r.t. $\bar{a}$, and $i:S \longrightarrow \bar{K}$ a homomorphism. Then the following hold:

1) if $i(1) = \bar{a}$ then there is no monomorphism $g:K \longrightarrow \bar{K}$ such that $g \circ f = i$.

2) if $i(1) \neq \bar{a}$ and $\bar{K}$ is full then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $g \circ f = i$.

Proof. Similar to the proof of Theorem 5.34. #

Corollary 5.37. Let S be an M.C. Classification IV semiring. Let 1 be the identity of $(S,\cdot)$. Then $(S,1)$ is a pointed semiring. Let K be the type II semifield w.r.t. a' given by the construction. Let e' be the identity of $(K-\{a'\},\cdot)$ then (K,e') is a pointed semifield. Let $f:S\longrightarrow K$ be the embedding given by the construction. Let $\mathcal{C}_{4,2}$ be the category whose objects are either pointed semirings $(S^*,1^*)$ where S^* is an M.C. Classification IV semiring and 1^* is the identity of $(S^*,\cdot)$ or a pointed semifields $(\bar{K},\bar{e})$ where $\bar{K}$ is a type II full semifields w.r.t. $\bar{a}$ and $\bar{e}$ is the identity of $(\bar{K}-\{\bar{a}\},\cdot)$ and whose morphisms are pointed semiring homomorphisms.
Then $((S,1),f,(K,e'))$ is a quotient semifield w.r.t. $\mathcal{C}_{4,2}$.

Theorem 5.38. Let K be any type III semifield w.r.t. a.
Let $d\in K-\{a\}$ be such that $ax = dx$ for all $x\in K$.
Let $\bar{K}$ be any type III semifield w.r.t. $\bar{a}$.
Let $\bar{d}\in\bar{K}-\{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ for all $x\in\bar{K}$.
If there exists a monomorphism $g:K\longrightarrow\bar{K}$ then $g(a) = \bar{a}$ and $g(d) = \bar{d}$.

Proof. Let $e,\bar{e}$ be the identities of $(K-\{a\},\cdot)$ and $(\bar{K}-\{\bar{a}\},\cdot)$ respectively. Suppose that $g(a)\neq\bar{a}$.
Then $g(a)g(a) = g(aa) = g(ad) = g(a)g(d)$ which implies that $g(a) = \bar{e}g(d)$. Since $g(d)\neq\bar{a}$ so $a = d$, a contradiction.
Thus $g(a) = \bar{a}$. Since $g(d)g(d) = g(a)g(d)$ so $g(d) = g(a)\bar{e} = \bar{a}\bar{e} = \bar{d}$. #

Theorem 5.39. Let S be an M.C. Classification III (IV) semiring w.r.t. a and let K be the type III semifield w.r.t. a' given by the construction. Let $[(d_1,d_2)]\in K-\{a'\}$ be such that $a'\alpha = [(d_1,d_2)]\alpha$ and $a'+\alpha = [(d\ ,d\ )]+\alpha$ for all $\alpha\in K$ and let $f:S\longrightarrow K$ be the embedding given by the construction.

Let $\bar{K}$ be ant type III semifield w.r.t. $\bar{a}$. Let $\bar{d} \in \bar{K}-\{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ for all $x \in \bar{K}$ and let $i:S \longrightarrow \bar{K}$ be a monomorphism Then the following hold:

1) if there exists a $y \in S$ such that $i(y) = \bar{d}$ but $f(x) = [(d_1,d_2)]$ for all $x \in S$, then there does not exist a monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

2) if there exist $y,u \in S$ such that $y \neq u$ and $i(y) = \bar{d}$, $f(u) = [(d_1,d_2)]$ then there does not exist a monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

3) if there exists a $u \in S$ such that $f(u) = [(d_1,d_2)]$ but $i(y) \neq \bar{d}$ for all $y \in S$ then there does not exist a monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

4) if $i(x) \neq \bar{d}$ and $f(x) \neq [(d_1,d_2)]$ for all $x \in S$ and $\frac{i(d_1)}{i(d_2)} \neq \bar{d}$ then there does not exist a monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

5) if $i(x) \neq \bar{d}$ and $f(x) \neq [(d_1,d_2)]$ for all $x \in S$ and $\frac{i(d_1)}{i(d_2)} = \bar{d}$ and $\bar{K}$ is a full then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

6) if there exists a $y \in S$ such that $i(y) = \bar{d}$ and $f(y) = [(d_1,d_2)]$ and $\bar{K}$ is a full then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

<u>Proof</u>. Note that $i(x) \neq \bar{a}$ for all $x \in S$ since if there exists an $x \in S$ such that $i(x) = \bar{a}$ then $\bar{a} = i(x) = i(xa) =$ $i(x)i(a) = \bar{a}i(a)$, a contradiction. Similarly, if S is a Classification IV semiring then $i(x) \neq \bar{a}$ for all $x \in S$.

We shall prove the Classification III semiring case the Classification IV semiring is proven similarly.

1) Suppose not. Then $gof(y) = i(y) = \bar{d} = g([(d_1,d_2)])$ By Theorem 5.38, we have that $f(y) = [(d_1,d_2)]$, a contradiction.

2) Suppose not. Then $i(u) = gof(u) = g([(d_1,d_2)]) = \bar{d} = i(y)$. Hence $u = y$, a contradiction.

3) Suppose not.
Then $i(u) = g(f(u)) = g([(d_1,d_2)]) = \bar{d}$, a contradiction.

4) Suppose not. Then $i(d_1) = g(f(d_1)) = g([(d_1,a)])$ and $i(d_2) = g(f(d_2)) = g([(d_2,a)])$. Hence $\frac{i(d_1)}{i(d_2)} = \frac{g([(d_1,a)])}{g([(d_2,a)])}$ $= g([(d_1,a)][(a,d_2)]) = g([(d_1,d_2)]) = \bar{d}$, a contradiction.

5) Define $g:K \longrightarrow \bar{K}$ as follows: for $\alpha \in K-\{a'\}$,
Choose $(x,y) \in \alpha$. Define $g(\alpha) = \frac{i(x)}{i(y)}$ and $g(a') = \bar{a}$.
Using the same proof as in Theorem 5.22, we can show that g is well-defined and 1-1. To show that g is a homomorphism, let $\alpha, \beta \in K$. Since $\bar{K}$ is full, $\bar{a}+x = \bar{d}+x$ for all $x \in \bar{K}$.

Case 1. $\alpha = \beta = a'$.
$g(\alpha+\beta) = g(a'+a') = g([(d_1,d_2)]+[(d_1,d_2)]) = g([(d_1d_2+d_1d_2, d_2d_2)])$
$= \frac{i(d_1)}{i(d_2)} + \frac{i(d_1)}{i(d_2)} = \bar{d}+\bar{d} = \bar{a}+\bar{a} = g(\alpha)+g(\beta)$.
$g(\alpha\beta) = g(a'a') = g([(d_1,d_2)][(d_1,d_2)]) = g([(d_1d_1, d_2d_2)]) =$
$\frac{i(d_1)}{i(d_2)} \frac{i(d_1)}{i(d_2)} = \bar{d}\bar{d} = \bar{a}\bar{a} = g(\alpha)g(\beta)$.

Case 2. $\alpha = a'$, $\beta \neq a'$. Choose $(z,w) \in \beta$.

$g(\alpha+\beta) = g(a'+\beta) = g([(d_1,d_2)]+[(z,w)]) = g([(d_1w+d_2z, d_2w)]) =$

$\frac{i(d_1)}{i(d_2)} + \frac{i(z)}{i(w)} = \bar{d}+\frac{i(z)}{i(w)} = \bar{a}+\frac{i(z)}{i(w)} = g(\alpha)+g(\beta)$.

$g(\alpha\beta) = g(a'[(z,w)]) = g([(d_1,d_2)][(z,w)]) = g([(d_1z,d_2w)]) =$

$\frac{i(d_1)i(z)}{i(d_2)i(w)} = \bar{d}g(\beta) = \bar{a}g(\beta) = g(\alpha)g(\beta)$.

Case 3. $\alpha \neq a'$, $\beta = a'$. The proof is similar to Case 2.

Case 4. $\alpha \neq a'$, $\beta \neq a'$. Choose $(x,y) \in \alpha$, $(z,w) \in \beta$.

$g(\alpha+\beta) = g([(xw+yz,yw)]) = \frac{i(xw+yz)}{i(yw)} = \frac{i(x)}{i(y)}+\frac{i(z)}{i(w)} = g(\alpha)+g(\beta)$.

$g(\alpha\beta) = g([(xz,yw)]) = \frac{i(x)}{i(y)}\frac{i(z)}{i(w)} = g(\alpha)g(\beta)$.

To show that $gof = i$, let $x \in S$.

Then $g(f(x)) = g([(x,a)]) = \frac{i(x)}{i(a)} = i(x)$. Thus $gof = i$.

Using a similar to the one used before, we have a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

6) Since $f(y) = [(d_1,d_2)]$, $[(y,a)] = [(d_1,d_2)]$.

Hence $yd_2 = d_1$ so $i(y)i(d_2) = i(d_1)$ which implies that

$\frac{i(d_1)}{i(d_2)} = i(y) = \bar{d}$. By Case 5, there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$. #

Theorem 5.40. Let S be an M.C. Classification V semiring, let K be the type II semifield given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction. Let $\bar{K}$ be any full type II semifield w.r.t. $\bar{a}$ and $i:S \longrightarrow \bar{K}$ a monomorphism. Then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

Proof. Claim that $i(x) \neq \bar{a}$ for all $x \in S$. To prove this, suppose not. Let $x \in S$ be such that $i(x) = \bar{a}$.
Then $i(x) = \bar{a} = \bar{a}\bar{a} = i(x)i(x)$. Hence $x = xx$. Since S is M.C., for all $y \in S$, $xxy = xy$ which implies that $xy = y$.
Thus x is the identity of $(S, \cdot)$, a contradiction.
Let $a' \in K$ be such that $(K-\{a'\}, \cdot)$ is a group.
Define $g:K \longrightarrow \bar{K}$ as follows, for $\alpha \in K-\{a'\}$, choose $(x,y) \in \alpha$.
Define $g(\alpha) = \frac{i(x)}{i(y)}$ and $g(a') = \bar{a}$. Using a proof similar to the one used in Theorem 5.34 (2), we get that g is the unique monomorphism such that $g \circ f = i$. #

Corollary 5.41. Let S be an M.C. Classification V semiring, let K be the type II semifield given by the construction and let $f:S \longrightarrow K$ embedding given by the construction.

Let $\mathcal{C}_{5,2}$ be the category whose objects are either M.C. Classification V semirings or full type II semifields and whose mophisms are semiring homomorphisms. Then (S,f,K) is a quotient semifield w.r.t. $\mathcal{C}_{5,2}$.

Theorem 5.42. Let S be an M.C. Classification V semiring w.r.t. a and let K be the type III semifield w.r.t. a' given by the construction. Let $[(d_1,d_2)] \in K-\{a'\}$ be such that $a'\alpha = [(d_1,d_2)]\alpha$ and $a'+\alpha = [(d_1,d_2)]+\alpha$ for all $\alpha \in K$ and let $f:S \longrightarrow K$ be the embedding given by the construction. Let $\bar{K}$ be any type III semifield w.r.t. $\bar{a}$, let $\bar{d} \in \bar{K}-\{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ for all $x \in \bar{K}$ and let $i:S \longrightarrow \bar{K}$ be a monomorphism. Then the following hold:

1) if there exists an $x \in S$ such that $i(x) = \bar{a}$ then there does not exist a monomorphism $g:K \longrightarrow \bar{K}$ such that $g \circ f = i$.
Assume that $i(x) \neq \bar{a}$ for all $x \in S$.

2) if there exists a $y \in S$ such that $i(y) = \bar{d}$ and $f(x) \neq [(d_1, d_2)]$ for all $x \in S$ then there does not exists a monomorphism $g: K \longrightarrow \bar{K}$ such that $gof = i$.

3) if there exists a $y \in S$ such that $i(y) = \bar{d}$ and there exists a $u \in S$ such that $f(u) = [(d\ , d\ )]$ and $u \neq y$ then there does not exist a monomorphism $g: K \longrightarrow \bar{K}$ such that $gof = i$.

4) if $i(y) \neq \bar{d}$ for all $y \in S$ and there exists a $u \in S$ such that $f(u) = [(d_1, d_2)]$ then there does not exists a monomorphism $g: K \rightarrow \bar{K}$ such that $gof = i$.

5) if $i(y) \neq \bar{d}$ for all $y \in S$ and $f(y) \neq [(d_1, d_2)]$ for all $y \in S$ and $\frac{i(d_1)}{i(d_2)} \neq \bar{d}$ then there does not exists a monomorphism $g: K \rightarrow \bar{K}$ such that $gof = i$.

6) if $i(y) \neq \bar{d}$ for all $y \in S$ and $f(y) \neq [(d_1, d_2)]$ for all $y \in S$ and $\frac{i(d_1)}{i(d_2)} = \bar{d}$ then there exists a unique monomorphism $g: K \longrightarrow \bar{K}$ such that $gof = i$.

7) if there exists a $y \in S$ such that $i(y) = \bar{d}$ and $f(y) = [(d_1, d_2)]$ and $\bar{K}$ is full then there exists a unique monomorphism $g: K \longrightarrow \bar{K}$ such that $gof = i$.

<u>Proof.</u> 1) Suppose not. Then $g(f(x)) = i(x) = \bar{a} = g(a')$ (by Theorem 5.38). Hence $f(x) = a'$, a contradiction.

2),3),4),5),6),7) are proven in a similar way to the proofs in Theorem 5.39. #

<u>Theorem 5.43.</u> Let S be a Classification V semiring w.r.t. a such that a is not M.C. in S. If there exists a monomorphism $i:S \longrightarrow \bar{K}$ where $\bar{K}$ is a type III semifield w.r.t. $\bar{a}$ and $\bar{d} \in \bar{K}-\{\bar{a}\}$ is such that $\bar{a}x = \bar{d}x$ for all $x \in \bar{K}$ then either $i(a) = \bar{d}$ or $i(a) = \bar{a}$.

<u>Proof.</u> Let $d \in S-\{a\}$ be such that $ax = dx$ for all $x \in S$. Suppose $i(a) \neq \bar{a}$ and $i(a) \neq \bar{d}$. Let $\bar{e}$ be the identity of $(\bar{K}-\{\bar{a}\},\cdot)$.

<u>Case 1.</u> There exists an $x \in S-\{a\}$ such that $i(x) = \bar{a}$.
Then $\bar{d}i(a) = \bar{a}i(a) = i(x)i(a) = i(xa) = i(xd) = i(x)i(d) = \bar{a}i(d) = \bar{d}i(d)$. Hence $i(a) = \bar{e}i(d)$. If $i(d) \neq \bar{a}$ then $i(a) = i(d)$ which implies that $a = d$, a contradiction. If $i(d) = a$ then $i(a) = \bar{d}$, a contradiction.

<u>Case 2.</u> $i(x) \neq \bar{a}$ for all $x \in S$. Then $i(a)i(a) = i(a)i(d)$. Hence $a = d$, a contradiction. #

We shall give an example of Theorem 5.43.

<u>Example 5.44.</u> By Example 4.38, $S-\{1\}$ is a Classification V semiring w.r.t. a. Consider $\mathbb{Q}^+$ with the usual addition and multiplication. Let $\bar{a}$ be a symbol not representing any element of $\mathbb{Q}^+$. Extend + and $\cdot$ from $\mathbb{Q}^+$ to $\bar{K} = \mathbb{Q}^+ \cup \{\bar{a}\}$ by $\bar{a}x = x\bar{a} = 2x$ for all $x \in \bar{K}$ and $\bar{a}+x = x+\bar{a} = 2+x$ for all $x \in \bar{K}$. Define $h:S-\{1\} \longrightarrow \bar{K}$ by $h(a) = \bar{a}$ and $h(x) = x$ for all $x \in S-\{1,a\}$. Clearly h is a monomorphism. Define $i:S-\{1\} \longrightarrow \bar{K}$ by $i(a) = 2$, $i(2) = \bar{a}$ and $i(x) = x$ for all $x \in S-\{1,2,a\}$ clearly i is 1-1. We must show that i is a homomorphism. Let $x,y \in S-\{1\}$.

<u>Case 1.</u> $x = y = a$.
$i(x+y) = i(a+a) = i(2+2) = 2+2 = i(a)+i(a) = i(x)+i(y)$.
$i(xy) = i(aa) = i(22) = 2\cdot 2 = i(a)i(a)$.

Case 2. $x = a$, $y \neq a$.

Subcase 2.1. $y = 2$. $i(x+y) = i(a+2) = i(2+2) = 2+2 = 2+a = i(a)+i(2) = i(x)+i(y)$.

$i(xy) = i(a2) = i(2\cdot 2) = 2\cdot 2 = 2a = i(a)i(2) = i(x)i(y)$.

Subcase 2.2. $y \neq 2$. $i(x+y) = i(a+y) = i(2+y) = 2+y = i(a)+i(y) = i(x)+i(y)$.

$i(xy) = i(2y) = 2y = i(a)i(y) = i(x)i(y)$.

Case 3. $x \neq a$, $y = a$. The proof is similar to the proof of Case 2.

Case 4. $x \neq a$, $y \neq a$.

Subcase 4.1. $x = y = 2$. $i(x+y) = i(2+2) = 2+2 = a+a = i(2)+i(2) = i(x)+i(y)$

$i(xy) = i(22) = 2\cdot 2 = aa = i(2)i(2) = i(x)i(y)$.

Subcase 4.2. $x = 2$, $y \neq 2$. $i(x+y) = i(2+y) = 2+y = a+y = i(2)+i(y) = i(x)+i(y)$.

$i(xy) = i(2y) = 2y = ay = i(2)i(y) = i(x)i(y)$.

Subcase 4.3. $x \neq 2$, $y = 2$. The proof is similar to the proof of Subcase 4.2.

Subcase 4.4. $x \neq 2$, $y \neq 2$. Done.

Hence i is a monomorphism. #

We shall give an example showing that there exists a Classification V semiring S w.r.t. a such that a is not M.C. in S and for all $x, y \in S$, $x+y \neq a$ and the type III semifield K given by the construction in Theorem 5.11 is not the smallest type III semifield containing S (i.e. it is possible that there exists a monomorphism $i:S \longrightarrow K'$ where K' is a type III semifield but there does not exist a monomorphism $g:K \longrightarrow K'$ such that $g \circ f = i$).

Example 5.45. Since $(\mathbb{Z}^+ - \{1\}, \max, \cdot)$ is an M.C. semiring. Let a be a symbol not representing any element of $\mathbb{Z}^+ - \{1\}$. Extend + and $\cdot$ from $\mathbb{Z}^+ - \{1\}$ to $S = (\mathbb{Z}^+ - \{1\}) \cup \{a\}$ by $ax = 2x$ for all $x \in S$ and $a+x = 2+x$ for all $x \in S$. Then $(S, +, \cdot)$ is a Classification V semiring w.r.t. a such that a is not M.C. in S and for all $x, y \in S$ $x+y \neq a$. Let $K = QR(S-\{a\}) \cup \{a'\}$ where $\alpha + a' = a' + \alpha = [(4,2)] + \alpha$ for all $\alpha \in K$ and $a'\alpha = \alpha a' = [(4,2)]\alpha$ for all $\alpha \in K$. Then K is the type III semifield given by the construction. Since $(\mathbb{Q}^+, \max, \cdot)$ is a ratio semiring. Let $\bar{a}$ be a symbol not representing any element of $\mathbb{Q}^+$. Let $T = \{x \in \mathbb{Q}^+ \mid x < 1\}$. Extend + and $\cdot$ from $\mathbb{Q}^+$ to $\bar{K} = \mathbb{Q}^+ \cup \{\bar{a}\}$ by

(1) $x\bar{a} = \bar{a}x = 2x$ for all $x \in \bar{K}$

(2) $\bar{a}+x = x+\bar{a} = \bar{a}$ for all $x \in T$

$\bar{a}+x = x+\bar{a} = 2+x$ for all $x \in \mathbb{Q}^+ - T$

$\bar{a}+\bar{a} = 2$

Then $\bar{K}$ is a type III semifield. Define $i:S \longrightarrow \bar{K}$ by $i(x) = x$ for all $x \in S-\{a\}$ and $i(a) = \bar{a}$. Clearly i is 1-1. To show that i is a homomorphism, let $x, y \in S$.

Case 1. $x = y = a$. Then $i(a+a) = i(2+2) = i(2) = 2 = \bar{a}+\bar{a} = i(a)+i(a)$. $i(aa) = i(2 \cdot 2) = 2 \cdot 2 = \bar{a}\bar{a} = i(a)i(a)$.

Case 2. $x = a$, $y \neq a$. Then $i(a+y) = i(2+y) = 2+y = \bar{a}+y = i(a)+i(y)$. $i(ay) = i(2y) = 2y = \bar{a}y = i(a)i(y)$.

Case 3. $x \neq a$, $y = a$. The proof is similar to the proof of Case 2.

Case 4. $x \neq a$, $y \neq a$. $i(x+y) = x+y = i(x)+i(y)$ and $i(xy) = xy = i(x)i(y)$. Hence i is a homomorphism claim that there is not a monomorphism $g:K \longrightarrow \bar{K}$ such that $g\circ f = i$.
To prove this suppose not. Then $\bar{a} = i(a) = g\circ f(a) = g(a')$
$5 = i(5) = g(f(5)) = g([(10,2)])$. $2 = g([(4,2)])$ so
$\bar{a} = \bar{a}+\frac{2}{5} = g(a')+g([(2,5)]) = g(a'+[(2,5)]) = g([(4,2)]+[(2,5)]) = g([(4,2)])$. Hence $a' = [(4,2)]$, a contradiction. #

Theorem 5.46. Let S be a Classification V semiring w.r.t. a such that a is not M.C. in S and for all $x,y \in S$ $x+y \neq a$.
Let K be the type III semifield given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction.
Let $\bar{K}$ be any type III semifield w.r.t. $\bar{a}$ and let $\bar{d} \in \bar{K}-\{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ for all $x \in \bar{K}$ and let $i:S \rightarrow \bar{K}$ be a monomorphism. Then the following hold:

1) if $i(a) = \bar{d}$ then there is no monomorphism $g:K \longrightarrow \bar{K}$ such that $g\circ f = i$.

2) if $i(a) = \bar{a}$ and $\bar{K}$ is full then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $g\circ f = i$.

Proof. 1) Since $i(a) = \bar{d}$, $i(d) = \bar{a}$ (if $i(d) \neq \bar{a}$ then $i(a)i(a) = i(a)i(d)$ which implies that $a = d$, a contradiction). Suppose that there exists a monomorphism $g:K \longrightarrow \bar{K}$ such that $g\circ f = i$. Then $g(f(d)) = i(d) = \bar{a} = g(a')$ (by Theorem 5.38). Hence we have that $f(d) = a'$, a contradiction.

2) Since $i(a) = \bar{a}$, $i(d)i(d) = i(d)i(a)$ which implies that $i(d) = \bar{e}i(a) = \bar{e}\bar{a} = \bar{d}$. Define $g:K \longrightarrow \bar{K}$ as follows: for $\alpha \in K-\{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = \frac{i(x)}{i(y)}$ and $g(a') = \bar{a}$. Using a proof similar to the one used in Theorem 5.39 (5) (substitute $f(d)$ for $[(d_1,d_2)]$ we get that g is the unique monomorphism such that $g \circ f = i$. #

Corollary 5.47. Let S be a Classification V semiring w.r.t. a such that a is not M.C. in S and for all $x,y \in S$, $x+y \neq a$. Let K be the type III semifield w.r.t. a' given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction. Let $\mathcal{C}_{5,3}$ be the category whose objects are either pointed semirings (S^*,a^*) where S^* is a Classification V semiring w.r.t. a^* and a^* is not M.C. in S^* and for all $x,y \in S^*$, $x+y \neq a^*$ or pointed semifields $(\bar{K},\bar{a})$ where $\bar{K}$ is a full type III semifield w.r.t. $\bar{a}$ and whose morphisms are pointed semiring homomorphisms. Then $((S,a),f,(K,a'))$ is a quotient semifield w.r.t. $\mathcal{C}_{5,3}$.

Theorem 5.48. Let S be a Classification V semiring w.r.t. a such that a is not M.C. in S. Let $d \in S-\{a\}$ be such that $ax = dx$ for all $x \in S$. Assume that for all $u,v \in S$, $u+v \neq d$ and $uv \neq d$. If there exists a monomorphism $i:S \longrightarrow \bar{K}$ where $\bar{K}$ is a type III semifield w.r.t. $\bar{a}$ then $i(d) = \bar{a}$ or $i(d) = \bar{d}$ where $\bar{d} \in \bar{K}-\{\bar{a}\}$ is such that $\bar{a}x = \bar{d}x$ for all $x \in \bar{K}$.

Proof. Let $\bar{e}$ be the identity of $(\bar{K}-\{\bar{a}\},\cdot)$. Suppose $i(d) \neq \bar{d}$ and $i(d) \neq \bar{a}$.

<u>Case 1</u>. There exists an $x \in S-\{d\}$ such that $i(x) = \bar{a}$.
Then $\bar{d}i(d) = \bar{a}i(d) = i(xd) = i(xa) = i(x)i(a) = \bar{a}i(a) = \bar{d}i(a)$.
Hence $i(d) = \bar{e}i(a)$. If $i(a) \neq \bar{a}$ then $i(d) = i(a)$, so $a = d$, a contradiction. If $i(a) = \bar{a}$ then $i(d) = \bar{e}i(a) = \bar{e}\bar{a} = \bar{d}$, a contradiction.

<u>Case 2</u>. $i(x) \neq a$ for all $x \in S$. Then $i(a)i(a) = i(a)i(d)$.
Hence $a = d$, a contradiction. #

<u>Theorem 5.49</u>. Let S be a Classification V semiring w.r.t. a such that a is not M.C. in S and there exist $x,y \in S-\{a\}$ such that $x+y = a$. Let $d \in S-\{a\}$ be such that $ax = dx$ for all $x \in S$ and assume that for all $u,v \in S$, $u+v \neq d$ and $uv \neq d$. Let K be the type III semifield given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction. Let $\bar{K}$ be any type III semifield w.r.t. $\bar{a}$ and $i:S \longrightarrow \bar{K}$ a monomorphism. Then the following hold:

(1) if $i(d) = \bar{d}$ then there is not monomorphism $g:K \rightarrow \bar{K}$ such that $gof = i$.

(2) if $i(d) = \bar{a}$ and $\bar{K}$ is full then there exists a unique monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$.

<u>Proof</u>. 1) Since $i(d) = \bar{d}$ then $i(a) = \bar{a}$ (if $i(a) \neq \bar{a}$ then $i(a)i(a) = i(a)i(d)$ which implies that $a = d$, a contradiction). Suppose there exists a monomorphism $g:K \longrightarrow \bar{K}$ such that $gof = i$. Then $g(f(a)) = i(a) = \bar{a} = g(a')$ (by Theorem 5.38). Hence $f(a) = a' = f(d)$, so $a = d$, a contradiction.

2) Since $i(d) = \bar{a}$, $i(a)i(a) = i(a)i(d)$ which implies that $i(a) = \bar{e}i(d) = \bar{e}\bar{a} = \bar{d}$. Define $g:K \longrightarrow \bar{K}$ as follows: for $\alpha \in K-\{a'\}$. choose $(x,y) \in \alpha$. Define $g(\alpha) = \frac{i(x)}{i(y)}$ and $g(a') = \bar{a}$. Using a proof similar to the one in Theorem 5.39 (5) (substitute $f(a)$ for $[(d_1,d_2)]$) we get that g is the unique monomorphism such that $g \circ f = i$. #

<u>Corollary 5.50</u>. Let S be a Classification V semiring w.r.t. a such that a is not M.C. in S and there exist $x,y \in S-\{a\}$ such that $x+y = a$. Let $d \in S-\{a\}$ be such that $ax = dx$ for all $x \in S$ and assume that for all $u,v \in S$, $u+v \neq d$ and $uv \neq d$. Let K be the type III semifield w.r.t. a' given by the construction and let $f:S \longrightarrow K$ be the embedding given by the construction.
Let $\mathcal{C}^*_{5,3}$ be the category whose objects are either pointed semirings (S^*,d^*) where S^* is a Classification V semiring w.r.t. a^* such that a^* is not M.C. in S and there exist $x,y \in S^*-\{a^*\}$ such that $x+y = a^*$ and for all $u,v \in S^*$ $u+v \neq d^*$ and $uv \neq d^*$ where $d^* \in S^*-\{a^*\}$ is such that $a^*x = d^*x$ for all $x \in S^*$ or pointed semifields $(\bar{K},\bar{a})$ where $\bar{K}$ is a full type III semifield w.r.t. $\bar{a}$ and whose morphisms are pointed semiring homomorphisms.
Then $((S,d),f,(K,a'))$ is a quotient semifield w.r.t. $\mathcal{C}^*_{5,3}$.