INTRODUCTION

The problem of expressing certain entities "in finite terms", such as, the computation of roots of polynomials in terms of radicals, the solving of differential equations in terms of elementary functions, arises frequently in Mathematics. One such problem known as "integration in finite terms" is dealt with in this thesis. Roughly speaking, the problem of integration in finite terms is that given a γ in a differential field F with derivation D, we ask when a solution of $D(y) = \gamma$ can be expressed in certain special forms. Historically, Joseph Liouville (see e.g. Ritt [1]) first systematically worked on the question of when an algebraic function has an algebraic integral and he later gave conditions relating to when an algebraic function has an integral of a special form called "elementary" , this particular result is generally known as Liouville's theorem on integration in finite terms. In its simplified form, it reads : if $\gamma(x)$ is an algebraic function whose integral is elementary, then

$$\int \gamma(x)dx = v_0(x) + c_1\log v_1(x) + \cdots + c_n\log v_n(x),$$

where n is a positive integer, each $v_i(x)$ an algebraic function, and each c_i a constant. The works of Liouville were subsequently extended by a number of other people such as D.D. Mordukhai-Boltovskoi [2], A. Ostrowski [3], J.F. Ritt [1], and M.Rosenlicht [4], [5], [6]. A proof of Liouville's theorem can be found in Ritt's classic exposition [1]; the proof is a combination of clever observations and is analytic in nature. In 1946, Ostrowski gave for the first time in [3] a proof of Liouville's theorem in the context of differential fields of complex meromorphic functions. In 1968, M. Rosenlicht found a completely algebraic proof of Liouville's theorem as described in his series of papers [4], [5], and [6].

To date, one of the most generalized forms of Liouville's theorem is due to M.F. Singer, B.D. Saunders and B.F. Caviness [7]. Singer, Saunders and Caviness generalized Liouville's theorem by enlarging the class of fields from elementary to a special class of fields, called $\mathcal{EL}$ - elementary, which includes elementary functions as well as special functions such as error function and logarithmic integral.

A natural question arises whether extensions to other classes of fields containing functions not previously covered are possible.

The first objective of this thesis is to affirmatively answer this question by establishing two more classes, namely, Ei and Gamma extensions.

Elementary functions not only enjoy some interesting properties but among them there appear some useful intrinsic algebraic relations, as witnessed through the structure theorem of Risch [8], which shows that if an algebraic relation holds among a set of elementary functions, then such functions must satisfy an algebraic relation of a special kind. In 1979, M. Rothstein and B.F. Caviness [9] generalized the structure theorem by enlarging the class of fields from elementary to a special class of fields, called generalized log-explicit extension.

The work of Rothstein and Caviness is not a straightforward generalization of Risch's result. In fact, it improves upon Risch's result subject to certain additional restrictions.

The second objective of this thesis is to re-consider and extend Risch's structure theorem to other class of fields.

The thesis is organized as follows:

Chapter I contains basic definitions and theorems relating to differential fields and their extensions. All results, except for Theorem 1.6 that involves the module of differentials, are given with proofs either complete or sketches. This is indeed done throughout the thesis so as to make the exposition as self-contained as possible. Emphases are called upon Theorems 1.7, 1.8 and 1.9 for they provide main artillery for the proofs of principal theorems in the last two chapters.

Chapter II deals with classical Liouville's theorem about integration in terms of elementary functions, and its recent generalization to the class of $\mathcal{EL}$ -elementary functions, due to Singer, Saunders and Caviness. The proof of Liouville's theorem(Theorem 2.1.1) given here is that of Rosenlicht [4], [5],[6] mentioned above, and it is entirely algebraic. The proof is by induction on the number of generators and it reduces to considering just one simple extension of each kind, exponential, logarithmic or algebraic separately. The main ideas of the proof are to apply appropriate automorphisms to the inductively proposed form in an extended field and then sum up in order to get the form of desired shape in the lower field.

The proof of $\mathcal{EL}$ -elementary extension of Liouville's theorem (Theorem 2.2.1) given here is that of Singer, Saunders and Caviness [7]. It follows the same line as that of Theorem 2.1.1 mentioned above, with much more complicated analysis arising from the wider class of functions adjoined. This line of attack is what we adopt for the proof of our main reslts in Chapter IV.

In Chapter III, we review a result, called structure theorem of Risch, which exhibits two close algebraic relations, one among exponentials, and the other among logarithms in elementary extension. The proof of the main theorem (Theorem 3.2.1) given here is due to Rothstein and Caviness [9]. It is done via induction on the number of transcendental extensions. By re-arranging transcendental elements appropriately, and analyzing linear dependence of differentials via Theorems 1.7 and 1.8, desired relations can be obtained in extended field and then can be pulled down to lower field by applying relevant automorphisms.

In Chapter IV, we give two main results extending those in Chapter II. First, we establish a Liouville type theorem (Theorem 4.1.2) by enlarging the class of function to an extension; called Ei-extension, which encompasses the $\mathcal{EL}$ -elementary extension of Singer, Saunders and Caviness. This generalization is natural in the sense that two more exponential and logarithmic like elements are adjoined to $\mathcal{EL}$ -elementary extension. The proof is along the same line as that of Theorem 2.2.1, but of course with more analysis.

Second, we establish another Liouville type theorem (Theorem 4.1.4) by enlarging the class of function to an extension; called Gamma extension, which encompasses the Gamma function not previously considered anywhere. The proof is also along the same line as that of Theorem 2.2.1 but with different analysis which involves rational power of the element adjoined.

The details of both proofs are displayed in steps so that one can easily see the logical flows and their inter-connections.

In the last chapter, we give a generalization of Risch's structure theorem to general elementary extension enlarging the elementary extension by adjoining nonelementary integral to it. This is perhaps the broadest one can hope for. The ideas of the proof resemble that given in Chapter III.

Notation. The following notation will be fixed throughout the entire exposition.

$\mathbf{Z}^+$ is the set of positive integers.

$\mathbf{Z}$ is the ring of integers.

$\mathbf{Q}$ is the field of rational numbers.

$\mathbf{R}$ is the field of real numbers.