CHAPTER II

DIFFERENTIAL FIELDS AND LIOUVILLE TYPE THEOREMS

2.1 Basic Definitions and Liouville's Theorem

Let F be a differential field. If x and y are elements of F, with $y \neq 0$, then x is called a logarithm of y, or y an exponential of x, if $D(x) = D(y)/y$ for each given derivation D of F. We write "x is a logarithm of y" or "y is an exponential of x" as $x = \log(y)$ or $y = \exp(x)$ respectively.

We say that a differential extension field K of F is an elementary extension of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for i, with $1 \leq i \leq n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1}.

Example. Let **C** be the field of complex numbers and let $F = \mathbf{C}(x)$ be the set of rational functions with coefficients in **C**. Then F is a differential field under the usual derivation: $D = d/dx$.

Thus $F_3 = \mathbf{C}(\, x, \log(x), \exp(-x^2), \exp(x\log(x)+\exp(-x^2))\,)$ is an elementary extension of F, because

$$F = F_0 \subset F_1 = F_0(t_1) \subset F_2 = F_1(t_2) \subset F_3 = F_2(t_3)$$

where $t_1 = \log(x)$,

$t_2 = \exp(-x^2)$,

$t_3 = \exp(x\log(x) + \exp(-x^2))$.

Remarks

(1) The t_i might satisfy at least 2 conditions in the definition, for example, let F = **R**(x) with the usual derivation : d/dx. Adjoining exp(x), produces the differential field F(exp(x)). Let t = exp(x/2). Then t is both algebraic over F(exp(x)) and is an exponential of some element in F(exp(x)).

(2) We say that y is elementary over F if y belongs to an elementary extension of F having the same subfield of constants.

We can now state Liouville's Theorem.

Theorem 2.1.1 ([4], [5], [6]). Let F be a differential field of characteristic zero with derivation D. Let γ be an element of F. Suppose there is an element y in an elementary extension of F having the same subfield of constants such that $D(y) = \gamma$. Then there exist constants $c_1,\dots,c_n$ in F and elements $u_1,\dots,u_n$, v in F, with $u_1,\dots,u_n$ nonzero, such that

$$\gamma = D(v) + \sum_{i=1}^{n} c_i D(u_i)/u_i .$$

Remarks. There are a number of variations and proofs of this theorem in the literature, (see e.g. [4], [5], [6]). Interestingly, the proofs have become progressively more algebraic.

Proof. Let E be an elementary extension of F having the same subfields of constants and E contains an element y satisfying $D(y) = \gamma$. So there is a finite tower of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_m = E$$

such that for each i = 1,...,m, $F_i = F_{i-1}(t_i)$, where t_i is a logarithm or an exponential of an element of F_{i-1} or algebraic over F_{i-1}. Thus $E = F(t_1,\dots,t_m)$. Clearly, $F = F_o$, $F_1,\dots,F_m = E$ have the same subfields of constants.

The proof is by induction on m. For $m = 0$, the theorem is obviously true. Let $m > 0$. Assume that the theorem is true for the case that the number of generators in E is less than m. We can apply the induction hypothesis to $\gamma \in F_1$ and the tower

$$F(t_1) = F_1 \subset F_2 \subset \cdots \subset F_m = F(t_1,\ldots,t_m)$$

to get $\gamma = D(v) + \sum_{i=1}^{n} c_i D(u_i)/u_i$ where $c_1,\ldots,c_n$ are constants and $v, u_1,\ldots,u_n$ are elements in $F(t_1)$ with $u_1,\ldots,u_n$ nonzero.

Setting $t_1 = t$. Then t is algebraic over F, or is a logarithm or is an exponential of an element of F. What we now have to do is to find a similar expression for γ, possibly with a different n, but with the elements $u_1,\ldots,u_n$, v belonging to F.

If t is algebraic over F, there exist a finite normal extension K of F containing F(t). Then for each $\sigma \in \mathrm{Aut}(K|F)$ we have $\gamma = D(\sigma v) + \sum_{i=1}^{n} c_i D(\sigma u_i)/(\sigma u_i)$ and summing over all σ we get

$$[K : F]\gamma = D\left(\sum_{\sigma} \sigma v\right) + \sum_{i=1}^{n} c_i D\left(\prod_{\sigma} \sigma u_i\right) \Big/ \left(\prod_{\sigma} \sigma u_i\right),$$

with each element $\sum_{\sigma} \sigma v$ and $\prod_{\sigma} \sigma u_i$ in F.

Thus we may assume t transcendental over F. Without loss of generality, we assume that $c_1,\ldots,c_n$ are linearly independent over $\mathbf{Q}$.

If t is a logarithm of an element of F, say $D(t) = D(a)/a$ for some nonzero a in F, then it is an immediate consequence of Theorem 1.9 that $u_1,\ldots,u_n$ are algebraic over F and $v = ct + w$ where c is a constant and w is algebraic over F, so that

$$\gamma = cD(a)/a + D(w) + \sum_{i=1}^{n} c_i D(u_i)/u_i,$$

the same situtation as in the case where t is algebraic over F we get an expression γ of the type desired.

If t is an exponential of an element of F, say $D(t)/t = D(b)$ for some $b \in F$, then by Theorem 1.9 we have v is algebraic over F and there are integers $m_0, m_1, \ldots, m_n$, with $m_0 \neq 0$, such that each $u_i^{m_0} t^{m_i}$ is algebraic over F. Thus we get

$$D(u_i)/u_i = (1/m_0)D(u_i^{m_0}t^{m_i})/(u_i^{m_0}t^{m_i}) - (m_i/m_0)D(t)/t.$$

Again we have the expression of γ as in the algebraic case and proceeding the same manner, we obtain the correct sum of γ. #

2.2 A Recent Extension of Liouville's Theorem

The following is one of the most generalized versions of Liouville's Theorem. It was due to M.F. Singer, B.D. Saunders and B.F. Caviness [7].

Let F be a differential field with derivation D and subfield of constants C. Let A and B be finite indexing sets and let

$$\mathcal{E} = \{ G_\alpha(\exp R_\alpha(Y)) \mid \alpha \in A\},$$
$$\mathcal{L} = \{ H_\beta(\log S_\beta(Y)) \mid \beta \in B\},$$

be sets of expressions where:

(1) $G_\alpha, R_\alpha, H_\beta, S_\beta$ are in C(Y) for all $\alpha \in A$, $\beta \in B$,

(2) for all $\beta \in B$, if $H_\beta(Y) = P_\beta(Y)/Q_\beta(Y)$ with P_β, Q_β in C[Y] and $Q_\beta \neq 0$, then $\deg P_\beta \leq \deg Q_\beta + 1$.

We say that a differential extension K of F is an <u>$\mathcal{E}\mathcal{L}$ - elementary extension</u> of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for i, with $1 \leq i \leq n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) for some $\alpha \in A$, there are u,and nonzero v in F_{i-1} such that $D(t_i) = (D(u))G_\alpha(v)$ where $v = \exp R_\alpha(u)$, (for brevity, $t_i = \int G_\alpha(\exp R_\alpha(u))D(u))$,

(v) for some $\beta \in B$, there are u, v in F_{i-1} such that $D(t_i) = (D(u))H_\beta(v)$ where $v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$,
(for brevity, $t_i = \int H_\beta(\log S_\beta(u))D(u))$.

Note that (i), (ii) and (iii) constitute the elementary extension while (iv) and (v) extend this notion and allow us to adjoin more special functions.

Example. Let **C** be the field of complex numbers and let F = **C**(x) be the set of rational functions with coefficients in **C**. Then F is a differential field under the usual derivation $D = d/dx$.

Let $G(Y) = Y$, $R(Y) = -Y^2$, $H(Y) = 1/Y$, $S(Y) = Y$.

Let $\mathcal{E} = \{G(\exp(-Y^2))\} = \{\exp(-Y^2)\}$ and

$\mathcal{L} = \{H(\log S(Y))\} = \{1/\log(Y)\}$.

Thus $F_3 = F(\exp(-x^2), \int\exp(-x^2), \int 1/\log(x))$ is an $\mathcal{EL}$-elementary extension of F,

since $F_0 = C(x) \subset F_1 = F_0(t_1) \subset F_2 = F_1(t_2) \subset F_3 = F_2(t_3)$,

where $t_1 = \exp(-x^2)$ is such that $D(t_1)/t_1 = D(-x^2)$,

$t_2 = \int\exp(-x^2)$ is such that $D(t_2) = G(\exp(-x^2))D(x)$,

$t_3 = \int 1/\log(x)$ is such that $D(t_3) = H(\log(x))D(x)$.

The definition of $\mathcal{EL}$ - elementary is broad enough to include the following functions:

(i) The error function is defined by

$$\mathrm{erf}(u) = \int (Du)\exp(-u^2)$$

where $G_\alpha(\exp R_\alpha(Y)) = \exp(-Y^2)$ with $G_\alpha(Y) = Y$ and $R_\alpha(Y) = -Y^2$.

(ii) The logarithmic integral is defined by

$$\mathrm{li}(u) = \int D(u)/\log(u)$$

with $H_\beta(Y) = 1/Y$ and $S_\beta(Y) = Y$.

$\mathcal{EL}$ - elementary extension does not include the dilogarithm defined by

$$Li_2(u) = \int (Du) \log(u)/(1-u)$$

nor the exponential integral

$$Ei(u) = \int (Du)\exp(u)/u,$$

since they both violate condition (1) of the definition. Of course, Ei(u) = li(exp(u)), so the exponential integral is implicitly covered in the definition of $\mathcal{EL}$-elementary extension; however, a theory that explicitly includes these functions is probably more useful.

Singer, Saunders and Caviness's theorem is as follows:

Theorem 2.2.1 ([7]). Let F be a differential field of characteristic zero with derivation D and algebraically closed subfield of constants C. Let γ be in F. Assume that there exist an $\mathcal{EL}$- elementary extension K of F and an element y in K such that $D(y) = \gamma$. Then there exist

(1) $a_1,\ldots,a_n$ in C,

(2) $w_0, w_1,\ldots,w_n$ in F, with nonzero $w_1,\ldots,w_n$,

(3) $b_{i\alpha} \in C$, $u_{i\alpha}$ and $v_{i\alpha}$ algebraic over F for all $\alpha \in A$ and $i \in I_\alpha$,

(4) $c_{i\beta} \in C$, $u_{i\beta}$ and $v_{i\beta}$ algebraic over F for all $\beta \in B$ and $i \in I_\beta$,

such that

$$\gamma = D(w_0) + \sum_{i=1}^{n} a_i D(w_i)/w_i + \sum_{\alpha\in A} \sum_{i\in I_\alpha} b_{i\alpha} (Du_{i\alpha}) G_\alpha(v_{i\alpha})$$

$$+ \sum_{\beta\in B} \sum_{i\in J_\beta} c_{i\beta}(Du_{i\beta})H_\beta(v_{i\beta}),$$

where I_α and J_β are finite sets of integers for all α and β and

$$D(v_{i\alpha}) = (DR_\alpha(u_{i\alpha}))v_{i\alpha} ,$$

$$D(v_{i\beta}) = (DS_\beta(u_{i\beta}))/S_\beta(u_{i\beta}), \; S_\beta(u_{i\beta}) \neq 0,$$

for all α , β and i.

The proof of the theorem is long. We state here an outline of proof; for a complete proof we refer to [7].

An outline of proof of the theorem. First, we consider the case where F is algebraically closed. The proof is done by induction on the transcendence degree of the $\mathcal{EL}$- elementary extension of F; when the transcendence degree is zero, the result is trivial. If the transcendence degree is positive we apply induction and the problem is reduced to showing:

Let K be an algebraic extension of F(t) where t is transcendental over F and satisfies either conditions (ii), or (iii), or (iv), or (v) in the definition of the $\mathcal{EL}$- elementary extension. Let $\gamma \in$ F and assume that K has no new constants and that there exist w_i, $u_{i\alpha}$, $u_{i\beta}$, $v_{i\alpha}$, $v_{i\beta}$ in K and constants a_i, $b_{i\alpha}$, $c_{i\beta}$ such that

$$(2.1) \qquad \gamma = D(w_o) + \sum a_i D(w_i)/w_i + \sum\sum b_{i\alpha}(Du_{i\alpha})G_\alpha(v_{i\alpha})$$

$$+ \sum\sum c_{i\beta}(Du_{i\beta}) H_\beta(v_{i\beta}),$$

where

$$D(v_{i\alpha}) = (DR_\alpha(u_{i\alpha}))v_{i\alpha} \qquad \text{and}$$

$$D(v_{i\beta}) = (DS_\beta(u_{i\beta}))/S_\beta(u_{i\beta}), \; S_\beta(u_{i\beta}) \neq 0.$$

Then there exist $\overline{w_i}$, $\overline{u_{i\alpha}}$, $\overline{u_{i\beta}}$, $\overline{v_{i\alpha}}$, $\overline{v_{i\beta}}$ in F and constants $\overline{a_i}$, $\overline{b_{i\alpha}}$, $\overline{c_{i\beta}}$ in F such that

$$\gamma = D(\overline{w_o}) + \sum \overline{a_i} D(\overline{w_i})/\overline{w_i} + \sum\sum \overline{b_{i\alpha}} D(\overline{u_{i\alpha}}) G_\alpha(\overline{v_{i\alpha}})$$

$$+ \sum\sum \overline{c_{i\beta}} D(\overline{u_{i\beta}}) H_\beta(\overline{v_{i\beta}}),$$

where $D(\bar{v}_{i\alpha}) = D(R_\alpha(\bar{u}_{i\alpha}))\bar{v}_{i\alpha}$ and

$$D(\bar{v}_{i\beta}) = D(S_\beta(\bar{u}_{i\beta}))/S_\beta(\bar{u}_{i\beta}), \quad S_\beta(\bar{u}_{i\beta}) \neq 0.$$

Next we consider each of the cases (ii) - (v), the main idea is to take the trace of both sides of (2.1) to force everything down F(t), and equate terms in the partial fraction decomposition (with respect to t) and then show that the term not depending on t on the right - hand side can be put in the prescribed form.

Finally, we remove the assumption that F is algebraically closed. The above arguments show that the shape of γ obtained in the conclusion of the theorem holds with a_i, $b_{i\alpha}, c_{i\beta}$ in C and w_i, $u_{i\alpha}$, $u_{i\beta}$, $v_{i\alpha}$, $v_{i\beta}$ algebraic over F. Taking automorphisms of a finite normal extension of F containing w_i, $u_{i\alpha}$, $u_{i\beta}$, $v_{i\alpha}$, $v_{i\beta}$ on both sides of the equation of γ. Then summing over all the automorphisms and this shows that γ has the correct form. #

Remarks. The complete proof of Theorem 2.2.1 given by Singer, Saunders and Caviness requires the subfield of constants of the $\mathcal{EL}$ - elementary extension of F to be the same as that of F. The proof of this fact is not simple because it involves the notion of constrained extension and other concepts from differential algebra. In most practical applications, it is not difficult to verify directly that the field of constants is the same. Thus in what follows it seems more convenient to adopt this requirement as a given hypothesis.