CHAPTER IV

EXTENSIONS OF LIOUVILLE'S THEOREM

4.1 Statements of the Main Theorems

Definition 4.1.1. Let F be a differential field with derivation D and subfield of constants C. Let A and B be finite indexing sets and let

$$\mathcal{E} = \{ G_\alpha(\exp R_\alpha(Y)) \mid \alpha \in A\},$$
$$\mathcal{L} = \{ H_\beta(\log S_\beta(Y)) \mid \beta \in B\},$$

be sets of expressions where:

(1) G_α, R_α, H_β, S_β are in $C(Y)$ for all $\alpha \in A$, $\beta \in B$,

(2) for all $\beta \in B$, if $H_\beta(Y) = P_\beta(Y)/Q_\beta(Y)$ with P_β, Q_β in $C[Y]$ and $Q_\beta \neq 0$, then $\deg P_\beta \le \deg Q_\beta$.

We say that a differential extension K of F is an <u>Ei-extension</u> of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for each $i = 1,\ldots,n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1}

(ii) $t_i = \exp(u)$ for some u in F_{i-1} ,

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) for some $\alpha \in A$, there are u and nonzero v in F_{i-1} such that
$D(t_i) = D(u)G_\alpha(v)$ where $v = \exp R_\alpha(u)$,
(for brevity, $t_i = \int G_\alpha(\exp R_\alpha(u))D(u)$),

(v) for some $\beta \in B$, there are u, v in F_{i-1} such that
$D(t_i) = D(u) H_\beta(v)$ where $v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$,
(for brevity, $t_i = \int H_\beta(\log S_\beta(u))D(u)$),

(vi) for some $\alpha \in A$, there are nonzero u, v in F_{i-1} such that

$D(t_i) = (D(u)/u)G_\alpha(v)$ where $v = \exp R_\alpha(u)$,

(for brevity, $t_i = \int G_\alpha(\exp R_\alpha(u))D(u)/u$),

(vii) for some $\beta \in B$, there are nonzero u and v in F_{i-1} such that

$D(t_i) = (D(u)/u)H_\beta(v)$ where $v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$,

(for brevity, $t_i = \int H_\beta(\log S_\beta(u))D(u)/u$).

Remark. The differential extension K of F equipped with cases (i)-(v) is an $\mathcal{E}\ell$- elementary extension of F.

Example. Let **C** be the field of complex numbers and let F = **C**(x) be the set of rational functions with coefficients in **C**. Then F is a differential field under the usual derivation D = d/dx.

Let $G(Y) = Y$, $R(Y) = Y$, $H(Y) = 1/(Y+2)$, $S(Y) = Y+1$.

Let $\mathcal{E} = \{ G(\exp R(Y)) \} = \{ \exp Y \}$ and

$\ell = \{ H(\log S(Y)) \} = \{ 1/(\log(Y+1) + 2) \}$.

Hence $K = F(\exp(x), \log(x+1), \int(D(x)/x)\exp(x), \int(D(x)/x)(1/(\log(x+1)+2)))$ is an Ei-extension of F, since

$$F = F_0 \subset F_1 = F_0(t_1) \subset F_2 = F_1(t_2) \subset F_3 = F_2(t_3) \subset F_4 = F_3(t_4) = K$$

where $t_1 = \exp(x)$, $t_2 = \log(x+1)$,

$t_3 = \int(D(x)/x)\exp(x)$ or $D(t_3) = (D(x)/x)\exp(x)$,

and $t_4 = \int(D(x)/x)(1/(\log(x+1)+2))$ or $D(t_4) = (D(x)/x)(1/(\log(x+1)+2))$.

Our first main theorem reads:

Theorem 4.1.2. Let F be a differential field of characteristic zero with derivation D and an algebraically closed subfield of constants C. Let $\gamma \in F$. Assume that there exist an Ei- extension K of F whose subfield of constants is C and $y \in K$ such that $D(y) = \gamma$. Then there exist

(1) $b_i \in C$, $v_0 \in F$ and $v_i \in F\backslash\{0\}$ for all $i \in J$,

(2) $c_{i\alpha}$, $d_{i\alpha} \in C$, nonzero elements $w_{i\alpha}$, $x_{i\alpha}$ algebraic over F for all $i \in I_\alpha$, $\alpha \in A$,

(3) $e_{i\beta}$, $f_{i\beta}$ $\in$ C, nonzero elements $y_{i\beta}$, $z_{i\beta}$ algebraic over F for all $i \in J_\beta$, $\beta \in B$, such that

$$\gamma = D(v_o) + \sum_{i \in J} b_i D(v_i)/v_i$$

$$+ \sum_{\alpha \in A} \sum_{i \in I_\alpha} [c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}] G_\alpha(x_{i\alpha})$$

$$+ \sum_{\beta \in B} \sum_{i \in J_\beta} [e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}] H_\beta(z_{i\beta}),$$

where A, B, J, I_α and J_β are all finite indexing sets,

$x_{i\alpha} = \exp R_\alpha(w_{i\alpha})$ for all $i \in I_\alpha, \alpha \in A$, and

$z_{i\beta} = \log S_\beta(y_{i\beta})$ and $S_\beta(y_{i\beta}) \neq 0$ for all $i \in J_\beta$, $\beta \in B$.

Definition 4.1.3. Let F be a differential field with derivation D and the subfield of constants C. We say that a differential extension K of F is a Gamma extension of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for each i, $1 \le i \le n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1}

(ii) $t_i = \exp(u)$ for some u in F_{i-1} ,

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) there are $G \in C(Y)$, u and nonzero v in F_{i-1} , $r \in \mathbf{Q}$ with $-1 \le r \le 1$ such that $D(t_i) = D(u^r)G(v)$ where $v = \exp(u)$.

Remarks.

(1) The differential extension K of F equipped with cases (i)-(iii) is an elementary extension of F.

(2) In case (iv), if $r = 1$ then such Gamma extension is also an Ei - extension.

The definition of Gamma extension contains the Gamma function which is defined as follows: Let **C** be the field of complex numbers. Then **C**(x) is a differential field with the usual derivation $D = d/dx$.

The Gamma function is defined by

$$\Gamma(x) = \int \exp(-x) D(x^r) \quad \text{where } r \in \mathbf{Q},\ 0 < r \leq 1.$$

Our second main theorem reads:

Theorem 4.1.4. Let F be a differential field of characteristic zero with derivation D and an algebraically closed subfield of constants C. Let $\gamma \in F$. Assume that there exist a Gamma extension K of F whose subfield of constants is C and $y \in K$ such that $D(y) = \gamma$. Then there exist

(1) $b_i \in C$, v_0 algebraic over F and nonzero elements v_i algebraic over F for all $i \in I$,

(2) $c_i \in C$, $r_i \in \mathbf{Q}$ with $-1 \leq r_i \leq 1$, nonzero elements w_i, x_i algebraic over F and $G_i \in C(Y)$ for all $i \in J$,

such that

$$\gamma = D(v_0) + \sum_{i \in I} b_i D(v_i)/v_i + \sum_{i \in J} c_i D(w_i^{r_i}) G_i(x_i),$$

where I, J are finite indexing sets, $D(x_i)/x_i = D(w_i)$ for all $i \in J$.

4.2 Preliminary Lemmas

We first state some results that are used in the proofs of the last two lemmas in this section.

Lemma 4.2.1 ([13,pp. 221-223]). Let F be a field and n an integer ≥ 2. Let $a \in F$, $a \neq 0$. Assume that for all prime numbers p such that $p \mid n$ we have $a \notin F^p$, and if $4 \mid n$ then $a \notin -4F^4$. Then $X^n - a$ is irreducible in F[X].

Lemma 4.2.2([14, pp. 163-164]). Let D be a unique factorization domain with quotient field F and f a primitive polynomial of positive degree in D[X]. Then f is irreducible in D[X] if and only if f is irreducible in F[X].

Lemma 4.2.3 ([7]). Let F be a field containing the algebraic closure of the rationals and let X and Y be indeterminates. Let A(Y) and B(Y) $\neq 0$ be relatively prime elements of F[Y]. Furthermore, assume A/B is not an n^{th} power in F(Y) for any positive integer $n \geq 2$. Then the polynomial $B(Y)X^m - A(Y)$ is irreducible in F(X)[Y] for any positive integer m.

Proof. Let $m \in Z^+$. By Lemma 4.2.1, $BX^m - A = B(X^m - (A/B))$ is irreducible in F(Y)[X]. By Lemma 4.2.2 , $BX^m - A$ is irreducible in F[Y][X] and so irreducible in F[X][Y]. Again by Lemma 4.2.2, $BX^m - A$ is irreducible in F(X)[Y]. #

Lemma 4.2.4 ([7]). Let F be a field, X and Y indeterminates, and A(Y) and B(Y) relatively prime elements of F[Y]. If a and b are elements of F with $a \neq 0$, then A(Y) - (aX+b)B(Y) is irreducible in F(X)[Y].

Proof. This again follows from two applications of Lemma 4.2.2 and the fact that aX + b - A(Y)/B(Y) is irreducible in F(Y)[X]. #

4.3 Ei - Extension

Before proving the main lemma, it will be convenient to define the following term:

If f and g are polynomials over a field F, and $g \neq 0$, then there exist unique polynomials $q(X) = a_0 + a_1X + \cdots + a_nX^n$ and r(X) over F such that $f(X)/g(X) = q(X) + r(X)/g(X)$, where $r(X) = 0$ or $\deg r(X) < \deg g(X)$. Call the unique element a_0 the <u>head</u> of f/g.

Lemma 4.3.1. Let F be a differential field of characteristic zero with derivation D and C its algebraically closed subfield of constants. Let A and B be finite indexing sets and assume that G_α, R_α, H_β, S_β are in C(Y) for all $\alpha \in A$, $\beta \in B$. Let t be transcendental over F such that $D(t) = D(u)t$ for some u in F. Let E be a finite algebraic differential extension of F(t) with extended derivation D. Assume that the subfield of constants of E is C. Let $\gamma \in F$. Assume that there exist

(1) $b_i \in C$, $v_0 \in E$, $v_i \in E\setminus\{0\}$ for all $i \in J$,

(2) $c_{i\alpha}, d_{i\alpha} \in C$, $w_{i\alpha}, x_{i\alpha} \in E\setminus\{0\}$ for all $i \in I_\alpha, \alpha \in A$,

(3) $e_{i\beta}, f_{i\beta} \in C$, $y_{i\beta}, z_{i\beta} \in E\setminus\{0\}$ for all $i \in J_\beta, \beta \in B$,

such that

$$\gamma = D(v_0) + \sum_{i\in J} b_i D(v_i)/v_i$$

$$+ \sum_{\alpha\in A} \sum_{i\in I_\alpha} [c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}] G_\alpha(x_{i\alpha})$$

$$+ \sum_{\beta\in B} \sum_{i\in J_\beta} [e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}] H_\beta(z_{i\beta}),$$

where J, I_α and J_β are all finite indexing sets,

$x_{i\alpha} = \exp R_\alpha(w_{i\alpha})$ for all $i \in I_\alpha, \alpha \in A$, and

$z_{i\beta} = \log S_\beta(y_{i\beta})$ and $S_\beta(y_{i\beta}) \neq 0$ for all $i \in J_\beta, \beta \in B$.

Then there exist

(1) $\bar{b}_i \in C$, $\bar{v}_0 \in F$, $\bar{v}_i \in F\setminus\{0\}$ for all $i \in \bar{J}$,

(2) $\bar{c}_{i\alpha}, \bar{d}_{i\alpha} \in C$, nonzero elements $\bar{w}_{i\alpha}, \bar{x}_{i\alpha}$ algebraic over F for all $i \in \bar{I}_\alpha, \alpha \in \bar{A}$,

(3) $\bar{e}_{i\beta}, \bar{f}_{i\beta} \in C$, nonzero elements $\bar{y}_{i\beta}, \bar{z}_{i\beta}$ algebraic over F for all $i \in \bar{J}_\beta, \beta \in \bar{B}$,

such that

$$\gamma = D(\bar{v}_0) + \sum_{i\in\bar{J}} \bar{b}_i D(\bar{v}_i)/\bar{v}_i$$

$$+ \sum_{\alpha\in\bar{A}} \sum_{i\in\bar{I}_\alpha} [\bar{c}_{i\alpha} D(\bar{w}_{i\alpha}) + \bar{d}_{i\alpha} D(\bar{w}_{i\alpha})/\bar{w}_{i\alpha}] G_\alpha(\bar{x}_{i\alpha})$$

$$+ \sum_{\beta\in\bar{B}} \sum_{i\in\bar{J}_\beta} [\bar{e}_{i\beta} D(\bar{y}_{i\beta}) + \bar{f}_{i\beta} D(\bar{y}_{i\beta})/\bar{y}_{i\beta}] H_\beta(\bar{z}_{i\beta}),$$

where $\bar{A}, \bar{B}, \bar{J}, \bar{I}_\alpha$ and $\bar{J}_\beta$ are all finite indexing sets,

$\bar{x}_{i\alpha} = \exp R_\alpha(\bar{w}_{i\alpha})$ for all $i \in \bar{I}_\alpha$, $\alpha \in \bar{A}$ and

$\bar{z}_{i\beta} = \log S_\beta(\bar{y}_{i\beta})$ and $S_\beta(\bar{y}_{i\beta}) \neq 0$ for all $i \in \bar{J}_\beta$, $\beta \in \bar{B}$.

Proof. Part I. Assume F is algebraically closed.

Step 1. We may assume that for all α in A, $R_\alpha \notin C$; for if $R_{\alpha_0} \in C$ for some $\alpha_0 \in A$, then for each $i \in I_{\alpha_0}$, $G_{\alpha_0}(x_{i\alpha_0}) \in C$.

Thus $\sum_{i\in I_{\alpha_0}} \left(c_{i\alpha_0} D(w_{i\alpha_0}) + d_{i\alpha_0} D(w_{i\alpha_0})/w_{i\alpha_0}\right) G_{\alpha_0}(x_{i\alpha_0})$ is of form

$D(v_0) + \sum b_i D(v_i)/v_i$ which can be included into the first two terms of γ.

Step 2. For each $\alpha \in A$, $i \in I_\alpha$ we have $D(x_{i\alpha}) = D(R_\alpha(w_{i\alpha}))x_{i\alpha}$, then by Theorem 1.9 we have that $R_\alpha(w_{i\alpha}) \in F$ and there exist rational integers $r_{i\alpha}$ and $p_{i\alpha}$ in F such that $x_{i\alpha} = p_{i\alpha} t^{r_{i\alpha}}$. Since $R_\alpha(w_{i\alpha}) \in F$ and F is algebraically closed, $w_{i\alpha} \in F$.

Step 3. For each $\beta \in B$, $i \in J_\beta$, we have $D(z_{i\beta}) = D(S_\beta(y_{i\beta}))/S_\beta(y_{i\beta})$.
We may assume that for all β in B, $S_\beta(Y)$ is not an m^{th} power in C(Y) for any positive integer m. If some $S_\beta(Y) = (\bar{S}_\beta(Y))^m$ then

$D(z_{i\beta}) = D(S_\beta(y_{i\beta}))/S_\beta(y_{i\beta}) = mD(\bar{S}_\beta(y_{i\beta}))/\bar{S}_\beta(y_{i\beta})$. For this case we could replace $S_\beta(Y)$ by $\bar{S}_\beta(Y)$. By Theorem 1.9, we have that $z_{i\beta} \in F$ and there exist rational integers $s_{i\beta}$ and $q_{i\beta}$ in F such that $S_\beta(y_{i\beta}) = q_{i\beta} t^{s_{i\beta}}$.

Note that we can arrange so that $r_{i\alpha}$ and $s_{i\beta}$ are actually integers. To see this, let $r_{i\alpha} = g_{i\alpha}/m$ and $s_{i\beta} = k_{i\beta}/m$, where $g_{i\alpha}$, $k_{i\beta}$ and m are integers. Let $\bar{t} = t^{1/m}$. Hence $D(\bar{t}) = D(u/m)\bar{t}$ and $F \subset F(\bar{t}) \subset E(\bar{t})$. If we replace E by $E(\bar{t})$ and t by $\bar{t}$, we still have fields of the appropriate form and furthermore, $x_{i\alpha} = p_{i\alpha}(\bar{t})^{g_{i\alpha}}$, and $S_\beta(y_{i\beta}) = q_{i\beta}(\bar{t})^{k_{i\beta}}$, where $g_{i\alpha}$ and $k_{i\beta}$ are integers. We shall use the old notation but from now on assume that $r_{i\alpha}$ and $s_{i\beta}$ are integers.

Step 4. Let K be an extension of E such that K is Galois over F(t) and let σ be an element of the Galois group of K over F(t). Then

$$\gamma = \sigma(\gamma) = D(\sigma v_0) + \sum_{i\in J} b_i D(\sigma v_i)/(\sigma v_i)$$

$$+ \sum_{\alpha\in A} \sum_{i\in I_\alpha} [c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}] G(x_{i\alpha})$$

$$+ \sum_{\beta\in B} \sum_{i\in J_\beta} [e_{i\beta} D(\sigma y_{i\beta}) + f_{i\beta} D(\sigma y_{i\beta})/(\sigma y_{i\beta})] H_\beta(z_{i\beta}).$$

Summing over all σ yields, for some M in $\mathbf{Z}$,

$$(4.1) \qquad M\gamma = D(Tv_0) + \sum_{i\in J} b_i D(Nv_i)/(Nv_i) + M\mathcal{E}_1 + \mathcal{E}_2,$$

where
$$\mathcal{E}_1 = \sum_{\alpha\in A} \sum_{i\in I_\alpha} [c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}] G(x_{i\alpha}),$$

$$\mathcal{E}_2 = \sum_{\beta\in B} \sum_{i\in J_\beta} [e_{i\beta} D(Ty_{i\beta}) + f_{i\beta} D(Ny_{i\beta})/(Ny_{i\beta})] H_\beta(z_{i\beta}),$$

and T and N denote the trace and norm respectively. We now consider the head of the right hand side of (4.1).

Step 5. Write $Tv_o = \sum_{i=0}^{n} h_i t^i + \sum\sum(a_{ij}/(t-t_i)^j)$,

where h_i, a_{ij} and t_i are in F. Hence the head of $D(Tv_o)$ is $D(h_o)$.

Step 6. For each $i \in J$ write $Nv_i = k_i \prod_{j=1}^{\alpha_i}(t-\mu_i)^{n_{ij}}$

where the $\alpha_i \in Z^+$, the $k_i \in F\backslash\{0\}$, the $\mu_i \in F$ and the $n_{ij} \in \mathbf{Z}$.

Therefore the head of $\sum_{i\in J} b_i D(Nv_i)/(Nv_i)$ is $\sum_{i\in J} b_i D(k_i)/k_i + \sum_{i\in J}\sum_{j=1}^{\alpha_i} b_i n_{ij} D(u)$.

Step 7. We find the head of $\mathcal{E}_1$.

For each $i \in I_\alpha$, $\alpha \in A$, recall $x_{i\alpha} = p_{i\alpha} t^{r_{i\alpha}}$. If $r_{i\alpha} = 0$, then $x_{i\alpha} \in F$ and hence $G_\alpha(x_{i\alpha}) \in F$. Assume that $r_{i\alpha} \neq 0$. Let $d_{\alpha 0}$ be the head of $G_\alpha(Y)$. Hence $d_{\alpha 0} \in C$. So the head of $G_\alpha(x_{i\alpha})$ is $d_{\alpha 0}$.

Therefore the head of $\mathcal{E}_1$ is

$$\sum\sum_{r_{i\alpha}=0} [c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}] G_\alpha(x_{i\alpha})$$

$$+ \quad \sum\sum_{r_{i\alpha}\neq 0} d_{\alpha 0}[c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}].$$

Step 8. We find the head of $\mathcal{E}_2$. For each $i \in J_\beta$, $\beta \in B$, recall $S_\beta(y_{i\beta}) = q_{i\beta} t^{s_{i\beta}}$.

Case 8.1. If $s_{i\beta} = 0$, then $S_\beta(y_{i\beta}) \in F$. Since F is algebraically closed and $y_{i\beta}$ is algebraic over F, $y_{i\beta} \in F$ Thus $Ty_{i\beta} = My_{i\beta}$ and $Ny_{i\beta} = y_{i\beta}^M$.

So $D(Ty_{i\beta}) = MD(y_{i\beta})$ and $D(Ny_{i\beta})/Ny_{i\beta} = M\, D(y_{i\beta})/y_{i\beta}$.

Case 8.2. Assume that $s_{i\beta} \neq 0$. Calculate the trace and norm of the $y_{i\beta}$. Write $S_\beta(Y) = A_\beta(Y)/B_\beta(Y)$ where $A_\beta, B_\beta \in C[Y]$, $B_\beta \neq 0$ and A_β and B_β are relatively prime in C[Y]. Each $y_{i\beta}$ satisfies $q_{i\beta} t^{s_{i\beta}} B_\beta(Y) - A_\beta(Y) = 0$.

By Lemma 4.2.3, $q_{i\beta} t^{s_{i\beta}} B_\beta(Y) - A_\beta(Y)$ is irreducible over F(t). So the trace and norm can be read of from its coefficients. The coefficients of $q_{i\beta} t^{s_{i\beta}} B_\beta(Y) - A_\beta(Y)$

are all of the form $\delta_{i\beta} q_{i\beta} t^{s_{i\beta}} + \varepsilon_{i\beta}$ where $\delta_{i\beta}, \varepsilon_{i\beta} \in C$.

Dividing by the leading coefficient, we get

$$Ty_{i\beta} = m_{i\beta}\left(\frac{\delta_{i\beta} q_{i\beta} t^{s_{i\beta}} + \varepsilon_{i\beta}}{\mu_{i\beta} q_{i\beta} t^{s_{i\beta}} + \upsilon_{i\beta}}\right), \quad \text{and}$$

$$Ny_{i\beta} = \left(\frac{\omega_{i\beta} q_{i\beta} t^{s_{i\beta}} + \zeta_{i\beta}}{\mu_{i\beta} q_{i\beta} t^{s_{i\beta}} + \upsilon_{i\beta}}\right)^{m_{i\beta}},$$

where $m_{i\beta} \in Z^+, \delta_{i\beta}, \mu_{i\beta}, \omega_{i\beta}, \varepsilon_{i\beta}, \upsilon_{i\beta}, \zeta_{i\beta} \in C$. Hence

$$D(Ty_{i\beta}) = \frac{m_{i\beta}\left(\delta_{i\beta}\upsilon_{i\beta} - \varepsilon_{i\beta}\mu_{i\beta}\right)\left(D(q_{i\beta}) + q_{i\beta}s_{i\beta}D(u)\right)t^{s_{i\beta}}}{\left(\mu_{i\beta} q_{i\beta} t^{s_{i\beta}} + \upsilon_{i\beta}\right)^2}$$

and

$$\frac{D(Ny_{i\beta})}{Ny_{i\beta}} = \frac{m_{i\beta}\left(\omega_{i\beta}\upsilon_{i\beta} - \zeta_{i\beta}\mu_{i\beta}\right)\left(D(q_{i\beta}) + q_{i\beta}s_{i\beta}D(u)\right)t^{s_{i\beta}}}{\left(\omega_{i\beta} q_{i\beta} t^{s_{i\beta}} + \zeta_{i\beta}\right)\left(\mu_{i\beta} q_{i\beta} t^{s_{i\beta}} + \upsilon_{i\beta}\right)}.$$

Thus the head of $D(Ty_{i\beta})$ is 0 and the head of $\dfrac{D(Ny_{i\beta})}{Ny_{i\beta}}$ is $\overline{m}_{i\beta} D(z_{i\beta})$ where $\overline{m}_{i\beta} \in \mathbf{Z}$.

Hence $\sum_{s_{i\beta} \neq 0}\sum f_{i\beta}\overline{m}_{i\beta} D(z_{i\beta}) H_\beta(z_{i\beta})$ is of form $D(\hat{v}_0) + \sum \hat{b}_i D(\hat{v}_i)/\hat{v}_i$ where $\hat{v}_0 \in F$ and $\hat{v}_i \in F\setminus\{0\}, \hat{b}_i \in C$.

Therefore the head of $\mathcal{E}_2$ is

$$M\sum_{s_{i\beta}=0}\sum \left[e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}\right] H_\beta(z_{i\beta}) + D(\hat{v}_0) + \sum \hat{b}_i D(\hat{v}_i)/\hat{v}_i.$$

Step 9. We conclude that the head of the right hand side of (4.1) is

$$D(\bar{v}_0) + \sum \bar{b}_i D(\bar{v}_i)/\bar{v}_i$$

$$+ M \sum\sum [c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}] G_\alpha(w_{i\alpha})$$

$$+ M \sum\sum [e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}] H_\beta(z_{i\beta}),$$

where $\bar{v}_0 \in F$, $\bar{v}_i \in F\backslash\{0\}$, $\bar{b}_i \in C$.

Then comparing the head of (4.1) and dividing by M, we get the correct sum of γ.

Part II. Assume that F is not algebraically closed.

Let $\bar{F}$ be an algebraic closure of F.

Let $S = \{v_0\} \cup \{v_i \mid i \in J\} \cup \{w_{i\alpha} \mid i \in I_\alpha, \alpha \in A\} \cup \{x_{i\alpha} \mid i \in I_\alpha, \alpha \in A\} \cup$
$\{y_{i\beta} \mid i \in J_\beta, \beta \in B\} \cup \{z_{i\beta} \mid i \in J_\beta, \beta \in B\}$.

Hence $\bar{F}(t, S)$ is algebraic over $\bar{F}(t)$.

Now $\bar{F}(t, S)$, $\bar{F}(t)$, $\bar{F}$ and F have the same subfield of constants C. (See Appcndix A for details). By Part I there exist

(1) $\bar{b}_i \in C$, $\bar{v}_0 \in \bar{F}$, $\bar{v}_i \in \bar{F}\backslash\{0\}$ for all $i \in \bar{J}$,

(2) $\bar{c}_{i\alpha}$, $\bar{d}_{i\alpha} \in C$, $\bar{w}_{i\alpha}$, $\bar{x}_{i\alpha} \in \bar{F}\backslash\{0\}$, for all $i \in \bar{I}_\alpha$, $\alpha \in \bar{A}$,

(3) $\bar{e}_{i\beta}$, $\bar{f}_{i\beta} \in C$, $\bar{y}_{i\beta}$, $\bar{z}_{i\beta} \in \bar{F}\backslash\{0\}$, for all $i \in \bar{J}_\beta$, $\beta \in \bar{B}$,

such that

$$\gamma = D(\bar{v}_0) + \sum_{i \in \bar{J}} \bar{b}_i D(\bar{v}_i)/\bar{v}_i \tag{4.2}$$

$$+ \sum_{\alpha \in \bar{A}} \sum_{i \in \bar{I}_\alpha} [\bar{c}_{i\alpha} D(\bar{w}_{i\alpha}) + \bar{d}_{i\alpha} D(\bar{w}_{i\alpha})/\bar{w}_{i\alpha}] G_\alpha(\bar{x}_{i\alpha})$$

$$+ \sum_{\beta \in \bar{B}} \sum_{i \in \bar{J}_\beta} [\bar{e}_{i\beta} D(\bar{y}_{i\beta}) + \bar{f}_{i\beta} D(\bar{y}_{i\beta})/\bar{y}_{i\beta}] H_\beta(\bar{z}_{i\beta}),$$

where $\bar{A}$, $\bar{B}$, $\bar{J}$, $\bar{I}_\alpha$ and $\bar{J}_\beta$ are all finite indexing sets,

$\bar{x}_{i\alpha} = \exp R_\alpha(\bar{w}_{i\alpha})$ for all $i \in \bar{I}_\alpha$, $\alpha \in \bar{A}$ and

$\bar{z}_{i\beta} = \log S_\beta(\bar{y}_{i\beta})$ and $S_\beta(\bar{y}_{i\beta}) \neq 0$ for all $i \in \bar{J}_\beta$, $\beta \in \bar{B}$.

Let K be a finite Galois extension of F containing

$\{\bar{v}_0\} \cup \{\bar{v}_i \mid i \in \bar{J}\} \cup \{\bar{w}_{i\alpha}, \bar{x}_{i\alpha} \mid i \in \bar{I}_\alpha, \alpha \in \bar{A}\} \cup \{\bar{y}_{i\beta}, \bar{z}_{i\beta} \mid i \in \bar{J}_\beta, \beta \in \bar{B}\}$

Applying σ an element of the Galois group of K over F in (4.2) and then summing over all σ, we get, for some M in **Z**,

$$\begin{aligned} M\gamma &= D(T\bar{v}_0) + \sum_{i \in \bar{J}} \bar{b}_i D(N\bar{v}_i)/(N\bar{v}_i) \\ &\quad + \sum_{\sigma} \sum_{\alpha \in \bar{A}} \sum_{i \in \bar{I}_\alpha} [\bar{c}_{i\alpha} D(\sigma\bar{w}_{i\alpha}) + \bar{d}_{i\alpha} D(\sigma\bar{w}_{i\alpha})/(\sigma\bar{w}_{i\alpha})] G_\alpha(\sigma\bar{x}_{i\alpha}) \\ &\quad + \sum_{\sigma} \sum_{\beta \in \bar{B}} \sum_{i \in \bar{J}_\beta} [\bar{e}_{i\beta} D(\sigma\bar{y}_{i\beta}) + \bar{f}_{i\beta} D(\sigma\bar{y}_{i\beta})/(\sigma\bar{y}_{i\beta})] H_\beta(\sigma\bar{z}_{i\beta}), \end{aligned}$$

where T and N denote the trace and norm respectively.

Note that $D(\sigma\bar{x}_{i\alpha}) = D(R_\alpha(\sigma\bar{w}_{i\alpha}))/(\sigma\bar{x}_{i\alpha})$ for all $i \in \bar{I}_\alpha$, $\alpha \in \bar{A}$ and

$D(\sigma\bar{z}_{i\beta}) = D(S_\beta(\sigma\bar{y}_{i\beta}))/(S_\beta(\sigma\bar{y}_{i\beta}))$ and $S_\beta(\sigma\bar{y}_{i\beta}) \neq 0$ for all $i \in \bar{J}_\beta$, $\beta \in \bar{B}$.

Since $T\bar{v}_0$ and $N\bar{v}_i$ are in F, this yields the final conclusion of the lemma. #

Lemma 4.3.2. Let F be a differential field of characteristic zero with derivation D and C being its algebraically closed subfield of constants. Let A and B be finite indexing sets and assume that

(1) $G_\alpha, R_\alpha, H_\beta, S_\beta$ are in C(Y) for all $\alpha \in A$, $\beta \in B$,

(2) for all $\beta \in B$, if $H_\beta(Y) = P_\beta(Y)/Q_\beta(Y)$ with P_β, Q_β in C[Y] and $Q_\beta \neq 0$, then $\deg P_\beta \leq \deg Q_\beta$.

Let t be transcendental over F satisfying one of the following conditions:

(i) $D(t) = D(u)/u$ for some nonzero u in F,

(ii) $D(t) = D(u)G_\alpha(v)$ for some α in A and some u, v in F, $v \neq 0$ such that $v = \exp R_\alpha(u)$,

(iii) $D(t) = D(u)H_\beta(v)$ for some β in B and some u, v in F such that

$v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$,

(iv) $D(t) = (D(u)/u)G_\alpha(v)$ for some α in A and some u, v in $F\setminus\{0\}$ such that

$v = \exp R_\alpha(u)$,

(v) $D(t) = (D(u)/u)H_\beta(v)$ for some β in B and some u, v in F, $u \neq 0$ such that

$v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$.

Let E be a finite algebraic differential extension of F(t) with extended derivation D. Assume that the subfield of constants of E is C. Let $\gamma \in F$. Assume that there exist

(1) $b_i \in C$, $v_0 \in E$, $v_i \in E\setminus\{0\}$ for all $i \in J$,

(2) $c_{i\alpha}, d_{i\alpha} \in C$, $w_{i\alpha}, x_{i\alpha} \in E\setminus\{0\}$ for all $i \in I_\alpha$, $\alpha \in A$,

(3) $e_{i\beta}, f_{i\beta} \in C$, $y_{i\beta}, z_{i\beta} \in E\setminus\{0\}$ for all $i \in J_\beta$, $\beta \in B$,

such that

$$\gamma = D(v_0) + \sum_{i\in J} b_i D(v_i)/v_i$$
$$+ \sum_{\alpha\in A}\sum_{i\in I_\alpha} [c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}]G_\alpha(x_{i\alpha})$$
$$+ \sum_{\beta\in B}\sum_{i\in J_\beta} [e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}]H_\beta(z_{i\beta}),$$

where J, I_α and J_β are all finite indexing sets,

$x_{i\alpha} = \exp R_\alpha(w_{i\alpha})$ for all $i \in I_\alpha$, $\alpha \in A$, and

$z_{i\beta} = \log S_\beta(y_{i\beta})$ and $S_\beta(y_{i\beta}) \neq 0$ for all $i \in J_\beta$, $\beta \in B$.

Then there exist

(1) $\bar{b}_i \in C$, $\bar{v}_0 \in F$, $\bar{v}_i \in F\setminus\{0\}$ for all $i \in \bar{J}$,

(2) $\bar{c}_{i\alpha}$, $\bar{d}_{i\alpha} \in C$, nonzero elements $\bar{w}_{i\alpha}$, $\bar{x}_{i\alpha}$ algebraic over F

for all $i \in \bar{I}_\alpha$, $\alpha \in \bar{A}$,

(3) $\bar{e}_{i\beta}$, $\bar{f}_{i\beta} \in C$, nonzero elements $\bar{y}_{i\beta}$, $\bar{z}_{i\beta}$ algebraic over F

for all $i \in \bar{J}_\beta$, $\beta \in \bar{B}$,

such that

$$\gamma \;=\; D(\bar{v}_0) \;+\; \sum_{i\in\bar{J}} \bar{b}_i D(\bar{v}_i)/\bar{v}_i$$

$$+ \sum_{\alpha\in\bar{A}} \sum_{i\in\bar{I}_\alpha} [\bar{c}_{i\alpha} D(\bar{w}_{i\alpha}) + \bar{d}_{i\alpha} D(\bar{w}_{i\alpha})/\bar{w}_{i\alpha}] G_\alpha(\bar{x}_{i\alpha})$$

$$+ \sum_{\beta\in\bar{B}} \sum_{i\in\bar{J}_\beta} [\bar{e}_{i\beta} D(\bar{y}_{i\beta}) + \bar{f}_{i\beta} D(\bar{y}_{i\beta})/\bar{y}_{i\beta}] H_\beta(\bar{z}_{i\beta}),$$

where $\bar{A}$, $\bar{B}$, $\bar{J}$, $\bar{I}_\alpha$ and $\bar{J}_\beta$ are all finite indexing sets,

$\bar{x}_{i\alpha} \;=\; \exp R_\alpha(\bar{w}_{i\alpha})$ for all $i \in \bar{I}_\alpha$, $\alpha \in \bar{A}$ and

$\bar{z}_{i\beta} \;=\; \log S_\beta(\bar{y}_{i\beta})$ and $S_\beta(\bar{y}_{i\beta}) \neq 0$ for all $i \in \bar{J}_\beta$, $\beta \in \bar{B}$.

Proof. Part I. Assume F is algebraically closed.

Step 1. We may assume that $R_\alpha \notin C$ for all $\alpha \in A$, by the same reasoning as in Lemma 4.3.1.

Step 2. For each $\alpha \in A$, $i \in I_\alpha$, we have that $D(x_{i\alpha}) = D(R_\alpha(w_{i\alpha}))x_{i\alpha}$, then by Theorem 1.9, we get $x_{i\alpha} \in F$ and there exist $\lambda_{i\alpha} \in C$, $p_{i\alpha} \in F$ such that $R_\alpha(w_{i\alpha}) = \lambda_{i\alpha} t + p_{i\alpha}$.

Step 3. For each $\beta \in B$, $i \in J_\beta$, we have that $D(z_{i\beta}) = D(S_\beta(y_{i\beta}))/S_\beta(y_{i\beta})$, then by Theorem 1.9, we get $S_\beta(y_{i\beta}) \in F$ and there exist $\bar{\lambda}_{i\beta} \in C$, $q_{i\beta} \in F$ such that $z_{i\beta} = \bar{\lambda}_{i\beta} t + q_{i\beta}$. Since $S_\beta(y_{i\beta}) \in F$ and F is algebraically closed, $y_{i\beta} \in F$.

Step 4. Let K be an extension field of E such that K is Galois over F(t) and let σ be an element of the Galois group of K over F(t). Then

$$\gamma = \sigma(\gamma) = D(\sigma v_0) + \sum_{i\in J} b_i D(\sigma v_i)/(\sigma v_i)$$

$$+ \sum_{\alpha\in A} \sum_{i\in I_\alpha} [c_{i\alpha} D(\sigma w_{i\alpha}) + d_{i\alpha} D(\sigma w_{i\alpha})/(\sigma w_{i\alpha})] G_\alpha(x_{i\alpha})$$

$$+ \sum_{\beta\in B} \sum_{i\in J_\beta} [e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}] H_\beta(z_{i\beta}).$$

Summing over all σ yields, for some M in **Z**,

(4.3) $$M\gamma = D(Tv_0) + \sum_{i\in J} b_i D(Nv_i)/(Nv_i) + \mathcal{E}_1 + M\mathcal{E}_2,$$

where $$\mathcal{E}_1 = \sum_{\alpha\in A} \sum_{i\in I_\alpha} [c_{i\alpha} D(Tw_{i\alpha}) + d_{i\alpha} D(Nw_{i\alpha})/(Nw_{i\alpha})] G_\alpha(x_{i\alpha}),$$

$$\mathcal{E}_2 = \sum_{\beta\in B} \sum_{i\in J_\beta} [e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}] H_\beta(z_{i\beta}),$$

and T and N denote the trace and norm respectively.

<u>Step 5</u>. Consider $\sum_{i\in J} b_i D(Nv_i)/(Nv_i)$.

Write $Nv_i = k_i \prod_{j=1}^{\alpha_i} (t-\mu_j)^{n_{ij}}$ where the $n_{ij} \in \mathbf{Z}$, the $k_i \in F\backslash\{0\}$, the $\mu_j \in F$ and the $\alpha_i \in Z^+$.

So $\sum_{i\in J} b_i D(Nv_i)/(Nv_i) = \sum_{i\in J} b_i D(k_i)/k_i$ + an element in $F(t)\backslash F[t]$.

<u>Step 6</u>. Next, consider $\mathcal{E}_1$.

Recall $R_\alpha(w_{i\alpha}) = \lambda_{i\alpha} t + p_{i\alpha}$ for all $i \in I_\alpha, \alpha \in A$.

<u>Case 6.1</u>. Assume that $\lambda_{i\alpha} = 0$.

For these α, i, $R_\alpha(w_{i\alpha}) \in F$ and thus $w_{i\alpha} \in F$.

So $Tw_{i\alpha} = Mw_{i\alpha}$ and $Nw_{i\alpha} = w_{i\alpha}^{M}$.

Hence $D(Tw_{i\alpha}) = M\,D(w_{i\alpha})$ and $D(Nw_{i\alpha})/(Nw_{i\alpha}) = M\,D(w_{i\alpha})/w_{i\alpha}$.

Case 6.2. Assume that $\lambda_{i\alpha} \neq 0$.

Write $R_\alpha(Y) = A_\alpha(Y)/B_\alpha(Y)$ where A_α and B_α are relatively prime in C[Y] and $B_\alpha \neq 0$. Each $w_{i\alpha}$ satisfies $A_\alpha(Y) - (\lambda_{i\alpha}t + p_{i\alpha})B_\alpha(Y) = 0$. By Lemma 4.2.4, $A_\alpha(Y) - (\lambda_{i\alpha}t + p_{i\alpha})B_\alpha(Y)$ is irreducible over F(t). So the trace and norm can be read of its coefficients. Therefore

$$Tw_{i\alpha} = m_{i\alpha}\left(\frac{\delta_{i\alpha}(\lambda_{i\alpha}t+p_{i\alpha})+\varepsilon_{i\alpha}}{\mu_{i\alpha}(\lambda_{i\alpha}t+p_{i\alpha})+\upsilon_{i\alpha}}\right)$$

and

$$Nw_{i\alpha} = \left(\frac{\zeta_{i\alpha}(\lambda_{i\alpha}t+p_{i\alpha})+\eta_{i\alpha}}{\mu_{i\alpha}(\lambda_{i\alpha}t+p_{i\alpha})+\upsilon_{i\alpha}}\right)^{m_{i\alpha}},$$

where $\delta_{i\alpha}, \varepsilon_{i\alpha}, \zeta_{i\alpha}, \eta_{i\alpha}, \mu_{i\alpha}, \upsilon_{i\alpha} \in C$ and $m_{i\alpha} \in Z^{+}$.

Therefore $$D(Tw_{i\alpha}) = m_{i\alpha}\frac{(\upsilon_{i\alpha}\delta_{i\alpha} - \varepsilon_{i\alpha}\mu_{i\alpha})(\lambda_{i\alpha}D(t) - D(p_{i\alpha}))}{(\mu_{i\alpha}(\lambda_{i\alpha}t+p_{i\alpha})+\upsilon_{i\alpha})^2}$$ and

$$\frac{D(Nw_{i\alpha})}{(Nw_{i\alpha})} = m_{i\alpha}\lambda_{i\alpha}D(t)\left[\frac{\zeta_{i\alpha}}{\zeta_{i\alpha}(\lambda_{i\alpha}t+p_{i\alpha})+\eta_{i\alpha}} - \frac{\mu_{i\alpha}}{\mu_{i\alpha}(\lambda_{i\alpha}t+p_{i\alpha})+\upsilon_{i\alpha}}\right].$$

The head of $D(Nw_{i\alpha})/(Nw_{i\alpha})$ is 0.

Now, consider $D(Tw_{i\alpha})$.

If $\mu_{i\alpha} \neq 0$, then the head of $D(Tw_{i\alpha})$ is 0. Assume that $\mu_{i\alpha} = 0$.

Hence $D(Tw_{i\alpha}) = \left(\frac{m_{i\alpha}\delta_{i\alpha}}{\upsilon_{i\alpha}}\right)\left(\frac{D(x_{i\alpha})}{x_{i\alpha}}\right)$, and so

$$\sum_{\substack{\alpha\in A \\ \lambda_{i\alpha}\neq 0, \mu_{i\alpha}=0}} \sum_{i\in I_\alpha} c_{i\alpha} D(Tw_{i\alpha})G_\alpha(x_{i\alpha}) = \sum_{\substack{\alpha\in A \\ \lambda_{i\alpha}\neq 0, \mu_{i\alpha}=0}} \sum_{i\in I_\alpha} \left(\frac{c_{i\alpha} m_{i\alpha}\delta_{i\alpha}}{\upsilon_{i\alpha}}\right)\left(\frac{D(x_{i\alpha})}{x_{i\alpha}}\right) G_\alpha(x_{i\alpha})$$

$$= D(\overline{w}_0) + \sum_{i\in \overline{J}} \overline{c}_i D(\overline{w}_i)/\overline{w}_i,$$

where $\overline{c}_i \in C$, the $\overline{w}_i$ are in F and $\overline{J}$ is the finite indexing set. This last equality follows from the fact that $\dfrac{G_\alpha(x_{i\alpha})}{x_{i\alpha}}$ is a rational function of $x_{i\alpha}$ with constant coefficients.

Therefore $\mathcal{E}_1 = \sum_{\alpha\in A} \sum_{i\in I_\alpha} [c_{i\alpha} D(Tw_{i\alpha}) + d_{i\alpha} D(Nw_{i\alpha})/(Nw_{i\alpha})]G_\alpha(x_{i\alpha})$

$$= M \sum_{\substack{\alpha\in A \\ \lambda_{i\alpha}=0}} \sum_{i\in I_\alpha} [c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}]G_\alpha(x_{i\alpha})$$

$$+ D(\overline{w}_0) + \sum_{i\in \overline{J}} \overline{c}_i D(\overline{w}_i)/\overline{w}_i \quad + \text{ an element in } F(t)\backslash F[t].$$

Step 7. Finally, consider $\mathcal{E}_2$. For each $i \in J_\beta$, $\beta \in B$, recall $z_{i\beta} = \overline{\lambda}_{i\beta} t + q_{i\beta}$.

Case 7.1. Assume $\overline{\lambda}_{i\beta} = 0$.

Hence $z_{i\beta} \in F$ and so $H_\beta(z_{i\beta}) \in F$.

Clearly, $\displaystyle\sum_{\substack{\beta\in B \\ \overline{\lambda}_{i\beta}=0}} \sum_{i\in J_\beta} (e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}) H_\beta(z_{i\beta}) \in F.$

Case 7.2. Assume that $\overline{\lambda}_{i\beta} \neq 0$.

Since deg(numerator of H_β) $\leq$ deg(denominator of H_β),

$$H_\beta(Y) = \sum_{i=1}^{n_\beta} \sum_{j=1}^{r_i} \left(\frac{a_{ij}}{(Y-\alpha_i)^j}\right) + q_\beta,$$

where n_β, $r_i \in Z^+$, and a_{ij}, α_i, $q_\beta \in C$.

Hence $$\sum_{\beta\in B}\sum_{\substack{i\in J_\beta \\ \bar{\lambda}_{i\beta}\neq 0}} (e_{i\beta}D(y_{i\beta}) + f_{i\beta}D(y_{i\beta})/y_{i\beta})H_\beta(z_{i\beta})$$

$$= \sum_{\beta\in B}\sum_{\substack{i\in J_\beta \\ \bar{\lambda}_{i\beta}\neq 0}} (e_{i\beta}D(y_{i\beta}) + f_{i\beta}D(y_{i\beta})/y_{i\beta})q_\beta \ + \text{ an element in } F(t)\backslash F[t].$$

Step 8. From (4.3) we conclude that

(4.4) $$M\gamma = D(Tv_0) + \sum_{i\in J} b_i D(k_i)/k_i$$

$$+ M \sum_{\alpha\in A}\sum_{\substack{i\in I_\alpha \\ \lambda_{i\alpha}=0}} [c_{i\alpha}D(w_{i\alpha}) + d_{i\alpha}D(w_{i\alpha})/w_{i\alpha}]G_\alpha(x_{i\alpha})$$

$$+ D(\bar{w}_0) + \sum_{i\in \bar{J}} \bar{c}_i D(\bar{w}_i)/\bar{w}_i$$

$$+ M \sum_{\beta\in B}\sum_{\substack{i\in J_\beta \\ \bar{\lambda}_{i\beta}=0}} (e_{i\beta}D(y_{i\beta}) + f_{i\beta}D(y_{i\beta})/y_{i\beta})H_\beta(z_{i\beta})$$

$$+ M \sum_{\beta\in B}\sum_{\substack{i\in J_\beta \\ \bar{\lambda}_{i\beta}\neq 0}} (e_{i\beta}D(y_{i\beta}) + f_{i\beta}D(y_{i\beta})/y_{i\beta})q_\beta$$

$$+ \text{ an element in } F(t)\backslash F[t].$$

Step 9. Now consider $D(Tv_0)$.

Write $Tv_0 = \sum_{j=0}^{n} \bar{v}_j t^j$ + an element in $F(t)\backslash F[t]$, where $n \in Z^+$ and the $\bar{v}_j \in F$. So

(4.5) $D(Tv_o) = D(\bar{v}_n)t^n + \sum_{j=1}^{n}(j\bar{v}_jD(t)+D(\bar{v}_{j-1}))t^{j-1}$ + an element in $F(t)\backslash F[t]$.

Claim that $n \leq 1$. Suppose that $n > 1$, hence $\bar{v}_n \neq 0$. Replacing (4.5) in (4.4), we have that the right hand side of (4.4) would contain an expression of the form t^i with $i \geq 1$. Comparing terms of degree n and n-1 in (4.4), $D(\bar{v}_n) = 0$ and $(n\,\bar{v}_nD(t)+D(\bar{v}_{n-1})) = 0$, Since $D(\bar{v}_n) = 0$, $\bar{v}_n \in C$.

Thus $D(n\,\bar{v}_nt+\bar{v}_{n-1}) = n\,\bar{v}_nD(t)+D(\bar{v}_{n-1}) = 0$.

So $n\,\bar{v}_nt + \bar{v}_{n-1} \in C$. Thus t is algebraic over F, a contradiction. So we have the claim.

From (4.5) , we get $D(Tv_o) = D(\bar{v}_1)t + (\bar{v}_1D(t)+D(\bar{v}_0))$ + an element in $F(t)\backslash F[t]$.

Clearly, $\bar{v}_1 \in C$. Hence $D(Tv_o) = \bar{v}_1D(t)+D(\bar{v}_0)$ + an element in $F(t)\backslash F[t]$.

<u>Step 10</u>. Replacing $D(Tv_o)$ in (4.4) and comparing the head, we get

$$M\gamma = \bar{v}_1D(t) + D(\bar{v}_0+\bar{w}_0) + \sum_{i\in J} c_iD(\bar{w}_i)/\bar{w}_i + \sum_{i\in J} b_iD(k_i)/k_i$$

$$+ M\sum_{\alpha\in A}\sum_{\substack{i\in I_\alpha\\ \lambda_{i\alpha}=0}}[c_{i\alpha}D(w_{i\alpha})+d_{i\alpha}D(w_{i\alpha})/w_{i\alpha}]G_\alpha(x_{i\alpha})$$

$$+ M\sum_{\beta\in B}\sum_{\substack{i\in J_\beta\\ \bar{\lambda}_{i\beta}=0}}(e_{i\beta}D(y_{i\beta})+f_{i\beta}D(y_{i\beta})/y_{i\beta})H_\beta(z_{i\beta}).$$

$$+ M\sum_{\beta\in B}\sum_{\substack{i\in J_\beta\\ \bar{\lambda}_{i\beta}\neq0}}(e_{i\beta}D(y_{i\beta})+f_{i\beta}D(y_{i\beta})/y_{i\beta})q_\beta$$

+ an element in $F(t)\backslash F[t]$.

Dividing by M, we obtain the correct sum of γ.

Part II. To remove the assumption that F is algebraically closed, we proceed as in the proof of Part II of Lemma 4.3.1. #

Proof of Theorem 4.1.2. Let m = tr.deg. K/F. The proof is by induction on m. If m = 0, then K is algebraic over F. Let E be an extension of K such that E is Galois over F and let σ be an element of the Galois group of E over F. Thus

$$\gamma = \sigma(\gamma) = D(\sigma y).$$

Summing over all σ yields, for some M in **Z**,

$$M\gamma = D(Ty),$$

where T denote the trace.

Hence $\gamma = D(Ty/M)$ and $Ty/M \in F$. Therefore the theorem is true in this case. Assume that m > 0. Suppose that the theorem is true for any Ei- extension L of a field F′ such that tr.deg.L/F′ < m. Since tr.deg.K/F = m, we can choose a transcendence basis $t_1,\ldots,t_m$ of K over F such that

$$F = F_0 \subset F(t_1) = F_1 \subset \cdots \subset F(t_1,\ldots,t_m) = F_m \subset K$$

and each t_i satisfies one of the following six conditions:

(1) $t_i = \exp(u)$ for some $u \in \overline{F}_{i-1} \cap K$, ($\overline{F}_{i-1}$ denote the algebraic closure of F_{i-1}),

(2) $t_i = \log(u)$ for some nonzero $u \in \overline{F}_{i-1} \cap K$,

(3) for some $\alpha \in A$, there are u and nonzero v in $\overline{F}_{i-1} \cap K$ such that $D(t_i) = D(u)G_\alpha(v)$ where $v = \exp R_\alpha(u)$,

(4) for some $\beta \in B$, there are u, v in $\overline{F}_{i-1} \cap K$ such that $D(t_i) = D(u)H_\beta(v)$ where $v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$,

(5) for some $\alpha \in A$, there are nonzero u, v in $\overline{F}_{i-1} \cap K$ such that $D(t_i) = (D(u)/u)G_\alpha(v)$ where $v = \exp R_\alpha(u)$,

(6) for some $\beta \in B$, there are nonzero u, v in $\overline{F}_{i-1} \cap K$ such that $D(t_i) = (D(u)/u)H_\beta(v)$ where $v = \log S_\beta(u)$ and $S_\beta(u) \neq 0$.

Note that K is also an Ei - extension of F_1 and tr.deg.K/F_1 = m-1 < m.

By induction hypothesis, there exist

(1) $b_i \in C$, $v_0 \in F_1$ and $v_i \in F_1 \setminus \{0\}$ for all $i \in J$,

(2) $c_{i\alpha}$, $d_{i\alpha} \in C$, nonzero elements $w_{i\alpha}$, $x_{i\alpha}$ algebraic over F_1 for all $i \in I_\alpha, \alpha \in A$,

(3) $e_{i\beta}$, $f_{i\beta} \in C$, nonzero elements $y_{i\beta}$, $z_{i\beta}$ algebraic over F_1 for all $i \in J_\beta$, $\beta \in B$,

such that

$$\gamma = D(v_0) + \sum_{i \in J} b_i D(v_i)/v_i$$

$$+ \sum_{\alpha \in A} \sum_{i \in I_\alpha} (c_{i\alpha} D(w_{i\alpha}) + d_{i\alpha} D(w_{i\alpha})/w_{i\alpha}) G_\alpha(x_{i\alpha})$$

$$+ \sum_{\beta \in B} \sum_{i \in I_\beta} (e_{i\beta} D(y_{i\beta}) + f_{i\beta} D(y_{i\beta})/y_{i\beta}) H_\beta(z_{i\beta}),$$

where A, B, J, I_α and J_β are all finite indexing sets,

$x_{i\alpha} = \exp R_\alpha(w_{i\alpha})$ for all $i \in I_\alpha$, $\alpha \in A$, and

$z_{i\beta} = \log S_\beta(y_{i\beta})$ and $S_\beta(y_{i\beta}) \neq 0$ for all $i \in J_\beta$, $\beta \in B$.

By Lemma 4.3.1 and 4.3.2, we get the correct sum of γ. #

Theorem 4.1.2 is false without the condition (2) in the definition of Ei- extension as seen in the following example.

Example. Let **C** be the field of complex numbers and let $F = \mathbf{C}(x, \log(x))$ with the usual derivation $D = d/dx$. Let $\mathcal{E} = \varnothing$ and $\mathcal{L} = \{\log Y(Y-1)\}$.

In this case the index set B is a singleton, $H(Y) = Y$ and $S(Y) = Y(Y-1)$.

This is excluded by condition (2) since deg (numerator of H) >deg(denominator of H).

Clearly, $$\frac{\log x}{x+1} = \frac{\log x(x+1)}{x+1} - \frac{\log(x+1)}{x+1}.$$

Hence $\int \frac{\log x}{x+1}$ lies in an Ei - extension of F.

Claim that

$$(4.6) \qquad \frac{\log x}{x+1} \neq D(v_o) + \sum_{i=1}^{n} b_i D(v_i)/v_i + \sum_{i=1}^{m} (e_i D(y_i) + f_i D(y_i)/y_i) z_i ,$$

for any $v_o \in F$, the $v_i \in F \backslash \{0\}$, the y_i, z_i algebraic over F with $z_i = \log y_i(y_i - 1)$ and $y_i(y_i - 1) \neq 0$ and constants b_i, e_i, $f_i \in \mathbf{C}$.

Suppose equality holds in (4.6).

By Theorem 3.2.1, we have for each $i = 1,\ldots,m$, that $z_i = r_i \log(x) + k_i$ for some $r_i \in \mathbf{Q}$, $k_i \in \mathbf{C}$. We also have, for each $i = 1,\ldots,m$, $y_i(y_i-1) = c_i x^{r_i}$ for some $c_i \in \mathbf{C}$. We can assume that each c_i is not zero. (If $c_i = 0$ for some i, then $y_i(y_i-1) = 0$, a contradiction.) Each y_i is algebraic over $K = \mathbf{C}(x, \log(x), x^{r_1},\ldots,x^{r_m})$ and satisfies the irreducible equation $Y(Y-1) - c_i x^{r_i} = 0$. Taking automorphisms over some appropriate extension (containing $y_1,\ldots,y_m$) of K and then summing over all the automorphisms, we obtain

$$t \frac{\log x}{x+1} = D(Tv_o) + \sum_{i=1}^{n} b_i D(Nv_i)/(Nv_i)$$
$$+ \sum_{i=1}^{m} (e_i D(Ty_i) + f_i D(Ny_i)/(Ny_i))(r_i \log(x) + k_i) ,$$

where $t \in Z^+$, T and N denote trace and norm respectively.

Now, consider $\sum_{i=1}^{m} (e_i D(Ty_i) + f_i D(Ny_i)/(Ny_i))(r_i \log(x) + k_i)$.

For each $i = 1,\ldots,m$, since $T y_i \in \mathbf{Z}$, $D(Ty_i) = 0$.

Since $Ny_i = (c_i x^{r_i})^t$, $D(Ny_i)/Ny_i = r_i\, t\, D(x)/x$.

Hence $\sum_{i=1}^{m} (e_i D(Ty_i) + f_i D(Ny_i)/(Ny_i))(r_i \log(x) + k_i)$ is of the form

$s\, D(\log^2 x) + r\, D(x)/x$ where $s, r \in \mathbf{C}$.

Therefore $\dfrac{\log x}{x+1} = D(w_o) + \sum_{i \in I} d_i D(w_i)/w_i$, where $w_o \in K$, $w_i \in K \backslash \{0\}$,

$d_i \in \mathbf{C}$ and I is a finite indexing set.

Take $\sigma \in$ Aut (K /C(x, log(x)) to the last equation and then summing over all σ, we get

$$t_1 \frac{\log x}{x+1} = D(Tw_0) + \sum_{i \in I} d_i D(Nw_i)/(Nw_i), \quad \text{where } t_1 \in \mathbf{Z}.$$

This contradicts the fact that $\int \frac{\log x}{x+1}$ is not elementary (for proof see Appendix B). #

4.4 Gamma Extension

Lemma 4.4.1. Let F be a differential field of characteristic zero with derivation D, and C being its algebraically closed subfield of constants. Let t be transcendental over F such that

(4.7) $\quad D(t) = D(u)t$ for some u in F.

Let E be a finite algebraic differential extension of F(t) with extended derivation D. Assume that the subfield of constants of E is C. Let $\gamma \in F$. Assume that there exist

(1) $b_i \in C$, $v_0 \in E$, $v_i \in E\backslash\{0\}$ for all $i \in I$,

(2) $c_i \in C$, $r_i \in \mathbf{Q}$ with $-1 \le r_i \le 1$, w_i, $x_i \in E\backslash\{0\}$ and $G_i \in C(Y)$ for all $i \in J$

such that

$$\gamma = D(v_0) + \sum_{i \in I} b_i D(v_i)/v_i + \sum_{i \in J} c_i D(w_i^{r_i}) G_i(x_i),$$

where I and J are finite indexing sets, $x_i = \exp(w_i)$ for all $i \in J$.

Then there exist

(1) $\bar{b}_i \in C$, $\bar{v}_0$ algebraic over F, and nonzero elements $\bar{v}_i$ algebraic over F for all $i \in \bar{I}$,

(2) $\bar{c}_i \in C$, $\bar{r}_i \in \mathbf{Q}$ with $-1 \le \bar{r}_i \le 1$, nonzero elements $\bar{w}_i$, $\bar{x}_i$ algebraic over F for all $i \in \bar{J}$,

such that

$$\gamma = D(\bar{v}_0) + \sum_{i\in\bar{I}} \bar{b}_i D(\bar{v}_i)/\bar{v}_i + \sum_{i\in\bar{J}} \bar{c}_i D(\overline{w_i^{r_i}}) G_i(\bar{x}_i),$$

where $\bar{I}$ and $\bar{J}$ are all finite indexing sets, $\bar{x}_i = \exp(\bar{w}_i)$ for all $i \in \bar{J}$.

Proof. Part I. Assume that F is algebraically closed.

For each $i \in J$, we have $D(x_i) = D(w_i)x_i$, then by Theorem 1.9, we have that $w_i \in F$ and there exist $\upsilon_i \in \mathbf{Q}$ and $p_i \in F$ such that $x_i = p_i t^{\upsilon_i}$. Without loss of generality, we may assume that υ_i are actually integers.

Let K be an extension of E such that K is Galois over F(t) and let σ be an element of the Galois group of K over F(t). Then

$$\gamma = \sigma(\gamma) = D(\sigma v_0) + \sum_{i\in I} b_i D(\sigma v_i)/(\sigma v_i) + \sum_{i\in J} c_i D(w_i^{r_i}) G_i(x_i).$$

Summing over all σ yields, for some M in $\mathbf{Z}$,

$$(4.8)\quad M\gamma = D(Tv_0) + \sum_{i\in I} b_i D(Nv_i)/(Nv_i) + M\sum_{i\in J} c_i D(w_i^{r_i}) G_i(x_i),$$

where T and N denote the trace and norm respectively.

We now consider the head of the right hand side of (4.8). It is straightforward to verify that

$$D(Tv_0) + \sum_{i\in I} b_i D(Nv_i)/Nv_i = D(\bar{v}_0) + \sum_{i\in I} b_i D(\bar{v}_i)/\bar{v}_i + aD(u)$$
$$+ \text{ elements in } F(t)\backslash F,$$

where $a \in C$, $\bar{v}_0 \in F$ and $\bar{v}_i \in F\backslash\{0\}$ for all $i \in I$.

For each $i \in J$, recall $x_i = p_i t^{\upsilon_i}$.

Write $$\sum_{i\in J} c_i D(w_i^{r_i}) G_i(x_i) = \sum_{\substack{i\in J\\ \upsilon_i=0}} c_i D(w_i^{r_i}) G_i(x_i) + \sum_{\substack{i\in J\\ \upsilon_i\neq 0}} c_i D(w_i^{r_i}) G_i(x_i).$$

Clearly, $$\sum_{\substack{i\in J\\ \upsilon_i=0}} c_i D(w_i^{r_i}) G_i(x_i) \in F.$$

It is easy to see that

$$\sum_{\substack{i\in J\\ \upsilon_i\neq 0}} c_i D(w_i^{r_i})G_i(x_i) = \sum_{\substack{i\in J\\ \upsilon_i\neq 0}} \bar{c}_i D(w_i^{r_i}) + \text{ elements in } F(t)\backslash F,$$

where the $\bar{c}_i \in C$.

From (4.8), we equate the head to get,

$$M\gamma = D(\bar{v}_0) + \sum_{i\in I} b_i D(\bar{v}_i)/\bar{v}_i + aD(u)$$

$$+ M \sum_{\substack{i\in J\\ \upsilon_i = 0}} c_i D(w_i^{r_i})G_i(x_i)$$

$$+ M \sum_{\substack{i\in J\\ \upsilon_i\neq 0}} \bar{c}_i D(w_i^{r_i}).$$

Dividing by M, we get the required result.

Part II. Assume that F is not algebraically closed.

Let $\bar{F}$ be an algebraic closure of F.

Let $S = \{v_0\}\cup\{v_i \mid i\in I\}\cup\{w_i \mid i\in J\}\cup\{x_i \mid i\in J\}$.

Hence $\bar{F}(t,S)$ is algebraic over $\bar{F}(t)$.

Note that $\bar{F}(t,S)$, $\bar{F}(t)$, $\bar{F}$ and F have the same subfield of constants C.(See Appendix A for details.) By Part I, there exist

(1) $\bar{b}_i \in C$, $\bar{v}_0 \in \bar{F}$, $\bar{v}_i \in \bar{F}\backslash\{0\}$ for all $i \in \bar{I}$,

(2) $\bar{c}_i \in C$, $\bar{r}_i \in \mathbf{Q}$ with $-1 \le \bar{r}_i \le 1$, $\bar{w}_i$, $\bar{x}_i \in \bar{F}\backslash\{0\}$ for all $i \in \bar{J}$,

such that

$$\gamma = D(\bar{v}_0) + \sum_{i\in \bar{I}} \bar{b}_i D(\bar{v}_i)/\bar{v}_i + \sum_{i\in \bar{J}} \bar{c}_i D(\bar{w}_i^{\bar{r}_i})G_i(\bar{x}_i),$$

where $\bar{I}$, $\bar{J}$ are finite indexing sets, $D(\bar{x}_i)/\bar{x}_i = D(\bar{w}_i)$ for all $i \in \bar{J}$. #

Lemma 4.4.2. Lemma 4.4.1 holds if equation (4.7) is replaced by one of the following conditions:

(i) $D(t) = D(u)/u$ for some nonzero u in F,

(ii) $D(t) = D(u^r)G(v)$ for some $r \in \mathbf{Q}$ with $-1 \leq r \leq 1$, $G \in C(Y)$ and $u, v \in F$ such that $v \neq 0$, $D(v)/v = D(u)$.

Proof. Part I. Assume that F is algebraically closed.

For each $i \in J$, we have that $D(x_i) = D(w_i)x_i$, then by Theorem 1.9, we get $x_i \in F$ and there exist $\lambda_i \in C$, $p_i \in F$ such that $w_i = \lambda_i t + p_i$.

Let K be an extension of E containing $\{w_i^{r_i} / i \in J\}$ such that K is Galois over F(t).

Let σ be an element of the Galois group of K over F(t). Then

$$\gamma = \sigma(\gamma) = D(\sigma v_0) + \sum_{i \in I} b_i D(\sigma v_i)/(\sigma v_i) + \sum_{i \in J} c_i D(\sigma w_i^{r_i}) G_i(x_i)$$

Summing over all σ yields, for some M in $\mathbf{Z}$.

$$(4.9) \quad M\gamma = D(Tv_0) + \sum_{i \in I} b_i D(Nv_i)/(Nv_i) + \sum_{i \in J} c_i D(Tw_i^{r_i}) G_i(x_i),$$

where T and N denote the trace and norm respectively.

Now consider $\sum_{i \in J} c_i D(Tw_i^{r_i}) G_i(x_i)$.

For each $i \in J$, recall $w_i = \lambda_i t + p_i$.

Write $$\sum_{i \in J} c_i D(Tw_i^{r_i}) G_i(x_i) = \sum_{\substack{i \in J \\ \lambda_i = 0}} c_i D(Tw_i^{r_i}) G_i(x_i) + \sum_{\substack{i \in J \\ \lambda_i \neq 0}} c_i D(Tw_i^{r_i}) G_i(x_i)$$

Consider $i \in J$ for which $\lambda_i = 0$. Hence $w_i \in F$ and also $w_i^{r_i} \in F$.

So $T w_i^{r_i} = M w_i^{r_i}$.

Thus $\sum_{\substack{i\in J\\ \lambda_i=0}} c_i D(Tw_i^{r_i})G_i(x_i) = \sum_{\substack{i\in J\\ \lambda_i=0}} c_i M D(w_i^{r_i})G_i(x_i)$.

Now consider $i \in J$ for which $\lambda_i \neq 0$. For these i ,$-1 \leq r_i \leq 1$.

If $r_i = 1$, then $Tw_i = Mw_i$ and $D(Tw_i) = M D(x_i)/x_i$.

If $r_i = -1$, then $Tw_i^{-1} = Mw_i^{-1}$ and $D(Tw_i^{-1}) = -MD(w_i)/(\lambda_i t+p_i)^2$.

If $-1 < r_i < 1$, then write $r_i = s_i/h_i$ where s_i and h_i are relatively prime in **Z**.

Here $w_i^{r_i}$ satisfy $Y^{h_i} - (\lambda_i t + p_i)^{s_i} = 0$. By Lemma 4.2.1, $Y^{h_i} - (\lambda_i t + p_i)^{s_i}$ is irreducible over F(t).

Hence $Tw_i^{r_i} = 0$. Thus $D(Tw_i^{r_i}) = 0$.

$$\text{Therefore } \sum_{\substack{i\in J\\ \lambda_i\neq 0}} c_i D(Tw_i^{r_i})G_i(x_i) = \sum_{\substack{i\in J\\ \lambda_i\neq 0, r_i=1}} c_i M \frac{D(x_i)}{x_i} G_i(x_i)$$

$$+ \text{ elements in } F(t)\backslash F[t],$$

$$= D(u_o) + \sum_{i\in \bar{J}} d_i D(u_i)/u_i + \text{ elements in } F(t)\backslash F[t],$$

where $u_o \in F$, $d_i \in C$, $u_i \in F\backslash\{0\}$ for all $i \in \bar{J}$ and $\bar{J}$ is a finite indexing set. This last equality follows from the fact that $G_i(x_i)/x_i$ is a rational function of x_i with constant coefficients. So

$$\sum_{i\in J} c_i D(Tw_i^{r_i})G_i(x_i) = M \sum_{\substack{i\in J\\ \lambda_i=0}} c_i D(w_i^{r_i})G_i(x_i) + D(u_o) + \sum_{i\in \bar{J}} d_i D(u_i)/u_i$$

$$+ \text{ elements in } F(t)\backslash F[t].$$

It is straightforward to see that

$$\sum_{i\in I} b_i D(Nv_i)/(Nv_i) = \sum_{i\in I} b_i D(k_i)/k_i + \text{ elements in } F(t)\backslash F[t],$$

where $k_i \in F\backslash\{0\}$ for all $i \in I$.

From (4.9), we can conclude that

$$(4.10)\qquad M\gamma = D(Tv_o) + \sum_{i\in I} b_i D(k_i)/k_i + M \sum_{\substack{i\in J\\ \lambda_i=0}} c_i D(w_i^{r_i}) G_i(x_i)$$

$$+ D(u_o) + \sum_{i\in \bar{J}} d_i D(u_i)/u_i + \text{elements in } F(t)\backslash F[t].$$

Now consider $D(Tv_o)$.

Write $Tv_o = \sum_{j=0}^{n} \bar{v}_j t^j$ + elements in $F(t)\backslash F[t]$, where $n \in Z^+$ and $\bar{v}_j \in F$ for all $j = 0, 1, \ldots, n$, we have

$$D(Tv_o) = D(\bar{v}_n)t^n + \sum_{j=1}^{n} (j\bar{v}_j D(t) + D(\bar{v}_{j-1}))t^{j-1}$$

$$+ \text{elements in } F(t)\backslash F[t].$$

Claim that $n \le 1$. Suppose that $n \ge 2$, hence $\bar{v}_n \ne 0$. Replacing $D(Tv_o)$ in (4.10), we have that the right hand side of (4.10) would contain an expression of the form t^i with $i \ge 1$. Comparing terms of degree n and n-1 in (4.10), we get $D(\bar{v}_n) = 0$ and $n\bar{v}_n D(t) + D(\bar{v}_{n-1}) = 0$. Since $D(\bar{v}_n) = 0$, $\bar{v}_n \in C$.

Hence $D(n\bar{v}_n t + \bar{v}_{n-1}) = n\bar{v}_n D(t) + D(\bar{v}_{n-1}) = 0$. So $n\bar{v}_n t + \bar{v}_{n-1} \in C$.

Thus t is algebraic over F, a contradiction. So we have the cliam.

Therefore $D(Tv_o) = D(\bar{v}_1)t + \bar{v}_1 D(t) + D(\bar{v}_o)$ + elements in $F(t)\backslash F[t]$.

Clearly, $\bar{v}_1 \in C$. Hence $D(Tv_o) = \bar{v}_1 D(t) + D(\bar{v}_o)$ + elements in $F(t)\backslash F[t]$.

Again replacing $D(Tv_o)$ in (4.10) and comparing the head , we get

$$M\gamma = \bar{v}_1 D(t) + D(\bar{v}_o + u_o) + \sum_{i\in I} b_i D(k_i)/k_i + \sum_{i\in \bar{J}} d_i D(u_i)/u_i$$

$$+ M \sum_{\substack{i\in J\\ \lambda_i=0}} c_i D(w_i^{r_i}) G_i(x_i)$$

Dividing by M, we obtain the correct sum of γ.

Part II. To remove the assumption that F is algebraically closed,we proceed as in the proof of Part II of Lemma 4.4.1. #

Proof of Theorem 4.1.4. Let m = tr.deg.K/F. The proof is by induction on m. If m = 0, then K is algebraic over F, and the theorem is trivially true. Assume that m > 0. Suppose that the theorem is true for any Gamma extension L of a field F′ such that tr.deg. L/F′ < m. Since tr.deg. K/F = m, we can choose a transcendence basis $t_1,\ldots,t_m$ of K over F such that $F = F_0 \subset F(t_1) = F_1 \subset \cdots \subset F(t_1,\ldots,t_m) = F_m \subset K$ and each t_i satisfies one of the following three conditions:

(1) $D(t_i) = D(u)/u$ for some nonzero u in $\overline{F}_{i-1} \cap K$,

(2) $D(t_i) = D(u)t_i$ for some u in $\overline{F}_{i-1} \cap K$,

(3) $D(t_i) = D(u^r)G(v)$ for some $r \in \mathbf{Q}$, $-1 \le r \le 1$, $G \in C(Y)$ and u, v in $\overline{F}_{i-1} \cap K$ such that $v \neq 0$, $D(v)/v = D(u)$.

($\overline{F}_{i-1}$ denote the algebraic closure of F_{i-1}).

Note that K is also a Gamma extension of F_1 and tr.deg. $K/F_1 = m-1 < m$. So by induction hypothesis, there exist

(1) $b_i \in C$, v_0 algebraic over F_1 and nonzero elements v_i algebraic over F_1 for all $i \in I$,

(2) $c_i \in C$, $r_i \in \mathbf{Q}$ with $-1 \le r_i \le 1$, nonzero elements w_i, x_i algebraic over F_1 and $G_i \in C(Y)$ for all $i \in J$,

such that

$$\gamma = D(v_0) + \sum_{i \in I} b_i D(v_i)/v_i + \sum_{i \in J} c_i D(w_i^{r_i}) G_i(x_i),$$

where I, J are finite indexing sets, $D(x_i)/x_i = D(w_i)$ for all $i \in J$.

By Lemma 4.4.1 and 4.4.2, we get the result of the theorem. #