CHAPTER IV

MATHEMATICAL MODELS OF FLUID FLOW IN DUCTS

The general governing equations of fluid dynamics for three-dimensional flow in rectangular coordinates

- The mass conservation or continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{4.1}$$

where **V** is the vector velocity field in cartesian space and given by

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

where **i**, **j** and **k** are the unit vectors along the x , y and z axes respectively.

- The three momentum conservation equations are given by

x-component :

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \tag{4.2}$$

y-component :

$$\frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{V}) = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \tag{4.3}$$

z-component :

$$\frac{\partial (\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{V}) = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z \tag{4.4}$$

equations (4.2) - (4.4) are called the *compressible Navier-Stokes equations.*

4.1 ASSUMPTIONS OF THE MODEL

1. The system is steady state.

2. The fluid is Newtonian.

Newton stated that shear stress in a fluid is proportional to the time rate of strain. Such fluids are called *Newtonian* fluids. Thus, the expressions of the various stresses in terms of velocity gradients and fluid properties are

$$\tau_{xx} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial u}{\partial x} \tag{4.5}$$

$$\tau_{yy} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial v}{\partial y} \tag{4.6}$$

$$\tau_{zz} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial w}{\partial z} \tag{4.7}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{4.8}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{4.9}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{4.10}$$

where μ is the *molecular viscosity coefficient* and λ is the *second viscosity coefficient*. Stokes made the hypothesis that

$$\lambda = -\frac{2}{3}\mu \tag{4.11}$$

The further assumption of μ is to be constant throughout the flow.

3. The density of fluid (ρ) is constant.

Thus , equation (4.1) becomes

$$\nabla \cdot \mathbf{V} = 0 \qquad (4.12)$$

4. Body forces are negligible.

Thus , the terms f_x , f_y and f_z in equations (4.2) - (4.4) which stand for the body forces per unit mass acting on fluid element as its component (such as gravitational , electric , magnetic forces.) are set to zero.

With these assumptions , the general governing equations are reduced to the specific governing equations for this study as follows

continuity :

$$\nabla \cdot \mathbf{V} = 0 \qquad (4.13)$$

x-momentum :

$$\rho \nabla \cdot (u\mathbf{V}) = - \frac{\partial P}{\partial x} + \mu \nabla^2 u \qquad (4.14)$$

y-momentum :

$$\rho \nabla \cdot (v\mathbf{V}) = - \frac{\partial P}{\partial y} + \mu \nabla^2 v \qquad (4.15)$$

z-momentum :

$$\rho \nabla \cdot (w\mathbf{V}) = - \frac{\partial P}{\partial z} + \mu \nabla^2 w \qquad (4.16)$$

equations (4.14) - (4.16) are called the *incompressible Navier-Stokes equations.*

4.2 TURBULENCE MODEL

One source of turbulence generating eddies is found in surfaces of flow discontinuity which occur whenever two fluid streams come together in such a way as to leave a sharp jump in velocity between adjacent layers. Such as at the tips of sharp projections , at the edges of bluff bodies , at zones of boundary-layer separation. (Daily and Harleman , 1966) Because of turbulent motion has a random nature , so it can be described by a set of statistical properties.

For this purpose , it is convenient to set the instantaneous value equal to the sum of a mean value plus a fluctuating component.

Thus , for the xyz-coordinate directions :

$$u = \overline{u} + u' \tag{4.17a}$$

$$v = \overline{v} + v' \tag{4.17b}$$

$$w = \overline{w} + w' \tag{4.17c}$$

where

$$\overline{u} = \frac{1}{T}\int_0^T u\,dt \qquad \text{etc. for } \overline{v}\ ,\ \overline{w} \tag{4.18}$$

with T being a long time in comparison to the time scale of the turbulence that is examined. Due to the fluctuations are both plus and minus , the mean of u' is

$$\overline{u'} = \frac{1}{T}\int_0^T u'\,dt \equiv 0 \tag{4.19}$$

and similarly for the y- and z-components.

Continuity must be satisfied for turbulent as for laminar motion. For incompressible fluids , the divergence of the velocity equals zero. Using the relations of equations (4.17) onto equation (4.13) , then obtain

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (4.20)$$

Taking averages of each term and using the relations such as equation (4.19) , then obtain

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (4.21a)$$

or

$$\nabla \cdot \bar{V} = 0 \qquad (4.21b)$$

and also

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (4.22)$$

Thus , both the means and the fluctuations must satisfy the continuity condition.

The equations of motion in the incompressible form of equations (4.14) - (4.16) can also be used the relations of equations (4.17) and including additional relations.

$$P = \bar{P} + P' \qquad (4.23)$$

$$\bar{P} = \frac{1}{T} \int_0^T P \, dt \qquad (4.24)$$

and

$$\overline{P'} = \frac{1}{T} \int_0^T P' \, dt \equiv 0 \qquad (4.25)$$

After rearrangement , then the three momentum equations are converted for application to turbulence as following equations.

x-component :

$$\rho\,\nabla\cdot\left(\overline{u}\overline{\mathbf{V}}\right) = -\frac{\partial \overline{P}}{\partial x} + \mu\,\nabla^2\overline{u} - \rho\left(\frac{\partial\,\overline{u'u'}}{\partial x} + \frac{\partial\,\overline{u'v'}}{\partial y} + \frac{\partial\,\overline{u'w'}}{\partial z}\right) \qquad (4.26)$$

y-component :

$$\rho\,\nabla\cdot\left(\overline{v}\overline{\mathbf{V}}\right) = -\frac{\partial \overline{P}}{\partial y} + \mu\,\nabla^2\overline{v} - \rho\left(\frac{\partial\,\overline{v'u'}}{\partial x} + \frac{\partial\,\overline{v'v'}}{\partial y} + \frac{\partial\,\overline{v'w'}}{\partial z}\right) \qquad (4.27)$$

z-component :

$$\rho\,\nabla\cdot\left(\overline{w}\overline{\mathbf{V}}\right) = -\frac{\partial \overline{P}}{\partial z} + \mu\,\nabla^2\overline{w} - \rho\left(\frac{\partial\,\overline{w'u'}}{\partial x} + \frac{\partial\,\overline{w'v'}}{\partial y} + \frac{\partial\,\overline{w'w'}}{\partial z}\right) \qquad (4.28)$$

In the equations of motion , u , v , w and P are everywhere replaced by $\overline{u}$. , $\overline{v}$, $\overline{w}$ and $\overline{P}$; but , in addition , new terms arise in equations (4.26) - (4.28) , which are associated with the turbulent velocity fluctuations. For convenience , the notation are introduced

$$\overline{\tau}^{(t)}_{xx} = -\rho\,\overline{u'u'} \qquad (4.29a)$$

$$\overline{\tau}^{(t)}_{xy} = -\rho\,\overline{u'v'} \qquad (4.29b)$$

and etc.

These terms are the components of the turbulent momentum flux $\overline{\tau}^{(t)}$; they are usually referred to as the *Reynolds stresses.* (Bird , Stewart and Lightfoot , 1960)

In order to adopts equations (4.26) - (4.28) to get the velocity profiles , some expression for $\bar{\tau}^{(t)}$ has to be inserted. In this study , the semiempirical relation of Boussinesq's eddy viscosity is applied.

This proposal was that one write (Boussinesq , 1877)

$$\bar{\tau}_{xy}^{(t)} = \mu_t \frac{d\bar{u}}{dy} \tag{4.30}$$

By analogy with Newton's law of viscosity ; μ_t is a *turbulent viscosity coefficient* or *eddy viscosity* and usually depends strongly on position.

Thus , equations (4.26) - (4.28) become

x-momentum : $$\rho \frac{\partial\left(\bar{u}_j \bar{u}\right)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x} + \frac{\partial}{\partial x_j}\left(\mu_{eff} \frac{\partial \bar{u}}{\partial x_j}\right) \tag{4.31}$$

y-momentum : $$\rho \frac{\partial\left(\bar{u}_j \bar{v}\right)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial y} + \frac{\partial}{\partial x_j}\left(\mu_{eff} \frac{\partial \bar{v}}{\partial x_j}\right) \tag{4.32}$$

z-momentum : $$\rho \frac{\partial\left(\bar{u}_j \bar{w}\right)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial z} + \frac{\partial}{\partial x_j}\left(\mu_{eff} \frac{\partial \bar{w}}{\partial x_j}\right) \tag{4.33}$$

where $\frac{\partial \bar{u}_j}{\partial x_j}$ is equivalent to $\nabla \cdot \bar{\mathbf{V}}$.

x_j is coordinate direction.

$\bar{u}_j$ is mean velocity component in x_j direction.

and μ_{eff} is the *effective viscosity coefficient* (Hjertager and Magnussen , 1981) which is expressed as

$$\mu_{eff} = \mu + \mu_t \tag{4.34}$$

The next aim is to find some expression for μ_t. In this study , the two transport equations model of turbulence kinetic energy (k) and rate of dissipation of turbulence kinetic energy (ε) is adopted to determine the turbulent viscosity as following relations. (Launder and Spalding , 1974)

$$\mu_t = \frac{\rho C_\mu k^2}{\varepsilon} \tag{4.35}$$

where C_μ is an empirical constant.

k and ε are obtained by solving the following differential equations.

$$\rho \frac{\partial\left(\bar{u}_j k\right)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\frac{\mu_t}{\sigma_k} + \mu\right)\frac{\partial k}{\partial x_j}\right] + G - \rho\varepsilon \tag{4.36}$$

and

$$\rho \frac{\partial\left(\bar{u}_j \varepsilon\right)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\frac{\mu_t}{\sigma_\varepsilon} + \mu\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_1 \frac{\varepsilon}{k} G - C_2 \rho \frac{\varepsilon^2}{k} \tag{4.37}$$

where C_1 , C_2 are coefficients in approximated turbulent transport equation.

σ_k , σ_ε are effective turbulent Prandtl numbers for transport of k and ε respectively.

and G is the rate of production of turbulence from the mean motion. This is given by

$$G = \mu_t \left\{ \begin{array}{l} 2\left[\left(\frac{\partial \overline{u}}{\partial x}\right)^2 + \left(\frac{\partial \overline{v}}{\partial y}\right)^2 + \left(\frac{\partial \overline{w}}{\partial z}\right)^2 \right] + \left[\frac{\partial \overline{v}}{\partial x} + \frac{\partial \overline{u}}{\partial y} \right]^2 \\ + \left[\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{u}}{\partial z} \right]^2 + \left[\frac{\partial \overline{w}}{\partial y} + \frac{\partial \overline{v}}{\partial z} \right]^2 \end{array} \right\} \quad (4.38)$$

After extensive examination of free turbulent flows , Launder , Morse , Rodi and Spalding (1972) suggested to follow values of the constants appearing in equations (4.35) - (4.37) as given in table 4.1.

C_μ	C_1	C_2	σ_k	σ_ε
0.09	1.44	1.92	1.0	1.3

Table 4.1 The values of the constants in the $k \sim \varepsilon$ model.

4.3 THE WALL FUNCTION METHOD

The form of the model which has been presented in section 4.2 is valid only for fully turbulent flows. Close to solid walls and some other interfaces , there are inevitably regions that viscous effects dominate.

To be able to account for the large gradients of the dependent variables near the walls in a coarse grid computation , special schemes must be employed. The method used in this study is the *wall function method* proposed by Launder and Spalding (1972). In this method the variation of the dependent variables in near wall regions is taken to be similar to that found in two-dimensional turbulent boundary-layers. This means that the wall shear stresses in x_j direction which appear in the momentum equations are calculated from

$$\tau_{w,j} = \frac{\kappa \; \mu \; y^{+}_{nw} \; \overline{u}_{j,nw}}{\ln\left(E \, y^{+}_{nw} \right) . \, \Delta y_{nw}} \tag{4.39}$$

where κ is the Von Karmann constant.

E is a roughness parameter.

$\overline{u}_{j,nw}$ is the absolute value of the resultant velocity parallel to the wall at the first grid node in x_j direction.

Δy_{nw} is the normal distance of the first grid point from the wall.

y^{+}_{nw} is normalized-coordinate near a wall , which is expressed as

$$y^{+}_{nw} = \frac{\rho \left(\sqrt{C_\mu} \, . \, k_{nw} \right)^{0.5} . \, \Delta y_{nw}}{\mu} \tag{4.40}$$

where C_μ equals to 0.09 as in the standard $k \sim \varepsilon$ model.

Equation (4.39) is the well-known logarithmic law of the wall , and strictly this law should be applied to a point whose y^{+}_{nw} value is in the range

$$30 \quad < \quad y^{+}_{nw} \quad < \quad 130$$

k_{nw} , the value of k for the grid point , can be calculated from the regular balance equation with the diffusion of turbulence kinetic energy being set to zero. When calculating k_{nw} , it is necessary to assign a value for the average turbulence kinetic energy-dissipation rate over the control volume between the wall and near wall point.

$$\int_0^{y_{nw}} \varepsilon \ dy = \left(\sqrt{C_\mu} \, . \, k_{nw} \right)^{1.5} \, . \, \frac{1}{\kappa} \ln \left(E \, y^{+}_{nw} \right) \qquad (4.41)$$

For the rate of dissipation of turbulence kinetic energy , the following assumption is made in the wall region.

$$\varepsilon_{nw} = \frac{C_\mu^{0.75} \ k_{nw}^{1.5}}{\kappa \, . \, \Delta y_{nw}} \qquad (4.42)$$

For a smooth wall , the constants in the wall functions are given by $\kappa = 0.41$ and $E = 8.6$.