การวิเคราะห์ฮาร์มอนิกของการแปลงแบบริสซ์ที่ปรากฏในสมการเชิงอนุพันธ์ย่อยไม่เฉพาะที่

นางสาวศศิกานต์ ยี่โป้

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HARMONIC ANALYSIS OF RIESZ-TYPE TRANSFORMS ARISING IN NONLOCAL PARTIAL DIFFERENTIAL EQUATIONS

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สมการไม่เฉพาะที่

$$\mathcal{L}_K^s u(x) := \mathrm{P.V.} \int_{\mathbb{R}^n} 2K(x,y) \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy = g(x)$$

โดยที่ $s \in (0,1)$ และ g เป็นฟังก์ชันที่กำหนดให้ ได้รับการศึกษาอย่างแพร่หลายภายใต้สมมติฐานที่แตกต่างกันของเคอร์เนล K ในงานวิทยานิพนธ์ฉบับนี้ เราแนะนำตัวดำเนินการที่สัมพันธ์กับสมการไม่เฉพาะที่ข้างต้นผ่านศักย์ริสซ์ ภายใต้สมมติฐานที่ว่า K สามารถวัดได้ สมมาตรและมีสมบัติเชิงวงรี เราได้ว่าตัวดำเนินดังกล่าวคือตัวดำเนินการคัลเดรอน-ซิกมุนด์

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The nonlocal equation

$$\mathcal{L}_K^s u(x) := \text{P.V.} \int_{\mathbb{R}^n} 2K(x,y) \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = g(x),$$

where $s \in (0,1)$ and g is a given function, has been widely studied under different assumptions on the kernel K. In this work, we introduce an operator associated to the above nonlocal equation via Riesz potential. Under assumptions that K is measurable, symmetric and elliptic, we obtain that such an operator is a Calderón-Zygmund operator.

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CONTENTS

CHAPTER I
INTRODUCTION

Riesz transforms arise naturally in linear PDEs of divergence form

$$\sum_{i,j=1}^{n} \partial_{x_j}(A(x)\partial_{x_i}u) = g \quad \text{in } \mathbb{R}^n \tag{1.1}$$

or nondivergence form

$$\sum_{i,j=1}^{n} A(x)\partial_{x_i}(\partial_{x_j}u) = g \quad \text{in } \mathbb{R}^n \tag{1.2}$$

where $A(x) = (a_{ij}(x)) \in \mathbb{R}^{n\times n}$ is a given coefficient function. They are the "zero-order" structural part of the PDE. In 1998, T. Iwaniec and C. Sbordone [10] proposed elegant methods in solving (1.1) via Riesz transforms. Under assumption that A is elliptic, bounded and of vanishing mean oscillation, they obtained the existence, uniqueness and also the Calderón-Zygmund L^p-theory of a solution.

We now adapt techniques of T. Iwaniec and C. Sbordone to the equation

$$\sum_{i=1}^{n} \partial_{x_i}(A(x)\partial_{x_i}u) = g \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

If we set $f := (-\Delta)^{\frac{1}{2}}u$ and apply the Riesz potential I^1 to equation (1.3), then (1.3) at least formally is equivalent to

$$\sum_{i=1}^{n} \mathcal{R}_i(A(x)\mathcal{R}_i f) = I^1 g \quad \text{in } \mathbb{R}^n.$$

In particular, if we define the operator

$$Tf := \sum_{i=1}^{n} \mathcal{R}_i(A(x)\mathcal{R}_i f)$$

then (1.3) is equivalent to the equation

$$T((-\Delta)^{\frac{1}{2}}u) = I^1 g \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

Thus, existence, regularity and uniqueness of a solution to (1.3) are basically related to harmonic analysis results on boundedness and invertibility of the operator T. Since the Riesz transforms are bounded on any L^p space, the most basic property is that T is a bounded linear operator from L^p to itself under the assumption that A is bounded. Additionally, if A is elliptic and of vanishing mean oscillation, then T has a bounded inverse which was proved by T. Iwaniec and C. Sbordone [10] as mentioned before.

In this work, we extend the previous method to a popular nonlocal equation

$$\mathcal{L}_K^s u = g \tag{1.5}$$

where $s \in (0,1)$ and the operator $\mathcal{L}_K^s$ is given by

$$\mathcal{L}_K^s u(x) := \text{P.V.} \int_{\mathbb{R}^n} 2K(x,y) \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy.$$

Notation P.V. stands for the principal value of the integral. Here we assume $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ to be measurable in x and y, symmetric $K(x,y) = K(y,x)$ and also elliptic, i.e. there exist $\lambda, \Lambda > 0$ such that $\lambda < K(x,y) < \Lambda$ for all $x, y \in \mathbb{R}^n$.

Notice that in the case $K = 1$, the operator $\mathcal{L}_K^s$ corresponds to the fractional Laplacian operator $(-\Delta)^s$.

Such equation has been studied by numerous authors, to name just a few, [1, 2, 3, 4, 5, 6, 7, 11, 12, 15, 16]. Recently, T. Mengesha, A. Schikorra and SY [13] introduced a natural analogue of the operator T associated to (1.5): set

$A_{K,s_1,s_2}(z_1,z_2)$ to be the following double integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(|x-z_1|^{s_1-n}-|y-z_1|^{s_1-n})\,(|x-z_2|^{s_2-n}-|y-z_2|^{s_2-n})}{|x-y|^{n+2s}}\,dx\,dy$$

where $0<s_1,s_2<1$ with $s_1+s_2=2s$, and set

$$T_{K,s_1,s_2}f(z_1) := \int_{\mathbb{R}^n} A_{K,s_1,s_2}(z_1,z_2)f(z_2)\,dz_2, \quad z_1\in\mathbb{R}^n. \tag{1.6}$$

The operator T_{K,s_1,s_2} is equivalent to (1.5) in the sense that solutions to (1.5) satisfy

$$T_{K,s_1,s_2}((-\Delta)^{\frac{s_1}{2}}u) = I^{s_2}g. \tag{1.7}$$

Again, existence, regularity and uniqueness of a solution to the equation (1.5) are related to boundedness and invertibility of T_{K,s_1,s_2}. In [13] – under Hölder continuity assumptions on the kernel K – it was shown that T_{K,s_1,s_2} is comparable (up to lower order terms) to the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$, and regularity theory was obtained as a distortion of the regularity theory known for the fractional Laplace equations.

On the contrary, in this work we want to show L^p boundedness of T_{K,s_1,s_2} without any continuity assumption on K. In view of [9, Theorem 4.2.2, Theorem 4.2.7], if an operator is a *Calderón-Zygmund operator*, we will immediately obtain weak type $(1,1)$ boundedness, L^p boundedness for any $p\in(1,\infty)$ and also $L^\infty\to BMO$ boundedness of such an operator. Consequently, our main result is stated as follows.

Theorem 1.1. *Let $K\in L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$. Then T_{K,s_1,s_2} is a Calderón-Zygmund operator. In particular, for any $p\in(1,\infty)$, there exists a constant $C>0$ such that*

1. $\|T_{K,s_1,s_2}f\|_{L^{1,\infty}(\mathbb{R}^n)} \le C\,\|K\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\,\|f\|_{L^1(\mathbb{R}^n)}$,
2. $\|T_{K,s_1,s_2}f\|_{L^p(\mathbb{R}^n)} \le C\,\|K\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\,\|f\|_{L^p(\mathbb{R}^n)}$,
3. $[T_{K,s_1,s_2}f]_{BMO(\mathbb{R}^n)} \le C\,\|K\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\,\|f\|_{L^\infty(\mathbb{R}^n)}$.

To prove that T_{K,s_1,s_2} is the Calderón-Zygmund operator, we first show that T_{K,s_1,s_2} is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and we then show that A_{K,s_1,s_2} is a *standard kernel*, i.e. the kernel A_{K,s_1,s_2} satisfies all properties in the following proposition.

Proposition 1.2. *For any $z_1 \neq z_2$ in $\mathbb{R}^n$, there exists a constant $C > 0$ such that A_{K,s_1,s_2} satisfies the size condition*

$$|A_{K,s_1,s_2}(z_1, z_2)| \leq \frac{C\,\|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}}{|z_1 - z_2|^n} \tag{1.8}$$

and for some $\alpha > 0$ the regularity conditions

$$|A_{K,s_1,s_2}(z_1 + h, z_2) - A_{K,s_1,s_2}(z_1, z_2)| \leq \frac{C\,|h|^\alpha \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}}{|z_1 - z_2|^{n+\alpha}} \tag{1.9}$$

whenever $|h| \leq \frac{1}{2}|z_1 - z_2|$ and

$$|A_{K,s_1,s_2}(z_1, z_2 + h) - A_{K,s_1,s_2}(z_1, z_2)| \leq \frac{C\,|h|^\alpha \|K\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}}{|z_1 - z_2|^{n+\alpha}} \tag{1.10}$$

whenever $|h| \leq \frac{1}{2}|z_1 - z_2|$.

An application of our main result, Theorem 1.1, is the following regularity results for "almost constant coefficients" (but without any further regularity assumption).

Theorem 1.3. *For any $s, s_1, s_2 \in (0, 1)$ with $s_1 + s_2 = 2s$, $s_1 \geq s$ and any $p \in [2, \infty)$, there exists $\varepsilon > 0$ such that the following holds. For any measurable kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)$ with $1 - \frac{\inf K}{\sup K} < \varepsilon$, if $u \in W^{s,2}(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$ satisfy*

$$\mathcal{L}_K^s u = (-\Delta)^{\frac{s}{2}} f \quad \text{in } \mathbb{R}^n$$

then there exists $C > 0$ such that

$$\|(-\Delta)^{\frac{s_1}{2}} u\|_{L^p(\mathbb{R}^n)} \leq C\,\|(-\Delta)^{\frac{s-s_2}{2}} f\|_{L^p(\mathbb{R}^n)}.$$

The small constant $\varepsilon > 0$ *is uniform in the following sense: if* $s, s_1, s_2 \in (\theta, 1-\theta)$ *and* $p \in [2, \frac{1}{\theta})$ *for some* $\theta > 0$*, then* ε *depends only on* θ *and the dimension.*

Observe that we obtain this estimate at all differentiability scales below 1. In Theorem 1.3 the conditions $s_1 \geq s$ and $p \geq 2$ means we restrict to higher integrability and differentiability for variational solutions, i.e. solutions $u \in W^{s,2}(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$ which can be obtained by the direct method of Calculus of Variations.

An outline of this work is as follows. We devote Chapter II to reviewing basic definitions and properties about function spaces and well-known singular integral operators, e.g. Fractional Laplacian, Riesz potential, Riesz operator and Calderón-Zygmund operator of nonconvolution type. After that we discuss how to define the operator T_{K,s_1,s_2} as in (1.6) and prove that such an operator is bounded from L^2 to itself in Chapter III. Next, we provide the computations that show that the kernel A_{K,s_1,s_2} is a standard kernel in Chapter IV. The application, Theorem 1.3, will be proved in the last chapter.

For convenience, throughout this work, C is used as a positive universal constant, usually depending on the dimension, and is often omitted it in the calculations by using the following symbols.

- $A \lesssim B$ means there exists a constant $C > 0$ not depending on A and B such that $A \leq CB$.

- $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

CHAPTER II

PRELIMILARIES

All definitions of function spaces and the operators mentioned in the introduction will be stated in the chapter. Moreover, we also present some known results which will be useful for our work.

2.1 Function spaces

We begin this section by introducing the space of Schwartz functions. Roughly speaking, it contains all functions such that its derivatives decay faster than the reciprocal of any polynomial at infinity and we then state the definition of space of tempered distributions which is its dual space. Lebesgue spaces, BMO space and Fractional Sobolev spaces will be given later.

2.1.1 Schwartz space and space of tempered distributions

The *Schwartz space*, denoted by $\mathcal{S}(\mathbb{R}^n)$, is the space of infinitely differentiable function f on $\mathbb{R}^n$ such that for every multi-indices α and β,

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty.$$

The quantities $p_{\alpha,\beta}(f)$ are called the *Schwartz seminorms* of f. Here

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $x = (x_1, \dots, x_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, i.e. α_i is nonnegative integers for all $i = 1, \dots, n$.

Let $\mathcal{S}'(\mathbb{R}^n)$ denote the dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, i.e. the space consists of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. Elements of $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions*. More precisely, a linear functional u is a tempered distribution if and only if there exist $C > 0$ and k, m integers such that

$$|u(f)| \leq C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq k}} p_{\alpha,\beta}(f)$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Definition 2.1. If $f \in \mathcal{S}(\mathbb{R}^n)$, the *Fourier transform* $\mathcal{F}f$ of f is given by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx$$

and the *inverse Fourier transform* $\mathcal{F}^{-1}f$ of f is given by

$$\mathcal{F}^{-1}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi}\, dx$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ with $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$.

2.1.2 Lebesgue spaces and BMO

For $1 < p < \infty$, $L^p(\mathbb{R}^n)$ denotes the set of all Lebesgue measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx \right)^{\frac{1}{p}} < \infty$$

and $L^\infty(\mathbb{R}^n)$ denotes the set of all Lebesgue measurable fuctions such that

$$\operatorname{ess\,sup} |f| := \inf\{C > 0 : |\{x \in \mathbb{R}^n : |f(x)| > C\}| = 0\} < \infty.$$

It is a well-known that the dual $(L^p)^*$ of L^p is isometric to $L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, the L^p norm of a function can be obtained via duality when $1 \leq p \leq \infty$ as follows:

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)} = 1} \left| \int_X f(x) g(x)\, dx \right|.$$

Since the space $C_c^\infty(\mathbb{R}^n)$ consisting of all compactly supported and infinitely differentiable functions on $\mathbb{R}^n$ is dense in $L^p(\mathbb{R}^n)$ for $p \in [1,\infty)$, we can weaken the condition a bit and obtain the following lemma.

Lemma 2.2. *Let $1 < p, p' < \infty$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^p(\mathbb{R}^n)$. Define $T_f : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ by*

$$T_f(g) = \int_{\mathbb{R}^n} fg$$

for $g \in C_c^\infty(\mathbb{R}^n)$. Then T_f is a bounded linear functional on $L^{p'}(\mathbb{R}^n)$ with

$$\|T_f\| = \|f\|_{L^p(\mathbb{R}^n)} = \sup_{\substack{g \in C_c^\infty(\mathbb{R}^n) \\ \|g\|_{L^{p'}(\mathbb{R}^n)} \le 1}} \int_{\mathbb{R}^n} fg.$$

Proof. It follows immediately from Hölder's inequality that T_f is a bounded linear operator on $C_c^\infty(\mathbb{R}^n)$ with $\|T_f\| \le \|f\|_{L^p(\mathbb{R}^n)}$. On the other hand, let

$$h = \frac{|f|^{p-1}\mathrm{sgn} f}{\|f\|^{p-1}_{L^p(\mathbb{R}^n)}}.$$

Then $h \in L^{p'}(\mathbb{R}^n)$ and $\|h\|_{L^{p'}(\mathbb{R}^n)} = 1$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p'}(\mathbb{R}^n)$, there is a sequence $(h_n) \in C_c^\infty(\mathbb{R}^n)$ such that $h_n \to h$ in $L^{p'}(\mathbb{R}^n)$. Thus,

$$\begin{aligned}
\|f\|_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} fh \\
&= \int_{\mathbb{R}^n} fh_n + \int_{\mathbb{R}^n} f(h - h_n) \\
&\le \|h_n\|_{L^{p'}(\mathbb{R}^n)} \int_{\mathbb{R}^n} f \frac{h_n}{\|h_n\|_{L^{p'}(\mathbb{R}^n)}} + \|f\|_{L^p(\mathbb{R}^n)} \|h - h_n\|_{L^{p'}(\mathbb{R}^n)} \\
&\le \|h_n - h\|_{L^{p'}(\mathbb{R}^n)} \int_{\mathbb{R}^n} f \frac{h_n}{\|h_n\|_{L^{p'}(\mathbb{R}^n)}} + \|h\|_{L^{p'}(\mathbb{R}^n)} \int_{\mathbb{R}^n} f \frac{h_n}{\|h_n\|_{L^{p'}(\mathbb{R}^n)}} \\
&\quad + \|f\|_{L^p(\mathbb{R}^n)} \|h - h_n\|_{L^{p'}(\mathbb{R}^n)} \\
&\le \|h_n - h\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n} f \frac{h_n}{\|h_n\|_{L^{p'}(\mathbb{R}^n)}} \\
&\quad + \|f\|_{L^p(\mathbb{R}^n)} \|h - h_n\|_{L^{p'}(\mathbb{R}^n)},
\end{aligned}$$

where the third and the last lines follows from Hölder's inequality. By taking

$n \to \infty$, the first and the third terms converges to zero and so

$$\|f\|_{L^p(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} f \frac{h_n}{\|h_n\|_{L^{p'}(\mathbb{R}^n)}} \leq \sup_{\substack{g \in C_c^\infty(\mathbb{R}^n) \\ \|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1}} \int_{\mathbb{R}^n} fg.$$

Hence $\|T_f\| \geq \|f\|_{L^p(\mathbb{R}^n)}$. The proof is now complete. □

We now give the definition of weak L^1 space which is larger than L^1 space.

Definition 2.3. The space *weak* L^1, denoted by $L^{1,\infty}(\mathbb{R}^n)$, is defined as the set of measurable function f such that

$$\|f\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| < \infty.$$

As the usual L^1 space, two functions in $L^{1,\infty}(\mathbb{R}^n)$ are considered equal if they are equal a.e.. However, $L^{1,\infty}(\mathbb{R}^n)$ is only a quasi-normed linear space because it do not satisfy the triangle inequality, that is,

$$\|f + g\|_{L^{1,\infty}(\mathbb{R}^n)} \leq 2 \left(\|f\|_{L^{1,\infty}(\mathbb{R}^n)} + \|g\|_{L^{1,\infty}(\mathbb{R}^n)} \right)$$

for every $f, g \in L^{1,\infty}(\mathbb{R}^n)$. We also can show that $\|f\|_{L^{1,\infty}(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$. This means $L^1(\mathbb{R}^n) \subseteq L^{1,\infty}(\mathbb{R}^n)$. Furthermore, the inclusion is strict. For example, $f(x) = |x|^{-n}$ is not integrable function but is in $L^{1,\infty}(\mathbb{R}^n)$ with $\|f\|_{L^{1,\infty}(\mathbb{R}^n)} = \nu_n$, where ν_n is the volume of the unite ball of $\mathbb{R}^n$.

The next space we introduce is called BMO space that plays role similar to the space L^∞ and often serves as substitute for it. For example, Riesz transforms (precise definition in section 2.2) do not map L^∞ to L^∞ but L^∞ to BMO.

Definition 2.4. The function f is of *bounded mean oscillation* if

$$[f]_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q \left| f(x) - \frac{1}{|Q|} \int_Q f(y)\, dy \right| dx < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ and $BMO(\mathbb{R}^n)$ denotes the set of all locally integrable functions on $\mathbb{R}^n$ with $[f]_{BMO(\mathbb{R}^n)} < \infty$.

It is obvious that every constant function c satisfies $[c]_{BMO(\mathbb{R}^n)} = 0$. Thus, $[\cdot]_{BMO(\mathbb{R}^n)}$ is only a seminorm. Moreover, $L^\infty(\mathbb{R}^n)$ is a proper subspace of $BMO(\mathbb{R}^n)$ with $[f]_{BMO(\mathbb{R}^n)} \leq 2\|f\|_{L^\infty(\mathbb{R}^n)}$ since $BMO(\mathbb{R}^n)$ contains the unbounded functions. For instance, $\log|x|$ is in $BMO(\mathbb{R}^n)$.

2.1.3 Fractional Sobolev spaces

There are many fractional Sobolev spaces, a particular one is induced by the *Gagliardo-Slobodeckij seminorm* : $s \in (0,1)$, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ open,

$$[f]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{\frac{1}{p}}.$$

We notice that $[f]_{W^{s,p}(\Omega)} < \infty$ for $s \geq 1$ if and only if f is constant. Hence, the definition is restricted to $s \in (0,1)$.

Definition 2.5. For any $s \in (0,1)$, $p \in (1,\infty)$ and Ω an open subset of $\mathbb{R}^n$, we define the *fractional Sobolev space* $W^{s,p}(\Omega)$ to be the space of all L^p functions f on Ω with $[f]_{W^{s,p}(\Omega)} < \infty$, endowed with the norm

$$\|f\|_{W^{s,p}(\Omega)} := \left(\|f\|^p_{L^p(\Omega)} + [f]^p_{W^{s,p}(\Omega)}\right)^{\frac{1}{p}}.$$

2.2 Fractional Laplacian, Riesz potential and Riesz operators

Let first consider the *Laplacian operator*

$$\Delta = \partial_1^2 + \cdots + \partial_n^2$$

whose Fourier transform is given by

$$-\mathcal{F}(\Delta f)(\xi) = (2\pi|\xi|)^2 \mathcal{F}(f)(\xi) \tag{2.1}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. By replacing the exponent 2 with a nonnegative real number s in the above equation, we define the *fractional Laplacian operator* $(-\Delta)^{\frac{s}{2}}$ by

$$((-\Delta)^{\frac{s}{2}}f)(x) = \mathcal{F}^{-1}\left((2\pi|\xi|)^s \mathcal{F}f(\xi)\right)(x) \tag{2.2}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. By calculating the inverse Fourier transform (2.2), we can write the fractional Laplacian operator as the integral representation for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0,2)$,

$$(-\Delta)^{\frac{s}{2}}f(x) = C(n,s)\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+s}}\,dy \tag{2.3}$$

where $C(n,s) = \frac{2^s\Gamma\left(\frac{n+s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{-s}{2}\right)}$. Observe that the kernel has a singularity. However, one can remove it by using the standard changing variable formula to have another following representation for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0,1)$,

$$(-\Delta)^{s}f(x) = \frac{1}{2}C(n,s)\int_{\mathbb{R}^n} \frac{2f(x)-f(x+y)-f(x-y)}{|y|^{n+2s}}\,dy. \tag{2.4}$$

More details can be found in [14]. Moreover, the fractionl Laplacian operator is also related to the fractional Sobolev space as stated in the following proposition.

Proposition 2.6. *Let $s \in (0,1)$ and let $f \in W^{s,2}(\mathbb{R}^n)$. Then*

$$[f]^2_{W^{s,2}(\mathbb{R}^n)} = C(n,s)^{-1}\,\|(-\Delta)^{\frac{s}{2}}f\|^2_{L^2(\mathbb{R}^n)}$$

where $C(n,s)$ is the same constant as (2.3).

Proof. We refer to [14, Proposition 3.6] for the proof. □

Observe that if $s > n$, the function $\xi \mapsto |\xi|^{-s}$ is not locally integrable on $\mathbb{R}^n$. However, such function is locally integrable for any $0 < s < n$. Thus, it makes sense to define the inverse operator of the fractional Laplacian operator for any $0 < s < n$. It is called the Riesz potential operator.

Definition 2.7. Let $0 < s < n$. The *Riesz potential operator* of order s is $I^s = (-\Delta)^{-\frac{s}{2}}$, i.e. for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$(I^s f)(x) = \mathcal{F}^{-1}\left((2\pi|\xi|)^{-s}\mathcal{F}f(\xi)\right)(x) \tag{2.5}$$

and so

$$(I^s f)(x) = C(n,s)\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-s}}\,dy \tag{2.6}$$

where $C(n,s) = \frac{\Gamma\left(\frac{n-s}{2}\right)}{2^s\pi^{\frac{n}{2}}\Gamma\left(\frac{s}{2}\right)}$.

The importance of the Riesz potential operators is that they improve the integrability of a function, more precisely, for $0 < s < n$, $1 < p < \infty$ and $sp < n$, there exists a constant $C = C(n,s,p) > 0$ such that

$$\|I^s(f)\|_{L^{\frac{np}{n-sp}}(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{2.7}$$

for all $f \in L^p(\mathbb{R}^n)$. In other words, the Riesz potential operators are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $q = \frac{np}{n-sp} > p$.

We next give the definition of the Riesz transforms $\mathcal{R}_i$, for $i = 1,\ldots,n$, which is the typical example of singular integral operators and it also appears naturally as the derivative of the Riesz potential I^1, i.e. $\mathcal{R}_i = \partial_i I^1$.

Definition 2.8. For $i = 1,\ldots,n$, the ith *Riesz transform* of f is given by

$$\mathcal{R}_i(f)(x) = C(n)\,\text{P.V.}\int_{\mathbb{R}^n}\frac{x_i - y_i}{|x-y|^{n+1}}f(y)\,dy$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Here $C(n) = \pi^{\frac{-n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)$.

Proposition 2.9. *For $i = 1,\ldots,n$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\mathcal{F}(\mathcal{R}_i f)(\xi) = -i\frac{\xi_i}{|\xi|}\mathcal{F}f(\xi) \tag{2.8}$$

and hence

$$\sum_{i=1}^{n} \mathcal{R}_i^2 = -I$$

where I is the identity operator. Furthermore, it is also bounded from $L^p(\mathbb{R}^n)$ to itself for every $1 < p < \infty$.

Remark 2.10. Due to the constants $C(n,s)$ and $C(n)$ in the definitions of the fractional Laplacian, Riesz potential and Riesz operators are not important to us, we will replace them by 1 from now on.

2.3 Calderón-Zygmund operator

In this section, we consider a singular integral

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$$

whose kernel function $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is singular along the diagonal $x = y$ and do not necessarily commute with translations (nonconvolution singular integral). We are interested in the question: what are sufficient conditions on function K so that the singular integral operators associated with K are bounded on $L^p(\mathbb{R}^n)$ for any $1 \leq p \leq \infty$? To answer this question, we introduce the *Calderón-Zygmund operator*.

Definition 2.11. A function $A(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ is called a *standard kernel* if it satisfies for some $C > 0$ the size condition

$$|A(x,y)| \leq \frac{C}{|x-y|^n} \tag{2.9}$$

and for some $\alpha > 0$ the regularity conditions

$$|A(x,y) - A(x',y)| \leq \frac{C|x-x'|^\alpha}{(|x-y| + |x'-y|)^{n+\alpha}} \tag{2.10}$$

whenever $|x - x'| \leq \frac{1}{2}\max(|x-y|, |x'-y|)$ and

$$|A(x,y) - A(x,y')| \leq \frac{C|y-y'|^{\alpha}}{(|x-y| + |x-y'|)^{n+\alpha}} \tag{2.11}$$

whenever $|y - y'| \leq \frac{1}{2}\max(|x-y|, |x-y'|)$.

Definition 2.12. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ defined by

$$T(f)(x) = \int_{\mathbb{R}^n} A(x,y)f(y)\,dy$$

is said to be *Calderón-Zygmund operator* if

- A is a stardard kernel
- T is a L^2-bounded operator, i.e. there exists $C > 0$ such that

$$\|T(f)\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

We end this section by addressing the question that we posed previously.

Theorem 2.13. *If T is a Calderón-Zygmund operator, then*

1. *T has a bounded extension that maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$:*

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)},$$

2. *T has a bounded extension that maps $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$:*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

3. *the distribution $T(f)$ can be identified with a BMO function that satisfies*

$$[Tf]_{BMO(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. See [9, Theorem 4.2.2 and Theorem 4.2.7] for the proof. □

CHAPTER III

T_{K,s_1,s_2} AND ITS L^2 BOUNDEDNESS

As we mentioned in the introduction that there is the operator T_{K,s_1,s_2} associated to the nonlocal equation

$$\mathcal{L}_K^s u = g \tag{3.1}$$

where $s \in (0,1)$ and the operator $\mathcal{L}_K^s$ is given by

$$\mathcal{L}_K^s u(x) := \text{P.V.} \int_{\mathbb{R}^n} 2K(x,y) \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy.$$

Recall that the function $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be measurable in x and y, symmetric $K(x,y) = K(y,x)$ and elliptic, which means there exist $\lambda, \Lambda > 0$ such that $\lambda < K(x,y) < \Lambda$ for all $x, y \in \mathbb{R}^n$.

In this chapter, we explain why the operator T_{K,s_1,s_2} should be defined as in (1.6) and we then prove L^2 boundedness of T_{K,s_1,s_2}.

Firstly, we show that the equation (3.1) is equivalent to the following integral equation

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dx \, dy = \int_{\mathbb{R}^n} g(x)\varphi(x) \, dx \tag{3.2}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. To show this, let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and we then consider

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dx \, dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(u(x) - u(y))\varphi(x)}{|x-y|^{n+2s}} \, dx \, dy \\
&\quad - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(u(x) - u(y))\varphi(y)}{|x-y|^{n+2s}} \, dx \, dy.
\end{aligned}$$

By Fubini's theorem and symmetry of K, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&= \int_{\mathbb{R}^n}\frac{1}{2}\mathcal{L}_K^s u(x)\varphi(x)\,dx - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(y,x)\frac{(u(x)-u(y))\varphi(y)}{|x-y|^{n+2s}}\,dx\,dy \\
&= \int_{\mathbb{R}^n}\frac{1}{2}\mathcal{L}_K^s u(x)\varphi(x)\,dx + \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K(y,x)\frac{(u(y)-u(x))}{|x-y|^{n+2s}}\,dx\right)\varphi(y)\,dy \\
&= \int_{\mathbb{R}^n}\mathcal{L}_K^s u(x)\varphi(x)\,dx \\
&= \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx.
\end{aligned}
$$

Secondly, we employ the integral representation of the Riesz potential which is the inverse of the fractional Laplacian with the equation (3.2). That is, for any $u \in \mathcal{S}(\mathbb{R}^n)$ and any $s_1, s_2 \in (0,1)$ so that $s_1 + s_2 = 2s$, we have

$$u(x) = I^{s_1}(-\Delta)^{\frac{s_1}{2}}u(x) = \int_{\mathbb{R}^n}|x-z|^{s_1-n}\,(-\Delta)^{\frac{s_1}{2}}u(z)\,dz$$

and

$$u(y) = I^{s_1}(-\Delta)^{\frac{s_1}{2}}u(x) = \int_{\mathbb{R}^n}|y-z|^{s_1-n}\,(-\Delta)^{\frac{s_1}{2}}u(z)\,dz.$$

It follows that

$$u(x)-u(y) = \int_{\mathbb{R}^n}\left(|x-z_1|^{s_1-n} - |y-z_1|^{s_1-n}\right)\,(-\Delta)^{\frac{s_1}{2}}u(z_1)\,dz_1.$$

Similarly, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, we also obtain

$$\varphi(x)-\varphi(y) = \int_{\mathbb{R}^n}\left(|x-z_2|^{s_2-n} - |y-z_2|^{s_2-n}\right)\,(-\Delta)^{\frac{s_2}{2}}\varphi(z_2)\,dz_2.$$

Consequently,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A_{K,s_1,s_2}(z_1,z_2)\,(-\Delta)^{\frac{s_1}{2}}u(z_1)\,(-\Delta)^{\frac{s_2}{2}}\varphi(z_2)\,dz_1\,dz_2
\end{aligned}
$$

where $A_{K,s_1,s_2}(z_1,z_2)$ is defined by

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{(|x-z_1|^{s_1-n}-|y-z_1|^{s_1-n})(|x-z_2|^{s_2-n}-|y-z_2|^{s_2-n})}{|x-y|^{n+2s}}\,dx\,dy.$$

On the other hand, we use integration by parts to obtain

$$\int_{\mathbb{R}^n} g(z)\varphi(z)\,dz = \int_{\mathbb{R}^n} g(z) I^{s_2}((-\Delta)^{\frac{s_2}{2}}\varphi)(z)\,dz = \int_{\mathbb{R}^n} I^{s_2}g(z)(-\Delta)^{\frac{s_2}{2}}\varphi(z)\,dz.$$

Thus, we can rewrite the equation (3.2) as

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A_{K,s_1,s_2}(z_1,z_2)\,(-\Delta)^{\frac{s_1}{2}}u(z_1)\,(-\Delta)^{\frac{s_2}{2}}\varphi(z_2)\,dz_1\,dz_2 = \int_{\mathbb{R}^n} I^{s_2}g(z)(-\Delta)^{\frac{s_2}{2}}\varphi(z)\,dz.$$

Since the above equation holds for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, this implies (in distributional sense)

$$\int_{\mathbb{R}^n} A_{K,s_1,s_2}(z_1,z_2)\,(-\Delta)^{\frac{s_1}{2}}u(z_1)\,dz_1 = I^{s_2}g(z_2).$$

Hence, the operator T_{K,s_1,s_2} is defined by

$$T_{K,s_1,s_2}f(z_2) := \int_{\mathbb{R}^n} A_{K,s_1,s_2}(z_1,z_2)f(z_1)\,dz_1,$$

for any $z_1 \in \mathbb{R}^n$ and for any $0 < s_1, s_2 < 1$ with $s_1 + s_2 = 2s$, so that solutions of (3.1) satisfy

$$T_{K,s_1,s_2}((-\Delta)^{\frac{s_1}{2}}u)(z) = I^{s_2}g(z)$$

for any z in $\mathbb{R}^n$.

Next we show that T_{K,s_1,s_2} is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Proposition 3.1. *Let $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then, for any $f \in L^2(\mathbb{R}^n)$, we have*

$$\|T_{K,s_1,s_2}f\|_{L^2(\mathbb{R}^n)} \lesssim \|K\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. By Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} (T_{K,s_1,s_2} f)(z)\varphi(z)\, dz \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(I^{s_1}\varphi(x) - I^{s_1}\varphi(y))\,(I^{s_2} f(x) - I^{s_2} f(y))}{|x-y|^{n+2s}}\, dx\, dy.
\end{aligned}$$

Then, using Hölder's inequality twice, we yield (recall that $s_1, s_2 \in (0,1)$ and $s_1 + s_2 = 2s$),

$$\begin{aligned}
&\int_{\mathbb{R}^n} (T_{K,s_1,s_2} f)(z)\varphi(z)\, dz \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) \frac{(I^{s_1}\varphi(x) - I^{s_1}\varphi(y))\,(I^{s_2} f(x) - I^{s_2} f(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&\lesssim \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|I^{s_1} f(x) - I^{s_1} f(y)|^2}{|x-y|^{n+2s_1}}\, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{|I^{s_1}\varphi(x) - I^{s_1}\varphi(y)|^2}{|x-y|^{n+2s_2}}\, dx \right)^{\frac{1}{2}} dy \\
&\lesssim \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|I^{s_1} f(x) - I^{s_1} f(y)|^2}{|x-y|^{n+2s_1}}\, dx\, dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|I^{s_2}\varphi(x) - I^{s_2}\varphi(y)|^2}{|x-y|^{n+2s_2}}\, dx\, dy \right)^{\frac{1}{2}} \\
&\approx [I^{s_1} f]_{W^{s_1,2}(\mathbb{R}^n)}\, [I^{s_2}\varphi]_{W^{s_2,2}(\mathbb{R}^n)}.
\end{aligned}$$

As a consequence of Proposition 2.6, we obtain the following estimate

$$\int_{\mathbb{R}^n} (T_{K,s_1,s_2} f)(z)\varphi(z)\, dz \lesssim \|f\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^2(\mathbb{R}^n)}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^n)$. By duality, Lemma 2.2, we deduce that

$$\|T_{K,s_1,s_2} f\|_{L^2(\mathbb{R}^n)} = \sup_{\substack{\varphi \in C_c^\infty(\mathbb{R}^n) \\ \|\varphi\|_{L^2(\mathbb{R}^n)} \le 1}} \int (T_{K,s_1,s_2} f)(z)\varphi(z)\, dz \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

This concludes the proof of proposition. □

CHAPTER IV

A_{K,s_1,s_2} IS A STANDARD KERNEL

The main result in this chapter is to prove that A_{K,s_1,s_2} is a *standard kernel* which satisfies the size and regularity conditions in Proposition 1.2. We organize this chapter into two sections. In the first section we prove some estimates which will be useful later and then prove that A_{K,s_1,s_2} is a standard kernel in the second section.

4.1 Some useful estimates

We begin this section with an estimate known as an application of the fundamental theorem of calculus.

Lemma 4.1. *For any $r \in \mathbb{R}$ and $\sigma \in [0,1]$, there exists a constant C depending on r such that the following holds. Let $a, b \in \mathbb{R}^n \backslash \{0\}$ with $|a-b| \lesssim \min\{|a|,|b|\}$. Then*

$$||a|^r - |b|^r| \leq C\,|a-b|^\sigma \min\left\{|a|^{r-\sigma}, |b|^{r-\sigma}\right\}.$$

Proof. We may assume $r \neq 0$, since the case $r = 0$ is trivial. Notice that if $|a-b| \lesssim \min\{|a|,|b|\}$, then $|a| \approx |b|$ (with a uniform constant) and so

$$\min\left\{|a|^{r-\sigma}, |b|^{r-\sigma}\right\} \approx |a|^{r-\sigma}.$$

Also, for any $\sigma \in [0,1]$, we have

$$|a-b| = |a-b|^\sigma |a-b|^{1-\sigma} \lesssim |a-b|^\sigma |a|^{1-\sigma}$$

and thus we obtain

$$|a-b|\,|a|^{r-1} \lesssim |a-b|^{\sigma}|a|^{1-\sigma}|a|^{r-1} = |a-b|^{\sigma}|a|^{r-\sigma}.$$

Hence, to complete the proof, it suffices to show that

$$||a|^r - |b|^r| \lesssim |a-b|\,|a|^{r-1}.$$

Dividing both sides of the above inequality by $|a|^r$, it is equivalent to show that

$$\left|\left|\frac{a}{|a|}\right|^r - \left|\frac{b}{|a|}\right|^r\right| \lesssim \left|\frac{a}{|a|} - \frac{b}{|a|}\right|.$$

Since $|a| \approx |b|$, there are uniform constants $0 < r_1 < 1 < r_2 < \infty$ such that both $\frac{a}{|a|}$ and $\frac{b}{|a|}$ are in $A := B_{r_2}(0)\backslash B_{r_1}(0)$. Thus, the estimate is now reduced to show

$$||u|^r - |v|^r| \lesssim |u-v|$$

for all $u, v \in A$. Since A is an annulus, for any $u, v \in A$ there exists a curve $\gamma \subset A$ with $\gamma(0) = u$, $\gamma(1) = v$, $|\gamma'| \approx |u-v|$ with constants depending only on r_1 and r_2. We define a function $\eta : [0,1] \to \mathbb{R}$ by $\eta(t) := |\gamma(t)|^r$. By the fundamental theorem of calculus, we obtain

$$||u|^r - |v|^r| = \int_0^1 \eta'(t)\,dt \leq \sup_{t\in[0,1]} |\eta'(t)| \lesssim \sup_{t\in[0,1]} |\gamma(t)|^{r-1}|\gamma'(t)| \lesssim |u-v|.$$

This completes the proof of the required inequality. $\square$

We now state and prove the following estimate which will help us in proving Lemma 4.3.

Lemma 4.2. *Let $\gamma \in \mathbb{R}$. Then there exists a constant $C = C(\gamma)$ such that the following holds. For any $t \in [0,1]$ and for any $a, b, h \in \mathbb{R}^n\backslash\{0\}$ such that*

$|h| \leq \frac{1}{2}\min\{|a|,|b|\}$, *we have*

$$||a+th|^\gamma - |b+th|^\gamma| \leq C(\gamma)\max\{||a|^\gamma - |b|^\gamma|\,, ||a+h|^\gamma - |b+h|^\gamma|\}. \qquad (4.1)$$

Proof. The inequality is valid obviously when $\gamma = 0$. The case $\gamma < 0$ can be deduced from $\gamma > 0$ as follows. If the inequality (4.1) holds for $\gamma > 0$, then

$$\begin{aligned}
&\left||a+th|^{-\gamma} - |b+th|^{-\gamma}\right| \\
&= |a+th|^{-\gamma}|b+th|^{-\gamma}\,||a+th|^\gamma - |b+th|^\gamma| \\
&\lesssim C(\gamma)\,|a+th|^{-\gamma}|b+th|^{-\gamma}\,\max\{||a|^\gamma - |b|^\gamma|\,, ||a+h|^\gamma - |b+h|^\gamma|\}.
\end{aligned}$$

By assumption, we obtain $|a| \approx |a+th| \approx |a+h|$ and also $|b| \approx |b+th| \approx |b+h|$. This implies that

$$|a+th|^{-\gamma}|b+th|^{-\gamma}\,||a|^\gamma - |b|^\gamma| \approx |a|^{-\gamma}|b|^{-\gamma}\,||a|^\gamma - |b|^\gamma| = \left||a|^{-\gamma} - |b|^{-\gamma}\right|,$$

and similarly, we obtain

$$\begin{aligned}
|a+th|^{-\gamma}|b+th|^{-\gamma}\,||a+h|^\gamma - |b+h|^\gamma| &\approx |a+h|^{-\gamma}|b+h|^{-\gamma}\,||a+h|^\gamma - |b+h|^\gamma| \\
&= \left||a+h|^{-\gamma} - |b+h|^{-\gamma}\right|.
\end{aligned}$$

Combining these estimates, we obtain that the inequality (4.1) also holds for $-\gamma$. Hence, we now need to show (4.1) only for $\gamma > 0$. We divide into three cases.

1. Case $\gamma = 2$: We first observe that

$$|a+th|^2 - |b+th|^2 = |a|^2 + 2t\langle h, a-b\rangle - |b|^2.$$

If $\langle h, a-b\rangle \geq 0$, we obtain that

$$|a+th|^2 - |b+th|^2 \leq |a|^2 + 2\langle h, a-b\rangle - |b|^2 = |a+h|^2 - |b+h|^2.$$

If $\langle h, a-b\rangle \leq 0$, we obtain that

$$|a+th|^2 - |b+th|^2 \leq |a|^2 - |b|^2.$$

Thus, for any $t \in [0,1]$, we have

$$|a+th|^2 - |b+th|^2 \leq \max\{\big||a|^2 - |b|^2\big|, \big||a+h|^2 - |b+h|^2\big|\}.$$

Similarly, we also obtain

$$|b+th|^2 - |a+th|^2 \leq \max\{\big||a|^2 - |b|^2\big|, \big||a+h|^2 - |b+h|^2\big|\}.$$

Consequently, the inequality (4.1) is established for $\gamma = 2$, i.e. we have shown

$$\big||a+th|^2 - |b+th|^2\big| \leq \max\{\big||a|^2 - |b|^2\big|, \big||a+h|^2 - |b+h|^2\big|\}. \tag{4.2}$$

To generalize this equation to all $\gamma > 0$, we observe that for any $p \geq 1$

$$|\tilde{A}^p - \tilde{B}^p| \approx |\tilde{A} - \tilde{B}|(\tilde{A}^{p-1} + \tilde{B}^{p-1}) \tag{4.3}$$

for all $\tilde{A}, \tilde{B} > 0$, with constants depending only on p. Indeed, taking $x := \frac{\min\{A,B\}}{\max\{A,B\}}$, (4.3) is equivalent to

$$1 - x^p \approx (1-x)(1+x^{p-1})$$

for all $x \in [0,1)$. Let

$$f(x) := \frac{1-x^p}{(1-x)(1+x^{p-1})}.$$

Then f is continuously extendable into $[0,1]$ with $f(1) = \frac{p}{2}$ because $p \geq 1$. Since $f(x) > 0$ for all $x \in [0,1]$, its infimum and supremum exist and also positive. Let $c(p) := \inf_{x\in[0,1]} f(x)$ and $C(p) := \sup_{x\in[0,1]} f(x)$. Then we have

$c(p) \leq f(x) \leq C(p)$ for all $x \in [0,1]$. That is, we get the estimate (4.3).

2. Case $\gamma > 2$: To use (4.3), we set $\tilde{A} = A^2$, $\tilde{B} = B^2$ and $p = \frac{\gamma}{2} > 1$. Thus,

$$|A^\gamma - B^\gamma| \approx |A^2 - B^2|(A^{\gamma-2} + B^{\gamma-2}) \tag{4.4}$$

for all $A, B > 0$. Consequently, with the help of (4.2) we find that for any $\gamma > 2$

$$\begin{aligned}&||a+th|^\gamma - |b+th|^\gamma| \\ &\lesssim \left(|a+th|^{\gamma-2} + |b+th|^{\gamma-2}\right) \max\{\left||a|^2 - |b|^2\right|, \left||a+h|^2 - |b+h|^2\right|\}.\end{aligned}$$

We again observe that $|a+th| \approx |a+h| \approx |a|$ and $|b+th| \approx |b+h| \approx |b|$. Thus, with the help of (4.4) we obtain

$$\begin{aligned}\left(|a+th|^{\gamma-2} + |b+th|^{\gamma-2}\right)\left||a|^2 - |b|^2\right| &\approx \left(|a|^{\gamma-2} + |b|^{\gamma-2}\right)\left||a|^2 - |b|^2\right| \\ &\approx ||a|^\gamma - |b|^\gamma|,\end{aligned}$$

and also

$$\left(|a+th|^{\gamma-2} + |b+th|^{\gamma-2}\right)\left||a+h|^2 - |b+h|^2\right| \approx ||a+h|^\gamma - |b+h|^\gamma|.$$

That is, we have established (4.1) for any $\gamma > 2$.

3. Case $\gamma \in (0,2)$: We apply (4.3) to $\tilde{A} = A^\gamma$, $\tilde{B} = B^\gamma$, and $p = \frac{2}{\gamma} > 1$ to obtain

$$|A^2 - B^2| \approx |A^\gamma - B^\gamma| \left(A^{2-\gamma} + B^{2-\gamma}\right)$$

for all $A, B > 0$, that is,

$$|A^\gamma - B^\gamma| \approx \frac{|A^2 - B^2|}{A^{2-\gamma} + B^{2-\gamma}} \tag{4.5}$$

for all $A, B > 0$. Now we argue as in the case $\gamma > 2$ to obtain the claim, namely by (4.2),

$$\begin{aligned}
&\left||a+th|^{\gamma}-|b+th|^{\gamma}\right| \\
&\lesssim \frac{1}{|a+th|^{2-\gamma}+|b+th|^{2-\gamma}} \max\{\left||a|^2-|b|^2\right|, \left||a+h|^2-|b+h|^2\right|\} \\
&\approx \max\{\left||a|^{\gamma}-|b|^{\gamma}\right|, \left||a+h|^{\gamma}-|b+h|^{\gamma}\right|\}.
\end{aligned}$$

The last line follows from (4.5) with $|a+th| \approx |a+h| \approx |a|$ and $|b+th| \approx |b+h| \approx |b|$.

Hence, the estimate (4.1) holds for every $\gamma \in \mathbb{R}$. □

In spirit of Lemma 4.1, the next lemma is obtained by the mean value theorem along with a few more technical arguments.

Lemma 4.3. *For any $s, \alpha, \sigma \in [0,1]$, there exists a constant $C > 0$ such that the following holds. Let $a, b, h \in \mathbb{R}^n \backslash \{0\}$ such that $a+h, b+h \neq 0$.*

(1) If $|h| < \frac{1}{2}\min\{|a|,|b|\}$ or $|h| < \frac{1}{2}\min\{|a+h|,|b+h|\}$, then

$$\begin{aligned}
&\left||a+h|^{s-n}-|b+h|^{s-n}-\left(|a|^{s-n}-|b|^{s-n}\right)\right| \\
&\leq C|h|^{\alpha}\left(\left||a+h|^{s-\alpha-n}-|b+h|^{s-\alpha-n}\right|+\left||a|^{s-\alpha-n}-|b|^{s-\alpha-n}\right|\right) \\
&+ C|h|^{\alpha}\min\{|a|^{s-\alpha-\sigma-n}, |b|^{s-\alpha-\sigma-n}\}\,|a-b|^{\sigma}.
\end{aligned}$$

(2) If $|h| > \frac{1}{2}\min\{|a|,|b|\}$ and $|h| > \frac{1}{2}\min\{|a+h|,|b+h|\}$, then

$$\begin{aligned}
&\left||a+h|^{s-n}-|b+h|^{s-n}\right|+\left||a|^{s-n}-|b|^{s-n}\right| \\
&\leq C|h|^{\alpha}\left||a+h|^{s-\alpha-n}-|b+h|^{s-\alpha-n}\right|+C|h|^{\alpha}\left||a|^{s-\alpha-n}-|b|^{s-\alpha-n}\right|.
\end{aligned}$$

Proof. (1) We assume that $|h| < \frac{1}{2}\min\{|a|,|b|\}$ or $|h| < \frac{1}{2}\min\{|a+h|,|b+h|\}$. Without loss of generality we may assume $|h| < \frac{1}{2}\min\{|a|,|b|\}$ since in this case $|a| \approx |a+\tilde{h}|$ and $|b| \approx |b+\tilde{h}|$ for any $|\tilde{h}| \leq |h|$. Thus, the latter case follows exactly the same way.

Let $f:\mathbb{R}^n\backslash\{0\}\to\mathbb{R}$ be defined by

$$f(h):=|a+h|^{s-n}-|b+h|^{s-n}.$$

Then, by the mean value theorem, we have that for some $\lambda\in(0,1)$, $\tilde{h}:=\lambda h$,

$$|f(h)-f(0)|\lesssim|\tilde{h}||Df(\tilde{h})|\le|h||Df(\tilde{h})|.$$

A calculation gives

$$\begin{aligned}Df(\tilde{h})&=(s-n)|a+\tilde{h}|^{s-1-n}\frac{a+\tilde{h}}{|a+\tilde{h}|}-(s-n)|b+\tilde{h}|^{s-1-n}\frac{b+\tilde{h}}{|b+\tilde{h}|}\\&=(s-n)\left(|a+\tilde{h}|^{s-1-n}-|b+\tilde{h}|^{s-1-n}\right)\frac{a+\tilde{h}}{|a+\tilde{h}|}\\&\quad+(s-n)|b+\tilde{h}|^{s-1-n}\left(\frac{a+\tilde{h}}{|a+\tilde{h}|}-\frac{b+\tilde{h}}{|b+\tilde{h}|}\right).\end{aligned}$$

To estimate the second term, we observe that for any $c,d\in\mathbb{R}^n\backslash\{0\}$,

$$\begin{aligned}\left|\frac{c}{|c|}-\frac{d}{|d|}\right|&=\frac{1}{|c|\,|d|}\big|c|d|-d|c|\big|\\&\le\frac{1}{|c|\,|d|}\left(|c-d||d|+|d|\,\big||d|-|c|\big|\right)\\&\le\frac{2}{|c|}|c-d|.\end{aligned}$$

Since we can interchange the role of c and d,

$$\left|\frac{c}{|c|}-\frac{d}{|d|}\right|\lesssim\min\{|c|^{-1},|d|^{-1}\}|c-d|.$$

It follows that for any $\sigma\in[0,1]$,

$$\left|\frac{c}{|c|}-\frac{d}{|d|}\right|=\left|\frac{c}{|c|}-\frac{d}{|d|}\right|^{\sigma}\left|\frac{c}{|c|}-\frac{d}{|d|}\right|^{1-\sigma}\lesssim2^{1-\sigma}\,\min\{|c|^{-\sigma},|d|^{-\sigma}\}|c-d|^{\sigma}.$$

Hence,

$$|h||Df(\tilde{h})| \lesssim |h| \left| |a+\tilde{h}|^{s-1-n} - |b+\tilde{h}|^{s-1-n} \right| + |h||b|^{s-1-\sigma-n} |a-b|^{\sigma}.$$

Since $|h| \lesssim |b|$ by assumption, the following estimate holds for any $\alpha \in [0,1]$

$$|h||Df(\tilde{h})| \lesssim |h| \left| |a+\tilde{h}|^{s-1-n} - |b+\tilde{h}|^{s-1-n} \right| + |h|^{\alpha}|b|^{s-\alpha-\sigma-n} |a-b|^{\sigma}.$$

We then interchange the role of $|a|$ and $|b|$ to get

$$\begin{aligned} |h||Df(\tilde{h})| &\lesssim |h| \left| |a+\tilde{h}|^{s-1-n} - |b+\tilde{h}|^{s-1-n} \right| \\ &\quad + |h|^{\alpha} \min\{|a|^{s-\alpha-\sigma-n}, |b|^{s-\alpha-\sigma-n}\} |a-b|^{\sigma}. \end{aligned}$$

It remains to estimate the first term. For this, we define $g : (0,\infty) \to \mathbb{R}$ by

$$g(t) := t^{\frac{s-n-1}{s-n-\alpha}} = t^{\frac{n+1-s}{n+\alpha-s}}.$$

Since $\alpha \in [0,1]$, we notice that $\frac{n+1-s}{n+\alpha-s} \geq 1$. Let $t_1, t_2 \in (0,\infty)$. By mean value theorem, there exists $c \in (t_1, t_2)$ such that

$$\begin{aligned} |g(t_1) - g(t_2)| &\leq g'(c)|t_1 - t_2| \\ &\lesssim \max\{(t_1)^{\frac{n+1-s}{n+\alpha-s}-1}, (t_2)^{\frac{n+1-s}{n+\alpha-s}-1}\} |t_1 - t_2|. \end{aligned}$$

We then apply the above inequality with $t_1 = |a+\tilde{h}|^{s-n-\alpha} \approx |a|^{s-n-\alpha}$ and $t_2 = |b+\tilde{h}|^{s-n-\alpha} \approx |b|^{s-n-\alpha}$. Thus, we obtain that

$$\begin{aligned} &\left| |a+\tilde{h}|^{s-1-n} - |b+\tilde{h}|^{s-1-n} \right| \\ &= \left| g\left(|a+\tilde{h}|^{s-n-\alpha}\right) - g\left(|b+\tilde{h}|^{s-n-\alpha}\right) \right| \\ &\lesssim \max\left\{ |a|^{(s-n-\alpha)\left(\frac{n+1-s}{n+\alpha-s}-1\right)}, |b|^{(s-n-\alpha)\left(\frac{n+1-s}{n+\alpha-s}-1\right)} \right\} \left| |a+\tilde{h}|^{s-\alpha-n} - |b+\tilde{h}|^{s-\alpha-n} \right| \\ &\approx \max\left\{ |a|^{\alpha-1}, |b|^{\alpha-1} \right\} \left| |a+\tilde{h}|^{s-\alpha-n} - |b+\tilde{h}|^{s-\alpha-n} \right|. \end{aligned}$$

Since $\alpha \le 1$, $\max\{|a|^{\alpha-1}, |b|^{\alpha-1}\} = \min\{|a|,|b|\}^{\alpha-1}$ and so we get

$$\begin{aligned}|h|\left||a+\tilde{h}|^{s-1-n}-|b+\tilde{h}|^{s-1-n}\right| &\lesssim |h|\min\{|a|,|b|\}^{\alpha-1}\left||a+\tilde{h}|^{s-\alpha-n}-|b+\tilde{h}|^{s-\alpha-n}\right| \\ &\lesssim |h|\,|h|^{\alpha-1}\left||a+\tilde{h}|^{s-\alpha-n}-|b+\tilde{h}|^{s-\alpha-n}\right| \\ &\lesssim |h|^{\alpha}\left||a+\tilde{h}|^{s-\alpha-n}-|b+\tilde{h}|^{s-\alpha-n}\right|.\end{aligned}$$

The second line above follows from the assumption that $|h| < \frac{1}{2}\min\{|a|,|b|\}$. Recall that $\tilde{h} = \lambda h$ for some $\lambda \in (0,1)$. Hence, the claim follows from Lemma 4.2.

(2) Assume that $|h| > \frac{1}{2}\min\{|a|,|b|\}$ and $|h| > \frac{1}{2}\min\{|a+h|,|b+h|\}$.

We only show the estimate of $||a+h|^{s-n}-|b+h|^{s-n}|$, the estimate for $||a|^{s-n}-|b|^{s-n}|$ is almost verbatim. There are two cases.

Case 1: $\min\{|a+h|,|b+h|\} \le \frac{1}{2}\max\{|a+h|,|b+h|\}$.

For any $\theta < n$, with a constant only depending on $\theta - n$, we observe that

$$\left||a+h|^{\theta-n}-|b+h|^{\theta-n}\right| \approx \min\{|a+h|,|b+h|\}^{\theta-n}.$$

Thus,

$$\begin{aligned}\left||a+h|^{s-n}-|b+h|^{s-n}\right| &\approx \min\{|a+h|,|b+h|\}^{s-n} \\ &= \min\{|a+h|,|b+h|\}^{s-n-\alpha+\alpha} \\ &\lesssim |h|^{\alpha}\min\{|a+h|,|b+h|\}^{s-n-\alpha} \\ &\approx |h|^{\alpha}\left||a+h|^{s-\alpha-n}-|b+h|^{s-\alpha-n}\right|.\end{aligned}$$

Case 2: $\min\{|a+h|,|b+h|\} \approx \max\{|a+h|,|b+h|\}$.

We define $g : [0,\infty) \to [0,\infty)$ by

$$g(t) := t^{\frac{s-n}{s-n-\alpha}}$$

for any $\alpha \in [0,1]$. Then, for any $t_1, t_2 \in (0,\infty)$, by mean value theorem, there exists $d \in (t_1, t_2)$ such that

$$|g(t_1) - g(t_2)| \leq g'(d)\,|t_1 - t_2|.$$

Thus,

$$|g(t_1) - g(t_2)| \leq \max\{t_1^{\frac{s-n}{s-n-\alpha}-1}, t_2^{\frac{s-n}{s-n-\alpha}-1}\}\,|t_1 - t_2|.$$

Taking $t_1 = |a+h|^{s-n-\alpha}$ and $t_2 = |b+h|^{s-n-\alpha}$, we then find

$$\begin{aligned}
&\big||a+h|^{s-n} - |b+h|^{s-n}\big| \\
&= \left|g\left(|a+h|^{s-n-\alpha}\right) - g\left(|b+h|^{s-n-\alpha}\right)\right| \\
&\lesssim \max\{|a+h|^{(s-n-\alpha)\left(\frac{s-n}{s-n-\alpha}-1\right)}, |b+h|^{(s-n-\alpha)\left(\frac{s-n}{s-n-\alpha}-1\right)}\}\ \big||a+h|^{s-\alpha-n} - |b+h|^{s-\alpha-n}\big| \\
&\approx \max\{|a+h|^{\alpha}, |b+h|^{\alpha}\}\ \big||a+h|^{s-\alpha-n} - |b+h|^{s-\alpha-n}\big| \\
&\approx \min\{|a+h|^{\alpha}, |b+h|^{\alpha}\}\ \big||a+h|^{s-\alpha-n} - |b+h|^{s-\alpha-n}\big| \\
&\lesssim |h|^{\alpha}\ \big||a+h|^{s-\alpha-n} - |b+h|^{s-\alpha-n}\big|.
\end{aligned}$$

This concludes the proof of the second claim. □

4.2 Standard kernel

The main result of this section is proving that A_{K,s_1,s_2} is a standard kernel which will be proved at the end of this section. For this purpose we use Lemma 4.3 to reduce the formula of A_{K,s_1,s_2} to a new kernel which will be given below. After having the estimate for the new kernel (Proposition 4.4 below), the main result will follows directly from Lemma 4.3 and Proposition 4.4.

Now we set $l = 1, 2$, $\alpha, \sigma \in [0,1]$, $s, s_1, s_2 \in (0,1)$ with $s_1 + s_2 = 2s$ and

$$M_l^{\alpha,\sigma}(z_1, z_2) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\,\kappa_l^{\alpha,\sigma}(x,y,z_1,z_2)\,dx\,dy$$

where

$$\kappa_1^{\alpha,\sigma}(x,y,z_1,z_2) := \frac{\left||x-z_1|^{s_1-\alpha-n} - |y-z_1|^{s_1-\alpha-n}\right| \left||x-z_2|^{s_2-n} - |y-z_2|^{s_2-n}\right|}{|x-y|^{n+2s}}$$

and

$$\kappa_2^{\alpha,\sigma}(x,y,z_1,z_2) := \frac{\min\{|x-z_1|^{s_1-\alpha-\sigma-n}, |y-z_1|^{s_1-\alpha-\sigma-n}\} \left||x-z_2|^{s_2-n} - |y-z_2|^{s_2-n}\right|}{|x-y|^{n+2s-\sigma}}.$$

Proposition 4.4. *Let $\theta \in (0, \frac{1}{10})$ be such that $10\theta < s, s_1, s_2 < 1 - 10\theta$. Then, for any $\alpha \in [0, \frac{1}{10}\theta)$, $\sigma \in (s_1 + \theta, 2s)$ and for any $l = 1, 2$, we have*

$$|M_l^{\alpha,\sigma}(z_1,z_2)| \leq C(\theta)\, \frac{\|K\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}}{|z_1-z_2|^{\alpha+n}}$$

for all $z_1 \neq z_2$.

Proof. Fix $z_1, z_2 \in \mathbb{R}^n$ such that $z_1 \neq z_2$ and let $\delta := |z_1 - z_2| > 0$. We split $\mathbb{R}^n \times \mathbb{R}^n$ into different cases

$$\mathbb{R}^n \times \mathbb{R}^n = \bigcup_{i=1}^{3} \mathcal{A}_i = \bigcup_{i=1}^{3} \mathcal{B}_i = \bigcup_{i=1}^{3} \mathcal{I}_i, \tag{4.6}$$

where

$$\mathcal{A}_1 \equiv \mathcal{A}_1(z_1) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \leq 10\min\{|x-z_1|, |y-z_1|\}\},$$
$$\mathcal{A}_2 \equiv \mathcal{A}_2(z_1) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-z_1| \leq 10\min\{|y-z_1|, |x-y|\}\},$$
$$\mathcal{A}_3 \equiv \mathcal{A}_3(z_1) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |y-z_1| \leq 10\min\{|x-z_1|, |x-y|\}\},$$

and $\mathcal{B}_i$ are the analogous cases involving z_2, namely

$$\mathcal{B}_1 \equiv \mathcal{B}_1(z_2) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \leq 10\min\{|x-z_2|, |y-z_2|\}\},$$
$$\mathcal{B}_2 \equiv \mathcal{B}_2(z_2) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-z_2| \leq 10\min\{|y-z_2|, |x-y|\}\},$$
$$\mathcal{B}_3 \equiv \mathcal{B}_3(z_2) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |y-z_2| \leq 10\min\{|x-z_2|, |x-y|\}\},$$

and lastly $\mathcal{I}_i$,

$$\mathcal{I}_1 \equiv \mathcal{I}_1(z_1, z_2) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in \mathbb{R}^n, |x - z_1| \leq 10\delta \text{ and } |x - z_2| \geq \frac{1}{10}\delta\},$$
$$\mathcal{I}_2 \equiv \mathcal{I}_2(z_1, z_2) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in \mathbb{R}^n, |x - z_2| \leq 10\delta \text{ and } |x - z_1| \geq \frac{1}{10}\delta\},$$
$$\mathcal{I}_3 \equiv \mathcal{I}_3(z_1, z_2)$$
$$= \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in \mathbb{R}^n, \frac{1}{100}|x - z_2| \leq |x - z_1| \leq 100|x - z_2| \text{ and } |x - z_1| \geq \frac{1}{100}\delta\right\}.$$

We note that there is no need for the sets to be disjoint. Hence,

$$M_l^{\alpha,\sigma}(z_1, z_2) \leq \sum_{i,j,k=1}^{3} \iint_{\mathcal{A}_i \cap \mathcal{B}_j \cap \mathcal{I}_k} K(x,y)\, \kappa_l^{\alpha,\sigma}(x, y, z_1, z_2)\, dx\, dy$$
$$=: \sum_{i,j,k=1}^{3} J_{i,j,k}^{\alpha,\sigma,l}(z_1, z_2).$$

Our strategy is now considering all combination of the cases above seperately. That is, we will prove that

$$J_{i,j,k}^{\alpha,\sigma,l}(z_1, z_2) \lesssim \delta^{-\alpha-n}$$

for all $i, j, k = 1, 2, 3$ and for all $l = 1, 2$.

Before we begin the lengthy proof of this proposition, let us warn the reader about an abuse of notation that we are going to use throughout the proof. While $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{I}_i$ are sets in $\mathbb{R}^n \times \mathbb{R}^n$, we will sometimes identify them with subsets of $\{x\} \times \mathbb{R}^n$ or $\mathbb{R}^n \times \{y\}$. It means that we will use the following convention

$$\int_{\mathcal{A}_i} f(x,y) dx = \int_{\mathcal{A}_i \cap (\mathbb{R}^n \times \{y\})} f(x,y) dx \equiv \int_{\mathbb{R}^n} \chi_{\mathcal{A}_i}(x,y) f(x,y) dx$$

and

$$\int_{\mathcal{A}_i} f(x,y) dy \equiv \int_{\mathcal{A}_i \cap (\{x\} \times \mathbb{R}^n)} f(x,y) dy \equiv \int_{\mathbb{R}^n} \chi_{\mathcal{A}_i}(x,y) f(x,y) dy.$$

Another observation which we always use is that if $|x - y| \lesssim \min\{|x - z|, |y - z|\}$ for any $x, y, z \in \mathbb{R}^n$, then $|x - z|$ and $|y - z|$ are comparable, i.e. $|x - z| \approx |y - z|$.

Estimating $J^{\alpha,\sigma,l}_{1,1,1}$, $J^{\alpha,\sigma,l}_{1,1,2}$ and $J^{\alpha,\sigma,l}_{1,1,3}$:

In this case we can use Lemma 4.1 to obtain the following estimates

$$\big||x-z_1|^{s_1-\alpha-n}-|y-z_1|^{s_1-\alpha-n}\big|\lesssim |x-z_1|^{s_1-\alpha-n-1}|x-y|$$

and

$$\big||x-z_2|^{s_2-n}-|y-z_2|^{s_2-n}\big|\lesssim |x-z_2|^{s_2-n-1}|x-y|.$$

It follows that

$$\kappa_1^{\alpha,\sigma}(x,y,z_1,z_2)\lesssim\frac{|x-z_1|^{s_1-\alpha-n-1}|x-z_2|^{s_2-n-1}}{|x-y|^{n+2s-2}}.$$

We observe that $(x,y)\in\mathcal{A}_1\cap\mathcal{B}_1$ implies that $|x-y|\lesssim\min\{|x-z_1|,|x-z_2|\}$. Thus, since $s<1$, we integrate $\kappa_1^{\alpha,\sigma}$ w.r.t. y and get

$$\int_{y\in\mathcal{A}_1\cap\mathcal{B}_1}\frac{1}{|x-y|^{n+2s-2}}dy\lesssim\min\{|x-z_1|,|x-z_2|\}^{2-2s}.$$

Thus, we have

$$\begin{aligned}J^{\alpha,\sigma,1}_{1,1,1}(z_1,z_2)&\lesssim\int_{\mathcal{A}_1\cap\mathcal{B}_1\cap\mathcal{I}_1}|x-z_1|^{s_1-\alpha-n-1}|x-z_2|^{s_2-n-1}|x-z_1|^{2-2s}\,dx\\&=\int_{\mathcal{A}_1\cap\mathcal{B}_1\cap\mathcal{I}_1}|x-z_1|^{1-\alpha-s_2-n}|x-z_2|^{s_2-n-1}\,dx.\end{aligned}$$

To use Hölder's inequality, we let $p>1$ be so small so that $(1-\alpha-s_2-n)p>-n$ and $(s_2-1-n)p'<-n$. Then

$$J^{\alpha,\sigma,1}_{1,1,1}(z_1,z_2)\lesssim\left(\int_{|x-z_1|\lesssim\delta}|x-z_1|^{(1-\alpha-s_2-n)p}\,dx\right)^{\frac{1}{p}}\left(\int_{|x-z_2|\gtrsim\delta}|x-z_2|^{(s_2-1-n)p'}\,dx\right)^{\frac{1}{p'}}.$$

Since $(1-\alpha-s_2-n)p>-n$ and $(s_2-1-n)p'<-n$, we compute

$$J^{\alpha,\sigma,1}_{1,1,1}(z_1,z_2)\lesssim\left(\delta^{(1-\alpha-s_2-n)p+n}\right)^{\frac{1}{p}}\left(\delta^{(s_2-1-n)p'+n}\right)^{\frac{1}{p'}}=\delta^{-\alpha-n}$$

which settles this case.

We omit the proof of $J_{1,1,2}^{\alpha,\sigma,1}(z_1,z_2)$ since it is similar to the above case essentially only interchanging the role of z_1 and z_2.

The proof of $J_{1,1,3}^{\alpha,\sigma,1}(z_1,z_2)$ is also similar to the above case but a bit simpler than that since $|x-z_1| \approx |x-z_2|$ when $(x,y) \in \mathcal{I}_3$. Hence, we arrive at

$$\begin{aligned} J_{1,1,3}^{\alpha,\sigma,1}(z_1,z_2) &\lesssim \int_{\mathcal{A}_1\cap\mathcal{B}_1\cap\mathcal{I}_3} |x-z_1|^{1-\alpha-s_2-n}\,|x-z_1|^{s_2-n-1}\,dx \\ &= \int_{|x-z_1|\gtrsim\delta} |x-z_1|^{-\alpha-2n}\,dx \\ &\approx \delta^{-\alpha-n}. \end{aligned}$$

Next, we will estimate $J_{1,1,k}^{\alpha,\sigma,2}$. Notice that for any $(x,y)\in\mathcal{A}_1$, we have $|x-z_1|$ and $|y-z_1|$ are comparable. Thus,

$$\kappa_2^{\alpha,\sigma}(x,y,z_1,z_2) \approx \frac{|x-z_1|^{s_1-\sigma-\alpha-n}\,\big||x-z_2|^{s_2-n}-|y-z_2|^{s_2-n}\big|}{|x-y|^{n+2s-\sigma}}.$$

Then applying Lemma 4.1 to the z_2-term leads to

$$\kappa_2^{\alpha,\sigma}(x,y,z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\sigma-\alpha-n}\,|x-z_2|^{s_2-n-1}}{|x-y|^{n+2s-\sigma-1}}.$$

Again, $|x-y| \lesssim \min\{|x-z_1|,|x-z_2|\}$ and, by assumption, we have $\sigma+1>2s$. Thus, we can integrate w.r.t. y to get

$$\begin{aligned} J_{1,1,1}^{\alpha,\sigma,2}(z_1,z_2) &\lesssim \int_{\mathcal{A}_1\cap\mathcal{B}_1\cap\mathcal{I}_1} |x-z_1|^{s_1-\sigma-\alpha-n}\,|x-z_2|^{s_2-n-1}\,|x-z_1|^{-2s+\sigma+1}\,dx \\ &= \int_{\mathcal{A}_1\cap\mathcal{B}_1\cap\mathcal{I}_1} |x-z_1|^{-s_2-\alpha+1-n}\,|x-z_2|^{s_2-n-1}\,dx. \end{aligned}$$

We observe that $-s_2-\alpha+1>0$ and $s_2-1<0$. Let $p>1$ be so small such that $(-s_2-\alpha+1-n)p>-n$ and $(s_2-1-n)p'<-n$. Then, by Hölder's inequality, we have

$$
\begin{aligned}
J_{1,1,1}^{\alpha,\sigma,2}(z_1,z_2) &\lesssim \left(\int_{|x-z_1|\lesssim\delta} |x-z_1|^{(-s_2-\alpha+1-n)p}\,dx\right)^{\frac{1}{p}} \left(\int_{|x-z_2|\gtrsim\delta} |x-z_2|^{(s_2-1-n)p'}\,dx\right)^{\frac{1}{p'}} \\
&\lesssim \left(\delta^{(-s_2-\alpha+1-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(s_2-1-n)p'+n}\right)^{\frac{1}{p'}} \\
&= \delta^{-\alpha-n}.
\end{aligned}
$$

Similarly, for $J_{1,1,2}^{\alpha,\sigma,2}(z_1,z_2)$, after integrating w.r.t. y (since $\sigma+1>2s$) we get

$$
\begin{aligned}
J_{1,1,2}^{\alpha,\sigma,2}(z_1,z_2) &\lesssim \int_{\mathcal{A}_1\cap\mathcal{B}_1\cap\mathcal{I}_2} |x-z_1|^{s_1-\sigma-\alpha-n}\,|x-z_2|^{s_2-n-1}\,|x-z_2|^{-2s+\sigma+1}\,dx \\
&= \int_{\mathcal{A}_1\cap\mathcal{B}_1\cap\mathcal{I}_2} |x-z_1|^{s_1-\sigma-\alpha-n}\,|x-z_2|^{-s_1+\sigma-n}\,dx.
\end{aligned}
$$

By assumption, $s_1-\sigma-\alpha<0$ and $-s_1+\sigma>0$. Then, using Hölder's inequality, we let $p>1$ be so small such that $(s_1-\sigma-\alpha-n)p<-n$ and $(-s_1+\sigma-n)p'>-n$. Then,

$$
\begin{aligned}
J_{1,1,2}^{\alpha,\sigma,2}(z_1,z_2) &\lesssim \left(\int_{|x-z_1|\gtrsim\delta} |x-z_1|^{(s_1-\sigma-\alpha-n)p}\,dx\right)^{\frac{1}{p}} \left(\int_{|x-z_2|\lesssim\delta} |x-z_2|^{(-s_1+\sigma-n)p'}\,dx\right)^{\frac{1}{p'}} \\
&\lesssim \left(\delta^{(s_1-\sigma-\alpha-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(-s_1+\sigma-n)p'+n}\right)^{\frac{1}{p'}} \\
&= \delta^{-\alpha-n}.
\end{aligned}
$$

As before we argue the same argument to estimate $J_{1,1,3}^{\alpha,\sigma,2}(z_1,z_2)$. Since $|x-z_1|\approx|x-z_2|$, we arrive at

$$
\begin{aligned}
J_{1,1,3}^{\alpha,\sigma,2}(z_1,z_2) &\lesssim \int_{\mathcal{A}_1\cap\mathcal{B}_1\cap\mathcal{I}_3} |x-z_1|^{-s_2-\alpha+1-n}\,|x-z_2|^{s_2-n-1}\,dx \\
&= \int_{|x-z_1|\gtrsim\delta} |x-z_1|^{-\alpha-2n}\,dx \\
&\approx \delta^{-\alpha-n}.
\end{aligned}
$$

Estimating $J^{\alpha,\sigma,l}_{1,2,1}$, $J^{\alpha,\sigma,l}_{1,2,2}$ and $J^{\alpha,\sigma,l}_{1,2,3}$:

Using Lemma 4.1, we obtain

$$||x-z_1|^{s_1-\alpha-n}-|y-z_1|^{s_1-\alpha-n}| \lesssim |x-y|^{\gamma}|x-z_1|^{s_1-\alpha-\gamma-n}$$

for any $\gamma \in [0,1]$. We then consider

$$||x-z_2|^{s_2-n}-|y-z_2|^{s_2-n}| \lesssim |x-z_2|^{s_2-n}.$$

Thus, for any $(x,y) \in \mathcal{A}_1 \cap \mathcal{B}_2$ and for any $\gamma \in [0,1]$, we have

$$\kappa_1^{\alpha,\sigma}(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-\gamma-n}\,|x-z_2|^{s_2-n}}{|x-y|^{n+2s-\gamma}}.$$

Moreover, since $|x-y| \lesssim |x-z_1|$ for any $(x,y) \in \mathcal{A}_1$, we also get the same estimate for the second type kernel

$$\kappa_2^{\alpha,\sigma}(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-\gamma-n}\,|x-z_2|^{s_2-n}}{|x-y|^{n+2s-\gamma}}$$

for any $\gamma \in [0,\sigma)$. Taking $\gamma < 2s$, we integrate w.r.t. y variable, observing that $(x,y) \in \mathcal{A}_1 \cap \mathcal{B}_2$ implies that $|x-y| \gtrsim |x-z_2|$ and thus

$$\int_{y \in \mathcal{A}_1 \cap \mathcal{B}_2} |x-y|^{-n-2s+\gamma}\,dy \lesssim |x-z_2|^{-2s+\gamma}.$$

Thus, for any $\gamma \in [0,\sigma) \cap [0,1]$, we have

$$\begin{aligned} J^{\alpha,\sigma,l}_{1,2,k}(z_1,z_2) &\lesssim \int_{\mathcal{I}_k} |x-z_1|^{s_1-\alpha-\gamma-n}\,|x-z_2|^{s_2-n}\,|x-z_2|^{\gamma-2s}\,dx \\ &= \int_{\mathcal{I}_k} |x-z_1|^{s_1-\alpha-\gamma-n}\,|x-z_2|^{\gamma-s_1-n}\,dx. \end{aligned}$$

Here, with a slight abuse of notation we identify $\mathcal{I}_k$ with the set of $x \in \mathbb{R}^n$ such that $\{x\} \times \mathbb{R}^n \subset \mathcal{I}_k$.

To calculate when $k = 1$, we take $\gamma = 0$ and thus

$$\begin{aligned} J^{\alpha,\sigma,l}_{1,2,1}(z_1,z_2) &\lesssim \int_{|x-z_1|\lesssim\delta, |x-z_2|\gtrsim\delta} |x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{-s_1-n}\,dx \\ &\lesssim \delta^{-s_1-n}\int_{|x-z_1|\lesssim\delta} |x-z_1|^{s_1-\alpha-n}\,dx \\ &\approx \delta^{-\alpha-n}. \end{aligned}$$

In the case $k = 3$, we again use $\gamma = 0$ to obtain

$$\begin{aligned} J^{\alpha,\sigma,l}_{1,2,3}(z_1,z_2) &\lesssim \int_{|x-z_1|\approx|x-z_2|\gtrsim\delta} |x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{-s_1-n}\,dx \\ &\lesssim \delta^{s_1-\alpha-n}\int_{|x-z_2|\gtrsim\delta} |x-z_2|^{-s_1-n}\,dx \\ &\approx \delta^{-\alpha-n}. \end{aligned}$$

For the remaining case $k = 2$, we choose $\gamma > s_1$ (which is possible with the restraints on γ above), and then get

$$\begin{aligned} J^{\alpha,\sigma,l}_{1,2,2}(z_1,z_2) &\lesssim \int_{|x-z_1|\gtrsim\delta, |x-z_2|\lesssim\delta} |x-z_1|^{s_1-\alpha-\gamma-n}\,|x-z_2|^{\gamma-s_1-n}\,dx \\ &\lesssim \delta^{s_1-\alpha-\gamma-n}\int_{|x-z_2|\lesssim\delta} |x-z_2|^{\gamma-s_1-n}\,dx \\ &\approx \delta^{-\alpha-n}. \end{aligned}$$

Estimating $J^{\alpha,\sigma,l}_{1,3,1}$, $J^{\alpha,\sigma,l}_{1,3,2}$ and $J^{\alpha,\sigma,l}_{1,3,3}$:

Let $(x,y) \in \mathcal{A}_1 \cap \mathcal{B}_3$. By Lemma 4.1 and $|x-y| \lesssim |x-z_1|$, it follows that

$$\kappa^{\alpha,\sigma}_1(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-1-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s-1}}$$

and we have a similar estimate for the second kernel.

$$\kappa^{\alpha,\sigma}_2(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-\sigma-1-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s-\sigma-1}} \lesssim \frac{|x-z_1|^{s_1-\alpha-1-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s-1}}.$$

We can treat these two kernels now almost verbatim. Since $|x-y| \lesssim |x-z_1|$ and

$|x-z_2| \approx |x-y|$ for $(x,y) \in \mathcal{A}_1 \cap \mathcal{B}_3$, we have

$$J_{1,3,1}^{\alpha,\sigma,l}(z_1,z_2) \lesssim \iint_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_1} \frac{|x-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s}}\,dx\,dy$$
$$\approx \iint_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_1} \frac{|x-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-z_2|^{n+2s}}\,dx\,dy.$$

Since $|y-z_2| \lesssim |x-z_2|$ and $s_2-n > -n$, we obtain

$$\int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_1} |y-z_2|^{s_2-n}\,dy \lesssim |x-z_2|^{s_2}$$

and so

$$J_{1,3,1}^{\alpha,\sigma,l}(z_1,z_2) \lesssim \int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_1} \frac{|x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2}}{|x-z_2|^{n+2s}}\,dx$$
$$\approx \int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_1} |x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{-s_1-n}\,dx.$$

Let $p>1$ so small such that $(s_1-\alpha-n)p > -n$ and $(-s_1-n)p' < -n$. Then, by Hölder's inequality,

$$J_{1,3,1}^{\alpha,\sigma,l}(z_1,z_2) \lesssim \left(\int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_1} |x-z_1|^{(s_1-\alpha-n)p}\;dx\right)^{\frac{1}{p}} \left(\int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_1} |x-z_2|^{(-s_1-n)p'}\;dx\right)^{\frac{1}{p'}}$$
$$\lesssim \left(\delta^{(s_1-\alpha-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(-s_1-n)p'+n}\right)^{\frac{1}{p'}}$$
$$\approx \delta^{-\alpha-n}.$$

In proving $J_{1,3,2}^{\alpha,\sigma,l}(z_1,z_2)$, we consider

$$J_{1,3,2}^{\alpha,\sigma,l}(z_1,z_2) \lesssim \iint_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_2} \frac{|x-z_1|^{s_1-\alpha-1-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s-1}}\,dx\,dy$$
$$\approx \iint_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_2} \frac{|x-z_1|^{s_1-\alpha-1-n}\,|y-z_2|^{s_2-n}}{|x-z_2|^{n+2s-1}}\,dx\,dy.$$

Since $|y-z_2| \lesssim |x-z_2|$ and $s_2-n > -n$, we obtain

$$\int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_2} |y-z_2|^{s_2-n}\,dy \lesssim |x-z_2|^{s_2}$$

and so

$$J^{\alpha,\sigma,l}_{1,3,2}(z_1,z_2) \lesssim \int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_2} \frac{|x-z_1|^{s_1-\alpha-1-n}\,|x-z_2|^{s_2}}{|x-z_2|^{n+2s-1}}\,dx$$
$$\approx \int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_2} |x-z_1|^{s_1-\alpha-1-n}\,|x-z_2|^{-s_1+1-n}\,dx.$$

Let $p>1$ so small such that $(s_1-\alpha-1-n)p<-n$ and $(-s_1+1-n)p'>-n$. Hölder's inequality gives that

$$J^{\alpha,\sigma,l}_{1,3,2}(z_1,z_2) \lesssim \left(\int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_2} |x-z_1|^{(s_1-\alpha-1-n)p}\,dx\right)^{\frac{1}{p}} \left(\int_{\mathcal{A}_1\cap\mathcal{B}_3\cap\mathcal{I}_2} |x-z_2|^{(-s_1+1-n)p'}\,dx\right)^{\frac{1}{p'}}$$
$$\lesssim \left(\delta^{(s_1-\alpha-1-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(-s_1+1-n)p'+n}\right)^{\frac{1}{p'}}$$
$$\approx \delta^{-\alpha-n}.$$

In the case $k=3$, we have $|x-z_1|\approx|x-z_2|$. Thus, we easily verifies that

$$J^{\alpha,\sigma,l}_{1,3,3}(z_1,z_2) \lesssim \int_{|x-z_1|\approx|x-z_2|\gtrsim\delta} |x-z_1|^{s_1-\alpha-1-n}\,|x-z_2|^{-s_1+1-n}\,dx \lesssim \delta^{-\alpha-n}.$$

Estimating $J^{\alpha,\sigma,l}_{2,1,1}$, $J^{\alpha,\sigma,l}_{2,1,2}$ and $J^{\alpha,\sigma,l}_{2,1,3}$:

For $(x,y)\in\mathcal{B}_1$, we use Lemma 4.1 to estimate that

$$\big||x-z_2|^{s_2-n}-|y-z_2|^{s_2-n}\big| \lesssim |x-z_2|^{s_2-n-\gamma}|x-y|^{\gamma}$$

for any $\gamma\in[0,1]$. Thus, for $(x,y)\in\mathcal{A}_2\cap\mathcal{B}_1$, the first type kernel is estimated by

$$\kappa^{\alpha,\sigma}_1(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2-n-\gamma}}{|x-y|^{n+2s-\gamma}}.$$

Since in this case $|y-z_1|\approx|x-y|$, we also have

$$\kappa^{\alpha,\sigma}_2(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-n}|y-z_1|^{-\sigma}\,|x-z_2|^{s_2-n-\gamma}}{|x-y|^{n+2s-\sigma-\gamma}} \approx \frac{|x-z_1|^{s_1-\alpha-n}|x-z_2|^{s_2-n-\gamma}}{|x-y|^{n+2s-\gamma}}.$$

Since $|x - z_1| \lesssim |x - y|$, we can further obtain

$$\kappa_l^{\alpha,\sigma}(z_1, z_2) \lesssim \frac{|x - z_1|^{s_1 - \alpha - t - n} |x - z_2|^{s_2 - n - \gamma}}{|x - y|^{n + 2s - \gamma - t}}$$

for any $t \geq 0$ and for any $\gamma \in [0, 1]$. To obtain the estimate when $(x, y) \in \mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_1$, we let $t := s_1 - 5\theta$ and choose $\gamma = 1$. Then $2s - 1 - t < 0$ and so we can integrate w.r.t. y to get

$$\begin{aligned} J_{2,1,1}^{\alpha,\sigma,l}(z_1, z_2) &\lesssim \iint_{\mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_1} \frac{|x - z_1|^{s_1 - \alpha - t - n} \, |x - z_2|^{s_2 - n - 1}}{|x - y|^{n + 2s - 1 - t}} \, dx \, dy \\ &\lesssim \int_{\mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_1} |x - z_1|^{s_1 - \alpha - t - n} \, |x - z_2|^{s_2 - n - 1} \, |x - z_2|^{1 + t - 2s} \, dx \\ &\lesssim \int_{\mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_1} |x - z_1|^{s_1 - \alpha - t - n} \, |x - z_2|^{-s_1 + t - n} \, dx \\ &\lesssim \delta^{-\alpha - n}, \end{aligned}$$

where we used Hölder's inequality in the last estimate together with $s_1 - \alpha - t > 0$ and $t - s_1 < 0$.

For any $(x, y) \in \mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_2$, we choose $t = 0$ and $\gamma < s_2$. Integrating in y, we obtain that

$$\begin{aligned} J_{2,1,2}^{\alpha,\sigma,l}(z_1, z_2) &\lesssim \iint_{\mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_2} \frac{|x - z_1|^{s_1 - \alpha - n} \, |x - z_2|^{s_2 - n - \gamma}}{|x - y|^{n + 2s - \gamma}} \, dx \, dy \\ &\lesssim \int_{\mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_2} |x - z_1|^{s_1 - \alpha - n} \, |x - z_2|^{s_2 - n - \gamma} \, |x - z_1|^{\gamma - 2s} \, dx \\ &\approx \int_{\mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_2} |x - z_1|^{\gamma - s_2 - \alpha - n} \, |x - z_2|^{s_2 - n - \gamma} \, dx \\ &\lesssim \delta^{-\alpha - n}. \end{aligned}$$

Again, the last line is a consequence of Hölder's inequality with $\gamma - s_2 - \alpha < 0$ and $s_2 - \gamma > 0$.

For any $(x, y) \in \mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{I}_3$, the proof follows in the same way as above but with the condition $|x-z_1| \approx |x-z_2|$ and $|x-z_1| \gtrsim \delta$. Hence,

$$\begin{aligned} J^{\alpha,\sigma,l}_{2,1,3}(z_1,z_2) &\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_1\cap\mathcal{I}_3} |x-z_1|^{s_1-\alpha-t-n}\,|x-z_2|^{-s_1+t-n}\,dx \\ &\approx \int_{|x-z_1|\gtrsim\delta} |x-z_1|^{-\alpha-2n}\,dx \\ &\approx \delta^{-\alpha-n}. \end{aligned}$$

Estimating $J^{\alpha,\sigma,l}_{2,2,1}$, $J^{\alpha,\sigma,l}_{2,2,2}$ and $J^{\alpha,\sigma,l}_{2,2,3}$:

Let $(x,y) \in \mathcal{A}_2 \cap \mathcal{B}_2$. Then we obtain

$$\kappa^{\alpha,\sigma}_1(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2-n}}{|x-y|^{n+2s}}.$$

and, since in this case $|y-z_1| \approx |x-y|$, we also get the same estimate

$$\kappa^{\alpha,\sigma}_2(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-n}|y-z_1|^{-\sigma}\,|x-z_2|^{s_2-n}}{|x-y|^{n+2s-\sigma}} \approx \frac{|x-z_1|^{s_1-\alpha-n}|x-z_2|^{s_2-n}}{|x-y|^{n+2s}}.$$

Since $|x-z_2| \lesssim |x-y|$, by integrating w.r.t. y leads to

$$\begin{aligned} J^{\alpha,\sigma,l}_{2,2,1}(z_1,z_2) &\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_2\cap\mathcal{I}_1} |x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2-n}\,|x-z_2|^{-2s}\,dx \\ &\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_2\cap\mathcal{I}_1} |x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{-s_1-n}\,dx. \end{aligned}$$

Let $p > 1$ be small enough so that $(s_1-\alpha-n)p > -n$ and $(-s_1-n)p' < -n$. Then, by Hölder's inequality, it follows that

$$\begin{aligned} J^{\alpha,\sigma,l}_{2,2,1}(z_1,z_2) &\lesssim \left(\int_{|x-z_1|\lesssim\delta} |x-z_1|^{(s_1-\alpha-n)p}\,dx\right)^{\frac{1}{p}} \left(\int_{|x-z_2|\gtrsim\delta} |x-z_1|^{(-s_1-n)p'}\,dx\right)^{\frac{1}{p'}} \\ &\lesssim \left(\delta^{(s_1-\alpha-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(-s_1-n)p'+n}\right)^{\frac{1}{p'}} \\ &\approx \delta^{-\alpha-n}. \end{aligned}$$

The same argument is also true for $J^{\alpha,\sigma,l}_{2,2,2}$ so we omit the proof.

For $(x,y) \in \mathcal{A}_2 \cap \mathcal{B}_2 \cap \mathcal{I}_3$, we follow by the same argument as before and after integrating w.r.t. y we have

$$J^{\alpha,\sigma,l}_{2,2,3}(z_1,z_2) \lesssim \int_{|x-z_1|\gtrsim\delta} |x-z_1|^{-\alpha-2n}\,dx \;\lesssim\; \delta^{-\alpha-n}.$$

Estimating $J^{\alpha,\sigma,l}_{2,3,1}$, $J^{\alpha,\sigma,l}_{2,3,2}$ and $J^{\alpha,\sigma,l}_{2,3,3}$:

Let $(x,y) \in \mathcal{A}_2 \cap \mathcal{B}_3$. Then

$$\kappa^{\alpha,\sigma}_1(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s}}$$

and, since $|y-z_1| \approx |x-y|$,

$$\kappa^{\alpha,\sigma}_2(z_1,z_2) \lesssim \frac{|x-z_1|^{s_1-\alpha-n}|y-z_1|^{-\sigma}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s-\sigma}} \lesssim \frac{|x-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s}}.$$

Recall that $|x-y| \approx |x-z_2|$ in our setting. Hence,

$$\begin{aligned}
&J^{\alpha,\sigma,l}_{2,3,1}(z_1,z_2)\\
&\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_1} |x-z_1|^{s_1-\alpha-n}\left(\int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_1} |y-z_2|^{s_2-n}\,|x-y|^{-2s-n}\,dy\right)dx\\
&\approx \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_1} |x-z_1|^{s_1-\alpha-n}|x-z_2|^{-2s-n}\int_{|y-z_2|\lesssim|x-z_2|} |y-z_2|^{s_2-n}\,dy\,dx\\
&\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_1} |x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{-s_1-n}\,dx.
\end{aligned}$$

By Hölder's inequality, let $p>1$ be small enough so that $(s_1-\alpha-n)p > -n$ and $(-s_1-n)p' < -n$ and then,

$$\begin{aligned}
J^{\alpha,\sigma,l}_{2,3,1}(z_1,z_2) &\lesssim \left(\int_{|x-z_1|\lesssim\delta} |x-z_1|^{(s_1-\alpha-n)p}\,dx\right)^{\frac{1}{p}} \left(\int_{|x-z_2|\gtrsim\delta} |x-z_2|^{(-s_1-n)p'}\,dx\right)^{\frac{1}{p'}}\\
&\lesssim \left(\delta^{(s_1-\alpha-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(-s_1-n)p'+n}\right)^{\frac{1}{p'}}\\
&\approx \delta^{-\alpha-n}.
\end{aligned}$$

We now focus on the case $k=2$. Let $p>1$ be such that $(s_2-n)p > -n$ and

$(-2s-n)p' < -n$. Then Hölder's inequality gives that

$$
\begin{aligned}
&J^{\alpha,\sigma,l}_{2,3,2}(z_1,z_2)\\
&\lesssim \iint_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_2} \frac{|x-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s}}\,dx\,dy\\
&\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_2} |x-z_1|^{s_1-\alpha-n}\left(\int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_2} |y-z_2|^{s_2-n}\,|x-y|^{-2s-n}\,dy\right)dx\\
&\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_2} |x-z_1|^{s_1-\alpha-n}\left(\int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_2} |y-z_2|^{(s_2-n)p}\,dy\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_2} |x-y|^{(-2s-n)p'}\,dy\right)^{\frac{1}{p'}} dx\\
&\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_2} |x-z_1|^{s_1-\alpha-n}\left(|x-z_2|^{(s_2-n)p+n}\right)^{\frac{1}{p}}\left(|x-z_1|^{(-2s-n)p'+n}\right)^{\frac{1}{p'}}\,dx\\
&\approx \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_2} |x-z_1|^{-s_2-\alpha-2n+\frac{n}{p'}}|x-z_2|^{s_2-n+\frac{n}{p}}\,dx.
\end{aligned}
$$

Again, by Hölder's inequality, let $q>1$ be small enough so that $(-s_2-\alpha-2n+\frac{n}{p'})q < -n$ and $(s_2-n+\frac{n}{p})q' > -n$ and then,

$$
\begin{aligned}
J^{\alpha,\sigma,l}_{2,3,2}(z_1,z_2) &\lesssim \left(\int_{|x-z_1|\gtrsim\delta} |x-z_1|^{(-s_2-\alpha-2n+\frac{n}{p'})q}\,dx\right)^{\frac{1}{q}}\left(\int_{|x-z_2|\lesssim\delta} |x-z_2|^{(s_2-n+\frac{n}{p})q'}\,dx\right)^{\frac{1}{q'}}\\
&\lesssim \left(\delta^{(-s_2-\alpha-2n+\frac{n}{p'})q+n}\right)^{\frac{1}{q}}\left(\delta^{(s_2-n+\frac{n}{p})q'+n}\right)^{\frac{1}{q'}} \approx \delta^{-\alpha-n}.
\end{aligned}
$$

For $(x,y)\in\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_3$, we do the same way with the case $\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_1$. However, it is easier than that because the set $\mathcal{I}_3$ gives us $|x-z_1|\approx|x-z_2|$.

$$
\begin{aligned}
&J^{\alpha,\sigma,l}_{2,3,3}(z_1,z_2)\\
&\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_3} |x-z_1|^{s_1-\alpha-n}\left(\int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_3} |y-z_2|^{s_2-n}\,|x-y|^{-2s-n}\,dy\right)dx\\
&\approx \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_3} |x-z_1|^{s_1-\alpha-n}|x-z_2|^{-2s-n}\int_{|y-z_2|\lesssim|x-z_2|} |y-z_2|^{s_2-n}\,dy\,dx\\
&\lesssim \int_{\mathcal{A}_2\cap\mathcal{B}_3\cap\mathcal{I}_3} |x-z_1|^{s_1-\alpha-n}\,|x-z_2|^{-s_1-n}\,dx\\
&\approx \int_{|x-z_1|\gtrsim\delta} |x-z_1|^{-\alpha-2n}\,dx \lesssim \delta^{-\alpha-n}.
\end{aligned}
$$

Estimating $J^{\alpha,\sigma,2}_{3,j,k}$, for $j = 1, 2, 3$ and $k = 1, 2, 3$: The proof of all these cases is the same as in the cases $J^{\alpha,\sigma,2}_{2,j,k}$ for every $j = 1, 2, 3$ and $k = 1, 2, 3$, since we observe that for $(x, y) \in \mathcal{A}_3$,

$$\min\{|x - z_1|^{s_1-\alpha-\sigma-n}, |y - z_1|^{s_1-\alpha-\sigma-n}\} = |x - z_1|^{s_1-\alpha-\sigma-n}$$

and $|x - z_1| \approx |x - y|$.

It therefore remains to verify for the first type kernel ($l = 1$).

Estimating $J^{\alpha,\sigma,1}_{3,1,1}$, $J^{\alpha,\sigma,1}_{3,1,2}$ and $J^{\alpha,\sigma,1}_{3,1,3}$:

Let $(x, y) \in \mathcal{A}_3 \cap \mathcal{B}_1$. As before the following estimate follows from Lemma 4.1

$$\big||x - z_2|^{s_2-n} - |y - z_2|^{s_2-n}\big| \lesssim |x - z_2|^{s_2-n-1}|x - y|.$$

Since $(x, y) \in \mathcal{A}_3$, we have $|x - y| \approx |x - z_1|$ and, thus,

$$\kappa^{\alpha,\sigma}_1(z_1, z_2) \lesssim \frac{|y - z_1|^{s_1-\alpha-n}\, |x - z_2|^{s_2-1-n}}{|x - y|^{n+2s-1}} \approx \frac{|y - z_1|^{s_1-\alpha-n}\, |x - z_2|^{s_2-1-n}}{|x - z_1|^{n+2s-1}}.$$

We now consider

$$\begin{aligned}
J^{\alpha,\sigma,1}_{3,1,1}(z_1, z_2) &\lesssim \iint_{\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_1} \frac{|y - z_1|^{s_1-\alpha-n}\, |x - z_2|^{s_2-1-n}}{|x - y|^{n+2s-1}}\, dx\, dy \\
&\approx \int_{\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_1} \left(\int_{\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_1} |y - z_1|^{s_1-\alpha-n}\, dy \right) \frac{|x - z_2|^{s_2-1-n}}{|x - z_1|^{n+2s-1}}\, dx \\
&\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_1} \frac{|x - z_1|^{s_1-\alpha}\, |x - z_2|^{s_2-1-n}}{|x - z_1|^{n+2s-1}}\, dx \\
&\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_1} |x - z_1|^{-s_2-\alpha+1-n}\, |x - z_2|^{s_2-1-n}\, dx.
\end{aligned}$$

Let $p > 1$ be small enough so that $(-s_2-\alpha+1-n)p > -n$ and $(s_2-1-n)p' < -n$.

By Hölder's inequality, we obtain

$$\begin{aligned}
J^{\alpha,\sigma,1}_{3,1,1}(z_1,z_2) &\lesssim \left(\int_{|x-z_1|\lesssim\delta} |x-z_1|^{(-s_2-\alpha+1-n)p}\,dx\right)^{\frac{1}{p}} \left(\int_{|x-z_2|\gtrsim\delta} |x-z_2|^{(s_2-1-n)p'}\,dx\right)^{\frac{1}{p'}} \\
&\lesssim \left(\delta^{(-s_2-\alpha+1-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(s_2-1-n)p'+n}\right)^{\frac{1}{p'}} \\
&\approx \delta^{-\alpha-n}.
\end{aligned}$$

Next, we consider $J^{\alpha,\sigma,1}_{3,1,2}$. Let $(x,y)\in\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_2$. Then

$$\kappa^{\alpha,\sigma}_1(z_1,z_2) \lesssim \frac{|y-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2-n}}{|x-y|^{n+2s}} \approx \frac{|y-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2-n}}{|x-z_1|^{n+2s}}.$$

Observe that $|y-z_1|\lesssim|x-z_1|$. This implies that

$$\int_{y\in\mathcal{A}_3\cap\mathcal{B}_1} |y-z_1|^{s_1-\alpha-n}\,dy \lesssim |x-z_1|^{s_1-\alpha}.$$

Hence,

$$\begin{aligned}
J^{\alpha,\sigma,1}_{3,1,2}(z_1,z_2) &\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_2} \frac{|x-z_1|^{s_1-\alpha}\,|x-z_2|^{s_2-n}}{|x-z_1|^{n+2s}}\,dx \\
&\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_2} |x-z_1|^{-s_2-\alpha-n}\,|x-z_2|^{s_2-n}\,dx.
\end{aligned}$$

By Hölder's inequality, let $p>1$ be small enough so that $(-s_2-\alpha-n)p<-n$ and $(s_2-n)p'>-n$. We obtain

$$\begin{aligned}
&J^{\alpha,\sigma,1}_{3,1,2}(z_1,z_2) \\
&\lesssim \left(\int_{|x-z_1|\gtrsim\delta} |x-z_1|^{(-s_2-\alpha-n)p}\,dx\right)^{\frac{1}{p}} \left(\int_{|x-z_2|\lesssim\delta} |x-z_2|^{(s_2-n)p'}\,dx\right)^{\frac{1}{p'}} \\
&\lesssim \left(\delta^{(-s_2-\alpha-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(s_2-n)p'+n}\right)^{\frac{1}{p'}} \\
&\approx \delta^{-\alpha-n}.
\end{aligned}$$

For $(x,y)\in\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_3$, after integrating w.r.t. y in case of $\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_1$ we arrive at

$$\begin{aligned} J^{\alpha,\sigma,1}_{3,1,3}(z_1,z_2) &\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_1\cap\mathcal{I}_3} |x-z_1|^{-s_2-\alpha+1-n}|x-z_2|^{s_2-1-n}\,dx \\ &\approx \int_{|x-z_1|\gtrsim\delta} |x-z_1|^{-\alpha-2n}\,dx \\ &\lesssim \delta^{-\alpha-n}. \end{aligned}$$

Estimating $J^{\alpha,\sigma,1}_{3,2,1}$, $J^{\alpha,\sigma,1}_{3,2,2}$ and $J^{\alpha,\sigma,1}_{3,2,3}$:

For $(x,y)\in\mathcal{A}_3\cap\mathcal{B}_2$, we have $|y-z_1|\lesssim|x-z_1|$ and $|x-z_2|\lesssim|y-z_2|$. It follows that

$$\kappa^{\alpha,\sigma}_1(z_1,z_2)\lesssim\frac{|y-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2-n}}{|x-y|^{n+2s}}.$$

That is, we now have

$$\begin{aligned} J^{\alpha,\sigma,1}_{3,2,1}(z_1,z_2) &\lesssim \iint_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_1} \frac{|y-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2-n}}{|x-y|^{n+2s}}\,dx\,dy \\ &\approx \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_1} |x-z_2|^{s_2-n}\left(\int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_1} |y-z_1|^{s_1-\alpha-n}\,|x-y|^{-n-2s}\,dy\right)dx. \end{aligned}$$

Let $p>1$ be so small such that $(s_1-\alpha-n)p>-n$ and $(-2s-n)p'<-n$. Then, by Hölder's inequality,

$$\begin{aligned} &J^{\alpha,\sigma,1}_{3,2,1}(z_1,z_2) \\ &\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_1} |x-z_2|^{s_2-n}\left(\int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_1} |y-z_1|^{(s_1-\alpha-n)p}\,dy\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_1} |x-y|^{(-2s-n)p'}\,dy\right)^{\frac{1}{p'}}\,dx \\ &\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_1} |x-z_2|^{s_2-n}\left(|x-z_1|^{(s_1-\alpha-n)p+n}\right)^{\frac{1}{p}}\left(|x-z_2|^{(-2s-n)p'+n}\right)^{\frac{1}{p'}}\,dx \\ &\approx \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_1} |x-z_2|^{-s_1-2n+\frac{n}{p'}}|x-z_1|^{s_1-\alpha-n+\frac{n}{p}}\,dx. \end{aligned}$$

Again, by Hölder's inequality, let $q>1$ be small so that $(s_1-\alpha-n+\frac{n}{p})q>-n$

and $(-s_1 - 2n + \frac{n}{p'})q' < -n$. Hence,

$$\begin{aligned}
&J_{3,2,1}^{\alpha,\sigma,1}(z_1,z_2)\\
&\lesssim \left(\int_{|x-z_1|\lesssim\delta} |x-z_1|^{(s_1-\alpha-n+\frac{n}{p})q}\,dx\right)^{\frac{1}{q}} \left(\int_{|x-z_2|\gtrsim\delta} |x-z_2|^{(-s_1-2n+\frac{n}{p'})q'}\,dx\right)^{\frac{1}{q'}}\\
&\lesssim \left(\delta^{(s_1-\alpha-n+\frac{n}{p})q+n}\right)^{\frac{1}{q}} \left(\delta^{(-s_1-2n+\frac{n}{p'})q'+n}\right)^{\frac{1}{q'}}\\
&\approx \delta^{-\alpha-n}.
\end{aligned}$$

The proof of $J_{3,2,2}^{\alpha,\sigma,1}$ is similar to the above case. We will also use Hölder's inequality twice. Let $p > 1$ be such that $(s_1-\alpha-n)p > -n$ and $(-2s-n)p' < -n$. Then Hölder's inequality gives that

$$\begin{aligned}
J_{3,2,2}^{\alpha,\sigma,1}(z_1,z_2) &\lesssim \iint_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_2} \frac{|y-z_1|^{s_1-\alpha-n}\,|x-z_2|^{s_2-n}}{|x-y|^{n+2s}}\,dx\,dy\\
&\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_2} |x-z_2|^{s_2-n}\left(\int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_2} |y-z_1|^{s_1-\alpha-n}\,|x-y|^{-n-2s}\,dy\right)dx\\
&\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_2} |x-z_2|^{s_2-n}\left(\int_{|y-z_1|\lesssim|x-z_1|} |y-z_1|^{(s_1-\alpha-n)p}\,dy\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{|x-y|\gtrsim|x-z_1|} |x-y|^{(-2s-n)p'}\,dy\right)^{\frac{1}{p'}}dx\\
&\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_2} |x-z_2|^{s_2-n}\left(|x-z_1|^{(s_1-\alpha-n)p+n}\right)^{\frac{1}{p}}\\
&\quad\times\left(|x-z_1|^{(-2s-n)p'+n}\right)^{\frac{1}{p'}}dx\\
&\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_2} |x-z_2|^{s_2-n}|x-z_1|^{-s_2-\alpha-n}\,dx.
\end{aligned}$$

Again, by Hölder's inequality, let $q > 1$ be small enough so that $(-s_2-\alpha-n)q < -n$ and $(s_2-n)q' > -n$ we obtain

$$\begin{aligned}
J_{3,2,2}^{\alpha,\sigma,1}(z_1,z_2) &\lesssim \left(\int_{|x-z_1|\gtrsim\delta} |x-z_1|^{(-s_2-\alpha-n)q}\,dx\right)^{\frac{1}{q}} \left(\int_{|x-z_2|\lesssim\delta} |x-z_2|^{(s_2-n)q'}\,dx\right)^{\frac{1}{q'}}\\
&\lesssim \left(\delta^{(-s_2-\alpha-n)q+n}\right)^{\frac{1}{q}} \left(\delta^{(s_2-n)q'+n}\right)^{\frac{1}{q'}}\\
&\approx \delta^{-\alpha-n}.
\end{aligned}$$

The case $(x,y) \in \mathcal{A}_3 \cap \mathcal{B}_2 \cap \mathcal{I}_3$ is easier since $|x-z_1|$ and $|x-z_2|$ are comparable. We employ Hölder's inequality once as in the case $\mathcal{A}_3 \cap \mathcal{B}_2 \cap \mathcal{I}_2$ by choosing the same p and p'. It follows that

$$\begin{aligned} J_{3,2,3}^{\alpha,\sigma,1}(z_1,z_2) &\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_2\cap\mathcal{I}_3} |x-z_2|^{s_2-n}\, |x-z_1|^{-s_2-\alpha-n}\, \chi_{\mathcal{I}_3}\, dx \\ &\approx \int_{|x-z_1|\gtrsim\delta} |x-z_1|^{-\alpha-2n}\, dx \\ &\lesssim \delta^{-\alpha-n}. \end{aligned}$$

<u>Estimating $J_{3,3,1}^{\alpha,\sigma,1}$, $J_{3,3,2}^{\alpha,\sigma,1}$ and $J_{3,3,3}^{\alpha,\sigma,1}$:</u>

There are several observations. First, we let $(x,y) \in \mathcal{A}_3 \cap \mathcal{B}_3$. Then $|x-y| \approx |x-z_1|$ and $|x-y| \approx |x-z_2|$, that is

$$|x-z_1| \approx |x-y| \approx |x-z_2|.$$

Moreover, if $(x,y) \in \mathcal{I}_1$, then $|x-z_1| \lesssim \delta$ and $|x-z_2| \gtrsim \delta$. Thus, $|x-z_1| \approx |x-z_2| \approx \delta$. This implies that $|y-z_1| \lesssim |x-z_1| \approx \delta$ and $|y-z_2| \lesssim |x-z_2| \approx \delta$. Hence, we have for all $(x,y) \in \mathcal{A}_3 \cap \mathcal{B}_3 \cap \mathcal{I}_1$ that

$$|y-z_2|, |y-z_1| \lesssim \delta.$$

However, $|z_1-z_2| = \delta$. Then either $|y-z_1| \approx \delta$ or $|y-z_2| \approx \delta$. Indeed, otherwise if $|y-z_1|, |y-z_2| \ll \delta$ then $\delta = |z_1-z_2| \le |y-z_1| + |y-z_2| \ll 2\delta$, which is a contradiction. Hence, for $(x,y) \in \mathcal{A}_3 \cap \mathcal{B}_3 \cap \mathcal{I}_1$, we have

$$\begin{aligned} \kappa_1^{\alpha,\sigma}(z_1,z_2) &\lesssim \frac{|y-z_1|^{s_1-\alpha-n}\, |y-z_2|^{s_2-n}}{|x-y|^{n+2s}} \\ &\lesssim \delta^{s_1-\alpha-n} \frac{|y-z_2|^{s_2-n}}{|x-y|^{n+2s}} + \delta^{s_2-n} \frac{|y-z_1|^{s_1-\alpha-n}}{|x-y|^{n+2s}} \end{aligned}$$

so that

$$\begin{aligned} J_{3,3,1}^{\alpha,\sigma,1}(z_1,z_2) &\lesssim \int_{|y-z_2|\lesssim\delta}\int_{|x-z_2|\gtrsim\delta} \delta^{s_1-\alpha-n}\frac{|y-z_2|^{s_2-n}}{|x-z_2|^{n+2s}}\,dx\,dy \\ &\quad + \int_{|y-z_1|\lesssim\delta}\int_{|x-z_2|\gtrsim\delta} \delta^{s_2-n}\frac{|y-z_1|^{s_2-n}}{|x-z_2|^{n+2s}}\,dx\,dy \\ &\lesssim \delta^{-n-\alpha}. \end{aligned}$$

The case that $(x,y)\in\mathcal{I}_2$ can be proved by the same technique as as in the proof of $(x,y)\in\mathcal{I}_1$. Thus, we omit the proof for that case.

The last estimate that we have to prove is $J_{3,3,3}^{\alpha,\sigma,1}(z_1,z_2)$. In order to do that, we have to consider three more subcases according to domain depending on y as follows:

$$\mathcal{J}_1 \equiv \mathcal{J}_1(z_1,z_2) = \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x\in\mathbb{R}^n, |y-z_1|\le 10\delta \text{ and } |y-z_2|\ge \frac{1}{10}\delta\},$$

$$\mathcal{J}_2 \equiv \mathcal{J}_2(z_1,z_2) = \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x\in\mathbb{R}^n, |y-z_2|\le 10\delta \text{ and } |y-z_1|\ge \frac{1}{10}\delta\},$$

$$\begin{aligned} &\mathcal{J}_3 \equiv \mathcal{J}_3(z_1,z_2) \\ &\quad = \left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x\in\mathbb{R}^n, \frac{1}{100}|y-z_2|\le|y-z_1|\le 100|y-z_2| \text{ and } |y-z_1|\ge\frac{1}{100}\delta\right\}. \end{aligned}$$

Firstly, let $(x,y)\in\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_1$. Since $|x-y|\approx|x-z_2|$,

$$\kappa_1^{\alpha,\sigma}(z_1,z_2) \lesssim \frac{|y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s}} \approx \frac{|y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-z_2|^{n+2s}}.$$

Then, integrating w.r.t. x, we obtain

$$\begin{aligned} J_{3,3,3}^{\alpha,\sigma,1}(z_1,z_2) &\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_1}\int_{|x-z_2|\gtrsim|y-z_2|} \frac{|y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-z_2|^{n+2s}}\,dx\,dy \\ &\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_1} |y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}\,|y-z_2|^{-2s}\,dy \\ &\approx \int_{\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_1} |y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{-s_1-n}\,dy. \end{aligned}$$

Let $p>1$ be small enough so that $(s_1-\alpha-n)p>-n$ and $(-s_1-n)p'<-n$.

Using Hölder's inequality, we have

$$J_{3,3,3}^{\alpha,\sigma,1}(z_1,z_2) \lesssim \left(\int_{|y-z_1|\lesssim\delta} |y-z_1|^{(s_1-\alpha-n)p}\,dy\right)^{\frac{1}{p}} \left(\int_{|y-z_2|\gtrsim\delta} |y-z_2|^{(-s_1-n)p'}\,dy\right)^{\frac{1}{p'}}$$
$$\lesssim \left(\delta^{(s_1-\alpha-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(-s_1-n)p'+n}\right)^{\frac{1}{p'}}$$
$$\approx \delta^{-\alpha-n}.$$

Next, we let $(x,y)\in\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_2$. Then

$$\kappa_1^{\alpha,\sigma}(z_1,z_2) \lesssim \frac{|y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s}} \approx \frac{|y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-z_1|^{n+2s}}$$

because $|x-y|\approx|x-z_1|$. Similarly as above case, we integrate w.r.t. x and use Hölder's inequality for $p>1$ such that $\frac{1}{p}+\frac{1}{p'}=1$, $(-s_2-\alpha-n)p<-n$ and $(s_2-n)p'>-n$. It follows that

$$J_{3,3,3}^{\alpha,\sigma,1}(z_1,z_2) \lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_2}\int_{|x-z_1|\gtrsim|y-z_1|} \frac{|y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-z_1|^{n+2s}}\,dx\,dy$$
$$\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_2} |y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}\,|y-z_1|^{-2s}\,dy$$
$$\approx \int_{\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_2} |y-z_1|^{-s_2-\alpha-n}\,|y-z_2|^{s_2-n}\,dy$$
$$\lesssim \left(\int_{|y-z_1|\gtrsim\delta} |y-z_1|^{(-s_2-\alpha-n)p}\,dy\right)^{\frac{1}{p}} \left(\int_{|y-z_2|\lesssim\delta} |y-z_2|^{(s_2-n)p'}\,dy\right)^{\frac{1}{p'}}$$
$$\lesssim \left(\delta^{(-s_2-\alpha-n)p+n}\right)^{\frac{1}{p}} \left(\delta^{(s_2-n)p'+n}\right)^{\frac{1}{p'}}$$
$$\approx \delta^{-\alpha-n}.$$

For $(x,y)\in\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_3$, we have $|y-z_2|\approx|y-z_1|\gtrsim\delta$ and so

$$\kappa_1^{\alpha,\sigma}(z_1,z_2) \lesssim \frac{|y-z_1|^{s_1-\alpha-n}\,|y-z_2|^{s_2-n}}{|x-y|^{n+2s}} \approx \frac{|y-z_1|^{2s-\alpha-2n}}{|x-z_1|^{n+2s}}.$$

Thus,

$$\begin{aligned} J^{\alpha,\sigma,1}_{3,3,3}(z_1,z_2) &\lesssim \int_{\mathcal{A}_3\cap\mathcal{B}_3\cap\mathcal{I}_3\cap\mathcal{J}_3} \int_{|x-z_1|\gtrsim|y-z_1|} \frac{|y-z_1|^{2s-\alpha-2n}}{|x-z_1|^{n+2s}}\,dx\,dy \\ &\lesssim \int_{|y-z_1|\gtrsim\delta} |y-z_1|^{2s-\alpha-2n}\,|y-z_1|^{-2s}\,dy \\ &\approx \int_{|y-z_1|\gtrsim\delta} |y-z_1|^{-\alpha-2n}\,dy \\ &\approx \delta^{-\alpha-n}. \end{aligned}$$

Finally, we have established all the cases. The proof is now complete. □

Having proved the above useful estimates, we are now ready to deduce from them to prove Proposition 1.2.

Proof of Proposition 1.2. Let $z_1 \neq z_2 \in \mathbb{R}^n$. The first inequality (1.8) is true by Proposition 4.4 with $\alpha = 0$. Next, we observe that A_{K,s_1,s_2} may not be symmetric in general (unless $s_1 = s_2 = s$). However, since for our setup the values of s_1 and s_2 are interchangeable, (1.9) and (1.10) are equivalent. This means it suffices to prove (1.9). To prove the inequality (1.9), we first use Lemma 4.3. Then the following estimate is valid for every $\alpha,\sigma \in [0,1]$,

$$\begin{aligned} &\left||x-z_1-h|^{s_1-n} - |y-z_1-h|^{s_1-n} - \left(|x-z_1|^{s-n} - |y-z_1|^{s-n}\right)\right| \\ &\lesssim |h|^{\alpha}\left(\left||x-z_1-h|^{s-\alpha-n} - |y-z_1-h|^{s-\alpha-n}\right| + \left||x-z_1|^{s-\alpha-n} - |y-z_1|^{s-\alpha-n}\right|\right) \\ &+ |h|^{\alpha}\min\{|x-z_1|^{s-\alpha-\sigma-n}, |y-z_1|^{s-\alpha-\sigma-n}\}\,|x-y|^{\sigma}. \end{aligned}$$

We choose σ large enough and α small enough so that the assumptions of Proposition 4.4 are satisfied and then use Proposition 4.4 to obtain

$$\begin{aligned} &|A_{K,s_1,s_2}(z_1+h,z_2) - A_{K,s_1,s_2}(z_1,z_2)| \\ &\lesssim |h|^{\alpha}\,|M^{\alpha,\sigma}_1(z_1+h,z_2) + M^{\alpha,\sigma}_1(z_1,z_2) + M^{\alpha,\sigma}_2(z_1,z_2)| \\ &\lesssim |h|^{\alpha}\left||z_1+h-z_2|^{-\alpha-n} + 2|z_1-z_2|^{-\alpha-n}\right| \\ &\lesssim |h|^{\alpha}|z_1-z_2|^{-\alpha-n}. \end{aligned}$$

The last line follows from $|z_1 + h - z_2| \approx |z_1 - z_2|$ because $|h| \leq \frac{1}{2}|z_1 - z_2|$ by assumption. This completes the proof. $\square$

CHAPTER V
APPLICATION TO NONLOCAL PDES

Proving the application, Theorem 1.3, is presented in this chapter. Before we start proving this, we first consider the operator T_{K,s_1,s_2} when $K = 1$.

Lemma 5.1. *If $K = 1$, then the operator T_{K,s_1,s_2} is an identity operator.*

Proof. To achieve this assertion, it suffices to verify that

$$\int_{\mathbb{R}^n} (T_{1,s_1,s_2} f)(z)\varphi(z)\, dz = \int_{\mathbb{R}^n} f(z)\varphi(z)\, dz$$

for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Let $\phi, g \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))\,(g(x)-g(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(x+h))\,(g(x)-g(x+h))}{|h|^{n+2s}}\, dx\, dh \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\phi(x)\,(g(x)-g(x+h))}{|h|^{n+2s}}\, dx\, dh - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\phi(x+h)\,(g(x)-g(x+h))}{|h|^{n+2s}}\, dx\, dh \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\phi(x)\,(g(x)-g(x+h))}{|h|^{n+2s}}\, dx\, dh - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\phi(x)\,(g(x-h)-g(x))}{|h|^{n+2s}}\, dx\, dh \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\phi(x)\,(2g(x)-g(x+h)-g(x-h))}{|h|^{n+2s}}\, dx\, dh
\end{aligned}$$

where the first and the third lines follow from changing variable formula. Using the integral representation of the fractional Laplacian operator (2.4), we obtain that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))\,(g(x)-g(y))}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} \phi(x)(-\Delta)^{\frac{2s}{2}} g(x)\, dx.$$

In view of Parseval's identiy (recall that $s_1 + s_2 = 2s$) we have

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(x)(-\Delta)^{\frac{2s}{2}} g(x)\, dx &= \int_{\mathbb{R}^n} \mathcal{F}(\phi)(\xi)\overline{|\xi|^{2s}\mathcal{F}(g)(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^n} |\xi|^{s_1}\mathcal{F}(\phi)(\xi)\overline{|\xi|^{s_2}\mathcal{F}(g)(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{s_1}{2}}\phi(x)(-\Delta)^{\frac{s_2}{2}} g(x)\, dx.\end{aligned}$$

That is, we have shown that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))\,(g(x)-g(y))}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s_1}{2}}\phi(x)(-\Delta)^{\frac{s_2}{2}} g(x)\, dx.$$

Next, let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then we use Fubini's theorem and apply the above equation with $\phi := I^{s_1}\varphi$ and $g := I^{s_2} f$ to get the following expression

$$\begin{aligned}&\int_{\mathbb{R}^n} (T_{1,s_1,s_2} f)(z)\varphi(z)\, dz \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(I^{s_1}\varphi(x) - I^{s_1}\varphi(y))\,(I^{s_2}f(x) - I^{s_2}f(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{s_1}{2}}(I^{s_1}\varphi(x))(-\Delta)^{\frac{s_2}{2}}(I^{s_2}f(x)) \\ &= \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx.\end{aligned}$$

This implies that T_{1,s_1,s_2} is an identity operator by approximation in $f_\ell, \varphi_k \in \mathcal{F}^{-1}(C_c^\infty(\mathbb{R}^n\backslash\{0\}))$. □

We now prove the Theorem 1.3. For the convenience we state the theorem again.

Theorem 1.3 *For any $s, s_1, s_2 \in (0,1)$ with $s_1 + s_2 = 2s$, $s_1 \geq s$ and any $p \in [2,\infty)$, there exists $\varepsilon > 0$ such that the following holds. For any measurable kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to (0,\infty)$ with $1 - \frac{\inf K}{\sup K} < \varepsilon$, if $u \in W^{s,2}(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$ satisfy*

$$\mathcal{L}_K^s u = (-\Delta)^{\frac{s}{2}} f \quad \text{in } \mathbb{R}^n$$

then there exists $C > 0$ such that

$$\|(-\Delta)^{\frac{s_1}{2}} u\|_{L^p(\mathbb{R}^n)} \leq C \,\|(-\Delta)^{\frac{s-s_2}{2}} f\|_{L^p(\mathbb{R}^n)}.$$

The small constant $\varepsilon > 0$ is uniform in the following sense: if $s, s_1, s_2 \in (\theta, 1-\theta)$ and $p \in [2, \frac{1}{\theta})$ for some $\theta > 0$, then ε depends only on θ and the dimension.

Proof. According to Chapter III, we have shown that the equation

$$\mathcal{L}_K^s u = g, \tag{5.1}$$

for a given function g, is equivalent to

$$T_{K,s_1,s_2}(-\Delta)^{\frac{s_1}{2}} u = I^{s_2} g$$

up to the multiplicative constant. Notice that the map $K \mapsto T_{K,s_1,s_2}$ is linear and K is positive. Thus, by dividing both side by $\sup K$, we obtain

$$T_{\frac{K}{\sup K},s_1,s_2}(-\Delta)^{\frac{s_1}{2}} u = \frac{1}{\sup K} I^{s_2} g.$$

Furthermore, we can write $\frac{K}{\sup K} = 1 - \left(1 - \frac{K}{\sup K}\right)$ and so

$$T_{1,s_1,s_2}(-\Delta)^{\frac{s_1}{2}} u - T_{1-\frac{K}{\sup K},s_1,s_2}(-\Delta)^{\frac{s_1}{2}} u = \frac{1}{\sup K} I^{s_2} g.$$

Since the operator T_{1,s_1,s_2} is an identity map by Lemma 5.1, the equation (5.1) can be rewritten as

$$\left(I - T_{\tilde{K},s_1,s_2}\right)((-\Delta)^{\frac{s_1}{2}} u) = \frac{1}{\sup K} I^{s_2} g.$$

by setting $\tilde{K} := 1 - \frac{K}{\sup K}$.

Now we show that $I - T_{\tilde{K},s_1,s_2} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is an invertible operator with continuous inverse. Indeed, under our assumption $1 - \frac{\inf K}{\sup K} < \varepsilon$ we yields that $\|\tilde{K}\|_{L^\infty(\mathbb{R}^n)} < \varepsilon$. If we take $\varepsilon > 0$ small enough, then we obtain by Theorem

1.1 that the operator $T_{\tilde{K},s_1,s_2} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ has the norm

$$\|T_{\tilde{K},s_1,s_2}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < 1$$

which implies that the Neumann series of $T_{\tilde{K},s_1,s_2}$ is convergent in the space of continuous linear maps from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Hence, $I - T_{\tilde{K},s_1,s_2}$ is invertible and its inverse is as follows

$$(I - T_{\tilde{K},s_1,s_2})^{-1} = \sum_{k=0}^{\infty} (T_{\tilde{K},s_1,s_2})^k.$$

Assume that $u \in W^{s,2}(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$ solve

$$\mathcal{L}_K^s u = (-\Delta)^{\frac{s}{2}} f.$$

Let $f_k \in C_c^\infty(\mathbb{R}^n)$ be an approximation of f such that

$$\|f_k - f\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{\frac{s-s_2}{2}}(f_k - f)\|_{L^p(\mathbb{R}^n)} \to 0 \qquad \text{as } k \to \infty.$$

For each $k \in \mathbb{N}$, we can solve the following equation

$$(-\Delta)^{\frac{s_1}{2}} \tilde{u}_k = (I - T_{\tilde{K},s_1,s_2})^{-1} \frac{1}{\sup K} (-\Delta)^{\frac{s-s_2}{2}} f_k$$

and

$$(-\Delta)^{\frac{s_1}{2}} \tilde{u} = (I - T_{\tilde{K},s_1,s_2})^{-1} \frac{1}{\sup K} (-\Delta)^{\frac{s-s_2}{2}} f.$$

Such a $\tilde{u}_k$ and $\tilde{u}$ is unique up to a constant and can be obtained from Fourier analysis since by assumption $(-\Delta)^{\frac{s-s_2}{2}} f_k \in L^p(\mathbb{R}^n)$, and thus

$$(I - T_{\tilde{K},s_1,s_2})^{-1} \frac{1}{\sup K} (-\Delta)^{\frac{s-s_2}{2}} f_k \in L^p(\mathbb{R}^n).$$

We also obtain the estimate

$$\|(-\Delta)^{\frac{s_1}{2}}\tilde{u}_k - (-\Delta)^{\frac{s_1}{2}}\tilde{u}\|_{L^p(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{s-s_2}{2}}(f_k - f)\|_{L^p(\mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty.$$

Also, by construction, we have

$$\mathcal{L}_K^s \tilde{u}_k = (-\Delta)^{\frac{s}{2}} f_k.$$

Consequently,

$$\mathcal{L}_K^s(\tilde{u}_k - u) = (-\Delta)^{\frac{s}{2}}(f_k - f).$$

Testing this equation with $\tilde{u}_k - u$ (observe that $\inf K > 0$ by assumption) we have

$$\|(-\Delta)^{\frac{s}{2}}\tilde{u}_k - (-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)} \lesssim \|f_k - f\|_{L^2(\mathbb{R}^n)} \to 0 \qquad \text{as } k \to \infty.$$

Since $(-\Delta)^{\frac{s_1}{2}}\tilde{u}_k$ converges to $(-\Delta)^{\frac{s_1}{2}}\tilde{u}$ in $L^p(\mathbb{R}^n)$ and $(-\Delta)^{\frac{s}{2}}\tilde{u}_k$ converges to $(-\Delta)^{\frac{s}{2}}u$ in $L^2(\mathbb{R}^n)$, we find that $(-\Delta)^{\frac{s_1}{2}}u = (-\Delta)^{\frac{s_1}{2}}\tilde{u} \in L^p(\mathbb{R}^n)$, and we have

$$\|(-\Delta)^{\frac{s_1}{2}}u\|_{L^p(\mathbb{R}^n)} = \|(-\Delta)^{\frac{s_1}{2}}\tilde{u}\|_{L^p(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{s-s_2}{2}}f\|_{L^p(\mathbb{R}^n)}. \qquad \square$$

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