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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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CONVERGENCE OF TRINOMIAL FORMULA FOR CALL OPTION PRICES

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สูตรทวินาม ซึ่งกำหนดโดย Cox, Ross และ Rubinstein ในปี 1979 คือเครื่องมือสำหรับการหาราคาคอลออปชั่น เป็นที่ทราบกันดีว่าราคาจากสูตรทวินามลู่เข้าสู่ราคาจากสูตรแบล็ค-โชลส์ซึ่งกำหนดโดย Black, Scholes และ Merton ในปี 1973 เมื่อจำนวนคาบ (n) ลู่เข้าสู่อนันต์

ในปี 1988 Boyle นำเสนอสูตรไตรนามซึ่งเป็นอีกเครื่องมือหนึ่งสำหรับการคำนวนราคาคอลออปชั่น เขาพิจารณาสูตรไตรนามในกรณีที่อัตราการขึ้นของหุ้นคือ $u = e^{\lambda\sigma\sqrt{\frac{T}{n}}}$ และอัตราการลงของหุ้นคือ $d = u^{-1}$ เมื่อ T คืออายุของออปชัน, σ คือความผันผวน และ $\lambda > 1$ หลังจากนั้น Entit และคณะ (ค.ศ. 2013) ได้ให้ตัวอย่างซึ่งแสดงว่าราคาคอลออปชันจากสูตรไตรนามเข้าใกล้ราคาคอลออปชั่นจากสูตรแบล็ค-โชลส์ ในวิทยานิพนธ์นี้เราให้การพิสูจน์ที่รัดกุมของข้อสันนิษฐานข้างต้นโดยการแสดงว่าสูตรไตรนามลู่เข้าสู่สูตรแบล็ค-โชลส์ ยิ่งไปกว่านั้นเราพิสูจน์ว่าอัตราของการลู่เข้านี้คือ $\frac{1}{\sqrt{n}}$

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The binomial formula given by Cox, Ross and Rubinstein (1979) is a tool for valuating the call option price. It is well known that the price from binomial formula converges to the price from Black-Scholes formula which was given by Black, Scholes and Merton (1973) as the number of periods (n) converges to infinity.

In 1988, Boyle introduced the trinomial formula which is another tool for calculating call option price. He considered the trinomial formula in the case that the rising rate of a stock price is $u = e^{\lambda\sigma\sqrt{\frac{T}{n}}}$ and the falling rate of the stock price is $d = u^{-1}$, where T is maturity time, σ is volatility and $\lambda > 1$. After that, Entit et al. (2013) gave examples which show that the call option price from trinomial formula is closed to the call option price from the Black-Scholes formula. In this thesis, we give the rigorous proof of this conjecture by showing that the trinomial formula converges to the Black-Scholes formula. Moreover, we prove that the rate of this convergence is $\frac{1}{\sqrt{n}}$.

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CONTENTS

CHAPTER I
INTRODUCTION

In finance, an option is a derivative that represents a contract rights to buy (call) or to sell (put) the underlying asset by the writer to the holder. The buyer has to pay premium for the rights granted. Two types of option widely used in applications are the American option and the European option. The American option can be exercised at any time prior to expiration and the other option can be exercised only at expiration. In this work, we are interested in the European call option. We denote

S_0 as the current stock price,

K as the strike price,

r as the risk-free rate of interest,

T as time to maturity

and σ as the volatility of the asset price.

A formula that has been widely used to calculate the theoretical option price in many stock markets is the Black–Scholes formula (C_{BS}). It was introduced by three economists, Black, Scholes and Merton ([5], 1973). This formula is given by

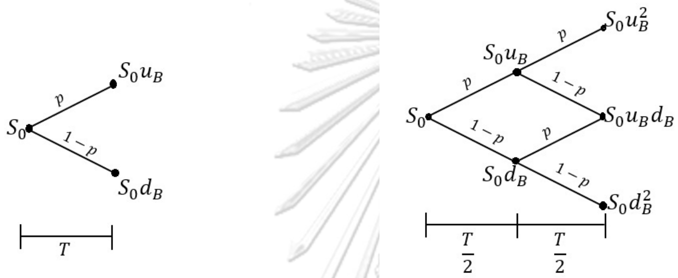
$$C_{BS} = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{1.1}$$

where
$$d_1 = \frac{\log(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt$ is the standard normal distribution function.

The binomial formula which is derived from the binomial model is another tool that is used to calculate the option price.

In the binomial model, we divide T into n periods. For $k = 0, 1, 2 \ldots, n$, let S_k be the stock price at the end of k^{th} period and assume that the current stock price S_k either rises to $S_k u_B$ with probability p or falls to $S_k d_B$ with probability $1 - p$ at the $(k+1)^{th}$ period, where $0 < p < 1$ and $0 < d_B < 1 < e^{\frac{rT}{n}} < u_B$. Figure 1.1 (a) and Figure 1.1 (b) are examples of the binomial model where $n = 1$ and $n = 2$, respectively.



(a) $n = 1$ (b) $n = 2$

Figure 1.1: Binomial model

Cox, Ross and Rubinste in ([9], 1979) showed that the binomial formula B_n for option price is

$$B_n = e^{-rT} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \max\left\{S_0 u_B^k d_B^{n-k} - K, 0\right\}, \tag{1.2}$$

where

$$p = \frac{e^{\frac{rT}{n}} - d_B}{u_B - d_B}, \quad u_B = e^{\sigma\sqrt{\frac{T}{n}}} \quad \text{and} \quad d_B = e^{-\sigma\sqrt{\frac{T}{n}}}. \tag{1.3}$$

It is well–known that the binomial formula converges to the Black–Scholes formula. Moreover, there are many researchers who found the rate of this convergence such as Leisen and Reimer ([17], 1996), Diener and Diener ([10], 2004), Heston and Zhou ([12], 2000), and Ratibenyakool and Neammanee ([20], 2019).

At the end of each period, the current stock price in the binomial model either rises or falls. In this work, we are interested in the model where the current stock price can steady at the end of period. That is the trinomial model. We assume that for $k = 0, 1, 2, \ldots, n-1$, the current stock price S_k either rises to $S_k u_T$ with probability p_u, falls to $S_k d_T$ with probability p_d or steadies at S_k with probability $p_m = 1 - p_u - p_d$, where $0 < p_u, p_d, p_m < 1$ and $0 < d_T < 1 < e^{\frac{rT}{n}} < u_T$. The example of the trinomial model where $n = 2$ is shown in Figure 1.2.

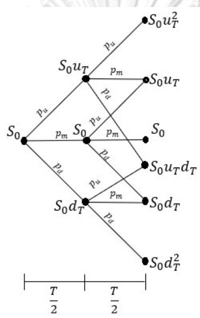


Figure 1.2: Trinomial model for $n = 2$

If $u_T d_T \neq 1$, then the pattern is quite complicated. We can simplify the trinomial model if we impose the condition $u_T d_T = 1$. The example is shown in Figure 1.3.

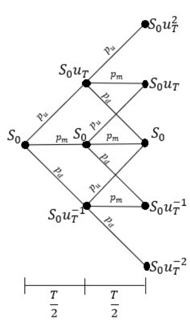


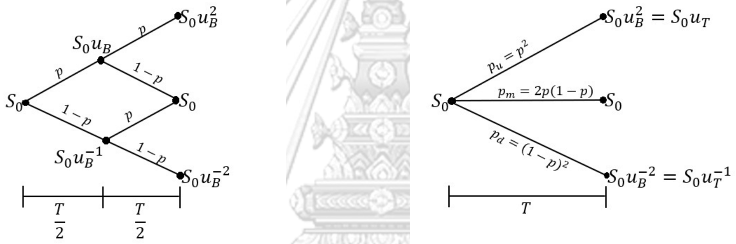
Figure 1.3: Trinomial model in case of $u_T d_T = 1$

Assuming $u_T d_T = 1$, Boyle ([6], 1988) showed that the trinomial formula T_n is given by

$$T_n = e^{-rT} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k, l, n-k-l} p_u^k p_d^l p_m^{n-k-l} \max\left\{S_0 u_T^{k-l} - K, 0\right\}, \tag{1.4}$$

where $\binom{n}{k, l, n-k-l} = \dfrac{n!}{k!l!(n-k-l)!}$.

In 2007, Ahn and Song ([2]) considered the trinomial model in case of $u_T = u_B^2$, $d_T = u_T^{-1}$, $p_u = p^2$ and $p_d = (1-p)^2$, where u_B and p are defined in (1.3). They gave an idea that $T_n = B_{2n}$ and converges to the Black–Scholes formula. We can see Figure 1.4 for $n = 1$.



(a) 2-step binomial model (b) 1-step trinomial model

Figure 1.4: Trinomial model of Ahn and Song

After that, Intarapanya and Neammanee ([14]) confirmed their conjecture by giving the rigorous proof in 2019. The result is stated in Theorem 1.1.

Theorem 1.1. *Let T_n be defined in (1.4) with $u_T = u_B^2$, $p_u = p^2$ and $p_d = (1-p)^2$, where u_B and p are defined in (1.3). Then,*

$$T_n = B_{2n} \quad \text{and} \quad \lim_{n \to \infty} T_n = C_{BS},$$

where C_{BS} and B_n are defined in (1.1) and (1.2), respectively.

In 1988, Boyle ([6]) gave the trinomial model in case of

$$u_T = e^{\lambda\sigma\sqrt{\frac{T}{n}}}, \tag{1.5}$$

$$p_u = \frac{(V + M^2 - M)\, u_T - (M - 1)}{(u_T - 1)\,(u_T^2 - 1)}, \tag{1.6}$$

$$p_d = \frac{(V + M^2 - M)\, u_T^2 - (M - 1)\, u_T^3}{(u_T - 1)\,(u_T^2 - 1)}, \tag{1.7}$$

and

$$p_m = 1 - p_u - p_d, \tag{1.8}$$

where $\lambda > 1$,

$$M = e^{\frac{rT}{n}} \quad \text{and} \quad V = \left(e^{\frac{\sigma^2 T}{n}} - 1\right) M^2$$

and Entit et al. ([11], 2013) presented an example to show that the price of this formula is closed to Black–Scholes formula.

In this work, we give the rigorous proof of this conjecture by showing that the trinomial formula converges to the Black–Scholes formula. We also provide the rate $\frac{1}{\sqrt{n}}$ of this convergence. Our result is stated in Theorem 1.2.

Theorem 1.2. *Let T_n be defined in* (1.4) *with u_T, p_u, p_d, and p_m be defined in* (1.5)–(1.8)*, respectively. If $K \geq S_0$, then*

$$\lim_{n\to\infty} T_n = C_{BS}, \tag{1.9}$$

where C_{BS} is defined in (1.1)*. Moreover, the rate of this convergence is $\frac{1}{\sqrt{n}}$. That is*

$$T_n = C_{BS} + O\left(\frac{1}{\sqrt{n}}\right).$$

To prove the theorem, we divide it into four parts. The first part is a basic knowledge of the formulas which is in Chapter 2. After that, we give the Berry-Esseen theorem for trinomial random vector in Chapter 3. The proof of (1.9) is presented in Chapter 4. For the last part, we show that the rate of the convergence is $\frac{1}{\sqrt{n}}$ in Chapter 5.

CHAPTER II
APPROXIMATION OF OPTION PRICES

In this chapter, we will present the formulas used in approximating the option price. In the first section, we will present the Black-Scholes formula which has been widely used to calculate the theoretical option price in many stock markets. Next, we will present the binomial formula in the second section and the trinomial formula which extends the concept of the binomial formula in the last section.

2.1 The Black-Scholes formula

In this section, we will present the source of a shortened version of the Black-Scholes formula (see more details in [7] and [13]). The Black-Scholes formula was introduced by three economists Fischer Black, Myron Scholes and Robert Merton in 1973 ([5]). It has been used to calculate the option price. The model makes certain assumptions, including:

1. the option can only be exercised at expiration;
2. no dividends are paid out during the life of the option;
3. the market movements cannot be predicted;
4. there is no commissions;
5. the risk-free rate and volatility of the underlying are constant; and
6. the returns on the underlying are normally distributed.

Assume that the stock price S_t obeys a stochastic process of the form

$$dS_t = \left(\mu + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dW_t,$$

where $t \geq 0$ and W_t is a standard Brownian motion.

Let $F(S_t, t)$ be the price of an option as a function of the underlying asset $S(t)$, at time t.

By Itô's lemma, we have

$$\begin{aligned} dF(S_t,t) = &\left(\left(\mu + \frac{\sigma^2}{2}\right) S_t \frac{\partial F}{\partial S_t}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2}(S_t,t) + \frac{\partial F}{\partial t}(S_t,t)\right) dt \\ &+ \sigma S_t \frac{\partial F}{\partial S_t}(S_t,t) dW_t. \end{aligned}$$

Let P be a value of the portfolio that is created by selling one option and buying δ stocks. Then, $P = F - \delta S_t$ which implies that

$$\begin{aligned} dP &= d\left(F - \delta S_t\right) \\ &= \left(\left(\mu + \frac{\sigma^2}{2}\right) S_t \frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} + \frac{\partial F}{\partial t}\right) dt + \sigma S_t \frac{\partial F}{\partial S_t} dW_t \\ &\quad - \delta\left(\left(\mu + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dW_t\right) \\ &= \left(\left(\mu + \frac{\sigma^2}{2}\right) S_t \left(\frac{\partial F}{\partial S_t} - \delta\right) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} + \frac{\partial F}{\partial t}\right) dt + \sigma S_t \left(\frac{\partial F}{\partial S_t} - \delta\right) dW_t. \end{aligned}$$

Note that the coefficient of dW_t contains the factor $\frac{\partial F}{\partial S_t} - \delta$. This equation can be simplified if we assume that $\delta = \frac{\partial F}{\partial S_t}$. Then,

$$dP = \left(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} + \frac{\partial F}{\partial t}\right) dt.$$

Since the return of the portfolio should be equal to the return of the riskless account, we have

$$dP = rPdt.$$

Then,

$$\left(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} + \frac{\partial F}{\partial t}\right) dt = \left(F - S_t \frac{\partial F}{\partial S_t}\right) rdt.$$

That is

$$rF = \frac{\partial F}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 F}{\partial S_t^2} + rS_t\frac{\partial F}{\partial S_t}. \tag{2.1}$$

This equation is called that the Black-Scholes equation.

If F is an option price, then the boundary conditions are

1. $F(S_T, T) = \max\{S_T - K, 0\}$
2. $F(0, t) = 0$ for all $t \in [0, T]$
3. For each $t \in [0, T]$, $F(S_t, t) \sim S_t$, where $S_t \to \infty$.

From these conditions, we can show that a solution of the Black-Scholes equation (2.1) is the Black–Scholes formula for an option price which is given by

$$C_{BS} = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{2.2}$$

where
$$d_1 = \frac{\log(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T},$$

and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt$ is the standard normal distribution function (see [7] for more details).

Example 2.1. Let S_0, K, r, σ and T be defined in Chapter 1.
Assume that $S_0 = \$42$, $K = \$40$, $r = 10\%$, $\sigma = 20\%$ and $T = 0.5$ years. Then,

$$d_1 = \frac{\log(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

and
$$d_2 = \frac{\log(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.6278.$$

Hence,

$$C_{BS} = 43\Phi(0.7693) - 40e^{-0.1\times 0.5}\Phi(0.6278) = \$4.76.$$

2.2 The binomial formula

The binomial formula was given by Cox, Ross and Rubinstein ([9], 1979). They divide T into n periods, where $n \in \mathbb{N}$. For each $k = 1, 2, \ldots, n$, let S_k be the stock price at the end of the k^{th} period. Assume that for $k = 0, 1, 2, \ldots, n-1$, the current stock price S_k either rises to $S_k u_B$ with probability p or falls to $S_k d_B$ with probability $1-p$ at the end of the $k+1$ period, where $0 < p < 1$ and $0 < d_B < 1 < u_B$. That is

$$S_{k+1} = \begin{cases} S_k u_B & \text{with probability } p \\ S_k d_B & \text{with probability } 1-p, \end{cases} \qquad \text{for } k = 0, 1, 2, \ldots, n-1.$$

From this fact, we see that

$$S_1 = \begin{cases} S_0 u_B & \text{with probability } p \\ S_0 d_B & \text{with probability } 1-p \end{cases}$$

and

$$S_2 = \begin{cases} S_1 u_B & \text{with probability } p \\ S_1 d_B & \text{with probability } 1-p. \end{cases}$$

Then,

$$S_2 = \begin{cases} S_0 u_B^2 & \text{with probability } p^2 \\ S_0 u_B d_B & \text{with probability } 2p(1-p) \\ S_0 d_B^2 & \text{with probability } (1-p)^2. \end{cases}$$

Hence, 2-step binomial formula is presented in Figure 2.1.

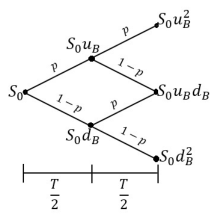


Figure 2.1: 2-step binomial model

In general case, the binomial formula satisfies the following diagram.

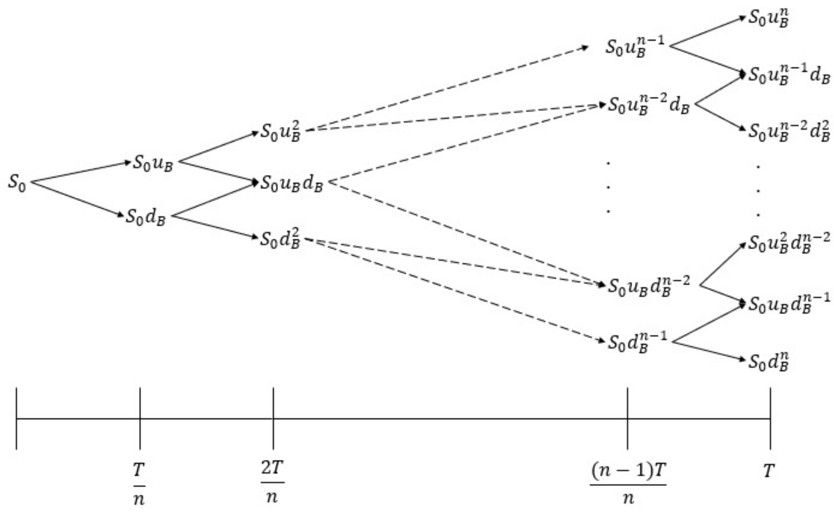


Figure 2.2: n-step binomial model

We can show that

$$S_n = S_0 u_B^j d_B^{n-j} \quad \text{with probability } \binom{n}{j} p^j (1-p)^{n-j},$$

where $\binom{n}{j} = \dfrac{n!}{(n-j)!j!}$ and $j = 0, 1, 2, \ldots, n.$
Let C_n be an option price at the end of the n^{th} period. Then,

$$C_n = \max\{S_n - K, 0\}.$$

That is

$$C_n = \max\left\{S_0 u_B^j d_B^{n-j} - K, 0\right\} \quad \text{with probability } \binom{n}{j} p^j (1-p)^{n-j},$$

for $j = 0, 1, 2, \ldots, n.$

Let $E[C_n]$ be the representative of the option price at the end of the n^{th} period and B_n be the current option price. Then, $B_n = e^{-rT}E[C_n]$, i.e.,

$$B_n = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max\left\{S_0 u_B^j d_B^{n-j} - K, 0\right\}. \tag{2.3}$$

From the risk neutrality hypothesis, we know that in a risk-neutral economy, the expected yield from all assets equals the risk-free rate of interest. Therefore, for $k = 0, 1, 2, \ldots, n-1$,

$$E[S_{k+1} \mid S_k] = S_k e^{\frac{rT}{n}} \tag{2.4}$$

which implies that $S_k u_B p + S_k d_B (1-p) = S_k e^{\frac{rT}{n}}$. That is

$$p = \frac{e^{\frac{rT}{n}} - d_B}{u_B - d_B} \tag{2.5}$$

(see [1], p.330 for more details).

If the binomial model is arbitrage free, then we need to assume that

$$0 < d_B < 1 < e^{\frac{rT}{n}} < u_B$$

which implies that $0 < p < 1$.

From (2.3), (2.4) and (2.5), Cox, Ross and Rubinstein ([9]) showed that

$$B_n = S_0 \sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j} - Ke^{-rT} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j}, \tag{2.6}$$

where $a = \min\{j \in \{0, 1, 2, \ldots, n\} \mid j \geq b\}$,

$$b = \frac{\log(K/S_0) - n \log d}{\log(u/d)}$$

and $q = pue^{-\frac{rT}{n}}$.

We observe that the formula p depends on u_B and d_B. There are many researchers who gave the formula of u_B and d_B. In 1976, Cox, Ross, and Rubinstein ([9]) obtained CRR formula by taking

$$u_B = e^{\sigma\sqrt{\frac{T}{n}}} \quad \text{and} \quad d_B = e^{-\sigma\sqrt{\frac{T}{n}}}. \tag{2.7}$$

In 1983, Jarrow and Rudd ([15]) defined

$$u_B = e^{\sigma\sqrt{\frac{T}{n}}+(r-\frac{1}{2}\sigma^2)\frac{T}{n}} \quad \text{and} \quad d_B = e^{-\sigma\sqrt{\frac{T}{n}}+(r-\frac{1}{2}\sigma^2)\frac{T}{n}}.$$

After that, Tian ([28], 1993) gave

$$u_B = \frac{e^{\frac{(r+\sigma^2)T}{n}}}{2}\left(e^{\frac{\sigma^2 T}{n}} + 1 + \sqrt{e^{\frac{2\sigma^2 T}{n}} + 2e^{\frac{\sigma^2 T}{n}-3}}\right)$$

and

$$d_B = \frac{e^{\frac{(r+\sigma^2)T}{n}}}{2}\left(e^{\frac{\sigma^2 T}{n}} + 1 - \sqrt{e^{\frac{2\sigma^2 T}{n}} + 2e^{\frac{\sigma^2 T}{n}-3}}\right)$$

and Walsh ([30]) gave

$$u_B = e^{\sigma\sqrt{\frac{T}{n}}+\frac{rT}{n}} \quad \text{and} \quad d_B = e^{-\sigma\sqrt{\frac{T}{n}}+\frac{rT}{n}}$$

in 2003.

In general, Chang and Palmer ([8], 2007) defined u_B and d_B by

$$u_B = e^{\sigma\sqrt{\frac{T}{n}}+\frac{\lambda_n\sigma^2 T}{n}} \quad \text{and} \quad d_B = e^{-\sigma\sqrt{\frac{T}{n}}+\frac{\lambda_n\sigma^2 T}{n}}, \tag{2.8}$$

where λ_n is a general bounded sequence.

We observe that Chang and Palmer model generalized CRR model, Jarrow and Rudd model and Walsh model by setting

$$\lambda_n = 0, \quad \lambda_n = \frac{r}{\sigma^2} - \frac{1}{2} \quad \text{and} \quad \lambda_n = \frac{r}{\sigma^2}, \quad \text{respectively.}$$

For CRR formula, Cox, Ross and Rubinstein showed that the binomial formula converges to the Black–Scholes formula. Their result is stated in Theorem 2.2.

Theorem 2.2. *Let* B_n *be defined in* (2.6) *with* p, u_B *and* d_B *are defined in* (2.5) *and* (2.7). *Then*

$$\lim_{n\to\infty} B_n = C_{BS}.$$

After that, Heston and Zhou ([12], 2000) showed that the rate of this convergence is $\dfrac{1}{\sqrt{n}}$ and Diener and Diener ([10], 2004) improved Heston and Zhou by adding an additional term in approximation B_n. Their result is stated in Theorem 2.3.

Theorem 2.3. *Let B_n be defined in* (2.6) *with p, u_B and d_B defined in* (2.5) *and* (2.7)*. Assume that $S_0 = 1$ and $T = 1$. Then, for large n, we have*

$$B_n = C_{BS} + \frac{e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi}} \frac{A - 12\sigma^2\left(\sigma_n^2 - 1\right)}{n} + O\left(\frac{1}{n\sqrt{n}}\right),$$

where $\sigma_n = 1 - 2\,\mathrm{frac}\left[\dfrac{\log(1/K) + n\log d}{\log(u/d)}\right]$ *and*
$A = -\sigma^2(6 + d_1^2 + d_2^2) + 4(d_1^2 - d_2^2)r - 12r^2$ *with* $\mathrm{frac}[x]$ *the fractional part of the real number x.*

Leisen and Reimer ([17], 1996) showed that the rate of convergence in Jarrow and Rudd formula is $\dfrac{1}{n}$. Their result is

$$B_n = C_{BS} + O\left(\frac{1}{n}\right).$$

In general, Chang and Palmer showed that B_n converges to C_{BS} at the rate $\dfrac{1}{n}$. Theorem 2.4 is their result.

Theorem 2.4. *Let B_n be defined in* (2.6) *with p, u_B and d_B defined in* (2.5) *and* (2.8)*. Then, for large n, we have*

$$B_n = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A_{2n} - 12\sigma^2 T\left(A_{1n}^2 - 1\right)}{n} + O\left(\frac{1}{n}\right),$$

where $\quad A_{1n} = 1 - 2\mathrm{frac}\left[\dfrac{\log(S_0/K) + n\log d}{\log(u/d)}\right]$

and $\quad A_{2n} = -\sigma^2 T\left(6 + d_1^2 + d_2^2\right) + 4T\left(d_1^2 - d_2^2\right)\left(r - \lambda_n\sigma^2\right) - 12T^2\left(r - \lambda_n\sigma^2\right)^2.$

In 2018, Ratibenyakool and Neammanee ([20]) improve Theorem 2.4 by giving the better rate of convergence which is showed in Theorem 2.5.

Theorem 2.5. *Let* u_B *and* d_B *be defined in* (2.8) *and* $0 < r, \sigma, T \leq 1$. *For large* n *such that*

$$n \geq \max\left\{100T, \frac{60}{\sigma^4}, \frac{1.2657\max\{d_1^2, d_2^2\}}{\sigma^4}, 30\max\{d_1^2, d_2^2\}\right\}$$

and $|r - \lambda_n \sigma^2| \leq 1$, *we have*

$$B_n = C_{BS} + \frac{S_0 e^{-d_1^2/2}}{24\sigma\sqrt{2\pi T}} \frac{A_{n2} - 12\sigma^2 T\left({A_{n1}}^2 - 1\right) + A_{n3}}{n} + \frac{C_0(d_1, d_2, \sigma)}{n\sqrt{n}},$$

where A_{n1} *and* A_{n2} *defined as in Theorem 2.4,*

$$\begin{aligned} A_{n3} = &-\sigma^2 T\left(2d_1^2 + 2d_2^2 - \sigma^2\sqrt{T}\right) + \left(24T^2 + 4T\sqrt{T}\right)\left(r - \lambda_n\sigma^2\right)^2 \\ &+ 4T\left(2\sigma d_1 + 4\sigma\sqrt{T}d_1 - \sigma^2\sqrt{T} + 4\sigma^2 T\right)\left(r - \lambda_n\sigma^2\right), \end{aligned}$$

$|C_0(d_1, d_2, \sigma)| \leq S_0 r(d_1) + K r(d_2)$ *and* $r(x) = \dfrac{1.7185|x|^3 + 19.3659}{\sigma^4} + 49.9851.$

2.3 The trinomial formula

The trinomial model is an extension of the binomial model. We know that the current stock price in the binomial formula either rises or falls at the end of each period. In the trinomial formula, we assume that for $k = 0, 1, 2, \ldots, n-1$, the current stock price S_k either rises to $S_k u_T$ with probability p_u, falls to $S_k d_T$ with probability p_d or steadies at S_k with probability $p_m = 1 - p_u - p_d$, where $0 < p_u, p_d, p_m < 1$ and $0 < d_T < 1 < u_T$. That is

$$S_{k+1} = \begin{cases} S_k u_T & \text{with probability } p_u \\ S_k & \text{with probability } p_m \\ S_k d_T & \text{with probability } p_d, \end{cases}$$

for $k = 0, 1, 2, \ldots, n-1$. Then

$$S_1 = \begin{cases} S_0 u_T & \text{with probability } p_u \\ S_0 & \text{with probability } p_m \\ S_0 d_T & \text{with probability } p_d \end{cases}$$

and

$$S_2 = \begin{cases} S_1 u_T & \text{with probability } p_u \\ S_1 & \text{with probability } p_m \\ S_1 d_T & \text{with probability } p_d \end{cases} = \begin{cases} S_0 u_T^2 & \text{with probability } p_u^2 \\ S_0 u_T & \text{with probability } 2p_u p_m \\ S_0 & \text{with probability } p_m^2 \\ S_0 u_T d_T & \text{with probability } 2p_u p_d \\ S_0 d_T & \text{with probability } 2p_m p_d \\ S_k d_T^2 & \text{with probability } p_d^2. \end{cases} \quad (2.9)$$

We note that if $u_T d_T \neq 1$, then the pattern is quite complicated as shown in the Figure 2.3.

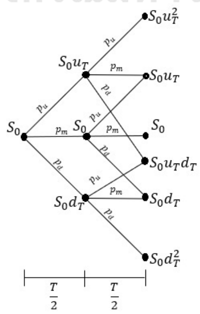


Figure 2.3: 2-step trinomial model in case of $u_T d_T \neq 1$

If we impose the condition $u_T d_T = 1$, then (2.9) can be simplified into this equation

$$S_2 = \begin{cases} S_0 u_T^2 & \text{with probability } p_u^2 \\ S_0 u_T & \text{with probability } 2p_u p_m \\ S_0 & \text{with probability } 2p_u p_d + p_m^2 \\ S_0 d_T & \text{with probability } 2p_m p_d \\ S_k d_T^2 & \text{with probability } p_d^2 \end{cases}$$

and the trinomial model in Figure 2.3 can be simplified into the trinomial model shown in Figure 2.4.

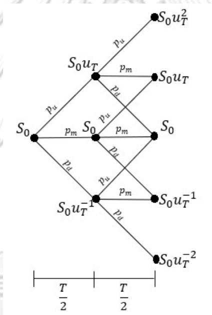


Figure 2.4: 2-step trinomial model in case of $u_T d_T = 1$

Under assumption $u_T d_T = 1$, we can show that

$$S_n = S_0 u_T^{j-l} \quad \text{with probability } \binom{n}{j, l, n-j-l} p_u^j p_d^l p_m^{n-j-l},$$

where

$$\binom{n}{j, l, n-j-l} = \frac{n!}{j!l!(n-j-l)!},$$

for $j, l = 0, 1, 2, \ldots, n$ and $j + l \leq n$.

Let $E\left[\max\left\{S_n - K, 0\right\}\right]$ represent an option price at the end of the n^{th} period and T_n be the current option price. Then,

$$\begin{aligned} T_n &= e^{-rT} E\left[\max\left\{S_n - K, 0\right\}\right] \\ &= e^{-rT} \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j, l, n-j-l} p_u^j p_d^l p_m^{n-j-l} \max\left\{S_0 u_T^{j-l} - K, 0\right\}. \end{aligned} \tag{2.10}$$

There are 2 approaches to define p_u, p_d and p_m. In 2007, Ahn and Song ([2]) considered a trinomial formula from the binomial formula with u_B and d_B defined in (2.7). From the 2-step binomial model as depicted in Figure 2.5.

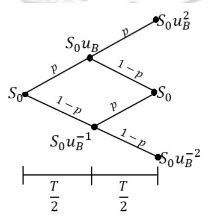


Figure 2.5: Binomial Model for $n = 2$

It is similar to the trinomial model with 1 period as depicted in Figure 2.6, where $p_u = p^2$, $p_d = (1-p)^2$ and $p_m = 2p(1-p)$ with $u_T = u_B^2$ and $d_T = d_B^2$.

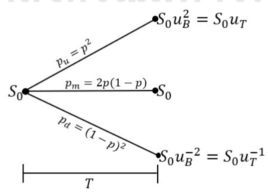


Figure 2.6: Trinomial Model for $n = 1$

They showed by example that the value T_n should be B_{2n} and converges to the Black–Scholes formula.

After that, Intarapanya and Neammanee ([14]) confirm the conjecture of Ahn and Song by giving the rigorous proof in 2019. The result is stated in Theorem 2.6.

Theorem 2.6. *Let T_n be defined in* (2.10) *with $u_T = u_B^2$, $p_u = p^2$ and $p_d = (1-p)^2$, where p and u_B are defined in* (2.5) *and* (2.7)*, respectively. Then*

$$T_n = B_{2n} \quad \text{and} \quad \lim_{n\to\infty} T_n = C_{BS},$$

where C_{BS} and B_n are defined in (2.2) *and* (2.6)*, respectively.*

Another approach is given by Boyle ([6], 1988). He assumed that the expected yield from all assets equals the risk-free rate of interest and the second moment from all assets equals the volatility. That is for $k = 0, 1, 2, \ldots, n-1$,

$$E[S_{k+1} \mid S_k] = S_k e^{\frac{rT}{n}} \tag{2.11}$$

and

$$E[S_{k+1}^2 \mid S_k] = S_k e^{\frac{\sigma^2 T}{n}}.$$

From these conditions, Boyle showed that

$$p_u = \frac{(V + M^2 - M)\, u_T - (M-1)}{(u_T - 1)\,(u_T^2 - 1)}, \tag{2.12}$$

$$p_d = \frac{(V + M^2 - M)\, u_T^2 - (M-1)\, u_T^3}{(u_T - 1)\,(u_T^2 - 1)}, \tag{2.13}$$

and

$$p_m = 1 - p_u - p_d,$$

where

$$M = e^{\frac{rT}{n}} \quad \text{and} \quad V = \left(e^{\frac{\sigma^2 T}{n}} - 1\right) M^2.$$

We observe that the formulas of p_u, p_d and p_m depend on u_T. Boyle gave an example to show that we can not use the rising rate $u_T = e^{\sigma\sqrt{\frac{T}{n}}}$ of Cox et al. If we let $\sigma = 0.2$, $r = 0.1$, $T = 1$ and $n = 20$, then $u_T = 1.045736$ which implies that $p_m = -0.0184 < 0$.

Boyle suggested to use

$$u_T = e^{\lambda\sigma\sqrt{\frac{T}{n}}} \tag{2.14}$$

where $\lambda > 1$. In this case, we show in Lemma 5.1 that the values of p_u, p_d and p_m are between 0 and 1.

After that, Entit et al. ([11], 2013) gave an example to show that an option price from Boyle's formula is closed to the price from Black–Scholes formula. Example 2.7 is an example of Entit et al.

Example 2.7. Let $S_0 = 100$, $K = 110$, $r = 0.05$, $\sigma = 0.3$ and $T = 1$. Table 2.2 presents the option prices from the trinomial formula of Boyle which is closed to the option prices from the Black–Scholes formula $C_{BS} = 10.0201$.

n	50	100	175	242
T_n	10.0274	10.0195	10.0263	10.0202

Table 2.1: Option price from Boyle's formula

From Example 2.7, Entit et al. gave the conjecture that the trinomial formula of Boyle should converge to the Black-Scholes formula. In our work, we give the rigorous proof of this conjecture.

CHAPTER III
BERRY-ESSEEN THEOREM FOR TRINOMIAL DISTRIBUTION

In the proof that the binomial formula B_n converges to the Black–Scholes formula C_{BS}, from (2.6), there are 2 terms of binomial probability, i.e.,

$$\sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j} \quad \text{and} \quad \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j}.$$

In order to prove Theorem 2.2, we need to show that that

$$\lim_{n\to\infty} \sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j} = \Phi(d_1) \quad \text{and} \quad \lim_{n\to\infty} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} = \Phi(d_2).$$

To do this, we need Berry-Esseen theorem for binomial distribution which is stated in Theorem 3.1.

Theorem 3.1 ([4], 1941)**.** *Let X be a binomial random variable with paramiter (n, q) and Z be the standard normal random variable. Then there exists a positive constant C such that for $a, b \in \mathbb{R}$,*

$$\left| P\left(a \le \frac{X - nq}{\sqrt{nq(1-q)}} \le b \right) - P\left(a \le Z \le b\right) \right| \le \frac{C}{\sqrt{n}}.$$

To prove that the trinomial formula T_n converges to the Black–Scholes formula C_{BS}, we also need the Berry-Esseen theorem for trinomial distribution. In this chapter, we will prove this theorem.

We will say that a random vector $(X_{n1}, X_{n2}, \ldots, X_{nk})$ has a multinomial distribution with parameters n and $(p_1, p_2, \ldots, p_k)$ such that $0 \le p_1, \ldots, p_k \le 1$ and

$\sum_{l=1}^{k} p_l = 1$ if the joint probability mass function of $X_{n1}, X_{n2}, \ldots, X_{nk}$ is

$$P(X_{n1} = x_1, \ldots, X_{nk} = x_k) = \begin{cases} \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} \cdots p_k^{x_k}, & \text{when} \quad \sum_{l=1}^{k} x_l = n \\ 0, & \text{ortherwise,} \end{cases}$$

where $x_1, x_2, \ldots, x_k$ are non-negative integers and $\binom{n}{x_1, x_2, \ldots, x_k} = \dfrac{n!}{x_1! x_2! \cdots x_k!}$. A random vector $(X_{n1}, X_{n2}, \ldots, X_{nk})$ is called a trinomial random vector if $k = 3$.

For a random vector $(X_1, X_2, \ldots, X_k)$ in $\mathbb{R}^k$, the characteristic function of $(X_1, X_2, \ldots, X_k)$ is defined by

$$E\left(e^{i(t_1, t_2, \ldots, t_k) \cdot (X_1, X_2, \ldots, X_k)}\right)$$

where $(t_1, t_2, \ldots, t_k) \in \mathbb{R}^k$ and $(t_1, t_2, \ldots, t_k) \cdot (X_1, X_2, \ldots, X_k) = \sum_{j=1}^{k} t_j X_j$.

It is showed that the characteristic function of the multinomial random vector $(X_{n1}, X_{n2}, \ldots, X_{nk})$ with parameters n and $(p_1, p_2, \ldots, p_k)$ is

$$\left(\sum_{l=1}^{k} p_l e^{it_l}\right)^n \tag{3.1}$$

for $(t_1, t_2, \ldots, t_k) \in \mathbb{R}^k$. ([24], p.82 for more details).

The covariance matrix for a random vector $(X_1, X_2, \ldots, X_k)$ is $\Sigma = [\sigma_{ij}]_{k \times k}$, where σ_{ij} is the covariance of X_i and X_j for $i, j = 1, 2, \ldots, k$.

We will give some properties of trinomial random vector in Lemma 3.2 and Proposition 3.3.

Lemma 3.2. *Let (X_{n1}, X_{n2}, X_{n3}) be a trinomial random vector with parameters n and (p_1, p_2, p_3). Then there exists a sequence of independent random vectors $Y_1, Y_2, \ldots, Y_n$ in $\mathbb{R}^3$ such that for each $j = 1, 2, \ldots, n$, $Y_j = (Y_{j1}, Y_{j2}, Y_{j3})$ and the random vectors $Y_1, Y_2, \ldots, Y_n$ satisfies the following conditions.*

1. $\sum_{j=1}^{n} Y_j \stackrel{d}{=} (X_{n1}, X_{n2}, X_{n3})$, *where* $X \stackrel{d}{=} Y$ *means that* X *and* Y *have the same distribution.*

2. *For each* $l = 1, 2, 3$, $\sum_{j=1}^{n} Y_{jl} \stackrel{d}{=} X_{nl}$.

3. *For each* $l = 1, 2, 3$, $Y_{1l}, Y_{2l}, \ldots, Y_{nl}$ *are independent Bernoulli random variables with parameter* p_l.

Proof. Let $Y_1, Y_2, \ldots, Y_n$ be independent random vectors in $\mathbb{R}^3$ such that for each $j = 1, 2, 3, \ldots, n$,

$$P(Y_j = (x_1, x_2, x_3)) = \begin{cases} p_1 & \text{if} \quad (x_1, x_2, x_3) = (1, 0, 0) \\ p_2 & \text{if} \quad (x_1, x_2, x_3) = (0, 1, 0) \\ p_3 & \text{if} \quad (x_1, x_2, x_3) = (0, 0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

1. For $j = 1, 2, \ldots, n$, let φ_j and φ be the characteristic functions of Y_j and $\sum_{j=1}^{n} Y_j$, respectively.

For $(t_1, t_2, t_3) \in \mathbb{R}^3$, we have

$$\begin{aligned} \varphi(t_1, t_2, t_3) &= \prod_{j=1}^{n} \varphi_j(t_1, t_2, t_3) \\ &= \prod_{j=1}^{n} E\left(e^{i(t_1, t_2, t_3)\cdot(Y_{j1}, Y_{j2}, Y_{j3})}\right) \\ &= \prod_{j=1}^{n} \Big(e^{i(t_1, t_2, t_3)\cdot(1,0,0)} P\left((Y_{j1}, Y_{j2}, Y_{j3}) = (1, 0, 0)\right) \\ &\quad + e^{i(t_1, t_2, t_3)\cdot(0,1,0)} P\left((Y_{j1}, Y_{j2}, Y_{j3}) = (0, 1, 0)\right) \\ &\quad + e^{i(t_1, t_2, t_3)\cdot(0,0,1)} P\left((Y_{j1}, Y_{j2}, Y_{j3}) = (0, 0, 1)\right)\Big) \\ &= \prod_{j=1}^{n} \left(\sum_{l=1}^{3} p_l e^{it_l}\right) \end{aligned}$$

$$= \left(\sum_{l=1}^{3} p_l e^{it_l} \right)^n .$$

From (3.1), we see that $\sum_{j=1}^{n} Y_j$ has a trinomial distibution with parameters n and (p_1, p_2, p_3), i.e, $\sum_{j=1}^{n} Y_j \overset{d}{=} (X_{n1}, X_{n2}, X_{n3})$.

2. Let $x \in \{0, 1, 2 \ldots, n\}$. Since

$$\sum_{j=1}^{n} Y_j \overset{d}{=} (X_{n1}, X_{n2}, X_{n3}),$$

we have

$$\begin{aligned}
P(X_{n1} \leq x) &= \sum_{y=0}^{x} P(X_{n1} = y) \\
&= \sum_{y=0}^{x} \sum_{z=0}^{n-y} P((X_{n1}, X_{n2}, X_{n3}) = (y, z, n-y-z)) \\
&= \sum_{y=0}^{x} \sum_{z=0}^{n-y} P\left(\sum_{j=1}^{n} Y_j = (y, z, n-y-z) \right) \\
&= \sum_{y=0}^{x} \sum_{z=0}^{n-y} P\left(\left(\sum_{j=1}^{n} Y_{j1}, \sum_{j=1}^{n} Y_{j2}, \sum_{j=1}^{n} Y_{j3} \right) = (y, z, n-y-z) \right) \\
&= \sum_{y=0}^{x} P\left(\sum_{j=1}^{n} Y_{j1} = y \right) \\
&= P\left(\sum_{j=1}^{n} Y_{j1} \leq x \right).
\end{aligned}$$

Then,

$$\sum_{j=1}^{n} Y_{j1} \overset{d}{=} X_{n1}.$$

Similarly, we can show that $\sum_{j=1}^{n} Y_{jl}$ and X_{nl} have the same distribution for $l = 2, 3$.

3. Let $x_1, x_2, \ldots, x_n \in \{0, 1, 2, \ldots, n\}$. Then,

$$P\left(Y_{11} \leq x_1, \ldots, Y_{n1} \leq x_n\right) = \sum_{y_1=0}^{x_1} \cdots \sum_{y_n=0}^{x_n} P\left(Y_{11} = y_1, \ldots, Y_{n1} = y_n\right) \tag{3.2}$$

Since $Y_1, Y_2, \ldots, Y_n$ are independent, we can show that

$$\begin{aligned}
&P\left(Y_{11} = y_1, Y_{21} = y_2, \ldots, Y_{n1} = y_n\right) \\
&= \sum_{z_1=0}^{n-y_1} \cdots \sum_{z_n=0}^{n-y_n} P\left(Y_1 = \left(y_1, z_1, n - y_1 - z_1\right), \ldots, Y_n = \left(y_n, z_n, n - y_n - z_n\right)\right) \\
&= \left(\sum_{z_1=0}^{n-y_1} P\left(Y_1 = \left(y_1, z_1, n - y_1 - z_1\right)\right)\right) \cdots \left(\sum_{z_n=0}^{n-y_n} P\left(Y_n = \left(y_n, z_n, n - y_n - z_n\right)\right)\right) \\
&= P\left(Y_{11} = y_1\right) P\left(Y_{21} = y_1\right) \cdots P\left(Y_{n1} = y_n\right).
\end{aligned}$$

From this fact and (3.2), we have

$$P\left(Y_{11} \leq x_1, \ldots, Y_{n1} \leq x_n\right) = \prod_{j=1}^{n} P\left(Y_{j1} \leq x_j\right).$$

Then, $Y_{11}, Y_{21}, \ldots, Y_{n1}$ are independent.
We see that

$$P\left(Y_{j1} = 1\right) = P\left(\left(Y_{j1}, Y_{j2}, Y_{j3}\right) = (1, 0, 0)\right) = p_1$$

and

$$\begin{aligned}
P\left(Y_{j1} = 0\right) &= P\left(\left(Y_{j1}, Y_{j2}, Y_{j3}\right) = (0, 1, 0)\right) + P\left(\left(Y_{j1}, Y_{j2}, Y_{j3}\right) = (0, 0, 1)\right) \\
&= p_2 + p_3 \\
&= 1 - p_1,
\end{aligned}$$

which implies that Y_{j1} is a Bernoulli independent random variable with parameter p_1, for all $j = 1, 2, \ldots, n$.
Similarly, for each $j = 1, 2, \ldots, n$, we can show that Y_{j2} and Y_{j3} are independent

Bernoulli random variables with parameter p_2 and p_3, respectively. $\square$

Proposition 3.3. *Let (X_{n1}, X_{n2}, X_{n3}) be a trinomial random vector with parameters n and (p_1, p_2, p_3). Then*

1. *$Var(X_{nl}) = np_l(1-p_l)$ for $l = 1, 2, 3$ and*

2. *the correlation between X_{n1} and X_{n2} is $\rho_n = -\sqrt{\dfrac{p_1p_2}{(1-p_1)(1-p_2)}}$.*

Proof. From Lemma 3.2, there exists a sequence of independent random vectors $\{Y_j = (Y_{j1}, Y_{j2}, Y_{j3})\}_{j\in\{1,2,\ldots,n\}}$ in $\mathbb{R}^3$ such that $\sum_{j=1}^{n} Y_j \stackrel{d}{=} (X_{n1}, X_{n2}, X_{n3})$, $\sum_{j=1}^{n} Y_{jl} \stackrel{d}{=} X_{nl}$ and $Y_{1l}, Y_{2l}, \ldots, Y_{nl}$ are independent Bernoulli random variables with parameter p_l, for $l = 1, 2, 3$.

1. Let $l \in \{1, 2, 3\}$.

We see that $\sum_{j=1}^{n} Y_{jl}$ is a binomial random variable with parameter (n, p_l).

Since $\sum_{j=1}^{n} Y_{jl} \stackrel{d}{=} X_{nl}$, for all $l = 1, 2, 3$, we have

$$\mathrm{Var}\left(X_{nl}\right) = \mathrm{Var}\left(\sum_{j=1}^{n} Y_{jl}\right) = np_l(1-p_l).$$

2. Let $j \in \{1, 2, \ldots, n\}$. Since $P\left(Y_{j1} = 1, Y_{j2} = 1\right) = 0$, we have

$$EY_{j1}Y_{j2} = \sum_{y_1,y_2\in\{0,1\}} y_1y_2P\left(Y_{j1} = y_1, Y_{j2} = y_2\right) = 0. \tag{3.3}$$

Let $j, l \in \{1, 2, \ldots, n\}$ such that $j \neq l$ and $x_1, x_2 \in \{0, 1\}$.

Since Y_j and Y_l are independent,

$$\begin{aligned}
&P\left(Y_{j1} = x_1, Y_{l2} = x_2\right) \\
&= \sum_{y_1=0}^{1-x_1}\sum_{y_2=0}^{1-x_2} P\left(Y_j = (x_1, y_1, 1-x_1-y_1), Y_l = (y_2, x_2, 1-x_2-y_2)\right) \\
&= \sum_{y_1=0}^{1-x_1}\sum_{y_2=0}^{1-x_2} P\left(Y_j = (x_1, y_1, 1-x_1-y_1)\right) P\left(Y_l = (y_2, x_2, 1-x_2-y_2)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{y_1=0}^{1-x_1} P\left(Y_j = (x_1, y_1, 1-x_1-y_1)\right)\right)\left(\sum_{y_2=0}^{1-x_2} P\left(Y_l = (y_2, x_2, 1-x_2-y_2)\right)\right) \\
&= P\left(Y_{j1} = x_1\right) P\left(Y_{l2} = x_2\right)
\end{aligned}
$$

which implies that Y_{j1} and Y_{l2} are independent.

Then,

$$
EY_{j1}Y_{l2} = EY_{j1}EY_{l2} = p_1p_2.
$$

From this fact and (3.3), we obtain

$$
\begin{aligned}
\mathrm{Cov}\left(X_{n1}, X_{n2}\right) &= \mathrm{Cov}\left(\sum_{j=1}^{n} Y_{j1}, \sum_{l=1}^{n} Y_{l2}\right) \\
&= E\left(\sum_{j=1}^{n} Y_{j1} - np_1\right)\left(\sum_{l=1}^{n} Y_{l2} - np_2\right) \\
&= E\left(\left(\sum_{j=1}^{n} Y_{j1}\right)\left(\sum_{l=1}^{n} Y_{l2}\right) - np_1\sum_{l=1}^{n} Y_{l2} - np_2\sum_{j=1}^{n} Y_{j1} + n^2p_1p_2\right) \\
&= \sum_{j=1}^{n}\sum_{l=1}^{n} EY_{j1}Y_{l2} - np_1\sum_{l=1}^{n} EY_{l2} - np_2\sum_{j=1}^{n} EY_{j1} + n^2p_1p_2 \\
&= \sum_{j=1}^{n} EY_{j1}Y_{j2} + \sum_{j=1}^{n}\sum_{\substack{l=1 \\ l\neq j}}^{n} EY_{j1}Y_{l2} - n^2p_1p_2 - n^2p_1p_2 + n^2p_1p_2 \\
&= n(n-1)p_1p_2 - n^2p_1p_2 \\
&= -np_1p_2.
\end{aligned}
$$

Hence

$$
\rho_n = \frac{\mathrm{Cov}\left(X_{n1}, X_{n2}\right)}{\sqrt{\mathrm{Var}\left(X_{n1}\right)\mathrm{Var}\left(X_{n2}\right)}} = -\sqrt{\frac{p_1p_2}{(1-p_1)(1-p_2)}}. \qquad \square
$$

A random vector $(Z_1, Z_2, \ldots, Z_k)$ in $\mathbb{R}^k$ is the multivariate normal random vector in $\mathbb{R}^k$ with mean vector μ in $\mathbb{R}^k$ and covariance matrix $\Sigma = [\sigma_{ij}]_{k\times k}$ if its probability density function is defined by

$$f(x) = \frac{1}{\left(\sqrt{2\pi}\right)^k \sqrt{\det(\Sigma)}} e^{-\frac{1}{2\det(\Sigma)}\left((x-\mu)\Sigma^{-1}(x-\mu)^T\right)},$$

for all $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$. In case of $\mu = \mathbf{0}$ and $\Sigma = \mathbf{I}_k$, where $\mathbf{0}$ is the zero vector and $\mathbf{I}_k$ is the identity matrix, we say that $(Z_1, Z_2, \ldots, Z_k)$ is the multivariate standard normal random vector. For a special case $k = 2$, (Z_1, Z_2) is said to be a bivariate standard normal random vector and the probability density function f of (Z_1, Z_2) is defined by $f(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}}$, for $(x_1, x_2) \in \mathbb{R}^2$.

We know that the binomial distribution converges to the normal distribution. We can find the rate of this convergence by the Berry-Esseen Theorem for the binomial distribution which is stated in Theorem 3.1 (see Korolev and Shevtsova ([16], 2010), Shevtsova ([26], 2011), Shevtsova ([25], 2013), Schulz ([23], 2016), and Zolotukhin, Nagaev and Chebotarev ([31], 2018) for more details). Moreover, some authors improved the rate of this convergence from $O\left(\frac{1}{\sqrt{n}}\right)$ to $O\left(\frac{1}{n}\right)$. For examples, see Uspensky ([29], 1937), Neammanee ([18], 2005). and Ratibenyakool and Neammanee ([21], 2017).

In order to prove the Berry-Esseen theorem for trinomial distribution in Theorem 3.7, we need following theorems.

Theorem 3.4 ([19], 2018)**.** *Let $W_1, W_2, \ldots, W_n$ be a sequence of independent random vectors in $\mathbb{R}^k$ and $(Z_1, Z_2, \ldots, Z_k)$ be the multivariate standard normal random vector. Assume that $EW_j = \mathbf{0}$ for all $j = 1, 2, \ldots, n$ and covariance matrix of $\sum_{j=1}^{n} W_j$ is $\mathbf{I}_k$. If A is a convex set in $\mathbb{R}^k$, then*

$$P\left(\sum_{j=1}^{n} W_j \in A\right) = P\left((Z_1, Z_2, \ldots, Z_k) \in A\right) + \Delta_n,$$

where $|\Delta_n| \le \left(42\sqrt[4]{k} + 16\right) \sum_{j=1}^{n} E\,||W_j||^3$ and $||\cdot||$ is the Euclidean norm.

Theorem 3.5. *Let (X_{n1}, X_{n2}, X_{n3}) be a trinomial random vector with parameters n and (p_1, p_2, p_3) and (Z_1, Z_2) be a bivariate standard normal random vector. For a convex subset A in $\mathbb{R}^3$, we define*

$$A^* = \{(x_1^*, x_2^*) \mid (x_1, x_2, n - x_1 - x_2) \in A\}, \tag{3.4}$$

where

$$x_1^* = \frac{x_1 - np_1}{\sqrt{2np_1(1-p_1)(1+\rho_n)}} + \frac{x_2 - np_2}{\sqrt{2np_2(1-p_2)(1+\rho_n)}}, \tag{3.5}$$

$$x_2^* = \frac{x_1 - np_1}{\sqrt{2np_1(1-p_1)(1-\rho_n)}} - \frac{x_2 - np_2}{\sqrt{2np_2(1-p_2)(1-\rho_n)}}, \tag{3.6}$$

ρ_n is the correlation between X_{n1} and X_{n2}. Then,

$$P((X_{n1}, X_{n2}, X_{n3}) \in A) = P((Z_1, Z_2) \in A^*) + \Delta_n(p_1, p_2),$$

where

$$|\Delta_n(p_1, p_2)| \leq \frac{\left(42\sqrt[4]{2} + 16\right)}{\sqrt{(1+\rho_n)^3}} \left(\frac{1}{\sqrt{np_1(1-p_1)}} + \frac{1}{\sqrt{np_2(1-p_2)}} \right). \tag{3.7}$$

Proof. Let $Y_1, Y_2, \ldots, Y_n$ be defined in Lemma 3.2.
For each $j = 1, 2, \ldots, n$, we define random variables W_{j1} and W_{j2} by

$$W_{j1} = \frac{Y_{j1} - p_1}{\sqrt{2np_1(1-p_1)(1+\rho_n)}} + \frac{Y_{j2} - p_2}{np_2(1-p_2)\sqrt{2(1+\rho_n)}}, \tag{3.8}$$

$$W_{j2} = \frac{Y_{j1} - p_1}{\sqrt{2np_1(1-p_1)(1-\rho_n)}} - \frac{Y_{j2} - p_2}{np_2(1-p_2)\sqrt{2(1-\rho_n)}} \tag{3.9}$$

and define a random vector W_j by

$$W_j = (W_{j1}, W_{j2}). \tag{3.10}$$

We see that $W_1, W_2, \ldots, W_n$ are independent and $EW_{j1} = EW_{j2} = 0$, for all

$j = 1, 2, \ldots, n$. Thus,

$$E\sum_{j=1}^{n} W_j = \left(E\sum_{j=1}^{n} W_{j1}, E\sum_{j=1}^{n} W_{j2}\right) = \left(\sum_{j=1}^{n} EW_{j1}, \sum_{j=1}^{n} EW_{j2}\right) = (0,0)\,. \quad (3.11)$$

Let $B = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2, n - x_1 - x_2) \in A\}$. Then,

$$A^* = \{(x_1^*, x_2^*) \mid (x_1, x_2) \in B\}\,.$$

We divide the proof into 3 steps as follows.

Step 1. We will show that

$$P\left((X_{n1}, X_{n2}, X_{n3}) \in A\right) = P\left(\sum_{j=1}^{n} W_j \in A^*\right).$$

We note that $P\left((X_{n1}, X_{n2}, X_{n3}) \in A\right) = P\left((X_{n1}, X_{n2}) \in B\right)$ and for $(x_1, x_2) \in B$, (x_1^*, x_2^*) which are defined by (3.5) and (3.6) is unique.
Hence, it is sufficient to prove that for all $(x_1, x_2) \in B$,

$$P\left((X_{n1}, X_{n2}) = (x_1, x_2)\right) = P\left(\sum_{j=1}^{n} W_j = (x_1^*, x_2^*)\right).$$

Let $(x_1, x_2) \in B$.
Since $\sum_{j=1}^{n} Y_j = \left(\sum_{j=1}^{n} Y_{j1}, \sum_{j=1}^{n} Y_{j2} \sum_{j=1}^{n} Y_{j3}\right) \stackrel{d}{=} (X_{n1}, X_{n2}, X_{n3})$, we have

$$\begin{aligned}
&P\left((X_{n1}, X_{n2}) = (x_1, x_2)\right) \\
&= P\left(\sum_{j=1}^{n} Y_{j1} = x_1,\ \sum_{j=1}^{n} Y_{j2} = x_2\right) \\
&= P\left(\frac{\sum_{j=1}^{n} Y_{j1} - np_1}{\sqrt{np_1(1-p_1)}} = \frac{x_1 - np_1}{\sqrt{np_1(1-p_1)}}, \frac{\sum_{j=1}^{n} Y_{j2} - np_2}{\sqrt{np_2(1-p_2)}} = \frac{x_2 - np_2}{\sqrt{np_2(1-p_2)}}\right).
\end{aligned} \quad (3.12)$$

We note that for random variables X and Y,

$$X = x \quad \text{and} \quad Y = y \quad \text{if and only if} \quad X + Y = x + y \quad \text{and} \quad X - Y = x - y.$$

Then,

$$P\left(X = x,\ Y = y\right) = P\left(X + Y = x + y \ \text{ and } \ X - Y = x - y\right).$$

From this fact and (3.12), we have

$$\begin{aligned}
&P\left((X_{n1}, X_{n2}) = (x_1, x_2)\right) \\
&= P\Bigg(\sqrt{2\left(1+\rho_n\right)}\sum_{j=1}^{n} W_{j1} = \frac{x_1 - np_1}{\sqrt{np_1(1-p_1)}} + \frac{x_2 - np_2}{\sqrt{np_2(1-p_2)}}, \\
&\qquad \sqrt{2\left(1-\rho_n\right)}\sum_{j=1}^{n} W_{j2} = \frac{x_1 - np_1}{\sqrt{np_1(1-p_1)}} - \frac{x_2 - np_2}{\sqrt{np_2(1-p_2)}}\Bigg) \\
&= P\left(\left(\sum_{j=1}^{n} W_{j1}, \sum_{j=1}^{n} W_{j2}\right) = (x_1^*, x_2^*)\right) \\
&= P\left(\sum_{j=1}^{n} W_j = (x_1^*, x_2^*)\right).
\end{aligned}$$

Thus,

$$P\left((X_{n1}, X_{n2}, X_{n3}) \in A\right) = P\left((X_{n1}, X_{n2}) \in B\right) = P\left(\sum_{j=1}^{n} W_j \in A^*\right).$$

Step 2. we will show that A^* is a convex set.

That is for all $\mathbf{x}^* = (x_1^*, x_2^*)$, $\mathbf{y}^* = (y_1^*, y_2^*) \in A^*$,

$$[\mathbf{x}^*, \mathbf{y}^*] = \{t\mathbf{x}^* + (1-t)\mathbf{y}^* \mid 0 < t < 1\} \subseteq A^*.$$

Let $\mathbf{x}^* = (x_1^*, x_2^*)$, $\mathbf{y}^* = (y_1^*, y_2^*) \in A^*$.

Then there exist $(x_1, x_2, n - x_1 - x_2)$ and $(y_1, y_2, n - y_1 - y_2)$ in A such that

$$x_1^* = \frac{x_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} + \frac{x_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}},$$
$$x_2^* = \frac{x_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1-\rho_n)}} - \frac{x_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1-\rho_n)}},$$
$$y_1^* = \frac{y_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} + \frac{y_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}}$$

and
$$y_2^* = \frac{y_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1-\rho_n)}} - \frac{y_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1-\rho_n)}}.$$

Let $0 < t < 1$. Then $t\mathbf{x}^* + (1-t)\mathbf{y}^* = (\tilde{u}_1, \tilde{u}_2)$, where

$$\begin{aligned}\tilde{u}_1 &= t\left(\frac{x_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} + \frac{x_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}}\right)\\ &\quad + (1-t)\left(\frac{y_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} + \frac{y_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}}\right)\\ &= \frac{tx_1 + (1-t)y_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} + \frac{tx_2 + (1-t)y_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}}\end{aligned}$$

and

$$\begin{aligned}\tilde{u}_2 &= t\left(\frac{x_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} - \frac{x_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}}\right)\\ &\quad + (1-t)\left(\frac{y_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} - \frac{y_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}}\right)\\ &= \frac{tx_1 + (1-t)y_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} - \frac{tx_2 + (1-t)y_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}}.\end{aligned}$$

Let $u_1 = tx_1 + (1-t)y_1$ and $u_2 = x_2 + (1-t)y_2$. Then

$$n - u_1 - u_2 = t(n - x_1 + x_2) + (1-t)(n - y_1 + y_2).$$

Since A is a convex set, we have

$$(u_1, u_2, n - u_1 - u_2) \in A.$$

Thus,

$$u_1^* = \frac{u_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} + \frac{u_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}} = \tilde{u}_1$$

and
$$u_2^* = \frac{u_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} - \frac{u_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}} = \tilde{u}_2.$$

Hence, $t\mathbf{x}^* + (1-t)\mathbf{y}^* = (u_1^*, u_2^*) \in A^*$ which implies that A^* is a convex set.

Step 3. We will show that

$$P\left(\sum_{j=1}^{n} W_j \in A^*\right) = P\left((Z_1, Z_2) \in A^*\right) + \Delta_n(p_1, p_2),$$

where $\Delta_n(p_1, p_2)$ is defined in (3.7).

To apply Theorem 3.4 for $\sum_{j=1}^{n} W_j$, where W_j is defined in (3.10), we have to show that the covariance matrix of $\sum_{j=1}^{n} W_j$ is $\mathbf{I}_2$.

For each $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
EW_{j1}^2 &= \frac{1}{2\,(1+\rho_n)} E\left(\frac{Y_{j1} - p_1}{\sqrt{np_1(1-p_1)}} + \frac{Y_{j2} - p_2}{\sqrt{np_2(1-p_2)}}\right)^2 \\
&= \frac{1}{2(1+\rho_n)} E\left(\frac{(Y_{j1} - p_1)^2}{np_1(1-p_1)} + \frac{2(Y_{j1} - p_1)(Y_{j2} - p_2)}{n\sqrt{p_1 p_2 (1-p_1)(1-p_2)}} + \frac{(Y_{j2} - p_2)^2}{np_1(1-p_1)}\right) \\
&= \frac{1}{2(1+\rho_n)} \left(\frac{1}{n} + \frac{2\rho_n}{n} + \frac{1}{n}\right) \\
&= \frac{1}{n}.
\end{aligned}$$

Then,

$$\mathrm{Var}\left(\sum_{j=1}^{n} W_{j1}\right) = \sum_{j=1}^{n} \mathrm{Var}(W_{j1}) = \sum_{j=1}^{n} EW_{j1}^2 = 1.$$

Similarly, we can show that $\mathrm{Var}\left(\sum_{j=1}^{n} W_{j2}\right) = 1$.

Let $j, l \in \{0, 1, 2, \ldots, n\}$. If $j = l$, then

$$
\begin{aligned}
&E\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}+\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)\left(\frac{Y_{l1}-p_1}{\sqrt{np_1(1-p_1)}}-\frac{Y_{l2}-p_2}{\sqrt{np_2(1-p_2)}}\right)\\
&=E\left(\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}\right)^2-\left(\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)^2\right)\\
&=\frac{1}{n}-\frac{1}{n}\\
&=0.
\end{aligned}
$$

Suppose that $j \neq l$. Then,

$$
\begin{aligned}
&E\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}+\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)\left(\frac{Y_{l1}-p_1}{\sqrt{np_1(1-p_1)}}-\frac{Y_{l2}-p_2}{\sqrt{np_2(1-p_2)}}\right)\\
&=E\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}+\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)E\left(\frac{Y_{l1}-p_1}{\sqrt{np_1(1-p_1)}}-\frac{Y_{l2}-p_2}{\sqrt{np_2(1-p_2)}}\right)\\
&=0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&EW_{j1}W_{l2}\\
&=\frac{E\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}+\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)\left(\frac{Y_{l1}-p_1}{\sqrt{np_1(1-p_1)}}-\frac{Y_{l2}-p_2}{\sqrt{np_2(1-p_2)}}\right)}{2\sqrt{1-\rho_n^2}}\\
&=0
\end{aligned}
$$

for all $j, l = 1, 2, \ldots, n$. This implies that

$$
\operatorname{Cov}\left(\sum_{j=1}^{n} W_{j1}, \sum_{l=1}^{n} W_{l2}\right) = 0.
$$

Then, the covariance matrix of $\sum_{j=1}^{n} W_j$ is $\mathbf{I}_2$.

From this fact, Step 2 and (3.11), we can apply Theorem 3.4 for $\sum_{j=1}^{n} W_j$. Then,

$$P\left(\sum_{j=1}^{n} W_j \in A^*\right) = P\left((Z_1, Z_2) \in A^*\right) + \Delta_n(p_1, p_2),$$

where $|\Delta_n(p_1, p_2)| \leq \left(42\sqrt[4]{2} + 16\right) \sum_{j=1}^{n} E\left|\left|(W_{j1}, W_{j2})\right|\right|^3$.

To show (3.7), we have to show that

$$\sum_{j=1}^{n} E\left|\left|(W_{j1}, W_{j2})\right|\right|^3 \leq \frac{1}{\sqrt{(1+\rho_n)^3}}\left(\frac{1}{\sqrt{np_1(1-p_1)}} + \frac{1}{\sqrt{np_2(1-p_2)}}\right).$$

For $j = 1, 2, \ldots, n$, we observe that

$$\begin{aligned} E\left|\frac{Y_{j1} - p_1}{\sqrt{np_1(1-p_1)}}\right|^3 &= \frac{1}{\sqrt{n^3 p_1^3(1-p_1^3)}}\left((1-p_1)^3 p_1 + p_1^3(1-p_1)\right) \\ &= \frac{(1-p_1)^2 + p_1^2}{n\sqrt{np_1(1-p_1)}} \\ &\leq \frac{1}{n\sqrt{np_1(1-p_1)}}. \end{aligned}$$

Similarly, we can show that

$$E\left|\frac{Y_{j2} - p_2}{\sqrt{np_2(1-p_2)}}\right|^3 \leq \frac{1}{n\sqrt{np_2(1-p_2)}}$$

for all $j = 1, 2, \ldots, n$.

From this fact, (3.8), (3.9) and Proposition 3.3 (2), we obtain $-1 < \rho_n < 0$ and

$$\begin{aligned} &E\left|\left|(W_{j1}, W_{j2})\right|\right|^3 \\ &= E\left(W_{j1}^2 + W_{j2}^2\right)^{\frac{3}{2}} \\ &= E\left[\frac{1}{2(1+\rho_n)}\left(\frac{Y_{j1} - p_1}{\sqrt{np_1(1-p_1)}} + \frac{Y_{j2} - p_2}{\sqrt{np_2(1-p_2)}}\right)^2\right. \end{aligned}$$

$$+\frac{1}{2(1-\rho_n)}\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}-\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)^2\Bigg]^{\frac{3}{2}}$$

$$\leq \frac{1}{\sqrt{8(1+\rho_n)^3}}E\Bigg[\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}+\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)^2$$

$$+\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}-\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)^2\Bigg]^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{8(1+\rho_n)^3}}E\left[\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}\right)^2+\left(\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)^2\right]^{\frac{3}{2}}$$

$$\leq \frac{1}{\sqrt{8(1+\rho_n)^3}}E\left[2\max\left\{\left(\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}\right)^2, \left(\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right)^2\right\}\right]^{\frac{3}{2}}$$

$$\leq \frac{1}{\sqrt{(1+\rho_n)^3}}\max\left\{E\left|\frac{Y_{j1}-p_1}{\sqrt{np_1(1-p_1)}}\right|^3, E\left|\frac{Y_{j2}-p_2}{\sqrt{np_2(1-p_2)}}\right|^3\right\}$$

$$\leq \frac{1}{n\sqrt{(1+\rho_n)^3}}\left(\frac{1}{\sqrt{np_1(1-p_1)}}+\frac{1}{\sqrt{np_2(1-p_2)}}\right)$$

for all $j = 1, 2, \ldots, n$.

Hence,

$$\sum_{j=1}^{n} E\,||(W_{j1}, W_{j2})||^3 \leq \sum_{j=1}^{n}\frac{1}{n\sqrt{(1+\rho_n)^3}}\left(\frac{1}{\sqrt{np_1(1-p_1)}}+\frac{1}{\sqrt{np_2(1-p_2)}}\right)$$

$$\leq \frac{1}{\sqrt{(1+\rho_n)^3}}\left(\frac{1}{\sqrt{np_1(1-p_1)}}+\frac{1}{\sqrt{np_2(1-p_2)}}\right).$$

From Step 1 and Step 3, we have

$$P\left((X_{n1}, X_{n2}, X_{n3}) \in A\right) = P\left((Z_1, Z_2) \in A^*\right) + \Delta_n(p_1, p_2),$$

where $\Delta_n(p_1, p_2)$ is defined in (3.7). □

Proposition 3.6. *For* $0 \le a_n \le n$, *let*

$$A_n = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2 \ge 0, \;\; x_1 + x_2 \le n \;\; and \;\; x_1 - x_2 \ge a_n\right\} \tag{3.13}$$

and A^* *be defined in* (3.4). *Then,*

$$P\left((Z_1, Z_2) \in A_n^*\right) = \frac{1}{2\pi} \int_{b_{n1}(p_2)}^{b_{n2}(p_2, a_n)} \int_{b_n(p_1,p_2,a_n)+c_n(p_1,p_2)u_2}^{e_n(p_1,p_2)-c_n(p_1,p_2)u_2} e^{-\frac{u_1^2+u_2^2}{2}} du_1 du_2,$$

where

$$b_{n1}(p_2) = \frac{-np_2}{\sqrt{np_2(1-p_2)}}, \tag{3.14}$$

$$b_{n2}(p_2, a_n) = \frac{n - a_n - 2np_2}{2\sqrt{np_2(1-p_2)}}, \tag{3.15}$$

$$b_n(p_1, p_2, a_n) = \frac{a_n - np_1 + np_2}{\sqrt{np_1(1-p_1)(1-\rho_n^2)}}, \tag{3.16}$$

$$c_n(p_1, p_2) = \frac{\sqrt{np_2(1-p_2)} - \rho_n\sqrt{np_1(1-p_1)}}{\sqrt{np_1(1-p_1)(1-\rho_n^2)}} \tag{3.17}$$

and

$$e_n(p_1, p_2) = \frac{n - np_1 - np_2}{\sqrt{np_1(1-p_1)(1-\rho_n^2)}}. \tag{3.18}$$

Proof. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$g(u_1, u_2) = \left(g_1(u_1, u_2), g_2(u_1, u_2)\right),$$

where

$$g_1(u_1, u_2) = \frac{u_1 + u_2}{\sqrt{2(1+\rho_n)}} \quad \text{and} \quad g_2(u_1, u_2) = \frac{u_1 - u_2}{\sqrt{2(1-\rho_n)}}.$$

Then,

$$P\left((Z_1, Z_2) \in A_n^*\right) = \int\int_{A_n^*} f(x_1^*, x_2^*) dx_1^* dx_2^* = \int\int_{g^{-1}(A_n^*)} f\left(g\left(u_1, u_2\right)\right) |D| \, du_1 du_2, \tag{3.19}$$

where $f(x_1^*, x_2^*) = \dfrac{1}{2\pi} e^{-\frac{x_1^{*2}+x_2^{*2}}{2}}$ is the probability density function of bivariate standard normal random vector (Z_1, Z_2) and

$$D = \det \begin{bmatrix} \dfrac{\partial g_1(u_1,u_2)}{\partial u_1} & \dfrac{\partial g_1(u_1,u_2)}{\partial u_2} \\ \dfrac{\partial g_2(u_1,u_2)}{\partial u_1} & \dfrac{\partial g_2(u_1,u_2)}{\partial u_2} \end{bmatrix} = \begin{vmatrix} \dfrac{1}{\sqrt{2(1+\rho_n)}} & \dfrac{1}{\sqrt{2(1+\rho_n)}} \\ \dfrac{1}{\sqrt{2(1-\rho_n)}} & -\dfrac{1}{\sqrt{2(1-\rho_n)}} \end{vmatrix} = -\frac{1}{\sqrt{1-\rho_n^2}}$$

(see [22] pp.153-154 for more details).

From (3.19) and the fact that

$$\begin{aligned}
(g_1(u_1,u_2))^2 + (g_2(u_1,u_2))^2 &= \left(\frac{u_1+u_2}{\sqrt{2(1+\rho_n)}}\right)^2 + \left(\frac{u_1-u_2}{\sqrt{2(1-\rho_n)}}\right)^2 \\
&= \frac{u_1^2+2u_1u_2+u_2^2}{2(1+\rho_n)} + \frac{u_1^2-2u_1u_2+u_2^2}{2(1-\rho_n)} \\
&= \frac{(u_1^2+2u_1u_2+u_2^2)(1-\rho_n)+(u_1^2-2u_1u_2+u_2^2)(1+\rho_n)}{2(1+\rho_n)(1-\rho_n)} \\
&= \frac{u_1^2-2\rho_n u_1u_2+u_2^2}{1-\rho_n^2} \\
&= \frac{(u_1-\rho_n u_2)^2+(1-\rho_n^2)u_2^2}{1-\rho_n^2},
\end{aligned}$$

we have

$$\begin{aligned}
P((Z_1,Z_2)\in A_n^*) &= \int\int_{g^{-1}(A_n^*)} f(g(u_1,u_2))\,|D|\,du_1du_2 \\
&= \frac{1}{2\pi\sqrt{1-\rho_n^2}}\int\int_{g^{-1}(A_n^*)} e^{-\frac{(g_1(u_1,u_2))^2+(g_2(u_1,u_2))^2}{2}}du_1du_2 \\
&= \frac{1}{2\pi\sqrt{1-\rho_n^2}}\int\int_{g^{-1}(A_n^*)} e^{-\frac{(u_1-\rho_n u_2)^2+\left(1-\rho_n^2\right)u_2^2}{2(1-\rho_n^2)}}du_1du_2. \qquad (3.20)
\end{aligned}$$

We know that $(x_1^*, x_2^*) \in A_n^*$ if and only if there exists a unique $(x_1, x_2, n-x_1-x_2) \in A_n$ such that

$$\begin{aligned}
x_1^* &= \frac{x_1-np_1}{\sqrt{2np_1(1-p_1)(1+\rho_n)}} + \frac{x_2-np_2}{\sqrt{2np_2(1-p_2)(1+\rho_n)}} \\
&= g_1\left(\frac{x_1-np_1}{\sqrt{np_1(1-p_1)}}, \frac{x_2-np_2}{\sqrt{np_2(1-p_2)}}\right)
\end{aligned}$$

and

$$\begin{aligned} x_2^* &= \frac{x_1 - np_1}{\sqrt{2np_1(1-p_1)\,(1+\rho_n)}} - \frac{x_2 - np_2}{\sqrt{2np_2(1-p_2)\,(1+\rho_n)}} \\ &= g_2\left(\frac{x_1 - np_1}{\sqrt{np_1(1-p_1)}}, \frac{x_2 - np_2}{\sqrt{np_2(1-p_2)}}\right). \end{aligned}$$

That is

$$g\left(\frac{x_1 - np_1}{\sqrt{np_1(1-p_1)}}, \frac{x_2 - np_2}{\sqrt{np_2(1-p_2)}}\right) = (x_1^*, x_2^*)\,.$$

Then,

$$g^{-1}\left(A_n^*\right) = \left\{\left(\frac{x_1 - np_1}{\sqrt{np_1(1-p_1)}}, \frac{x_2 - np_2}{\sqrt{np_2(1-p_2)}}\right) \mid (x_1, x_2, n - x_1 - x_2) \in A_n\right\}.$$

We know that for $(x_1, x_2, x_3) \in A_n$, $x_1, x_2 \geq 0$, $x_1 + x_2 \leq n$ and $x_2 - x_1 \leq -a_n$. Then

$$0 \leq x_2 \leq \frac{n - a_n}{2} \quad \text{and} \quad x_2 + a_n \leq x_1 \leq n - x_2.$$

We can show that if $0 \leq x_2 \leq \frac{n - a_n}{2}$ and $x_2 + a_n \leq x_1 \leq n - x_2$, then $x_1, x_2 \geq 0$, $x_1 + x_2 \leq n$ and $x_2 - x_1 \leq -a_n$.

Thus,

$$A_n = \left\{(x_1, x_2, n - x_1 - x_2) \in \mathbb{R}^3 \mid 0 \leq x_2 \leq \frac{n - a_n}{2}, \text{ and } a_n + x_2 \leq x_1 \leq n - x_2\right\}.$$

Note that for $(u_1, u_2) = \left(\frac{x_1 - np_1}{\sqrt{np_1(1-p_1)}}, \frac{x_2 - np_2}{\sqrt{np_2(1-p_2)}}\right) \in g^{-1}\left(A_n^*\right)$, we have

$$0 \leq x_2 \leq \frac{n - a_n}{2} \quad \text{and} \quad a_n + x_2 \leq x_1 \leq n - x_2.$$

These imply that

$$\frac{-np_2}{\sqrt{np_2(1-p_2)}} \le u_2 \le \frac{n-a_n-2np_2}{2\sqrt{np_2(1-p_2)}}, \text{ i.e., } b_{n1}(p_2) \le u_2 \le b_{n2}(p_2,a_n)$$

and

$$\frac{a_n - np_1 + np_2 + u_2\sqrt{np_2(1-p_2)}}{\sqrt{np_1(1-p_1)}} \le u_1 \le \frac{n - np_1 - np_2 - u_2\sqrt{np_2(1-p_2)}}{\sqrt{np_1(1-p_1)}}.$$

From this fact and (3.20), we obtain

$$\begin{aligned} &P\left((Z_1,Z_2)\in A_n^*\right) \\ &= \frac{1}{2\pi\sqrt{1-\rho_n^2}} \int_{b_{n1}}^{b_{n2}} \int_{\frac{a_n-np_1+np_2+u_2\sqrt{np_2(1-p_2)}}{\sqrt{np_1(1-p_1)}}}^{\frac{n-np_1-np_2-u_2\sqrt{np_2(1-p_2)}}{\sqrt{np_1(1-p_1)}}} e^{-\frac{(u_1-\rho_n u_2)^2+\left(1-\rho_n^2\right)u_2^2}{2(1-\rho_n^2)}} du_1 du_2. \end{aligned}$$

We can show that

$$\begin{aligned} &\int_{\frac{a_n-np_1+np_2+u_2\sqrt{np_2(1-p_2)}}{\sqrt{np_1(1-p_1)}}}^{\frac{n-np_1-np_2-u_2\sqrt{np_2(1-p_2)}}{\sqrt{np_1(1-p_1)}}} e^{-\frac{(u_1-\rho_n u_2)^2}{2(1-\rho_n^2)}} du_1 \\ &= \sqrt{1-\rho_n^2} \int_{b_n(p_1,p_2,a_n)+c_n(p_1,p_2)u_2}^{e_n(p_1,p_2)-c_n(p_1,p_2)u_2} e^{-\frac{u^2}{2}} du \quad \left(u = \frac{u_1-\rho_n u_2}{\sqrt{1-\rho_n^2}}\right), \end{aligned}$$

where b_n, c_n and e_n are defined in (3.16)–(3.18), respectively. Then,

$$P\left((Z_1,Z_2)\in A_n^*\right) = \frac{1}{2\pi} \int_{b_{n1}(p_2)}^{b_{n2}(p_2,a_n)} \int_{b_n(p_1,p_2,a_n)+c_n(p_1,p_2)u_2}^{e_n(p_1,p_2)-c_n(p_1,p_2)u_2} e^{-\frac{u_1^2+u_2^2}{2}} du_1 du_2.$$

□

Theorem 3.7. *Let A_n be defined in (3.13) and (X_{n1}, X_{n2}, X_{n3}) be a trinomial random vector with parameters n and (p_1, p_2, p_3). Then*

$$\begin{aligned} P\left((X_{n1},X_{n2},X_{n3})\in A_n\right) = \frac{1}{2\pi} &\int_{b_{n1}(p_2)}^{b_{n2}(p_2,a_n)} \int_{b_n(p_1,p_2,a_n)+c_n(p_1,p_2)u_2}^{e_n(p_1,p_2)-c_n(p_1,p_2)u_2} e^{-\frac{u_1^2+u_2^2}{2}} du_1 du_2 \\ &+ \Delta_n(p_1,p_2), \end{aligned}$$

where $b_{n1}(p_2),\ b_{n2}(p_2,a_n),\ b_n(p_1,p_2,a_n),\ c_n(p_1,p_2),\ e_n(p_1,p_2)$ *and* $\Delta_n(p_1,p_2)$ *are defined in* (3.14)–(3.18) *and* (3.7), *respectively.*

Proof. To prove this Theorem, it is sufficient to show that A_n is a convex set.
Let $t \in (0,1)$ and $(x_1,x_2,x_3)\,,(y_1,y_2,y_3) \in A_n$.
Then $x_1,x_2,y_1,y_2 \geq 0$, $x_1+x_2 \leq n$, $y_1+y_2 \leq n$, $x_1-x_2 \geq a_n$ and $y_1-y_2 \geq a_n$.
Let

$$\begin{aligned}(u_1,u_2,u_3) &= t\,(x_1,x_2,x_3) + (1-t)\,(y_1,y_2,y_3)\\ &= (tx_1+(1-t)y_1,\ tx_2+(1-t)y_2,\ tx_3+(1-t)y_3)\,.\end{aligned}$$

We see that $u_1,u_2 \geq 0$.
Since $x_1+x_2 \leq n$ and $y_1+y_2 \leq n$, we have

$$\begin{aligned} & u_1+u_2 = t(x_1+x_2)+(1-t)(y_1+y_2) \leq tn+(1-t)n = n\\ \text{and}\qquad & u_1-u_2 = t(x_1-x_2)+(1-t)(y_1-y_2) \geq ta_n+(1-t)a_n = a_n.\end{aligned}$$

Then, $(u_1,u_2,u_3) \in A_n$ which implies that A_n is a convex set.
From this fact, Theorem 3.5 and Proposition 3.6, the proof is complete. □

CHAPTER IV
CONVERGENCE OF TRINOMIAL FORMULA

In this chapter, we will show that the trinomial formula converges to the Black–Scholes formula, i.e., $\lim_{n\to\infty} T_n = C_{BS}$. In addition, we give examples of option prices from both trinomial formula and Black–Scholes formula.

4.1 Convergence of trinomial formula

In the showing that $\lim_{n\to\infty} B_n = C_{BS}$, we write B_n in form (2.6) which have 2 terms of binomial probability. To prove $\lim_{n\to\infty} T_n = C_{BS}$, we also need to write T_n in form which have 2 terms of trinomial probability as the lemma 4.1.

Let p_u and p_d be defined in (2.12) and (2.13) with u_T is defined in (2.14). Then,

$$\begin{aligned}
p_u &= \frac{\left(V+M^2-M\right)u_T-(M-1)}{(u_T-1)\left(u_T^2-1\right)} \\
&= \frac{\left(\left(e^{\frac{\sigma^2 T}{n}}-1\right)M^2+M^2-M\right)u_T-(M-1)}{u_T^3-u_T^2-u_T+1} \\
&= \frac{\left(M^2 e^{\frac{\sigma^2 T}{n}}-M\right)u_T^{-1}-(M-1)\,u_T^{-2}}{u_T-1-u_T^{-1}+u_T^{-2}} \\
&= \frac{\left(e^{\frac{2rT}{n}}e^{\frac{\sigma^2 T}{n}}-e^{\frac{rT}{n}}\right)e^{-\lambda\sigma\sqrt{\frac{T}{n}}}-\left(e^{\frac{rT}{n}}-1\right)e^{-2\lambda\sigma\sqrt{\frac{T}{n}}}}{e^{\frac{\lambda\sigma\sqrt{nT}}{n}}-1-e^{\frac{-\lambda\sigma\sqrt{nT}}{n}}+e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}} \\
&= \frac{e^{\frac{2rT+\sigma^2 T-\lambda\sigma\sqrt{nT}}{n}}-e^{\frac{rT-\lambda\sigma\sqrt{nT}}{n}}-e^{\frac{rT-2\lambda\sigma\sqrt{nT}}{n}}+e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}}{e^{\frac{\lambda\sigma\sqrt{nT}}{n}}-1-e^{\frac{-\lambda\sigma\sqrt{nT}}{n}}+e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}} \qquad (4.1)
\end{aligned}$$

and

$$\begin{aligned}
p_d &= \frac{\left(V+M^2-M\right)u_T^2-(M-1)\,u_T^3}{(u_T-1)\left(u_T^2-1\right)} \\
&= \frac{M^2 e^{\frac{\sigma^2 T}{n}}-M-(M-1)\,u_T}{u_T-1-u_T^{-1}+u_T^{-2}}
\end{aligned}$$

$$= \frac{e^{\frac{2rT}{n}} e^{\frac{\sigma^2 T}{n}} - e^{\frac{rT}{n}} - \left(e^{\frac{rT}{n}} - 1\right) e^{\lambda\sigma\sqrt{\frac{T}{n}}}}{e^{\frac{\lambda\sigma\sqrt{nT}}{n}} - 1 - e^{\frac{-\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}}$$

$$= \frac{e^{\frac{2rT+\sigma^2 T}{n}} - e^{\frac{rT}{n}} - e^{\frac{rT+\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{\lambda\sigma\sqrt{nT}}{n}}}{e^{\frac{\lambda\sigma\sqrt{nT}}{n}} - 1 - e^{\frac{-\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}}. \tag{4.2}$$

We defined q_u, q_d and q_m by

$$q_u = p_u u_T e^{-\frac{rT}{n}}, \tag{4.3}$$

$$q_d = p_d u_T^{-1} e^{-\frac{rT}{n}} \tag{4.4}$$

and

$$q_m = p_m e^{-\frac{rT}{n}}, \tag{4.5}$$

where $p_m = 1 - p_u - p_d$.

By (2.11), we have

$$S_k p_u u_T + S_k p_d u_T^{-1} + S_k p_m = S_k e^{\frac{rT}{n}}$$

which implies that

$$u_T p_u e^{-\frac{rT}{n}} + p_d u_T^{-1} e^{-\frac{rT}{n}} + p_m e^{-\frac{rT}{n}} = 1,$$

i.e., $q_u + q_d + q_m = 1$.

From this fact and the fact that $q_u, q_d, q_m > 0$, we obtain

$$0 < q_u, q_d, q_m < 1.$$

Lemma 4.1. *Let* (X_{n1}, X_{n2}, X_{n3}) *and* (Y_{n1}, Y_{n2}, Y_{n3}) *be trinomial random vectors with parameters* n *and* (q_u, q_d, q_m), *and* n *and* (p_u, p_d, p_m), *where* p_u, p_d, q_u, q_d *and* q_m *are defined in* (4.1)–(4.5) *and* $p_m = 1 - p_u - p_d$. *Let* T_n *be defined in* (2.10). *Then*

$$T_n = S_0 P\left((X_{n1}, X_{n2}, X_{n3}) \in A_n\right) - K e^{-rT} P\left((Y_{n1}, Y_{n2}, Y_{n3}) \in A_n\right),$$

where A_n is defined in (3.13) with

$$a_n = \frac{\sqrt{n}\log(K/S_0)}{\lambda\sigma\sqrt{T}}. \tag{4.6}$$

Proof. By (2.10), we note that

$$\begin{aligned} T_n &= e^{-rT}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j,l,n-j-l}p_u^j p_d^l p_m^{n-j-l}\max\left\{S_0u_T^{j-l}-K,0\right\} \\ &= e^{-rT}\sum_{(j,l,k)\in B}\binom{n}{j,l,k}p_u^j p_d^l p_m^k\max\left\{S_0u_T^{j-l}-K,0\right\}, \end{aligned}$$

where $B = \{(j,l,k)\in\mathbb{R}^3 \mid j,l,k\in\mathbb{N}\cup\{0\} \quad \text{and} \quad j+l+k=n\}$.
Since $u_T = e^{\lambda\sigma\sqrt{\frac{T}{n}}}$, we observe that for $(j,l,k)\in B$,

$$S_0u_T^{j-l}-K\geq 0 \quad \text{if and only if} \quad j-l\geq\frac{\log(K/S_0)}{\log u_T}=\frac{\sqrt{n}\log(K/S_0)}{\lambda\sigma\sqrt{T}}=a_n.$$

Then

$$\begin{aligned} T_n &= e^{-rT}\sum_{(j,l,k)\in C}\binom{n}{j,l,k}p_u^j p_d^l p_m^k\left(S_0u_T^{j-l}-K\right) \\ &= S_0\sum_{(j,l,k)\in C}\binom{n}{j,l,k}(p_uu_Te^{-\frac{T}{n}})^j(p_du_T^{-1}e^{-\frac{T}{n}})^l(p_me^{-\frac{T}{n}})^k \\ &\quad - Ke^{-rT}\sum_{(j,l,k)\in C}\binom{n}{j,l,k}p_u^j p_d^l p_m^k \\ &= S_0\sum_{(j,l,k)\in C}\binom{n}{j,l,k}q_u^j q_d^l q_m^k - Ke^{-rT}\sum_{(j,l,k)\in C}\binom{n}{j,l,k}p_u^j p_d^l p_m^k \\ &= S_0P\left((X_{n1},X_{n2},X_{n3})\in C\right) - Ke^{-rT}P\left((Y_{n1},Y_{n2},Y_{n3})\in C\right), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} C &= B\cap\left\{(j,l,k)\in\mathbb{R}^3 \mid j-l\geq a_n\right\} \\ &= \left\{(j,l,k)\in\mathbb{R}^3 \mid j,l,k\in\mathbb{N}\cup\{0\}, \quad j+l+k=n \quad \text{and} \quad j-l\geq a_n\right\}. \end{aligned}$$

We know that for $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 \neq n$,

$$P\left((X_{n1}, X_{n2}, X_{n3}) = (x_1, x_2, x_3)\right) = 0 = P\left((Y_{n1}, Y_{n2}, Y_{n3}) = (x_1, x_2, x_3)\right).$$

By (3.13), we have

$$A_n = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2 \geq 0, \;\; x_1 + x_2 \leq n \;\text{ and }\; x_1 - x_2 \geq a_n\right\}$$

with a_n is defined in (4.6).
We can see that

$$P\left((X_{n1}, X_{n2}, X_{n3}) \in A_n\right) = P\left((X_{n1}, X_{n2}, X_{n3}) \in C\right)$$

and

$$P\left((Y_{n1}, Y_{n2}, Y_{n3}) \in A_n\right) = P\left((Y_{n1}, Y_{n2}, Y_{n3}) \in C\right).$$

From these facts and (4.7), the proof of this lemma is complete. □

Lemma 4.2. *Let* p_u *and* p_d *be defined in* (4.1) *and* (4.2). *Then,*

1. $\lim_{n\to\infty} p_u = \dfrac{1}{2\lambda^2}$,
2. $\lim_{n\to\infty} p_d = \dfrac{1}{2\lambda^2}$, *and*
3. $\lim_{n\to\infty} \sqrt{n}\,(p_d - p_u) = -\dfrac{(2r - \sigma^2)\sqrt{T}}{2\lambda\sigma}$.

Proof. 1. Let $x = \sqrt{\dfrac{T}{n}}$. Then, by (4.1), we have

$$p_u = \frac{e^{2rx^2+\sigma^2x^2-\lambda\sigma x} - e^{rx^2-\lambda\sigma x} - e^{rx^2-2\lambda\sigma x} + e^{-2\lambda\sigma x}}{e^{\lambda\sigma x} - 1 - e^{-\lambda\sigma x} + e^{-2\lambda\sigma x}} = \frac{f(x)}{g(x)}, \tag{4.8}$$

where

$$f(x) = e^{2rx^2+\sigma^2x^2-\lambda\sigma x} - e^{rx^2-\lambda\sigma x} - e^{rx^2-2\lambda\sigma x} + e^{-2\lambda\sigma x} \tag{4.9}$$

and

$$g(x) = e^{\lambda\sigma x} - 1 - e^{-\lambda\sigma x} + e^{-2\lambda\sigma x}. \tag{4.10}$$

We see that

$$f'(x) = \left(4rx + 2\sigma^2 x - \lambda\sigma\right) e^{2rx^2+\sigma^2x^2-\lambda\sigma x} - (2rx - \lambda\sigma)\, e^{rx^2-\lambda\sigma x}$$
$$- (2rx - 2\lambda\sigma)\, e^{rx^2-2\lambda\sigma x} - 2\lambda\sigma e^{-2\lambda\sigma x}$$

and $$g^{(k)}(x) = \lambda^k\sigma^k e^{\lambda\sigma x} - (-1)^k\lambda^k\sigma^k e^{-\lambda\sigma x} + (-2)^k\lambda^k\sigma^k e^{-2\lambda\sigma x}. \quad (4.11)$$

Note that $\lim_{x\to 0}\frac{f(x)}{g(x)}$ and $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ are in the form $\frac{0}{0}$.

By the L'hopital's rule, we obtain

$$\lim_{n\to\infty} p_u = \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)}. \quad (4.12)$$

Since

$$f''(x) = \left(4rx + 2\sigma^2 x - \lambda\sigma\right)^2 e^{2rx^2+\sigma^2x^2-\lambda\sigma x} + \left(4r + 2\sigma^2\right) e^{2rx^2+\sigma^2x^2-\lambda\sigma x}$$
$$- (2rx - \lambda\sigma)^2\, e^{rx^2-\lambda\sigma x} - 2re^{rx^2-\lambda\sigma x} - (2rx - 2\lambda\sigma)^2\, e^{rx^2-2\lambda\sigma x}$$
$$- 2re^{rx^2-2\lambda\sigma x} + 4\lambda^2\sigma^2 e^{-2\lambda\sigma x}$$

and (4.11), we have

$$\lim_{x\to 0}\frac{f''(x)}{g''(x)} = \frac{\lambda^2\sigma^2 + 4r + 2\sigma^2 - \lambda^2\sigma^2 - 2r - 4\lambda^2\sigma^2 - 2r + 4\lambda^2\sigma^2}{4\lambda^2\sigma^2} = \frac{1}{2\lambda^2}.$$

From this fact and (4.12), we finish the proof.

2. By (4.2), we have

$$p_d = \frac{e^{\frac{2rT+\sigma^2T}{n}} - e^{\frac{rT}{n}} - e^{\frac{rT+\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{\lambda\sigma\sqrt{nT}}{n}}}{e^{\frac{\lambda\sigma\sqrt{nT}}{n}} - 1 - e^{\frac{-\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}} = \frac{h(x)}{g(x)}, \quad (4.13)$$

where $x = \sqrt{\frac{T}{n}}$,

$$h(x) = e^{(2r+\sigma^2)x^2} - e^{rx^2} - e^{rx^2+\lambda\sigma x} + e^{\lambda\sigma x} \quad (4.14)$$

and $g(x)$ is defined in (4.10).

By the same argument of 1., we can use the L'hopital's rule to show that

$$\lim_{n\to\infty} p_d = \lim_{x\to 0} \frac{h''(x)}{g''(x)} = \frac{1}{2\lambda^2}.$$

3. From (4.8) and (4.13), we have

$$\sqrt{\frac{n}{T}}\,(p_d - p_u) = \frac{h(x) - f(x)}{xg(x)},$$

where $f(x)$, $g(x)$ and $h(x)$ are defined in (4.9), (4.10) and (4.14) with $x = \sqrt{\frac{T}{n}}$.

We can see that $\lim_{x\to 0} \frac{h(x) - f(x)}{xg(x)}$, $\lim_{x\to 0} \frac{(h(x) - f(x))'}{(xg(x))'}$ and $\lim_{x\to 0} \frac{(h(x) - f(x))''}{(xg(x))''}$ are in the form $\frac{0}{0}$. Hence,

$$\begin{aligned}\lim_{n\to\infty} \sqrt{n}\,(p_d - p_u) &= \lim_{x\to 0} \frac{h(x) - f(x)}{xg(x)} \\ &= \lim_{x\to 0} \frac{h'''(x) - f'''(x)}{xg'''(x) + 3g''(x)} \\ &= -\frac{(2r - \sigma^2)\sqrt{T}}{2\lambda\sigma}.\end{aligned}$$

□

Lemma 4.3. *Let q_u and q_d be defined in (4.3) and (4.4). Then*

1. $\lim_{n\to\infty} q_u = \frac{1}{2\lambda^2}$.
2. $\lim_{n\to\infty} q_d = \frac{1}{2\lambda^2}$.
3. $\lim_{n\to\infty} \sqrt{n}\,(q_d - q_u) = -\frac{(2r + \sigma^2)\sqrt{T}}{2\lambda\sigma}$.

Proof. 1. We see that

$$\lim_{n\to\infty} e^{\lambda\sigma\sqrt{\frac{T}{n}}} e^{-\frac{rT}{n}} = 1.$$

From this fact, (4.3) and lemma 4.2 (1.), we have

$$\lim_{n\to\infty} q_u = \lim_{n\to\infty} p_u u_T e^{-\frac{rT}{n}} = \lim_{n\to\infty} p_u e^{\lambda\sigma\sqrt{\frac{T}{n}}} e^{-\frac{rT}{n}} = \frac{1}{2\lambda^2}.$$

2. Similar to 1., we can show that

$$\lim_{n\to\infty} q_d = \lim_{n\to\infty} p_d e^{-\lambda\sigma\sqrt{\frac{T}{n}}} e^{-\frac{rT}{n}} = \frac{1}{2\lambda^2}.$$

3. By (2.14), (4.3) and (4.4), we obtain

$$\begin{aligned}\sqrt{n}\,(q_d - q_u) &= e^{-\frac{rT}{n}}\sqrt{n}\left(p_d u_T^{-1} - p_u u_T\right)\\ &= u_T^{-1} e^{-\frac{rT}{n}}\sqrt{n}\,(p_d - p_u) + p_u e^{-\frac{rT}{n}}\sqrt{n}\left(u_T^{-1} - u_T\right)\\ &= e^{-\lambda\sigma\sqrt{\frac{T}{n}}} e^{-\frac{rT}{n}}\sqrt{n}\,(p_d - p_u) + p_u e^{-\frac{rT}{n}-\lambda\sigma\sqrt{\frac{T}{n}}}\sqrt{n}\left(1 - e^{2\lambda\sigma\sqrt{\frac{T}{n}}}\right).\end{aligned}$$

Note that $\lim_{n\to\infty}\sqrt{n}\left(1 - e^{2\lambda\sigma\sqrt{\frac{T}{n}}}\right) = \lim_{n\to\infty}\dfrac{1 - e^{2\lambda\sigma\sqrt{\frac{T}{n}}}}{n^{-\frac{1}{2}}}$ is in the form $\dfrac{0}{0}$.

We use the L'hopital's rule to show that

$$\lim_{n\to\infty}\sqrt{n}\left(1 - e^{2\lambda\sigma\sqrt{\frac{T}{n}}}\right) = \lim_{n\to\infty}\frac{\lambda^2\sigma\sqrt{T}n^{-\frac{3}{2}}e^{2\lambda\sigma\sqrt{\frac{T}{n}}}}{-\frac{1}{2}n^{-\frac{3}{2}}} = -2\lambda^2\sigma\sqrt{T}.$$

From this fact and lemma 4.2 (1.) and (3.), we have

$$\begin{aligned}\lim_{n\to\infty} p_u e^{-\frac{rT}{n}-\lambda\sigma\sqrt{\frac{T}{n}}}\sqrt{n}\left(1 - e^{2\lambda\sigma\sqrt{\frac{T}{n}}}\right) &= \left(\lim_{n\to\infty} p_u\right)\left(\lim_{n\to\infty}\sqrt{n}\left(1 - e^{2\lambda\sigma\sqrt{\frac{T}{n}}}\right)\right)\\ &= \left(\frac{1}{2\lambda^2}\right)\left(-2\lambda^2\sigma\sqrt{T}\right)\\ &= -\frac{\sigma\sqrt{T}}{\lambda}\end{aligned}$$

and

$$\lim_{n\to\infty} e^{-\lambda\sigma\sqrt{\frac{T}{n}}} e^{-\frac{rT}{n}}\sqrt{n}\,(p_d - p_u) = \lim_{n\to\infty}\sqrt{n}\,(p_d - p_u) = -\frac{(2r - \sigma^2)\sqrt{T}}{2\lambda\sigma}.$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty}\sqrt{n}\,(q_d-q_u) &= \lim_{n\to\infty} e^{-\lambda\sigma\sqrt{\frac{T}{n}}}e^{-\frac{rT}{n}}\sqrt{n}\,(p_d-p_u)\\
&\quad+\lim_{n\to\infty} p_u e^{-\frac{rT}{n}}\sqrt{n}\left(e^{-\lambda\sigma\sqrt{\frac{T}{n}}}-e^{\lambda\sigma\sqrt{\frac{T}{n}}}\right)\\
&= -\frac{(2r+\sigma^2)\sqrt{T}}{2\lambda\sigma}.
\end{aligned}$$

□

Lemma 4.4. *Let (Y_{n1},Y_{n2},Y_{n3}) be trinomial random vector with parameters n and $(p_u,p_d,1-p_u-p_d)$, where p_u and q_d are defined in (4.1) and (4.2). Let A_n be defined in (3.13) with a_n is defined in (4.6). If $K\geq S_0$, then*

$$\lim_{n\to\infty} P\left((Y_{n1},Y_{n2},Y_{n3})\in A_n\right)=\Phi\left(d_2\right),$$

where

$$d_2=\frac{\log(S_0/K)+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \tag{4.15}$$

Proof. Since $K\geq S_0$, we have $a_n\geq 0$. From Theorem 3.7, we have

$$\begin{aligned}
P\left((Y_{n1},Y_{n2},Y_{n3})\in A_n\right) &= \frac{1}{2\pi}\int_{b_{n1}(p_d)}^{b_{n2}(p_d,a_n)}\int_{b_n(p_u,p_d,a_n)+c_n(p_u,p_d)x_2}^{e_n(p_u,p_d)-c_n(p_u,p_d)x_2} e^{-\frac{x_1^2+x_2^2}{2}}dx_1dx_2\\
&\quad+\Delta_n(p_u,p_d),
\end{aligned} \tag{4.16}$$

where

$$|\Delta_n(p_u,p_d)|\leq\frac{\left(42\sqrt[4]{2}+16\right)}{\sqrt{(1+\rho_n)^3}}\left(\frac{1}{\sqrt{np_u(1-p_u)}}+\frac{1}{\sqrt{np_d(1-p_d)}}\right),$$

$b_{n1}(p_d)$, $b_{n2}(p_d,a_n)$, $b_n(p_u,p_d,a_n)$, $c_n(p_u,p_d)$ and $e_n(p_u,p_d)$ be defined in (3.14)–(3.18) with ρ_n is the correlation between Y_{n1} and Y_{n2}.

By Proposition 3.3 (2.), we have

$$\rho_n=-\sqrt{\frac{p_up_d}{(1-p_u)(1-p_d)}}. \tag{4.17}$$

From Lemma 4.2, we see that

$$\lim_{n\to\infty} \frac{\left(42\sqrt[4]{2}+16\right)}{\sqrt{(1+\rho_n)^3}} \left(\frac{1}{\sqrt{np_u(1-p_u)}} + \frac{1}{\sqrt{np_d(1-p_d)}} \right) = 0$$

which implies that

$$\lim_{n\to\infty} \Delta_n(p_u, p_d) = 0.$$

Next, we will show that

$$\frac{1}{2\pi} \int_{b_{n1}(p_d)}^{b_{n2}(p_d,a_n)} \int_{b_n(p_u,p_d,a_n)+c_n(p_u,p_d)x_2}^{e_n(p_u,p_d)-c_n(p_u,p_d)x_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 = \Phi\left(d_2\right).$$

For convenience, we write b_{n1}, b_{n2}, b_n, c_n and e_n instead of $b_{n1}(p_d)$, $b_{n2}(p_d, a_n)$, $b_n(p_u, p_d, a_n)$, $c_n(p_u, p_d)$ and $e_n(p_u, p_d)$, respectively.

By lemma 4.2, we have

$$\lim_{n\to\infty} b_{n1} = \lim_{n\to\infty} \frac{-np_d}{\sqrt{np_d(1-p_d)}} = -\infty, \tag{4.18}$$

$$\begin{aligned}
\lim_{n\to\infty} b_{n2} &= \lim_{n\to\infty} \frac{n-a_n-2np_d}{2\sqrt{np_d(1-p_d)}} \\
&= \lim_{n\to\infty} \frac{(1-2p_d)\sqrt{n}}{2\sqrt{p_d(1-p_d)}} - \lim_{n\to\infty} \frac{a_n}{2\sqrt{np_d(1-p_d)}} \\
&= \lim_{n\to\infty} \frac{(1-2p_d)\sqrt{n}}{2\sqrt{p_d(1-p_d)}} - \lim_{n\to\infty} \frac{\log\left(K/S_0\right)}{2\lambda\sigma\sqrt{Tp_d(1-p_d)}} \\
&= \infty
\end{aligned} \tag{4.19}$$

and $$\lim_{n\to\infty} \rho_n = -\lim_{n\to\infty} \sqrt{\frac{p_u p_d}{(1-p_u)(1-p_d)}} = \frac{1}{1-2\lambda^2}.$$

From this fact and lemma 4.2, we obtain

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} \frac{\sqrt{np_d(1-p_d)} - \rho_n\sqrt{np_u(1-p_u)}}{\sqrt{np_u(1-p_u)(1-\rho_n^2)}} \\
&= \left(\frac{-2\lambda^2}{1-2\lambda^2}\right)\left(\frac{2\lambda^2-1}{\sqrt{(1-2\lambda^2)^2-1}}\right)
\end{aligned}$$

$$= \frac{\lambda}{\sqrt{\lambda^2 - 1}}, \tag{4.20}$$

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} \frac{n\,(1 - p_u - p_d)}{\sqrt{np_u(1-p_u)(1-\rho_n^2)}} = \infty \tag{4.21}$$

and
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{p_u(1-p_u)(1-\rho_n^2)}} \left(\frac{a_n}{\sqrt{n}} + \sqrt{n}\,(p_d - p_u)\right)$$
$$= \frac{\lambda\sqrt{2\lambda^2-1}}{\sqrt{\lambda^2-1}} \left(\frac{\log{(K/S_0)}}{\lambda\sigma\sqrt{T}} - \frac{(2r-\sigma^2)\sqrt{T}}{2\lambda\sigma}\right)$$
$$= -\frac{d_2\sqrt{2\lambda^2-1}}{\sqrt{\lambda^2-1}}. \tag{4.22}$$

Note that

$$\begin{aligned}\int_{b_n+c_nx_2}^{e_n-c_nx_2} e^{-\frac{x_1^2}{2}} dx_1 &= -\int_{-b_n-c_nx_2}^{-e_n+c_nx_2} e^{-\frac{x_1^2}{2}} dx_1 \\ &= -\left(\int_{-\infty}^{-e_n+c_nx_2} e^{-\frac{x_1^2}{2}} dx_1 - \int_{-\infty}^{-b_n-c_nx_2} e^{-\frac{x_1^2}{2}} dx_1\right) \\ &= \int_{-\infty}^{-b_n-c_nx_2} e^{-\frac{x_1^2}{2}} dx_1 - \int_{-\infty}^{-e_n+c_nx_2} e^{-\frac{x_1^2}{2}} dx_1.\end{aligned}$$

Then,

$$\int_{b_{n1}}^{b_{n2}} \int_{b_n+c_nx_2}^{e_n-c_nx_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 = R_1 - R_2, \tag{4.23}$$

where
$$R_1 = \int_{b_{n1}}^{b_{n2}} \int_{-\infty}^{-b_n-c_nx_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2$$

and
$$R_2 = \int_{b_{n1}}^{b_{n2}} \int_{-\infty}^{-e_n+c_nx_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2.$$

We observe that

$$\begin{aligned}0 \le R_2 &= \int_{b_{n1}}^{b_{n2}} \int_{-\infty}^{-e_n} e^{-\frac{(x+c_nx_2)^2+x_2^2}{2}} dx dx_2 \quad (x = x_1 - c_nx_2) \\ &= \int_{-\infty}^{-e_n} \int_{b_{n1}}^{b_{n2}} e^{-\frac{x^2+2c_nx_2x+c_n^2x_2^2+x_2^2}{2}} dx_2 dx\end{aligned}$$

$$= \int_{-\infty}^{-e_n} \int_{b_{n1}}^{b_{n2}} e^{-\frac{\frac{x^2}{c_n^2+1} + \left(x_2\sqrt{c_n^2+1} + \frac{c_n x}{\sqrt{c_n^2+1}}\right)^2}{2}} dx_2 dx$$

$$= \int_{-\infty}^{-e_n} e^{-\frac{x^2}{2(c_n^2+1)}} \int_{b_{n1}}^{b_{n2}} e^{-\frac{\left(x_2\sqrt{c_n^2+1} + \frac{c_n x}{\sqrt{c_n^2+1}}\right)^2}{2}} dx_2 dx$$

$$\leq \int_{-\infty}^{-e_n} e^{-\frac{x^2}{2(c_n^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{\left(x_2\sqrt{c_n^2+1} + \frac{c_n x}{\sqrt{c_n^2+1}}\right)^2}{2}} dx_2 dx$$

$$= \int_{-\infty}^{-e_n} e^{-\frac{x^2}{2(c_n^2+1)}} \frac{1}{\sqrt{c_n^2+1}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du dx \quad \left(u = x_2\sqrt{c_n^2+1} + \frac{c_n x}{\sqrt{c_n^2+1}}\right)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{c_n^2+1}} \int_{-\infty}^{-e_n} e^{-\frac{x^2}{2(c_n^2+1)}} dx$$

$$= \sqrt{2\pi} \int_{-\infty}^{\frac{-e_n}{\sqrt{c_n^2+1}}} e^{-\frac{v^2}{2}} dv \quad \left(v = \frac{x}{\sqrt{c_n^2+1}}\right)$$

$$= 2\pi\Phi\left(\frac{-e_n}{\sqrt{c_n^2+1}}\right). \tag{4.24}$$

By (4.20) and (4.21), we have $\lim_{n\to\infty} \frac{e_n}{\sqrt{c_n^2+1}} = \infty$ which implies that

$$\lim_{n\to\infty} \Phi\left(\frac{-e_n}{\sqrt{c_n^2+1}}\right) = 0$$

and
$$\lim_{n\to\infty} R_2 = 0. \tag{4.25}$$

We can follow the arguments of (4.24) to show that

$$R_1 \leq 2\pi\Phi\left(\frac{-b_n}{\sqrt{c_n^2+1}}\right). \tag{4.26}$$

Note that

R_1

$$= \int_{b_{n1}}^{b_{n2}} \int_{-\infty}^{-b_n} e^{-\frac{(x-c_n x_2)^2 + x_2^2}{2}} dx dx_2 \quad (x = x_1 + c_n x_2)$$

$$= \int_{-\infty}^{-b_n} \int_{b_{n1}}^{b_{n2}} e^{-\frac{x^2 - 2c_n x_2 x + c_n^2 x_2^2 + x_2^2}{2}} dx_2 dx$$

$$= \int_{-\infty}^{-b_n} \int_{b_{n1}}^{b_{n2}} e^{-\frac{\frac{x^2}{c_n^2+1} + \left(x_2\sqrt{c_n^2+1} - \frac{c_n x}{\sqrt{c_n^2+1}} \right)^2}{2}} dx_2 dx$$

$$= \int_{-\infty}^{-b_n} e^{-\frac{x^2}{2(c_n^2+1)}} \int_{b_{n1}}^{b_{n2}} e^{-\frac{\left(x_2\sqrt{c_n^2+1} - \frac{c_n x}{\sqrt{c_n^2+1}} \right)^2}{2}} dx_2 dx$$

$$= \frac{1}{\sqrt{c_n^2+1}} \int_{-\infty}^{-b_n} e^{-\frac{x^2}{2(c_n^2+1)}} \int_{\gamma_n(x)}^{\beta_n(x)} e^{-\frac{u^2}{2}} du dx \quad \left(u = x_2\sqrt{c_n^2+1} - \frac{c_n x}{\sqrt{c_n^2+1}} \right)$$

$$\geq \frac{1}{\sqrt{c_n^2+1}} \int_{\frac{b_{n1}(c_n^2+1)}{2c_n}}^{-b_n} e^{-\frac{x^2}{2(c_n^2+1)}} \int_{\gamma_n(x)}^{\beta_n(x)} e^{-\frac{u^2}{2}} du dx, \tag{4.27}$$

where

$$\gamma_n(x) = b_{n1}\sqrt{c_n^2+1} - \frac{c_n x}{\sqrt{c_n^2+1}},$$

and
$$\beta_n(x) = b_{n2}\sqrt{c_n^2+1} - \frac{c_n x}{\sqrt{c_n^2+1}}.$$

Since $\rho_n = -\sqrt{\frac{p_u p_d}{(1-p_u)(1-p_d)}} < 0$, we have

$$c_n = \frac{\sqrt{np_2(1-p_2)} - \rho_n \sqrt{np_1(1-p_1)}}{\sqrt{np_1(1-p_1)(1-\rho_n^2)}} > 0$$

which implies that

$$\frac{b_{n1}\,(c_n^2+1)}{2} \leq c_n x \leq -c_n b_n, \quad \text{for} \quad \frac{b_{n1}\,(c_n^2+1)}{2c_n} \leq x \leq -b_n.$$

Then,

$$\gamma_n(x) = b_{n1}\sqrt{c_n^2+1} - \frac{c_n x}{\sqrt{c_n^2+1}} \leq b_{n1}\sqrt{c_n^2+1} - \frac{b_{n1}\,(c_n^2+1)}{2\sqrt{c_n^2+1}} = \frac{b_{n1}\sqrt{c_n^2+1}}{2} =: \gamma_n \tag{4.28}$$

and

$$\beta_n(x) = b_{n2}\sqrt{c_n^2+1} - \frac{c_n x}{\sqrt{c_n^2+1}} \geq b_{n2}\sqrt{c_n^2+1} + \frac{c_n b_n}{\sqrt{c_n^2+1}} =: \beta_n. \tag{4.29}$$

Hence, $[\gamma_n, \beta_n] \subseteq [\gamma_n(x), \beta_n(x)]$ and

$$\int_{\gamma_n(x)}^{\beta_n(x)} e^{-\frac{u^2}{2}} du \geq \int_{\gamma_n}^{\beta_n} e^{-\frac{u^2}{2}} du.$$

From this fact and (4.27), we have

$$\begin{aligned}
R_1 &\geq \frac{1}{\sqrt{c_n^2+1}} \int_{\frac{b_{n1}(c_n^2+1)}{2c_n}}^{-b_n} e^{-\frac{x^2}{2(c_n^2+1)}} \int_{\gamma_n}^{\beta_n} e^{-\frac{u^2}{2}} du dx \\
&= \left(\frac{1}{\sqrt{c_n^2+1}} \int_{\frac{b_{n1}(c_n^2+1)}{2c_n}}^{-b_n} e^{-\frac{x^2}{2(c_n^2+1)}} dx \right) \left(\int_{\gamma_n}^{\beta_n} e^{-\frac{u^2}{2}} du \right) \\
&= \left(\int_{\frac{b_{n1}\sqrt{c_n^2+1}}{2c_n}}^{\frac{-b_n}{\sqrt{c_n^2+1}}} e^{-\frac{v^2}{2}} dv \right) \left(\int_{\gamma_n}^{\beta_n} e^{-\frac{u^2}{2}} du \right) \qquad \left(v = \frac{x}{\sqrt{c_n^2+1}} \right) \\
&= 2\pi \left(\Phi\left(\frac{-b_n}{\sqrt{c_n^2+1}} \right) - \Phi\left(\frac{b_{n1}\sqrt{c_n^2+1}}{2c_n} \right) \right) \left(\Phi\left(\beta_n\right) - \Phi\left(\gamma_n\right) \right).
\end{aligned} \tag{4.30}$$

By (4.18), (4.19), (4.20) and (4.22), we obtain

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \frac{b_{n1}\sqrt{c_n^2+1}}{2} = -\infty,$$

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} b_{n2}\sqrt{c_n^2+1} + \frac{c_n b_n}{\sqrt{c_n^2+1}} = \infty,$$

$$\lim_{n\to\infty} \frac{b_{n1}\sqrt{c_n^2+1}}{2c_n} = -\infty$$

and $$\lim_{n\to\infty} \frac{-b_n}{\sqrt{c_n^2+1}} = d_2$$

which imply that

$$\lim_{n\to\infty} \Phi\left(\frac{-b_n}{\sqrt{c_n^2+1}} \right) = \Phi\left(d_2\right)$$

and

$$\lim_{n\to\infty}\left(\Phi\left(\frac{-b_n}{\sqrt{c_n^2+1}}\right)-\Phi\left(\frac{b_{n1}\sqrt{c_n^2+1}}{2c_n}\right)\right)\left(\Phi\left(\beta_n\right)-\Phi\left(\gamma_n\right)\right)=\Phi\left(d_2\right).$$

From these facts, (4.26) and (4.30), we have

$$\lim_{n\to\infty} R_1 = 2\pi\Phi\left(d_2\right). \tag{4.31}$$

Hence, by (4.23), (4.25) and (4.31),

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{b_{n1}}^{b_{n2}}\int_{b_n+c_nu_2}^{e_n-c_nu_2} e^{-\frac{u_1^2+u_2^2}{2}}du_1du_2=\lim_{n\to\infty}R_1-\lim_{n\to\infty}R_2=\Phi\left(d_2\right).$$

□

Lemma 4.5. *Let* (X_{n1}, X_{n2}, X_{n3}) *be trinomial random vector with parameters* n *and* (q_u, q_d, q_m)*, where* q_u*,* q_d *and* q_m *are defined in* (4.3), (4.4) *and* (4.5).
Let A_n *be defined in* (3.13) *with* a_n *is defined in* (4.6)*. If* $K \geq S_0$*, then*

$$\lim_{n\to\infty} P\left(\left(X_{n1}, X_{n2}, X_{n3}\right)\in A_n\right)=\Phi\left(d_1\right),$$

where

$$d_1=\frac{\log(S_0/K)+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \tag{4.32}$$

Proof. We can follow the arguments of the proof in lemma 4.4 and replace p_u and p_d by q_u and q_d to show that the conclusion of this lemma holds. □

Using lemma 4.1 and lemma 4.4-4.5, we have the following Theorem.

Theorem 4.6. *Let* T_n *be defined in* (2.10)*. If* $K \geq S_0$*, then*

$$\lim_{n\to\infty} T_n = C_{BS},$$

where C_{BS} *is defined in* (2.2).

4.2 Numerical examples

In the following examples, we compare the option prices from trinomial and Black–Scholes formulas for different values of n.

Example 4.7. The option prices obtained from the trinomial formula with parameters $S_0 = \$100$, $K = \$110$, $r = 5\%$, $\sigma = 30\%$, $T = 1$ year and $\lambda = 1.3$ are plotted in Figure 4.1 for $n = 5, 10, 15, \ldots, 170$. The black solid line indicates the corresponding Black–Scholes option price. Table 4.1 shows some explicit trinomial option prices along with the Black–Scholes option price.

n	20	40	60	80	100	120	140
T_n	10.0236	10.0633	10.0233	10.0293	10.0372	10.032	10.0215
C_{BS}	10.0201						

Table 4.1: Option price using trinomial formula and Black–Scholes formula with $S = \$100$, $K = \$110$, $r = 5\%$, $\sigma = 30\%$, $T = 1$ year and $\lambda = 1.3$

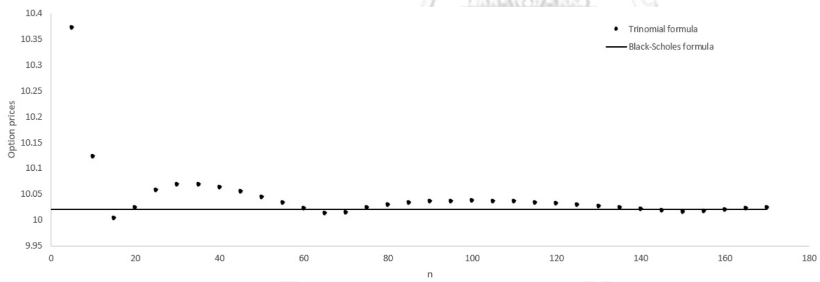


Figure 4.1: Option prices using trinomial formula and Black–Scholes formula with $S = \$100$, $K = \$110$, $r = 5\%$, $\sigma = 30\%$, $T = 1$ year and $\lambda = 1.3$

Example 4.8. Similarly, the results of an example with parameters $S_0 = \$110$, $K = \$120$, $r = 4\%$, $\sigma = 40\%$, $T = 1$ year and $\lambda = 1.5$ are shown in Figure 4.2 and Table 4.2.

n	20	40	60	80	100	120	140
T_n	15.4346	15.3067	15.3256	15.3517	15.3553	15.3504	15.3422
C_{BS}	15.3310						

Table 4.2: Option prices using trinomial formula and Black–Scholes formula with $S = \$110$, $K = \$120$, $r = 4\%$, $\sigma = 40\%$, $T = 1$ year and $\lambda = 1.5$

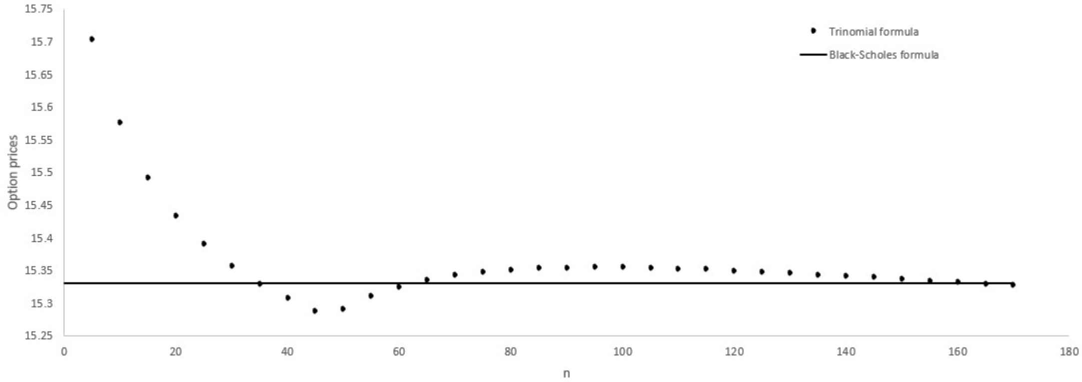


Figure 4.2: Option prices using trinomial formula and Black–Scholes formula with $S = \$110$, $K = \$120$, $r = 4\%$, $\sigma = 40\%$, $T = 1$ year and $\lambda = 1.5$

Based on Figure 4.1 and Figure 4.2, we can see that option prices from trinomial formula converges to the option price from the Black Scholes formula when n increases which is in accordance with theorem 4.6.

CHAPTER V
RATE OF CONVERGENCE OF TRINOMIAL FORMULA

In chapter 4, we know that the trinomial formula converges to the Black–Scholes formula, i.e., $\lim_{n\to\infty} T_n = C_{BS}$. In this chapter, we will show that the rate of this convergence is $\dfrac{1}{\sqrt{n}}$, i.e., $T_n = C_{BS} + O\left(\dfrac{1}{\sqrt{n}}\right)$.

Before we prove the main result in Theorem 5.6, we need the following 4 lemmas.

Lemma 5.1. *Let p_u and p_d be defined in* (4.1) *and* (4.2). *Then, for a large n,*

1. $p_u = \dfrac{1}{2\lambda^2} + \dfrac{2r\sqrt{T} - \sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\dfrac{1}{n}\right)$.

2. $p_d = \dfrac{1}{2\lambda^2} - \dfrac{2r\sqrt{T} - \sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\dfrac{1}{n}\right)$.

Proof. 1. For $a, b \in \mathbb{R}$, we have

$$e^{\frac{a}{n}+\frac{b}{\sqrt{n}}} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{a}{n}+\frac{b}{\sqrt{n}}\right)^k = 1 + \frac{b}{\sqrt{n}} + \frac{2a+b^2}{2n} + \frac{6ab+b^3}{6n\sqrt{n}} + r_n,$$

where

$$r_n = \frac{a^2}{2n^2} + \frac{1}{6}\left(\frac{a^3}{n^3} + \frac{3a^2 b}{n^2\sqrt{n}} + \frac{3ab^2}{n^2}\right) + \sum_{k=4}^{\infty}\frac{1}{k!}\left(\frac{a}{n}+\frac{b}{\sqrt{n}}\right)^k.$$

It is easy to show that, for large n, we have

$$\left|\sum_{k=4}^{\infty}\frac{1}{k!}\left(\frac{a}{n}+\frac{b}{\sqrt{n}}\right)^k\right| \leq \frac{1}{24}\left(\frac{a}{n}+\frac{b}{\sqrt{n}}\right)^4 \sum_{k=0}^{\infty}\left(\frac{|a|}{n}+\frac{|b|}{\sqrt{n}}\right)^k = O\left(\frac{1}{n^2}\right)$$

which implies that

$$|r_n| \leq \frac{a^2}{2n^2} + \frac{1}{6}\left(\frac{a^3}{n^3} + \frac{3a^2b}{n^2\sqrt{n}} + \frac{3ab^2}{n^2}\right) + O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{n^2}\right).$$

Hence,

$$e^{\frac{a}{n}+\frac{b}{\sqrt{n}}} = 1 + \frac{b}{\sqrt{n}} + \frac{2a+b^2}{2n} + \frac{6ab+b^3}{6n\sqrt{n}} + O\left(\frac{1}{n^2}\right). \tag{5.1}$$

We can apply this fact to show that

$$\begin{aligned}
e^{\frac{(2r+\sigma^2)T}{n}+\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}} &= 1 + \frac{\lambda\sigma\sqrt{T}}{\sqrt{n}} + \frac{(4r+2\sigma^2+\lambda^2\sigma^2)T}{2n} \\
&\quad + \frac{(12r+6\sigma^2+\lambda^2\sigma^2)\,\lambda\sigma T\sqrt{T}}{6n\sqrt{n}} + O\left(\frac{1}{n^2}\right), \\
e^{\frac{rT}{n}+\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}} &= 1 + \frac{\lambda\sigma\sqrt{T}}{\sqrt{n}} + \frac{(2r+\lambda^2\sigma^2)T}{2n} + \frac{(6r+\lambda^2\sigma^2)\lambda\sigma T\sqrt{T}}{6n\sqrt{n}} \\
&\quad + O\left(\frac{1}{n^2}\right), \\
e^{\frac{rT}{n}} &= 1 + \frac{rT}{n} + O\left(\frac{1}{n^2}\right)
\end{aligned}$$

and
$$e^{\frac{k\lambda\sigma\sqrt{T}}{\sqrt{n}}} = 1 + \frac{k\lambda\sigma\sqrt{T}}{\sqrt{n}} + \frac{k^2\lambda^2\sigma^2T}{2n} + \frac{k^3\lambda^3\sigma^3T\sqrt{T}}{6n\sqrt{n}} + O\left(\frac{1}{n^2}\right)$$

for all $k \in \mathbb{R}$.

From these facts and (4.1), we obtain

$$e^{\frac{3\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{2\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}} + 1 = \frac{2\lambda^2\sigma^2T}{n} + \frac{3\lambda^3\sigma^3T\sqrt{T}}{n\sqrt{n}} + O\left(\frac{1}{n^2}\right)$$

and

$$\begin{aligned}
p_u &= \frac{e^{\frac{2rT+\sigma^2T-\lambda\sigma\sqrt{nT}}{n}} - e^{\frac{rT-\lambda\sigma\sqrt{nT}}{n}} - e^{\frac{rT-2\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}}{e^{\frac{\lambda\sigma\sqrt{nT}}{n}} - 1 - e^{\frac{-\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}} \\
&= \frac{e^{\frac{(2r+\sigma^2)T}{n}+\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{rT}{n}+\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{rT}{n}} + 1}{e^{\frac{3\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{2\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}} + 1}
\end{aligned}$$

$$= \frac{\sigma^2 T + \dfrac{\lambda\sigma r T\sqrt{T} + \lambda\sigma^3 T\sqrt{T}}{\sqrt{n}} + O\left(\dfrac{1}{n^2}\right)}{2\lambda^2\sigma^2 T + \dfrac{3\lambda^3\sigma^3 T\sqrt{T}}{\sqrt{n}} + O\left(\dfrac{1}{n}\right)}$$

$$= \frac{1}{2\lambda^2} + \frac{2r\sqrt{T} - \sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + \frac{O\left(\dfrac{1}{n}\right)}{2\lambda^2\sigma^2 T + \dfrac{3\lambda^3\sigma^3 T\sqrt{T}}{\sqrt{n}} + O\left(\dfrac{1}{n}\right)}$$

$$= \frac{1}{2\lambda^2} + \frac{2r\sqrt{T} - \sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\frac{1}{n}\right).$$

2. Similarly to 1., we have

$$p_d = \frac{e^{\frac{2rT+\sigma^2 T}{n}} - e^{\frac{rT}{n}} - e^{\frac{rT+\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{\lambda\sigma\sqrt{nT}}{n}}}{e^{\frac{\lambda\sigma\sqrt{nT}}{n}} - 1 - e^{\frac{-\lambda\sigma\sqrt{nT}}{n}} + e^{\frac{-2\lambda\sigma\sqrt{nT}}{n}}}$$

$$= \frac{e^{\frac{(2r+\sigma^2)T}{n} + \frac{2\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{rT}{n} + \frac{2\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{rT}{n} + \frac{3\lambda\sigma\sqrt{T}}{\sqrt{n}}} + e^{\frac{3\lambda\sigma\sqrt{T}}{\sqrt{n}}}}{e^{\frac{3\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{2\lambda\sigma\sqrt{T}}{\sqrt{n}}} - e^{\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}} + 1}$$

$$= \frac{\sigma^2 T + \dfrac{2\lambda\sigma^3 T\sqrt{T} - \lambda\sigma r T\sqrt{T}}{\sqrt{n}} + O\left(\dfrac{1}{n^2}\right)}{2\lambda^2\sigma^2 T + \dfrac{3\lambda^3\sigma^3 T\sqrt{T}}{\sqrt{n}} + O\left(\dfrac{1}{n}\right)}$$

$$= \frac{1}{2\lambda^2} - \frac{2r\sqrt{T} - \sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + \frac{O\left(\dfrac{1}{n}\right)}{2\lambda^2\sigma^2 T + \dfrac{3\lambda^3\sigma^3 T\sqrt{T}}{\sqrt{n}} + O\left(\dfrac{1}{n}\right)}$$

$$= \frac{1}{2\lambda^2} - \frac{2r\sqrt{T} - \sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\frac{1}{n}\right).$$

$\square$

Remark 5.2. We see that for large n, p_u and p_d are between 0 and $\dfrac{1}{2}$ which imply that $p_m = 1 - p_u - p_d \in (0, 1)$.

Lemma 5.3. *Let q_u and q_d be defined in* (4.3) *and* (4.4) *with u_T, p_u and p_d defined in* (2.12), (2.13) *and* (2.14). *Then, for a large n,*

1. $q_u = \dfrac{1}{2\lambda^2} + \dfrac{2r\sqrt{T} + \sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\dfrac{1}{n}\right)$;

2. $q_d = \frac{1}{2\lambda^2} - \frac{2r\sqrt{T}+\sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\frac{1}{n}\right)$.

Proof. 1. We apply (5.1) to show that

$$e^{\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}-\frac{rT}{n}} = 1 + \frac{\lambda\sigma\sqrt{T}}{\sqrt{n}} + O\left(\frac{1}{n}\right).$$

From this fact, (4.3) and lemma 5.1 (1.), we obtain

$$\begin{aligned}
q_u &= p_u u_T e^{-\frac{rT}{n}} \\
&= e^{\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}-\frac{rT}{n}} \left(\frac{1}{2\lambda^2} + \frac{2r\sqrt{T}-\sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\frac{1}{n}\right)\right) \\
&= \frac{1}{2\lambda^2} + \frac{2r\sqrt{T}+\sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\frac{1}{n}\right).
\end{aligned}$$

2. Similarly to 1., we have

$$\begin{aligned}
q_d &= p_d u_T^{-1} e^{-\frac{rT}{n}} \\
&= e^{-\frac{\lambda\sigma\sqrt{T}}{\sqrt{n}}-\frac{rT}{n}} p_d \\
&= \left(1 - \frac{\lambda\sigma\sqrt{T}}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right)\left(\frac{1}{2\lambda^2} - \frac{2r\sqrt{T}-\sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\frac{1}{n}\right)\right) \\
&= \frac{1}{2\lambda^2} - \frac{2r\sqrt{T}+\sigma^2\sqrt{T}}{4\lambda\sigma\sqrt{n}} + O\left(\frac{1}{n}\right).
\end{aligned}$$

□

Lemma 5.4. *Let* (Y_{n1}, Y_{n2}, Y_{n3}) *be trinomial random vector with parameters* n *and* $(p_u, p_d, 1-p_u-p_d)$, *where* p_u *and* p_d *are defined in* (4.1) *and* (4.2). *Let* A_n *be defined in* (3.13) *with* a_n *is defined in* (4.6). *If* $K \geq S_0$, *then for large* n,

$$P\left((Y_{n1}, Y_{n2}, Y_{n3}) \in A_n\right) = \Phi(d_2) + O\left(\frac{1}{\sqrt{n}}\right),$$

where d_2 *is defined in* (4.15).

Proof. From (4.16), we have

$$P\left((Y_{n1}, Y_{n2}, Y_{n3}) \in A_n\right) = \frac{1}{2\pi}\int_{b_{n1}(p_d)}^{b_{n2}(p_d,a_n)} \int_{b_n(p_u,p_d,a_n)+c_n(p_u,p_d)x_2}^{e_n(p_u,p_d)-c_n(p_u,p_d)x_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 + \Delta_n(p_u, p_d),$$

where

$$|\Delta_n(p_u, p_d)| \leq \frac{\left(42\sqrt[4]{2}+16\right)}{\sqrt{(1+\rho_n)^3}}\left(\frac{1}{\sqrt{np_u(1-p_u)}} + \frac{1}{\sqrt{np_d(1-p_d)}}\right),$$

$b_{n1}(p_d)$, $b_{n2}(p_d, a_n)$, $b_n(p_u, p_d, a_n)$, $c_n(p_u, p_d)$, $e_n(p_u, p_d)$ and ρ_n be defined in (3.14)–(3.18) and (4.17).

Note that $\lim\limits_{n\to\infty} p_u$, $\lim\limits_{n\to\infty} p_d$ and $\lim\limits_{n\to\infty} \rho_n$ exist, then

$$\frac{\left(42\sqrt[4]{2}+16\right)}{\sqrt{(1+\rho_n)^3}}\left(\frac{1}{\sqrt{np_u(1-p_u)}} + \frac{1}{\sqrt{np_d(1-p_d)}}\right) = O\left(\frac{1}{\sqrt{n}}\right)$$

which implies that

$$\Delta_n(p_u, p_d) = O\left(\frac{1}{\sqrt{n}}\right).$$

Next, we will show that

$$\frac{1}{2\pi}\int_{b_{n1}(p_d)}^{b_{n2}(p_d,a_n)} \int_{b_n(p_u,p_d,a_n)+c_n(p_u,p_d)x_2}^{e_n(p_u,p_d)-c_n(p_u,p_d)x_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 = \Phi(d_2) + O\left(\frac{1}{\sqrt{n}}\right).$$

For convenience, we write b_{n1}, b_{n2}, b_n, c_n and e_n instead of $b_{n1}(p_d)$, $b_{n2}(p_d, a_n)$, $b_n(p_u, p_d, a_n)$, $c_n(p_u, p_d)$ and $e_n(p_u, p_d)$, respectively.

By (4.23) and (4.24), we have

$$\int_{b_{n1}}^{b_{n2}} \int_{b_n+c_nx_2}^{e_n-c_nx_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 = R_1 - R_2, \tag{5.2}$$

where

$$R_1 = \int_{b_{n1}}^{b_{n2}} \int_{-\infty}^{-b_n - c_n x_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2$$

and

$$0 \le R_2 = \int_{b_{n1}}^{b_{n2}} \int_{-\infty}^{-e_n + c_n x_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 \le 2\pi\Phi\left(\frac{-e_n}{\sqrt{c_n^2+1}}\right). \tag{5.3}$$

Since $\lim_{n\to\infty} p_u$, $\lim_{n\to\infty} p_d$ and $\lim_{n\to\infty} \rho_n$ exist and $\lim_{n\to\infty} p_u + \lim_{n\to\infty} p_d < 1$, we have

$$e_n = \sqrt{n}\left(\frac{1-p_u-p_d}{\sqrt{p_u(1-p_u)(1-\rho_n^2)}}\right) \ge C_1\sqrt{n}, \tag{5.4}$$

for some positive constant C_1. We know that for $t \in (-\infty, 0)$,

$$\Phi(t) \le -\frac{1}{t\sqrt{2\pi}} e^{-\frac{t^2}{2}} \le -\frac{1}{t}$$

([27], p.26) which implies that

$$\Phi(-C\sqrt{n}) = O\left(\frac{1}{\sqrt{n}}\right), \tag{5.5}$$

for all positive constant C. By (5.4) and $\lim_{n\to\infty} c_n$ exists, we have

$$\frac{-e_n}{\sqrt{c_n^2+1}} \le -C_2\sqrt{n},$$

for some positive constant C_2. From this fact, (5.3) and (5.5), we obtain

$$0 \le R_2 \le 2\pi\Phi\left(\frac{-e_n}{\sqrt{c_n^2+1}}\right) \le 2\pi\Phi\left(-C_2\sqrt{n}\right) = O\left(\frac{1}{\sqrt{n}}\right),$$

i.e.,

$$R_2 = O\left(\frac{1}{\sqrt{n}}\right). \tag{5.6}$$

By Lemma 5.1, we have

$$p_u = \frac{1}{2\lambda^2} + O\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad p_d = \frac{1}{2\lambda^2} + O\left(\frac{1}{\sqrt{n}}\right)$$

which implies that

$$p_u(1-p_u) + 2p_u p_d + p_d(1-p_d) = p_u + p_d - (p_u - p_d)^2 = \frac{1}{\lambda^2} + O\left(\frac{1}{\sqrt{n}}\right). \quad (5.7)$$

We know that for $\gamma, y \in \mathbb{R}$,

$$(1+y)^\gamma = \sum_{k=0}^{\infty} \binom{\gamma}{k} y^k, \quad (5.8)$$

where $$\binom{\gamma}{k} = \begin{cases} \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!} & if \quad k \in \mathbb{N} \\ 1 & if \quad k = 0 \end{cases}$$

(see [3], p. 356 for more details). Hence, for $k \in \mathbb{N}$,

$$\left|\binom{-\frac{1}{2}}{k+1}\right| = \left|\frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-k+1)(-\frac{1}{2}-k)}{(k+1)!}\right| = \left|\binom{-\frac{1}{2}}{k}\right| \frac{\frac{1}{2}+k}{k+1} \leq \left|\binom{-\frac{1}{2}}{k}\right|.$$

Then, $\left|\binom{-\frac{1}{2}}{k}\right| \leq \left|\binom{-\frac{1}{2}}{1}\right| = \frac{1}{2}$ for $k \geq 1$.
From this fact, (5.7) and (5.8), we have

$$\begin{aligned} \left(p_u + p_d - (p_u - p_d)^2\right)^{-\frac{1}{2}} &= \lambda\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)^{-\frac{1}{2}} \\ &= \lambda\left(\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\left(O\left(\frac{1}{\sqrt{n}}\right)\right)^k\right) \\ &= \lambda\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\left(O\left(\frac{1}{\sqrt{n}}\right)\right)^k\right) \\ &= \lambda + O\left(\frac{1}{\sqrt{n}}\right). \quad (5.9) \end{aligned}$$

Hence, by (5.9) and lemma 5.1,

$$-\frac{b_n}{\sqrt{c_n^2+1}} = \frac{1}{\sqrt{p_u+p_d-(p_u-p_d)^2}}\left(-\frac{a_n}{\sqrt{n}}+\sqrt{n}\,(p_u-p_d)\right)$$
$$= \left(\lambda + O\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{\log\left(S_0/K\right)}{\lambda\sigma\sqrt{T}} + \frac{2r\sqrt{T}-\sigma^2\sqrt{T}}{2\lambda\sigma} + O\left(\frac{1}{\sqrt{n}}\right)\right)$$
$$= d_2 + O\left(\frac{1}{\sqrt{n}}\right).$$

From this fact, we have

$$\left|\int_{d_2}^{-\frac{b_n}{\sqrt{c_n^2+1}}} e^{-\frac{u^2}{2}}\,du\right| \le \left|\frac{b_n}{\sqrt{c_n^2+1}}+d_2\right| = O\left(\frac{1}{\sqrt{n}}\right)$$

which implies that

$$\Phi\left(\frac{-b_n}{\sqrt{c_n^2+1}}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-b_n}{\sqrt{c_n^2+1}}} e^{-\frac{u^2}{2}}\,du$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{u^2}{2}}\,du + \frac{1}{\sqrt{2\pi}}\int_{d_2}^{-\frac{b_n}{\sqrt{c_n^2+1}}} e^{-\frac{u^2}{2}}\,du$$
$$= \Phi(d_2) + O\left(\frac{1}{\sqrt{n}}\right). \tag{5.10}$$

By (4.26), (4.30) and (5.10), we have

$$R_1 \le 2\pi\Phi\left(\frac{-b_n}{\sqrt{c_n^2+1}}\right) = 2\pi\Phi(d_2) + O\left(\frac{1}{\sqrt{n}}\right) \tag{5.11}$$

and

$$R_1 \ge 2\pi\left(\Phi\left(\frac{-b_n}{\sqrt{c_n^2+1}}\right) - \Phi\left(\frac{b_{n1}\sqrt{c_n^2+1}}{2c_n}\right)\right)\left(\Phi\left(\beta_n\right)-\Phi\left(\gamma_n\right)\right)$$
$$= 2\pi\left(\Phi(d_2) + O\left(\frac{1}{\sqrt{n}}\right) - \Phi\left(\frac{b_{n1}\sqrt{c_n^2+1}}{2c_n}\right)\right)\left(1-\Phi\left(-\beta_n\right)-\Phi\left(\gamma_n\right)\right), \tag{5.12}$$

where γ_n and β_n are defined in (4.28) and (4.29).

Since $\lim_{n\to\infty} p_d$ exists, we have

$$b_{n1} = -\frac{np_d}{\sqrt{np_d(1-p_d)}} \leq -C_3\sqrt{n},$$

for some positive constant C_3.

From this fact and the fact that $\lim_{n\to\infty} c_n$ exists, we obtain

$$\frac{b_{n1}\sqrt{c_n^2+1}}{2c_n} \leq -C_4\sqrt{n}. \tag{5.13}$$

and

$$\gamma_n = \frac{b_{n1}\sqrt{c_2^2+1}}{2} \leq -C_5\sqrt{n}, \tag{5.14}$$

for some positive constants C_4 and C_5.

By (5.5), (5.13) and (5.14), we have

$$\Phi\left(\frac{b_{n1}\sqrt{c_n^2+1}}{2c_n}\right) \leq \Phi\left(-C_4\sqrt{n}\right) = O\left(\frac{1}{\sqrt{n}}\right) \tag{5.15}$$

and

$$\Phi\left(\gamma_n\right) \leq \Phi\left(-C_5\sqrt{n}\right) = O\left(\frac{1}{\sqrt{n}}\right). \tag{5.16}$$

Since $\lim_{n\to\infty} p_d$, $\lim_{n\to\infty} c_n$ and $\lim_{n\to\infty} b_n$ exist, we have

$$b_{n2} = \frac{(1-2p_d)\sqrt{n}}{2\sqrt{p_d(1-p_d)}} - \frac{\log\left(K/S_0\right)}{2\lambda\sigma\sqrt{Tp_d(1-p_d)}} \geq C_6\sqrt{n}$$

and

$$\beta_n = b_{n2}\sqrt{c_n^2+1} + \frac{c_n b_n}{\sqrt{c_n^2+1}} \geq C_7\sqrt{n},$$

for some positive constants C_6 and C_7, i.e., $-\beta_n \leq -C_7\sqrt{n}$.

From this fact and (5.5), we have

$$\Phi\left(-\beta_n\right) \leq \Phi\left(-C_7\sqrt{n}\right) = O\left(\frac{1}{\sqrt{n}}\right). \tag{5.17}$$

By (5.12) and (5.15)–(5.17), we obtain

$$R_1 \geq 2\pi \left(\Phi(d_2) + O\left(\frac{1}{\sqrt{n}}\right)\right) \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) = 2\pi\Phi(d_2) + O\left(\frac{1}{\sqrt{n}}\right). \quad (5.18)$$

Hence, by (5.2), (5.6), (5.11) and (5.18),

$$\frac{1}{2\pi}\int_{b_{n1}(p_d)}^{b_{n2}(p_d,a_n)} \int_{b_n(p_u,p_d,a_n)+c_n(p_u,p_d)x_2}^{e_n(p_u,p_d)-c_n(p_u,p_d)x_2} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 = R_1 - R_2 = \Phi(d_2) + O\left(\frac{1}{\sqrt{n}}\right).$$

□

Lemma 5.5. *Let* (X_{n1}, X_{n2}, X_{n3}) *be trinomial random vector with parameters* n *and* (q_u, q_d, q_m)*, where* q_u*,* q_d *and* q_m *are defined in* (4.3)–(4.5)*. Let* A_n *be defined in* (3.13) *with* a_n *is defined in* (4.6)*. If* $K \geq S_0$*, then for large* n*,*

$$P\left((X_{n1}, X_{n2}, X_{n3}) \in A_n\right) = \Phi(d_1) + O\left(\frac{1}{\sqrt{n}}\right),$$

where d_1 *is defined in* (4.32)*.*

Proof. We can follow the arguments of the proof in lemma 5.4 and replace p_u and p_d by q_u and q_d to show that the conclusion of this lemma holds. □

Using lemma 4.1 and lemma 5.4-5.5, we have the following Theorem.

Theorem 5.6. *Let* T_n *and* C_{BS} *defined in* (2.10) *and* (2.2)*. If* $K \geq S_0$*, then, for large* n*,*

$$T_n = C_{BS} + O\left(\frac{1}{\sqrt{n}}\right).$$

CHAPTER VI
FUTURE RESEARCH

In this Chapter, we will give some idea of future research for trinomial formula. In case of binomial formula, Heston and Zhou ([12], 2000) showed that the rate of convergence is $\frac{1}{\sqrt{n}}$. After that, Diener and Diener ([10]) improved the rate to $\frac{1}{n}$ in 2004. To do this, they used the Berry-Esseen theorem for binomial distribution with a collection term, i.e.,

$$P\left(\frac{X-np}{\sqrt{np(1-p)}} \leq x\right) = \Phi\left(x\right) + \frac{C}{\sqrt{n}} + O\left(\frac{1}{n}\right), \tag{6.1}$$

where X is a binomial random variable with parameter (n, p), C is a known constant and Φ is the standard normal distribution function.

In [10], when they applied (6.1) to B_n, the term of order $\frac{1}{\sqrt{n}}$ is zero. Hence,

$$B_n = C_{BS} + O\left(\frac{1}{n}\right).$$

From Chapter 5, we know that the rate of convergence in case of trinomial formula is $\frac{1}{\sqrt{n}}$. We have an idea to use the same technique of [10] to improve the rate of convergence to $\frac{1}{n}$. To prove this conjecture, we need the Berry-Esseen theorem for trinomial distribution with a correction term.

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