ผลเฉลยโฮโลกราฟฟิกจากเกจซูปเปอร์กราวิตี N=6 ในสี่มิติ

นายจักรภัทร สียางนอก

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรวิทยาศาสตรมหาบัณฑิตมหาบัณฑิต
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HOLOGRAPHIC SOLUTIONS FROM N=6, D=4 GAUGED SUPERGRAVITY

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ในงานวิจัยนี้ เราศึกษาผลเฉลยที่มีซูปเปอร์ซิมเมทรี่จากเกจซูปเปอร์กราวิตี $N=6$ ในสี่มิติที่มีเกจกรุ๊ป $SO(6)$ การมีอยู่ของสุญญากาศซูปเปอร์ซิมเมทรี่ AdS_4 ที่รักษาสมมาตร $SO(6)$ ในทฤษฎีดังกล่าวสอดคล้องกับทฤษฎีสนามซูปเปอร์คอนฟอลมอลในสามมิติ ซึ่งอนุญาตให้เราศึกษาผลเฉลยในรูปของโดเมนวอล ที่เชื่อมโยงระหว่างสุญญากาศ AdS_4 นี้กับจุดเอกฐานในระดับ IR ทีมีสมมาตร $SO(2)\times SO(4), U(3), SO(2)\times SO(2)\times SO(2)$, และ $SO(3)$ ผลเฉลยเหล่านี้อธิบาย RG flows จากทฤษฎีสนาม N=6 ซูปเปอร์คอนฟอลมอลในระดับ UV ไปยังทฤษฎีสนามไม่คอนฟอลมอลที่ผิดรูปไปเนื่องจากมวลในระดับ IR นอกจากผลเฉลยที่มีสมมาตร $SO(3), U(3)$, และ $SO(2)\times SO(2)\times SO(2)$ ที่รักษาสมมาตรทั้งหมดของซูปเปอร์ซิมเมทรี่ $N=6$ แล้ว ผลเฉลยที่มีสมมาตร $SO(3)$ สามารถรักษาซูปเปอร์ซิมเมทรี่ $N=6$ หรือ $N=2$ ได้ขึ้นอยู่กับการปรากฏอยู่ของสเกลาร์เทียม นอกจากนี้ ผลเฉลยที่มีสมมาตร $SO(2)\times SO(4)$ ยังให้การผิดรูปไปเนื่องจากมวลที่รู้จักอยู่แล้วของทฤษฎีสนาม $N=6$ ซูปเปอร์คอนฟอลมอล เรายังได้ศึกษาผลเฉลยในรูปของเจนัสที่มีสมมาตร $SO(2)\times SO(4)$ ซึ่งอธิบายความผิดปกติเชิงคอนฟอลมอลในสองมิติในทฤษฎีสนาม $N=6$ ซูปเปอร์คอนฟอลมอลที่ไม่ทำลายซูปเปอร์ซิมเมทรี่ $N=(4,2)$ ในท้ายที่สุด เราศึกษาหลุมดำ AdS ที่มีซูปเปอร์ซิมเมทรี่และประจุแม่เหล็กโดยมีขอบฟ้าเหตุการณ์อยู่ในรูปของ $AdS_2\times H^2$ เราพบว่าผลเฉลยที่สมมาตรของ $SO(2)\times SO(4)$ และมีซูปเปอร์ซิมเมทรี่ $N=2$ ที่ควบคู่ระหว่าง $AdS_2\times H^2$ กับ สุญญากาศ AdS_4 ซึ่งอธิบายถึงการบีบอัดเชิงทวิตท์ของทฤษฎีสนาม $N=6$ ซูปเปอร์คอนฟอลมอลบนอวกาลไฮเปอร์โบลิค H^2

ภาควิชา ฟิสิกส์ .... ลายมือชื่อนิสิต ....

สาขาวิชา ฟิสิกส์ .... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....

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JAKKAPAT SEEYANGNOK : HOLOGRAPHIC SOLUTIONS FROM N=6, D=4 GAUGED SUPERGRAVITY. ADVISOR : PROF. PARINYA KARNDUMRI, Ph.D., 147 pp.

In this thesis, we study holographic solutions of four-dimensional $N = 6$ gauged supergravity with $SO(6)$ gauge group. The theory admits a unique $N = 6$ supersymmetric AdS_4 vacuum dual to a three-dimensional $N = 6$ SCFT and gives us a number of supersymmetric domain walls interpolating between this AdS_4 vacuum and singular geometries in IR with $SO(2) \times SO(4)$, $U(3)$, $SO(2) \times SO(2) \times SO(2)$, and $SO(3)$ symmetries. These solutions describe RG flows from $N = 6$ SCFT in UV to non-conformal field theories driven by mass deformations. In particular, the solution with $SO(2) \times SO(4)$ symmetry coincides with the known mass deformations of the dual $N = 6$ SCFT. Despite most of the solutions preserving $N = 6$ supersymmetry, the $SO(3)$ case preserves $N = 6$ or $N = 2$ depending on the absence of pseudoscalars. We also give an analytic form of supersymmetric Janus solution with $SO(2) \times SO(4)$ symmetry, which describes a two-dimensional conformal defect in the $N = 6$ SCFT with unbroken $N = (4, 2)$ supersymmetry. Finally, we study supersymmetric AdS black hole with magnetic charges and the horizon geometry of the form $AdS_2 \times H^2$. We find an N=2 supersymmetric solution with $SO(2) \times SO(4)$ symmetry interpolating between the AdS_4 vacuum and an $AdS_2 \times H^2$ fixed point, which is dual to a twisted compactification of $N = 6$ SCFT on hyperbolic space H^2.

Department : Physics .... Student's Signature ....

Field of Study : Physics .... Advisor's Signature ....

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Contents

page

Chapter **page**

List of Tables

Table **page**

List of Figures

Figure **page**

Figure **page**

CHAPTER I

Introduction

General Relativity describes the dynamics of gravity by the presence of curvature spacetime. This revolution leads to several fascinating phenomena and new understandings of astrophysics. Black holes are an object surrounded by a horizon from which even light cannot escape [67, 68]. Gravitational waves recently discovered in [70] is the ripples of spacetime traveling at the speed of light. Gravitational lensing is a manifest example of how matter disturbs spacetime. Spectacular un-

Figure 1.1: The first image of back hole at the center of galaxy M87 (Credits: Event Horizon Telescope collaboration et al.)

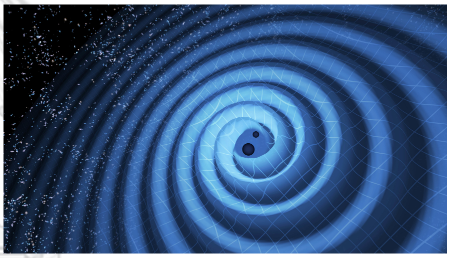

Figure 1.2: An illustration of two black holes merging which creates the ripples of spacetime called gravitational waves (credit: LIGO/T. Pyle)

derstandings of Big Bang cosmology also result from general relativity with the accidental discovery of cosmic microwave background (CMB) in 1940. The experimental discoveries of these consequences firmly support the idea of Einstein's gravity of general relativity. Despite these successes, physicists discovered some disfavor aspects of general relativity. The presence of matter in general relativity lacks a microscopic picture of quantum physics. As matters are squashed into a

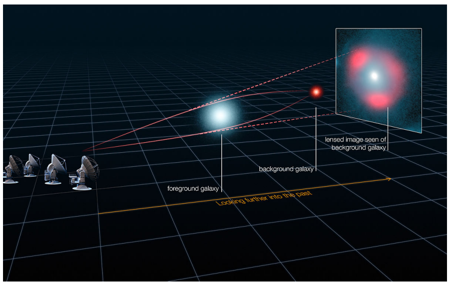

Figure 1.3: The image of a distant galaxies is distorted by nearer galaxies called the gravitational lensing. (Credit: ALMA (ESO/NRAO/NAOJ).)

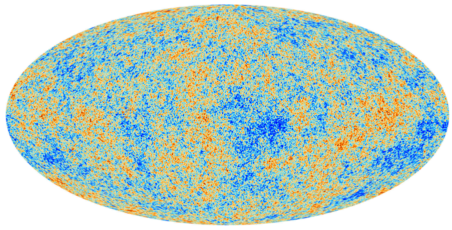

Figure 1.4: The snapshop of the universe when it was 380,000 years old is called CMB. (Credits: ESA and the Planck Collaboration)

tiny region of spacetime, the curvature of spacetime becomes infinity leading to a catastrophic situation for general relativity. As opposed to other fundamental interactions, electromagnetic, weak, and strong interactions, gravity is short of a quantum description, and cannot be quantized to quantum gravity in a usual way of canonical quantization due to uncontrollable divergences, see some reviews in [71, 72]. Therefore, we think that promoting gravity to quantum gravity might eventually lead to an aspect of the theory of everything.

String theory [74–79] is a quantum theory that contains one-dimensional vibrating strings and p-dimensional spatial extended objects called p-branes. The quantization of strings gives rise to different types of fields. It was first expected to be a theory of quantum gravity, furthermore, it has been demonstrated as a promising candidate for the theory of everything. There are five different versions of self-consistent string theories: type I, type IIA, type IIB, Heterotic $SO(32)$, and Heterotic $E_8 \times E_8$ living in ten-dimensional spacetime. Among these strings, M-theory is an eleven-dimensional non-perturbative theory connecting them via dualities shown in figure (1.5). As a promising candidate of the theory of everything, string/M-Theory should give effective field theories consistent with quantum field theories in four-dimensional spacetime. In principles, this can be studied by the

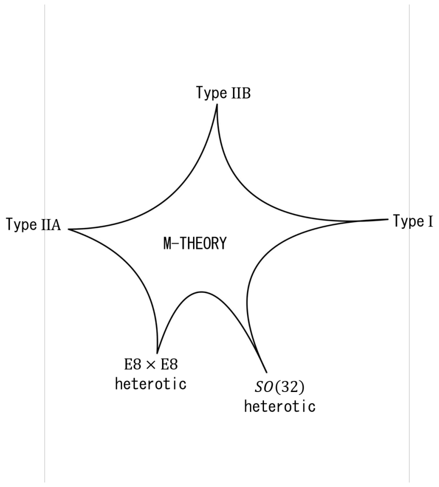

Figure 1.5: The image shows the five different string theories living in $D = 10$ which can be related to M-theory living in $D = 11$.

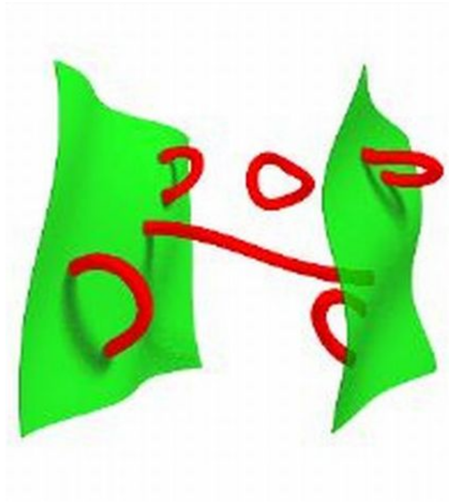

Figure 1.6: This imamge shows a closed string together with open strings with its ends being subject to p-branes (Credit: This image created by Steuard Jensen and brought to David Tong)

compactification of string theory on the product manifold $\mathbb{R}^{1,3} \times M^{D-4}$, where $\mathbb{R}^{1,3}$ is four-dimensional Minkowski spacetime and M^{D-4} is an internal compact manifold which plays an important role to determine interactions. Moreover, things get more interesting, when we study String/M-Theory in anti-de Sitter spacetime (AdS).

According to the AdS/CFT correspondence [1–3], string theory in $AdS_{d+1} \times M^{D-d-1}$ spacetime is dual to a superconformal field theory (SCFT) which lives on its boundary ∂AdS_{d+1} corresponding to d-dimensional Minkowski spacetime $\mathbb{R}^{1,d-1}$. In this sense, local fields $h_i(x,r)$[1] in the bulk spacetime is dual to local operators $\mathcal{O}_i(x)$ in the dual SCFT on the boundary. This allows us to calculate correlation functions describing interactions in the dual SCFT. To treat calcula-

[1]Where (x,r) is the coordinate of the bulk AdS spacetime, and x is the coordinate on the boudary of the AdS spacetime.

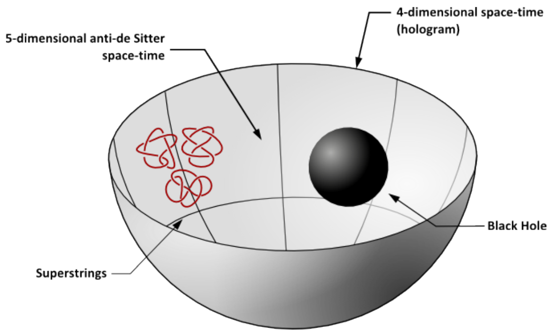

Figure 1.7: This image shows that AdS/CFT correspodence is the equivalence between physics of gravity in the bulk of AdS_5 and N=4 supersymmetric Yang-Mills theory (SYM) on the boundary of AdS_5. (Credit: `https://www.quantum-bits.org/?p=1134`)

tions more traceable, we apply the low energy limit and the large-N limit [1]. In this limit, the AdS/CFT duality suggests that strongly-coupled SCFTs is dual to gauged supergravities. This duality is also called strong/weak duality which is one of the most important results of string/M-Theory. We can generalize the duality by studying string/M-Theory in[2] $AAdS_{d+1} \times M^{D-d-1}$ dual to non-conformal field theories, quantum field theories. It is very important to have such a duality because if string/M-Theory is truly a theory of everything, it must satisfy all theoretical aspects in physics. One of the most famous example of the AdS/CFT correspondence is that of type IIB string theory in $AdS_5 \times S^5$ dual to four-dimensional $N = 4$ supersymmetric Yang-Mills field theory as shown in figure (1.7). This shows that type IIB string theory having its low-energy action as type IIB supergravity in $AdS_5 \times S^5$ containing some properties that are dynamically equivalent to four-dimensional $N = 4$ supersymmetric Yang-Mills field theory with gauge group $SU(N)$ and a coupling constant g_{YM}. Therefore, the free parameters of supersymmetric Yang-Mills theory are mapped to the free parameters obtained from type IIB supergravity which is weakly curved in $AdS_5 \times S^5$ spacetime.

On the other hand, supergravity is an extension theory of general relativity,

[2] $AAdS$ is an asymptotic anti-de Sitter space.

which is invariant under local supersymmetries. Supersymmetry gives the relation between the internal and spacetime symmetry. It introduces spinor generators called supercharges along with graded Lie algebra. The commutation relations of supercharges with internal and spacetime generators form Lie superalgebra. As a result, for an elementary particle, there exists the corresponding particle called superpartner forming a supermultiplet. For example, the superpartner of a massless spin-2 graviton is a massless spin-3/2 gravitino. The introduction of supersymmetry to general relativity also results in a better divergence behavior. Supergravity is a gauge theory of supersymmetry living in various dimensions ranging from two to eleven dimensions. We can also construct supergravity by coupling the gravity multiplet to matter multiplets, a chiral multiplet, or a vector multiplet. On the other hand, we can also increase the number of supersymmetry. The former theory is called matter-coupled supergravities. The latter is called extended supergravities. Those theories are called ungauged supergravity since the gravitini are not charged. We can further promote ungauged supergravities to more interesting theories called gauged supergravities where gravitini are charged under gauge fields of the theory by promoting a suitable subgroup of global symmetry group to a gauge group. Studying supersymmetric solutions of gauged supergravities in various dimensions also plays an important role in understanding string/M-theory. In the AdS/CFT correspondence, these solutions provide the holographic tool to investigate strongly-coupled systems of quantum field theories, conformal defects, and condensed matter systems AdS/CMT [73]. In many cases, solutions of lower-dimensional gauged supergravities can be uplifted to to higher-dimensional origins via consistent truncations resulting in a complete description of AdS/CFT dualities.

In this thesis, we study supersymmetric solutions from four-dimensional $N = 6$ gauged supergravity with $SO(6)$ gauge group. The theory has previously been constructed in [4] by using embedding tensor formalism which is obtained from a consistent truncation of the maximal $N = 8$ gauged supergravity [5], see also [6–8]. The N=6 gauged supergravity admits a unique $N = 6$ supersymmetric AdS_4 vacuum preserving $SO(6)$ symmetry dual to three-dimensional $N = 6$ SCFT. A recent result on supersymmetric AdS vacua [9] also confirms the uniqueness of $N = 6$ supersymmetric AdS_4 vacua. The reserach paper [4] has also pointed out that this AdS_4 vacuum describes a consistent truncation of type IIA theory on CP^3, so the AdS_4 vacuum can be uplifted to type IIA theory in the form of $AdS_4 \times CP^3$ space. This has been shown in [10] and more recent studies in [11–14]. The dual $N = 6$ SCFT of type IIA theory has been studied in [15]. In general, three-dimensional superconformal field theories (SCFTs) have the form of Chern-Simons-Matter (CSM) theories because the usual gauge theories with Yang-Mills gauge kinetic terms are not conformal. These SCFTs result from would-volume theories of M2-branes on various transverse spaces and they play an important role in understanding the dynamics of M2-branes. Studying supersymmetric solutions of gauged supergravities may be useful for the use of their holographic descriptions at least in the large-N limit.

Many supersymmetric solutions of gauged supergravities have been studied and interpreted in terms of their dual field theories. In this thesis, we will be studying these solutions from four-dimensional $N = 6$ gauged supergravities. We firstly look at supersymmetric domain wall solutions interpolating between the $N = 6$ supersymmetric AdS_4 vacuum and singular geometries. These solutions describe RG flows from the dual $N = 6$ SCFT in UV to non-conformal phases in IR resulting from mass deformations. There are a number of similar solutions having intensively studied in $N = 8$ and $N = 2$ gauged supergravities, [17–25], along with a recent studies of $N = 3, 4, 5$ gauged supergravities, [26–31]. We hopefully expect that this work would fill up the complete solutions of gauged supergravities in four dimensions. We will also study supersymmetric Janus solutions in which the spacetime takes the form of AdS_3-sliced domain walls interpolating between asymptotic AdS_4 spaces. Holographically, these solutions are dual to two-dimensional conformal defects within the $N = 6$ SCFT and break the superconformal symmetry in the three-dimensional bulk to a smaller one on the two-dimensional surfaces. There is also this kind of solution in four-dimensional gauged supergravities which has previously been studied, [28, 29, 31–35]. Finally, we will look for supersymmetric solutions that take the form of $AdS_2 \times \Sigma^2$ geometries with Σ^2 being a Riemann surface, and interpolate between these $AdS_2 \times \Sigma^2$ geometries and the supersymmetric AdS_4 vacuum. These solutions describe supersymmetric black holes in asymptotically AdS_4 space, and there is a number of these solutions which has already been investigated in other gauged supergravities, [31, 36–46].

$N = 6$ gauged supergravity in four dimensions has the global symmetry $SO^*(12)$ with the compact maximal subgroup of $U(6) \sim SU(6) \times U(1)$ together with thirty real scalars encoded in a coset manifold of $\mathcal{M}_{\text{scl}} = SO^*(12)/U(6)$. The gauge group $SO(6)$ of the $N = 6$ gauged supergravity can be obtained from a consistent truncation of the $SO(8)$ gauge group of the maximal $N = 8$ gauged supergravity. The $N = 8$ gauged theory appears as a consistent truncation of eleven-dimensional supergravity on S^7 [50–55]. The four-dimensional $N = 6$ gauged supergravity with $SO(6)$ gauge group can be uplifted via consistent truncations to eleven dimensions. On the other hand, the N=6 theory is also a consistent truncation of type IIA theory on CP^3. As a result, all solutions of the $N = 6$ theory which will be given here have higher-dimensional origins and can be embedded in ten- or eleven-dimensional supergravities. Besides, the scalar potential of the four-dimensional $N = 6$ gauged supergravity has already considered in [56] along with a recent version in a more general form of the embedding tensor formalism in [4] where the fermion-shift matrices and the scalar potential have been identified by consistent truncation of the $N = 8$ theory.

We will organize the thesis as follows. We will firstly review the important ingredient of supergravity in the first few chapters. In the introduction, we review all the relevant ideas of studying four-dimensional $N = 6$ gauged supergravity with $SO(6)$ gauge group. In chapter 2, we will review some properties of a manifold for studying general relativity. Then, we apply such tools to describe gravity in chapter 3. After that, we will recall some crucial properties of supersymmetry and supergravity in chapter 4. In chapter 5, we will discuss the structure of $N = 6$ gauged supergravity in four dimensions with $SO(6)$ gauge group. Finally, we will look at the holographic solutions of the $N = 6$ gauged theory in chapter 6. Then, we end this thesis with the conclusion and comments.

CHAPTER II

Differential Geometry

Differential geometry is the study of curved space. In physics, it is very important to define a locally flat space at every point on a curved space, so the curved space is called a manifold. To study physics on a curved manifold, we need additional structures such as a metric tensor and connections. A manifold equipped with the metric tensor is said to be (pseudo-)Riemannian manifold denoted by $(\mathcal{M}, g)$, see also [79–88].

2.1 Manifolds and Tensor Fields

A d-dimensional manifold is a locally flat topological space $\mathcal{M}$ with the differentiable structure of class C^k.The manifold $\mathcal{M}$, for every point on the manifold $p_\alpha \in \mathcal{M}$, has neighborhood homeomorphism from an open set u_α which can be mapped to a coordinate function $\mathbb{R}^d$ by Ψ_α as shown in figure (2.1). Ψ_α is the coordinate function represented by d variables $\{x^\mu(p)\} = \{x^1(p), x^2(p), ..., x^d(p)\}$ [1]. The pair of (u_α, Ψ_α) is called a coordinate chart.

$$\Psi_\alpha : u_\alpha \in \mathcal{M} \to \mathbb{R}^d \tag{2.1.1}$$

The collection of coordinate charts is called atlas "A".

It is also possible to have more than one coordinate representation, so we define the coordinate transformation $x^\mu(p) = y^\nu(p)$ as shown in figure (2.1) by

$$\Psi_\alpha \circ \Psi_\beta^{-1} : \mathbb{R}^d \to \mathbb{R}^d. \tag{2.1.2}$$

[1]If a d-dimensional manifold takes the form of $\mathcal{M}^{1,d-1}$, the local space becomes Lorentzian space of $\mathbb{R}^{1,d-1}$.

$(\Psi_\alpha \circ \Psi_\beta^{-1})$ is a composite map between coordinate charts, if $u_\alpha \cap u_\beta \neq \varnothing$.

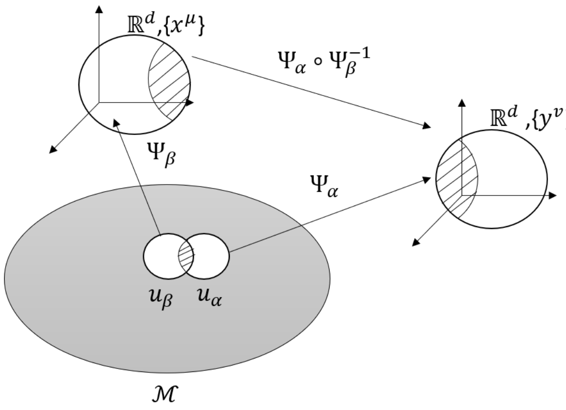

Figure 2.1: The map between overlapping coordinate charts

The definition of the differentiable structure of class $C^k(1 \leq k \leq \infty)$ of a d-dimensional manifold $\mathcal{M}$ is a collection of coordinate charts $\{(u_\alpha, \Phi_\alpha) \mid \alpha \in A\}$ satisfying

1. $\bigcup_{\alpha \in A} u_\alpha = \mathcal{M}$
2. $(\Psi_\alpha \circ \Psi_\beta^{-1})$ is C^k, $\alpha, \beta \in A$.

If $k \to \infty$, the manifold is said to be smooth manifold.

Let $f : \mathcal{M} \to \mathcal{N}$ be a map between an m-dimensional and an n-dimensional manifold as shown in figure (2.2) together with $\Phi(p) = \{x^\mu\}$, $\Psi(f(p)) = \{y^\alpha\}$, and $y = \Psi \circ \Phi^{-1}$, we can write a coordinate representation in terms of another coordinate representation as

$$y^\nu = f^\nu(x^\mu) \tag{2.1.3}$$

where f is C^k times differentiable with respect to x^μ around a point $p \in \mathcal{M}$ or $\Phi(p) = \{x^\mu\}$. If $k \to \infty$, f is said to be smooth. f is called a diffeomorphism if $\Phi \circ f \circ \Psi^{-1}$ is invertible and both $y = \Phi \circ f^{-1}\Psi^{-1}(x)$ and $x = \Psi \circ f \circ \Phi^{-1}(y)$ are smooth namely, C^∞. The manifold $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic regarded as the same manifold denoted by $\mathcal{M} \equiv \mathcal{N}$, and dim $\mathcal{M}$=dim $\mathcal{N}$.

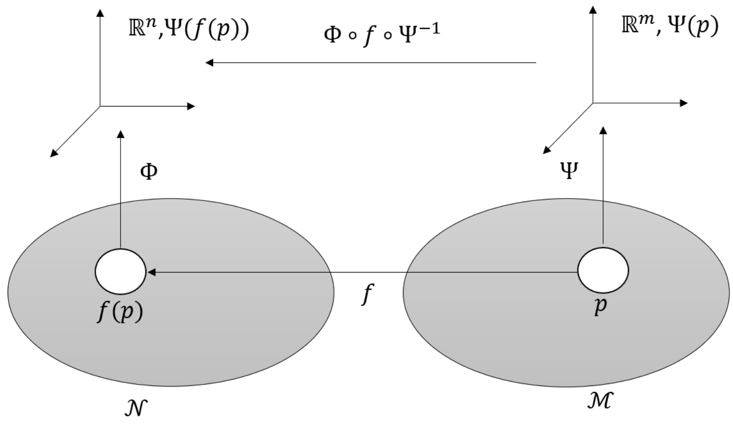

Figure 2.2: The map between manifolds

2.1.1 Product Manifold

If both $\mathcal{M}$ and $\mathcal{N}$ are an m- and n-dimensional manifold, a product manifold [2] $\mathcal{M} \times \mathcal{M}$ is an $(m+n)$-dimensional manifold. For example, the product manifold of two one-sphere manifolds is a torus, $T^2 = S^1 \times S^1$ as shown in figure (2.3).

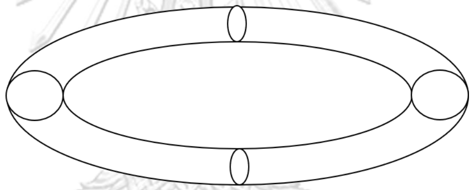

Figure 2.3: The product manifold of $\mathcal{M} \times \mathcal{N} = S^1 \times S^1 = T^2$.

2.1.2 Curves and Scalars

A curve $c(t)$ on a manifold shown in figure (2.4) which does not intersect with itself can be parameterized by a parameter t where $t \in (a,b)$ and (a,b) can be extended to $(-\infty, \infty)$. $c(t)$ can be written in the coordinate representaion as

$$x(t) = \Psi \circ c : \mathbb{R} \to \mathbb{R}^d \tag{2.1.4}$$

where we can write $\{x^\mu(t)\} = \{x^1(t), x^2(t), ..., x^d(t)\}$.

[2] A $(1,d)$-dimensional spacetime in supergravities can be studied by the compactification of D-dimensional string/M-theory on a (D-d-1)-dimensional compact manifold via the product

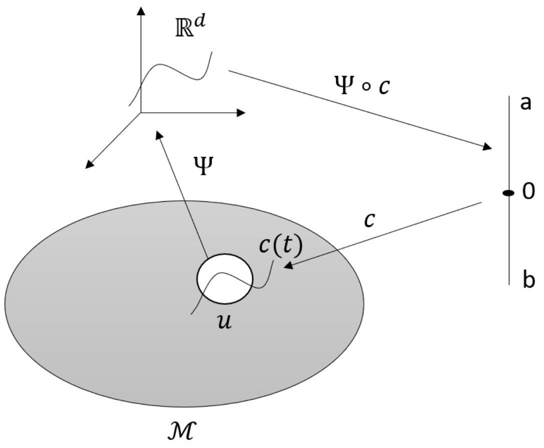

Figure 2.4: This figure shows the map between a curve on a manifold, coordinate chart, and real numbers

A scalar function f on a manifold $\mathcal{M}$ is a smooth map from the manifold $\mathcal{M}$ to real numbers $\mathbb{R}$ as shown in figure (2.5)

$$f : \mathcal{M} \to \mathbb{R} \tag{2.1.5}$$

where f can be parameterized by a coordinate chart of d variables written as $f \circ \Psi^{-1} : \mathbb{R}^d \to \mathbb{R}$

$$f(x) = f \circ \Psi^{-1}(x) \tag{2.1.6}$$

which defines a real-valued function on $\mathcal{M}$ in terms of coordinate representation. A set of smooth scalar functions on the manifold $\mathcal{M}$ is denoted by $\mathbb{F}(\mathcal{M})$.

2.1.3 Vectors and Dual Vectors

A vector on the manifold $\mathcal{M}$ can be defined as a tangent vector of a parametric curve $c(t)$ as shown in figure (2.6). The tangent vector at a point $p = c(t = 0)$ is determined by a directional derivative of a scalar function along the parametric

manifold of a non-compact manifold and the compact manifold of $\mathbb{R}^{1,d} \times \mathcal{M}^{D-d-1}_{\text{internal}}$.

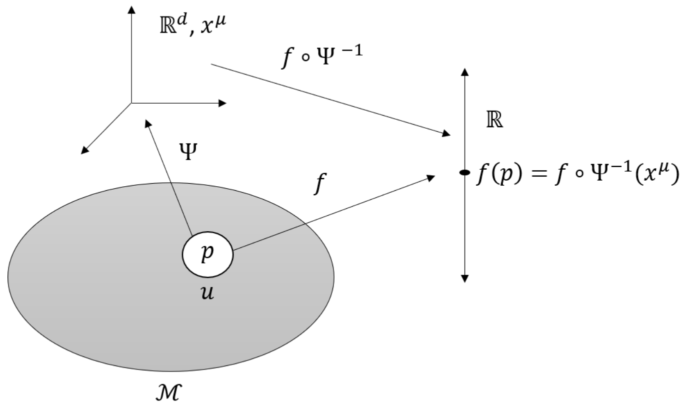

Figure 2.5: The map between a curve on manifold $c(t)$, coordinate chart Ψ, and real number $\mathbb{R}$

curve $f(c(t))$ given by

$$\frac{df(c(t))}{dt}\Big|_{t=0} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu(t)}{dt}\Big|_{t=0} \tag{2.1.7}$$

$$= V^\mu \partial_\mu f \tag{2.1.8}$$

where we have used the chain rule by writting $c(t)$ in terms of a coordinate representation and $V^\mu = \frac{dx^\mu(c(t))}{dt}\big|_{t=0}$.

We can define a tangent vector operator V on a manifold $\mathcal{M}$ at a point $p = c(t = 0)$ as

$$V = V^\mu \partial_\mu \tag{2.1.9}$$

where V^μ is called a vector component of a basis vectors $e_\mu = \partial_\mu$.

The action of the tangent vector operator V on a scalar f can written as

$$V[f] \equiv \frac{df(c(t))}{dt}\Big|_{t=0} = V^\mu \partial_\mu f \tag{2.1.10}$$

which coincides with the equation (2.1.7).

However, there usually exists more than one parametric curve at a point p which gives the same tangent vector, so we can define the equivalent class of parametric curves on a manifold if the curves satisfy

1. $c_1(t = 0) = c_2(t = 0) = p$

2. $\frac{dx^\mu(c_1(t))}{dt}\big|_{t=0} = \frac{dx^\mu(c_2(t))}{dt}\big|_{t=0}$.

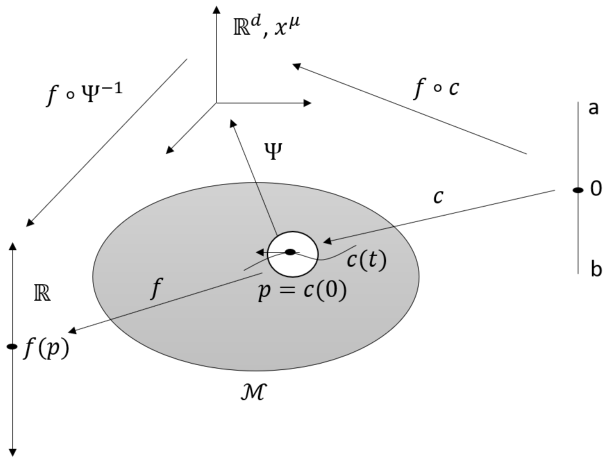

Figure 2.6: This figure shows the map between a curve on manifold, coordinate charts, and real number to define the general coordinate transformation

The curves $c_1(t)$ and $c_2(t)$ resulting in the same tangent vector V at a point p are in the same equivalence class, manely $c_1(t) \sim c_2(t)$. All of the equivalence classes of curves at a certain point $p \in \mathcal{M}$ characterize all of the tangent vectors forming a tangent vector space T_pM. The collection of tangent vectors spaces $\bigcup\limits_{p\in M} T_pM$ having the structure of a differentiable manifold is called the tangent bundle TM.

We can also define the transformation rule of the vector between different coordinate representations as

$$V = V^\mu \partial_\mu = V^{\mu'} \partial_{\mu'} \tag{2.1.11}$$

since the vector itself is a geometric object and is independent of coordinate representations, we can define the transformation of the vector component

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu. \tag{2.1.12}$$

and the transformation of the basis vector as

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \tag{2.1.13}$$

In general, $\frac{\partial x^{\mu'}}{\partial x^\mu}$ and $\frac{\partial x^\mu}{\partial x^{\mu'}}$ are not constant, so the transformation of the vector $V^\mu(x) \to V^{\mu'}(x')$ is called general coordinate transformation (GCT). The new basis $e_{\mu'}$ of T_pM is now the linear combination of the original one e_μ.

A dual vector (one-form) is a linear function mapping a vector to real numbers satisfying the linear properties

$$\omega : \mathbb{R}^n \rightarrow \mathbb{R}, \tag{2.1.14}$$

$$\omega(\alpha v + \beta u) = \alpha\omega(v) + \beta\omega(u) \tag{2.1.15}$$

where $v, w \in \mathbb{R}^n$. The dual vector space is a vector space $(\mathbb{R}^n)^*$ dual to the vector space $\mathbb{R}^n$. Therefore, the dual tangent vector space at a point $p \in \mathcal{M}$ of T_pM is called cotangent space T_p^*M. If $\omega_p \in T_p^*M$,

$$\omega_p : T_pM \rightarrow \mathbb{R}. \tag{2.1.16}$$

Regarding dx^μ as a dual basis vector of T_p^*M, we write a dual vector as

$$\omega = \omega_\mu dx^\mu. \tag{2.1.17}$$

The action between a dual basis vector of T_p^*M and a basis vector of T_pM can be defined as

$$\langle dx^\mu, \frac{\partial}{\partial x_\nu} \rangle = \delta^\mu_\nu. \tag{2.1.18}$$

We can define the inner product as $\langle \quad , \quad \rangle : T_p^*M \times T_pM \rightarrow \mathbb{R}$ which maps a vector and a dual vector to real numbers

$$\langle \omega, V \rangle = \omega_\mu V^\nu \langle dx^\mu, \frac{\partial}{\partial x_\nu} \rangle = \omega_\mu V^\mu. \tag{2.1.19}$$

Similar to the vector transformation (2.1.12), we can also define the dual vector transformation under the change of coordinate representation[3] given by

$$\omega = \omega_\mu dx^\mu = \omega_{\mu'} dx^{\mu'}. \tag{2.1.20}$$

The transformation of the dual vector component can be written as

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu \tag{2.1.21}$$

[3]This is true for an arbitrary tensor because a tensor is a geometrical object which is invariant under the change of coordinate representation. However, the component of a tensor does change due to the change of coordinate which we choose to represent. Then, whenever we write tensor transformation, we mean the tensor component.

and the transformation of dual basis vector reads

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} dx^{\mu}. \tag{2.1.22}$$

2.1.4 Tensors

A tensor is a multilinear map of r elements of tangent vector space T_pM and q elements of cotangent space T_p^*M to real numbers $\mathbb{R}$

$$T^{q,r} \in (T_1^*(M) \otimes ... \otimes T_r^*(M)) \otimes (T_1(M) \otimes ... \otimes T_p(M)) = T^q_{r,p}(M) \tag{2.1.23}$$

where $T^q_{r,p}(M)$ is an extended vector space at point $p \in \mathcal{M}$. $T^{q,r}$ is called a tensor of rank (q,r). We can express the rank $T^{q,r}$ tensor in terms of dual basis vectors and basis vectors as following

$$T^{q,k} = T^{\mu_1 ... \mu_q}{}_{\nu_1 ... \nu_r}(\frac{\partial}{\partial x^{\mu_1}} \otimes \frac{\partial}{\partial x^{\mu_2}} \otimes ... \otimes \frac{\partial}{\partial x^{\mu_q}} \otimes dx^{\nu_1} \otimes dx^{\nu_2} \otimes ... \otimes dx^{\nu_r}). \tag{2.1.24}$$

Similar to the (dual)-vector transformation (2.1.12), (2.1.21), a tensor transforms according to the vector and the dual vector component with the upper and lower indices, respectively

$$T^{\mu'_1 ... \mu'_r}{}_{\nu'_1 ... \nu'_k} = (\frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \frac{\partial x^{\mu'_2}}{\partial x^{\mu_2}} ... \frac{\partial x^{\mu'_r}}{\partial x^{\mu_r}})(\frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \frac{\partial x^{\nu_2}}{\partial x^{\nu'_2}} ... \frac{\partial x^{\nu_k}}{\partial x^{\nu'_k}}) T^{\mu_1 ... \mu_r}{}_{\nu_1 ... \nu_k}. \tag{2.1.25}$$

If a vector is smoothly distributed at a point $p \in \mathcal{M}$, the vector is said to be a vector field. Similarly, a tensor field of type (r,k) is a smooth distribution of element of $T^r_{k,p}(M)$ on each a point $p \in \mathcal{M}$. The set of the vector fields and tensor fields on the manifold $\mathcal{M}$ can be represented by $\chi(\mathcal{M})$ and $\tau^q_r(\mathcal{M})$, respectively.

2.1.5 Lagrangian on Manifolds

Let $\mathcal{M}$ be a differentiable manifold with its tangent bundle TM, Lagrangian is a map from the tangent bundle to real numbers

$$L : TM \to \mathbb{R}. \tag{2.1.26}$$

Lagrangian [4] is a function of position and velocity corresponding to a point on manifold $\mathcal{M}$ and a tangent vector at that point, respectively. A map $\gamma : \mathcal{M} \rightarrow \mathcal{M}$ is a curve of the trajectory of the Lagrangian on a manifold $\mathcal{M}$ if γ extremizes the functional action given by

$$S[\gamma, \dot{\gamma}] \equiv \int_{t_1}^{t_2} L(\gamma, \dot{\gamma}) dt \tag{2.1.27}$$

where $\dot{\gamma} \in TM_{\gamma(t)}$. The evolution of coordiante x^μ of a point in motion satisfies

$$\frac{d}{dt}\frac{\partial L}{\partial V^\mu} = \frac{\partial L}{\partial x^\mu}. \tag{2.1.28}$$

This is called Euler-Lagrange's equation where $\dot{\gamma}(t) = \frac{dx^\mu}{dt}\partial_\mu = V^\mu e_\mu$.

2.2 Differential Forms

2.2.1 p-form tensors

A p-form tensor or p-form is a totally anti-symmetric rank $(0, p)$ tensor defined as

$$\omega^p = \frac{1}{p!}\omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \tag{2.2.1}$$

where the wedge product is antisymmetric namely, $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$, and the space of p-forms is characterized by $\Lambda^p(M)$.

2.2.2 Wedge Product

The wedge product of p-form and q-form on $\Lambda^p(M)$ and $\Lambda^q(M)$ respectively, can be written as

$$\begin{aligned}(A \wedge B)_{p+q} &= \frac{1}{p!q!} A_{\mu_1 \ldots \mu_p} B_{\nu_1 \ldots \nu_q} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p} \\ &= \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\nu_1 \ldots \nu_q]} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p} \\ &= (A \wedge B)_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p}.\end{aligned} \tag{2.2.2}$$

[4] One of the most compact ways to encode the information of a physical system such as the field's equations, symmetries, etc.

So, the component of the wedge product reads

$$(A \wedge B)_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \dots \mu_p} B_{\nu_1 \dots \nu_q]}. \tag{2.2.3}$$

2.2.3 Exterior derivative

The exterior derivative d allows us to differentiate a p-form to get a $(p+1)$-form, or d is a map from $\Lambda^p \to \Lambda^{p+1}$ given by

$$\begin{aligned} d\omega_p &= d(\frac{1}{p!}\omega_{\mu_1\mu_2\dots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}) \\ &= \frac{1}{p!}\partial_\rho \omega_{\mu_1\mu_2\dots\mu_p} dx^\rho \wedge dx^{\mu_1} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p} \\ &= \frac{(p+1)!}{p!}\partial_{[\rho}\omega_{\mu_1\mu_2\dots\mu_p]} dx^\rho \wedge dx^{\mu_1} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p} \\ &= (d\omega)_{\rho\mu_1\mu_2\dots\mu_p} dx^\rho \wedge dx^{\mu_1} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}. \end{aligned} \tag{2.2.4}$$

So, we can read the component of the exterior derivative as

$$(d\omega)_{\rho\mu_1\mu_2\dots\mu_p} = (p+1)\partial_{[\rho}\omega_{\mu_1\mu_2\dots\mu_p]}. \tag{2.2.5}$$

2.2.4 Hodge Duality

The hodge duality $*$ is a map $* : \Lambda^p \to \Lambda^{m-p}$ given by

$$\begin{aligned} *\omega_p &= \frac{1}{p!}\omega_{\mu_1\mu_2\dots\mu_p} * (dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}) \\ &= \frac{1}{p!(n-p)!}\omega_{\mu_1\mu_2\dots\mu_p} \epsilon_{\nu_1\nu_2\dots\nu_{n-p}}{}^{\mu_1\mu_2\dots\mu_p} dx^{\nu_1} \wedge dx^{\nu_2} \wedge \dots \wedge dx^{\nu_{n-p}}. \end{aligned} \tag{2.2.6}$$

So, one can read the component of the hodge duality as

$$(*\omega_p)_{\nu_1\nu_2\dots\nu_{n-p}} = \frac{1}{p!}\epsilon_{\nu_1\nu_2\dots\nu_{n-p}}{}^{\mu_1\mu_2\dots\mu_p}\omega_{\mu_1\mu_2\dots\mu_p}. \tag{2.2.7}$$

2.3 Riemannian Manifolds

A Riemannian manifold is a manifold equipped with a metric tensor. In this section, we will review important quantities related to the metric tensor to study physics on a manifold. One of the most important aspects of the metric tensor is that it encodes the dynamics of gravity.

2.3.1 The Metric tensor and Vielbein

A metric tensor $g_{\mu\nu}$ is an additional structure on a manifold $\mathcal{M}$ so that we can define an infinitesimal distance between two neighboring points

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{2.3.1}$$

where $g_{\mu\nu}$ is a symmetric tensor, and it is a diagonalizable matrix under general coordinate transformation (GCT)

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x^\mu}\frac{\partial x^\sigma}{\partial x^\nu}g_{\rho\sigma}(x). \tag{2.3.2}$$

The type of Riemannian [5] manifold can be classified by the signature of the eigenvalues of the diagonalized metric tensor namely, $+1, 0, -1$

$$g_{\mu\nu} = \begin{cases} \mathrm{diag}(-1, ..., 1, ..., 0), & \text{If there exists a zero eigenvalue,} \\ & g_{\mu\nu} \text{ is degenerate, } g^{\mu\nu} \text{ does not exist.} \\ \mathrm{diag}(1, 1, ..., 1), & \text{If all eigenvalues are positive,} \\ & \text{the manifold is the Riemannian manifold, or Euclidean space.} \\ \mathrm{diag}(-1, 1, ..., 1), & \text{If there exists one negative and non-zero eigenvalue,} \\ & \text{the manifold is the pseudo-Riemannian manifold} \\ & \text{, or Lorentzian space.} \end{cases}$$

[5] The Euclidean manifold plays important role in the internal manifold of supergravities on which string/M-theory are compactified, and Lorentzian manifold is the property of spacetime in which we live especially four-dimensional spacetime.

On other hand, in theories such as supergravities where fermions coupled to gravity, we need to use a frame where the fermions belong called local Lorentz tangent space. The diagonal metric tensor is quadratically related to this frame in terms of a vielbein, e^a_μ, with the flat Minkowski metric $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$.

$$g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x). \tag{2.3.3}$$

The indices $[a,b,c..]$ and $[\mu,\nu,..,\lambda]$ are called tangent and spacetime indices, respectively. Sometimes, we also use $[\hat{\mu},\hat{\nu},..,\hat{\lambda}]$ to refer the tangent indices. Moreover, we can use the vielbien to map a vector of spacetime indices to tangent indices

$$V^a = V^\mu e^a_\mu \quad , \quad V^\mu = e^\mu_a V^a \tag{2.3.4}$$

$$V_a = V_\mu e^\mu_a \quad , \quad V_\mu = e^a_\mu V_a. \tag{2.3.5}$$

The vielbien transforms as dual vector of the local Lorentz tangent space

$$e'^a_\mu(x) = \Lambda^{-1a}{}_b(x)e^b_\mu(x). \tag{2.3.6}$$

The transformation is called local Lorentz transformation because $\Lambda(x)$ depends on spacetime location x. Moreover, The vielbien also transforms under diffeomorphism as

$$e'^a_\mu(x') = \frac{\partial x^\rho}{\partial x^\mu}e_\rho(x) \tag{2.3.7}$$

Therefore, This indicates that the vielbien can also be seen as vectors forming an orthonormal set in tangent space at each point

$$e^a_\mu g_{\mu\nu} e^\nu_b = \eta_{ab} \tag{2.3.8}$$

with e^ν_a being the inverse vielbien.

The vielbien can be also considered as the local Lorentz vector defining the local Lorentz basis 1-form

$$e^a = e^a_\mu(x)dx^\mu \tag{2.3.9}$$

which is dual to its inverse vielbien in the dual basis

$$e_a = e^\mu_a(x)\partial_\mu. \tag{2.3.10}$$

2.3.2 Covariant Derivatives

Tensors are geometrical objects on a manifold, so the differentiation of tensor should result in a tensor satisfying the tensor transformation (2.1.25). To make the consideration more traceable, we consider the derivative of a vector as a tensor of rank (1,1) and express the tensor transformation under general coordinate transformation as

$$\begin{aligned} \partial_\mu V^\nu \rightarrow \partial_{\mu'} V^{\nu'} &= (\frac{\partial x^\mu}{\partial^{\mu'}}\frac{\partial}{\partial x^\mu})(\frac{\partial x^{\nu'}}{\partial x^\nu}V^\nu) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu(\frac{\partial x^{\nu'}}{\partial x^{\nu'}})V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\partial_\mu V^\nu. \end{aligned} \tag{2.3.11}$$

The first term causes the transformation not to be the tensor transformation unless $\partial_\mu(\frac{\partial x^{\nu'}}{\partial x^{\nu'}}) = 0$. This implies the linear transformation on a flat manifold, which is not always the case.

Therefore, we will introduce a connection, which compensates the curvature effect and preserves the transformation rule. As a result, the ordinary derivative is now replaced by the covariant derivative given by

$$\nabla_\nu V^\mu = \frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\rho}(g)V^\rho \tag{2.3.12}$$

where $\Gamma^\mu{}_{\nu\rho}(g)$ is Christoffel symbol.

Moreover, we could also define the covariant derivative of dual vector [6] as

$$\nabla_\nu \omega_\mu = \partial_\nu \omega_\mu - \Gamma^\rho{}_{\nu\mu}(g)\omega_\rho. \tag{2.3.13}$$

In the absence of fermions, we have the torsion free condition $\Gamma^\rho{}_{\nu\mu} = \Gamma^\rho{}_{\mu\nu}$ and the metric compatibility[7] $\nabla_\rho g_{\mu\nu} = 0$, so we can define the Christoffel symbol in terms of the metric tensor as

$$\Gamma^\rho{}_{\nu\mu}(g) = \frac{1}{2}g^{\rho\lambda}(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\nu\mu}). \tag{2.3.14}$$

As a result, Christoffel symbol is not a free field but written in terms of the metric tensor. In particular, Christoffel symbol is not gauge field of gravitational

[6] We use the fact that $\nabla_\mu(\omega_\nu v^\nu) = \partial_\mu(\omega_\nu v^\nu)$

[7] A manifold admits locally flat space, so at a certain point $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ and $\nabla_\rho \eta_{\mu\nu} = 0$. This is a tensor equation which holds true in any coordinate.

interaction but the metric tensor, which is governed by Einstein's field equation.

In the presence of a torsion, Christoffel symbol is not symmetric and cannot be completely determined by the metric tensor. Then, we need an extra variable to describe spacetime instead of the metric tensor alone. We therefore define

$$T_{\mu\nu}{}^{\rho} = 2\Gamma^{\rho}{}_{[\mu\nu]} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} \tag{2.3.15}$$

where $T^{\rho}{}_{\mu\nu} = -T^{\rho}{}_{\nu\mu}$ is called torsion tensor. Then, the Christoffel symbol $\Gamma^{\rho}{}_{\mu\nu}(g)$ is replaced by a connection $\Gamma^{\rho}{}_{\mu\nu}$ given by

$$\Gamma^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu}(g) + \frac{1}{2}(T_{\nu}{}^{\rho}{}_{\mu} + T_{\mu}{}^{\rho}{}_{\nu} + T^{\rho}{}_{\mu\nu}) \tag{2.3.16}$$

where the first term is defined in(2.3.14) the last term is called the contorsion tensor written as

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2}(T_{\nu}{}^{\rho}{}_{\mu} + T_{\mu}{}^{\rho}{}_{\nu} + T^{\rho}{}_{\mu\nu}). \tag{2.3.17}$$

The connection is said to be Levi-civita connection or Christoffel symbol in the absence of the torsion tensor.

$$\Gamma^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu}(g) = \frac{1}{2}g^{\rho\lambda}(\partial_{\nu}g_{\lambda\mu} + \partial_{\mu}g_{\nu\lambda} - \partial_{\lambda}g_{\nu\mu}) \tag{2.3.18}$$

On the other hand, when we consider a two-form in the local Lorentz frame from the one-form (2.3.9)

$$de^{a} = \frac{1}{2}(\partial_{\mu}e^{a}_{\nu} - \partial_{\nu}e^{a}_{\mu})dx^{\mu} \wedge dx^{\nu} \tag{2.3.19}$$

which leads to the local Lorentz transformation as

$$d(\Lambda^{-1a}{}_{b}e^{a}) = \Lambda^{-1a}{}_{b}de^{a} + d\Lambda^{-1a}{}_{b}e^{a}. \tag{2.3.20}$$

Similar to the vector transformation under GCT, to compensate the extra second term in the transformation, we introduce the anti-symmetric two-form connection called the spin connection given in the first Cartan structure equation

$$T^{a} = de^{a} + \omega^{a}{}_{b} \wedge e^{b} \tag{2.3.21}$$

where T^{a} is the torsion 2-form.

We can define the covariant derivative for the local Lorentz frame called the Lorentz covariant derivative

$$D_{\mu}V^{a} = \partial_{\mu}V^{a} + \omega_{\mu}{}^{a}{}_{b}V^{a} \quad , \text{and} \quad D_{\mu}V_{a} = \partial_{\mu}V_{a} - \omega_{\mu}{}^{b}{}_{a}V_{b}. \tag{2.3.22}$$

The affine connection of Christoffel symbol is not an independent field but is written in terms of the metric tensor field as in (2.3.14) by imposing the metric compatible and torsionless condition.

Similarly, we assume that the spin connection is completely determined by the vielbien if we impose the torsionless condition.

$$D_\mu e^a{}_\nu - D_\nu e^a{}_\mu = \partial_\mu e^a{}_\nu - \partial_\mu e^a{}_\mu + \omega_\mu{}^a{}_b e^b{}_\nu - \omega_\nu{}^a{}_b e^b{}_\mu = 0 \tag{2.3.23}$$

The unique solution of (2.3.23) is

$$\omega_{\mu ab}[e] = \frac{1}{2}(e_a{}^\nu \omega_{\mu\nu b} - e_b{}^\nu \omega_{\mu\nu a} - e_a{}^\rho e_b{}^\sigma e_\rho{}^c \omega_{\rho\sigma c}) \tag{2.3.24}$$

where $\omega_{\mu\nu a} = \partial_\mu e_{\nu a} - \partial_\nu e_{\mu a}$.

The presence of the torsion in supergravity results from the existence of fermions not spacetime itself, and we can rewrite the torsion free spin connection with the torsion contribution from fermions to the torsion free spin connection via the contorsion tensor as

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}[e] + K^a{}_\mu{}^b \tag{2.3.25}$$

where $K^a{}_\mu{}^b$ is called the contorsion (2.3.17).

In calculations, we always assume spacetime to be torsion free since we physically think that the intrinsic properties of spacetime can be completely characterized by the spacetime interval through the metric tensor defined in (2.3.1). However, in the presence of fermions, we can think of the presence of torsion as a direct result of the fermions' properties, which equivalently turns out to be the presence of the higher order of fermions entering the torsion free spin connection via the contorsion tensor mentioned in (2.3.25).

2.3.3 Curvature Tensors

In the absence of fermions, the intrinsic curvature of the geometry can be defined by the commutation of covariant derivative given by

$$[\nabla_\rho, \nabla_\sigma] V^\mu = R^\mu{}_{\nu\rho\sigma} V^\nu. \tag{2.3.26}$$

Moreover, we can express Riemann tensor in terms of the metric tensor as

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\nu\rho}. \tag{2.3.27}$$

Riemann tensor is the only tensor that can be constructed from the metric tensor with the properties

$$\begin{aligned} R_{\rho[\sigma\mu\nu]} &= 0, \\ R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma}, \\ R_{\rho\sigma\mu\nu} &= -R_{\rho\sigma\nu\mu}, \\ R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu}, \\ \nabla_{[\lambda}R_{\rho\sigma]\mu\nu} &= 0. \end{aligned} \tag{2.3.28}$$

The fourth equation is called Bianchi's identity. For an n-dimensional manifold, Riemann tensor has $n^2(n^2-1)/12$ independent components.

We can contract Riemann tensor with the metric tesnor, so we obtain Ricci tensor $R_{\mu\nu}$

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = g^{\lambda\rho}g_{\lambda\sigma}R^{\sigma}{}_{\mu\rho\nu} \tag{2.3.29}$$

Finally, we can do the contraction of Ricci tensor with the metric tensor, and we get Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} \tag{2.3.30}$$

which is invariant under GCT.

The curvature tensor can also be evaluated by a two-form curvature written in the second Cartan's structure equation given by

$$R^{ab} = d\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b} \tag{2.3.31}$$

where the components in spacetime read

$$R^{ab} = \frac{1}{2}R^{ab}{}_{\mu\nu}dx^\mu \wedge dx^\nu \tag{2.3.32}$$

which can be written in terms of the spin connection

$$R_{\mu\nu}{}^{a}{}_{b} = \partial_\mu \omega_\nu{}^{a}{}_{b} - \partial_\nu \omega_\mu{}^{a}{}_{b} + \omega_\mu{}^{a}{}_{c}\omega_\nu{}^{c}{}_{b} - \omega_\nu{}^{a}{}_{c}\omega_\mu{}^{c}{}_{b}. \tag{2.3.33}$$

Ricci identities are the commutation relations of Lorentz covariant derivative on various fields given by

$$\begin{aligned}[D_\mu, D_\nu]\phi &= \frac{1}{2}R_{\mu\nu ab}M^{ab}\psi,\\ [D_\mu, D_\nu]V^a &= R_{\mu\nu}{}^{a}{}_{b}V^b,\\ [D_\mu, D_\nu]\psi &= \frac{1}{4}R_{\mu\nu ab}\gamma^{ab}\psi\end{aligned} \tag{2.3.34}$$

with M^{ab}, γ^{ab},ϕ, and ψ being Lorentz generators, gamma matrices, scalar field, and fermion field respectively. These equations give rise to a generalization of spacetime covariant identity (2.3.26) as

$$[\nabla_\mu, \nabla_\nu]V^\rho = R_{\mu\nu}{}^{\rho}{}_{\sigma}V^\sigma - T_{\mu\nu}{}^{\sigma}D_\sigma V^\rho. \tag{2.3.35}$$

2.3.4 Parallel Transport and Geodesics

In a curved manifold, there is no well-defined way to say whether or not two vectors at different points are parallel because we can only compare vectors at the same point.

In a flat manifold, the transport of a vector along a curve $x^\mu(\lambda)$ preserves the direction of the vector since there is no effect from the curvature as shown in figure (2.7). This statement can be extended to an arbitrary tensor, so we can write the equation of parallel transport in a flat manifold as

$$0 = \frac{d}{d\lambda}T^{\alpha_1\alpha_2\ldots\alpha_m}{}_{\beta_1\beta_2\ldots\beta_n} = \frac{dx^\mu}{d\lambda}\partial_\mu T^{\alpha_1\alpha_2\ldots\alpha_m}{}_{\beta_1\beta_2\ldots\beta_n}. \tag{2.3.36}$$

In curved manifold, the vector is manifestly influenced by the curvature effect as shown in figure (2.8), so we will replace the ordinary derivative with the covariant derivative

$$0 = \frac{dx^\mu}{d\lambda}\nabla_\mu T^{\alpha_1\alpha_2\ldots\alpha_m}{}_{\beta_1\beta_2\ldots\beta_n}. \tag{2.3.37}$$

To obtain a path in which a free particle travels in spacetime called geodesics, we need to generalize a straight line on a flat manifold to a curved manifold. The

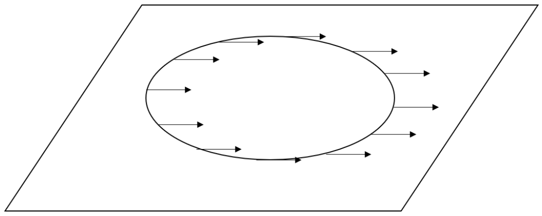

Figure 2.7: This figure shows the parallel transport on flat plane, so the vector along the curve will not be affected by the curvature.

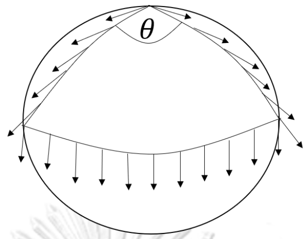

Figure 2.8: This figure shows that the vector transported along the closed loop is affected by the curvature of the sphere.

geodesic path is a curve $x^\mu(\lambda)$ that parallel transports its tangent vector. We can write the geodesic equation as

$$\begin{aligned} 0 &= \frac{dx^\mu}{d\lambda}\nabla_\mu(\frac{dx^\nu}{d\lambda}) \\ &= \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}. \end{aligned} \tag{2.3.38}$$

If we have the initial conditions of position $x^\mu(\lambda_0)$ and the direction $\frac{dx^\mu}{d\lambda}|_{\lambda_0}$, we can compute a unique geodesic path. The direction condition can be also considered as an inertial velocity if we choose $\lambda = \tau$ where τ is a proper time of the moving particle.

2.3.5 Integration on Manifold

The invariant volume element denoted by Ω_M is given by

$$\Omega_M = \sqrt{-g}dx^1 \wedge dx^2 \wedge ... \wedge dx^n = edx^1 \wedge dx^2 \wedge ... \wedge dx^n \tag{2.3.39}$$

where g is the determinant of the metric tensor and e is the determinant of the vielbein. Then, we can define the integration of scalar functions $f \in \mathbb{F}(M)$ over the manifold as

$$\int_M f\Omega_M = \int_M \sqrt{-g}dx^1 \wedge dx^2 \wedge ... \wedge dx^m. \tag{2.3.40}$$

Moreover, We can also express the invariant volume element in terms of hodge star as

$$*1 = \frac{\sqrt{-g}}{m!}\tilde{\epsilon}_{\mu_1\mu_2...\mu_m}dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_m} = \sqrt{-g}dx^1 \wedge dx^2 \wedge ... \wedge dx^m \tag{2.3.41}$$

where $\tilde{\epsilon}_{\mu_1\mu_2...\mu_m}$ is called Levi-Civita symbol,

$$\tilde{\epsilon}_{\mu_1\mu_2...\mu_m} = \begin{cases} +1, & \text{if } (\mu_1\mu_2...\mu_m) \text{ is an even permutation of 012...m} \\ -1, & \text{if } (\mu_1\mu_2...\mu_m) \text{ is an odd permutation of 012...m} \\ 0, & \text{otherwise.} \end{cases}$$

The Levi-Civita symbol can be related to the Levi-Civita tensor given by

$$\epsilon_{\mu_1\mu_2...\mu_m} = \sqrt{-g}\tilde{\epsilon}_{\mu_1\mu_2...\mu_m}. \tag{2.3.42}$$

On the other hand, we can write

$$\epsilon^{\mu_1\mu_2...\mu_m} = \frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\mu_1\mu_2...\mu_m}. \tag{2.3.43}$$

We can also write the volume form of spacetime indices in terms of tangent indices as

$$\epsilon_{\mu_1...\mu_n}dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \epsilon_{a_1...a_n}dx^{a_1} \wedge \cdots \wedge dx^{a_n} \tag{2.3.44}$$

where $\epsilon_{a_1...a_n}$ coincides with the Levi-Civita symbol $\tilde{\epsilon}_{\mu_1\mu_2...\mu_n}$,denoted by $\epsilon_{\hat{0}\hat{1}\hat{2}\hat{3}} = 1$.

We can also define an interval distance from the infinitesimal length

$$l = \int \sqrt{g_{\mu\nu}\dot{x^\nu}\dot{x^\mu}}d\lambda \tag{2.3.45}$$

which leads to the geodesic equation parameterized by a parameter λ

$$\ddot{x^\mu} + \frac{1}{2}g^{\mu\lambda}(\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\rho\lambda} - \partial_\lambda g_{\rho\sigma})\dot{x^\rho}\dot{x^\sigma} = 0. \tag{2.3.46}$$

This equation coincides with the geodesic equation obtained in (2.3.38) where $\dot{x^\nu} = \frac{dx^\nu}{d\lambda}$, $\ddot{x^\mu} = \frac{d^2x^\nu}{d\lambda^2}$, and $\Gamma^\mu{}_{\rho\sigma}(g) = \frac{1}{2}g^{\mu\lambda}(\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\rho\lambda} - \partial_\lambda g_{\rho\sigma})$.

2.4 Symmetry On Manifold

Symmetry on a manifold is a set of map which leaves geometry invariant or maps the manifold to itself. We first review a map between manifolds. We define a

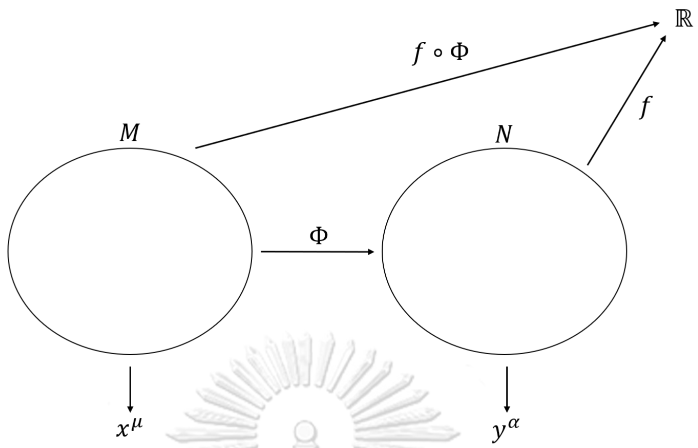

Figure 2.9: Φ is a map between a manifold $\mathcal{M}$ and $\mathcal{N}$

pullback of f on $\mathcal{N}$ to $\mathcal{M}$ by Φ as shown in (2.9) given by

$$\Phi^* f \equiv f \circ \Phi : \mathcal{M} \to \mathbb{R}. \tag{2.4.1}$$

The pullback Φ^* is a composite map of Φ and f. We define the pushforward of a vector as

$$(\Phi_* V)(f) = V(\Phi^* f). \tag{2.4.2}$$

This gives a generalization of the vector transformation under the general coordinate transformation because $\mathcal{M}$ and $\mathcal{N}$ are not necessarily the same manifold, and the component reads

$$(\Phi_* V)^\alpha \partial_\alpha = V^\mu \frac{\partial y^\alpha}{\partial x^\mu}. \tag{2.4.3}$$

Then, we can also define pullback of one-form as

$$(\Phi^* \omega)(V) = \omega(\Phi_* V) \tag{2.4.4}$$

with the component

$$(\Phi^* \omega)_\mu = \frac{\partial y^\alpha}{\partial x^\mu} \omega_\alpha. \tag{2.4.5}$$

Therefore, we can pushforward any arbitrary tensor fields of rank $(k,0)$ as

$$(\Phi_* T)^{\alpha_1...\alpha_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}...\frac{\partial y^{\alpha_1}}{\partial x^{\mu_N}}T^{\mu_1...\mu_N} \tag{2.4.6}$$

and pullback any arbitrary tensor fields of rank $(0,l)$ as

$$(\Phi^* S)_{\mu_1...\mu_N} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}...\frac{\partial y^{\alpha_1}}{\partial x^{\mu_N}}S_{\alpha_1...\alpha_l} \tag{2.4.7}$$

but not for any arbitrary tensor fields of rank (k,l).

However, if Φ is invertible, or Φ^{-1} exists, the manifold $\mathcal{M}$ and $\mathcal{N}$ are then diffeomorphic or they are the same manifold. As a result, we can define both pushforward and pullback on an arbitrary vectors and an arbitrary one-form at the same time. In general, Φ and Φ^{-1} allow us to move any tensor fields of rank (k,l) by pullback or pushforward as

$$(\Phi_* T)(\omega^{(1)},...,\omega^{(k)},V^{(1)},...,V^{(k)}) = T(\Phi^*\omega^{(1)},...,\Phi^*\omega^{(k)},[\Phi^{-1}]_*V^{(1)},...,[\Phi^{-1}]_*V^{(l)}). \tag{2.4.8}$$

The component of the tensor reads

$$(\Phi_* T)^{\alpha_1...\alpha_k}{}_{\beta_1...\beta_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}...\frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial y^{\beta_1}}...\frac{\partial x^{\nu_l}}{\partial y^{\beta_l}}T^{\alpha_1...\alpha_k}{}_{\beta_1...\beta_l} \tag{2.4.9}$$

where $\Phi^* = [\Phi^{-1}]_*$. Pullback is the inverse of pushforward. This gives us the advantage that we can use the diffeomorphism map of Φ to move tensor fields from one point to another point on the manifold. This can be also considered as active coordinate transformations.

Then, we are able to compare tensor fields at different points on the manifold. Therefore, we can define Lie derivative as

$$\mathcal{L}_V T^{\mu_1...\mu_k}{}_{\nu_1...\nu_l} = \lim_{t\to 0}(\frac{\nabla_t T^{\mu_1...\mu_k}}{t}) \tag{2.4.10}$$

where

$$\nabla_t T^{\mu_1...\mu_k}{}_{\nu_1...\nu_l}(p) = \Phi_t^*[T^{\mu_1...\nu_k}{}_{\nu_1...\nu_k}(\Phi_t(p))] - T^{\mu_1...\mu_k}{}_{\nu_1...\nu_l}.$$

Lie derivative indicates the rate of change of tensor field along the tangent vector V^μ of Φ_t a curve parametrized by t and it does not change the rank of the tensor.

The component of Lie derivative on an arbitrary tensor field of rank (k,l) can be defined as

$$\begin{aligned}\mathcal{L}_V T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} &= V^\sigma\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \\ &- (\nabla_\lambda V^{\mu_1})T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \\ &- (\nabla_\lambda V^{\mu_2})T^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} - \ldots \\ &+ (\nabla_{\nu_1} V^{\lambda})T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} \\ &+ (\nabla_{\nu_2} V^{\lambda})T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\lambda\ldots\mu_l} + \ldots\end{aligned} \tag{2.4.11}$$

Now, we can define the symmetry on manifold that the map Φ_t, diffeomorphism, is a symmetry of any tensor fields if the tensors are invariant under pullback under Φ_t along the integral curve of V^μ, which can be written as

$$\Phi_t^* T = T \tag{2.4.12}$$

and the symmetry of metric tensor is called isometry defined as

$$\mathcal{L}_V g_{\mu\nu} = 2\nabla_{(\mu}K_{\nu)} = \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{2.4.13}$$

This equation is called Killing equation, and Φ_t and K^μ are called isometry and Killing vector, respectively.

The maximally symmetric space admits the maximal number of Killing vectors. It is homogeneous and isotropic space with $n(n+1)/2$ Killing vectors and a constant scalar curvature. In Euclidean space, these spaces are $\mathbb{R}^n$, S^n, and H^n corresponding to a flat, spherical, and hyperbolic space respectively. In Lorentzian space, these spaces are Minkowski, de Sitter(dS), and anti-de Sitter(AdS) space.

2.5 Anti-de Sitter space

An d-dimensional anti-de Sitter Space (AdS_d) is the maximally symmetric space with d translations and $\binom{d}{2} = \frac{d(d-1)}{2}$ rotations with a constant negative scalar curvature. Due to the symmetries, the Riemann curvature of a maximally symmetric space is invariant under both translations and rotations.

As a result, we can write the Riemann curvature tensor of a maximally symmetric space in terms of the metric tensor as

$$R_{\mu\nu\rho\sigma} = \frac{\kappa}{L^2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \tag{2.5.1}$$

where κ defines the type of space as

$$\kappa = \begin{cases} +1, & \text{de-Sitter Space of positive scalar curvature.} \\ -1, & \text{anti-de Sitter space of negative scalar curvature.} \\ 0, & \text{Euclidean or Minkowski space.} \end{cases}$$

L is called AdS radius which appears in the embedding space where it is more convenient to study AdS_{d+1} by embedding it in $\mathbf{R}^{2,d}$ with a metric $\eta_{AB} = \text{diag}(-,+,+,...,+,-)$.

The coordinate for the embedding space is given by Y^A, $A = 0,1,2,...,d,d+1$. Therefore, any points on AdS can be defined by

$$\text{AdS}_{d+1} := \{x \in \mathbf{R}^{2,d} | -(Y^0)^2 + \sum_{i=1}^{d}(Y^i)^2 - (Y^{d+1})^2 = -L^2\}. \tag{2.5.2}$$

We can also determine Ricci tensor as

$$R_{\mu\nu} = \frac{\kappa}{L^2} d g_{\mu\nu} \tag{2.5.3}$$

and Ricci scalar as

$$R = \frac{\kappa}{L^2}(d+1)d. \tag{2.5.4}$$

Moreover, we can also use the Poincare coordinate by firstly defining $Y^A \to (x^0, x^i, u)$,

$$\begin{aligned} Y^0 &= Lux^0, \\ Y^i &= Lux^i, \\ Y^d &= \frac{1}{2u}[u^2(L^2 - x^2) - 1], \\ Y^{d+1} &= \frac{1}{2u}[u^2(L^2 + x^2) + 1] \end{aligned} \tag{2.5.5}$$

where $i = 1,2,...,d-1$, $x^2 = \eta_{\alpha\beta}x^\alpha x^\beta$ with $\alpha, \beta = 0,1,2,...,d-1$, and $\eta_{\alpha\beta}$ is d-dimensiomal Minkowski metric. Therefore, we can write the metric as

$$ds^2 = L^2[\frac{du^2}{u^2} + u^2\eta_{\alpha\beta}dx^\alpha dx^\beta]. \tag{2.5.6}$$

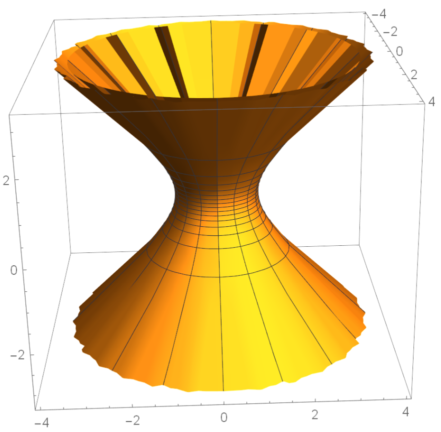

Figure 2.10: The image shows Anti-de Sitter space of constant negative scalar curvature.

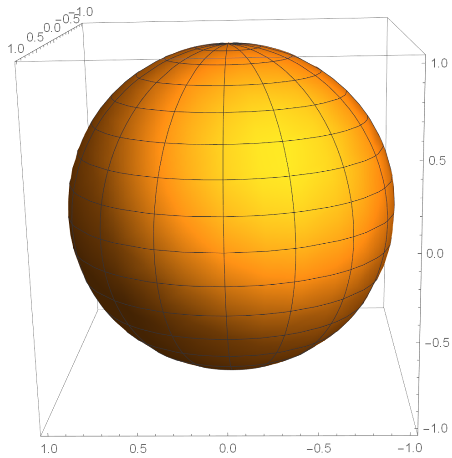

Figure 2.11: The image shows de Sitter space of constant positive scalar curvature.

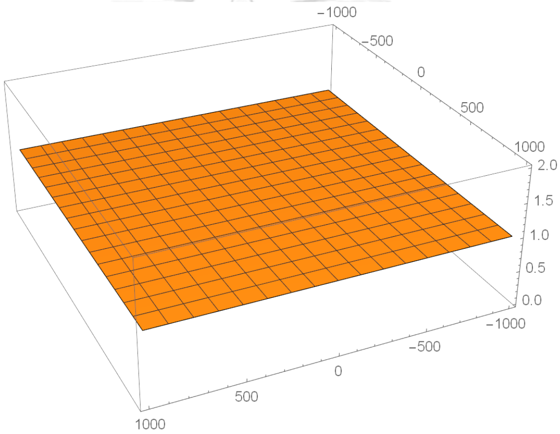

Figure 2.12: The image shows Minkowski space of zero scalar curvature.

We can rewrite the metric in the Poincare coordinate (x^α, z) as

$$ds^2 = \frac{L^2}{z^2}[\eta_{\alpha\beta}dx^{\alpha\beta} + dz^2] \tag{2.5.7}$$

where $u = \frac{1}{z}$.

Finally, we use a coordinate which we will use to study holographic solutions by defining

$$e^{\frac{r}{L}} = \frac{L}{z}. \tag{2.5.8}$$

Therefore, we get the *AdS* metric of the form

$$ds^2 = e^{\frac{2r}{L}} \eta_{\alpha\beta} dx^\alpha dx^\beta + dr^2. \tag{2.5.9}$$

Not only is *AdS* the maximally symmetric space, but also it preverses all supersymmetries of theory. However, when we consider the asymptotically anti-de Sitter (AAdS) space which becomes AdS at some certain points called *AdS* fixed points. We can rewrite the metric in general form as

$$ds^2 = e^{2A(r)} \eta_{\alpha\beta} dx^\alpha dx^\beta + dr^2. \tag{2.5.10}$$

This spacetime is called "Domain wall" and the *AdS* fixed point is a point that $A(r)$ becomes r/L.

CHAPTER III

General Relativity

General relativity is a theory of gravity that describes the dynamics of gravity in terms of spacetime curvature. This results from Einstein's brilliant thought experiment so-called the principle of equivalence. The principle of equivalence states that in a sufficiently small region of spacetime we can remove the effect of gravitation so that physics obeys special relativity. This interpretation coincides with the mathematical idea of the manifold where the geometry is locally flat. In this case, the spacetime is represented by a pseudo-Riemannian manifold. Due to redundancy of coordinate representations on a manifold, the symmetry underlying general relativity is a general coordinate transformation (GCT), see also for an incomplete list of review on general relativity, [79, 82–84, 87, 88, 105].

3.1 Minkowski Spacetime

We have already mentioned that General Relativity is a generalization of Special Relativity, so we first review some crucial features about special relativity or a flat Minkowski spacetime. Special relativity is a theory that proposes the relation between space and time giving rise to a description of spacetime. Special relativity is a theory which invariant under Poincare group $ISO(d,1)$ called "the maximal isometry group of $d+1$-dimensional Minkowski spacetime". Poincare group can be written as a semi-direct product of translation group and Lorentz group of

$$ISO(d,1) = \left(\mathbb{R} \oplus \mathbb{R}^d\right) \rtimes O(d,1) \tag{3.1.1}$$

respectively. The Lorentz group $O(d,1)$ contains a subgroup of $SO(d,1)$ with positive determinant forming a proper Lorentz group of boosts and rotations and the other elements with negative determinant corresponding to parity and time reversal transformation.

Physically, special relativity is based on two postulates

1. The laws of physics take the same form in all inertial frames.
2. The speed of light is the same for all observers in all inertial frames.

These postulates lead to an important result called "simultaneity". It states that events that occur at the same time in an inertial frame do not necessarily occur at the same time in another inertial frame moving relative to the first one.

To characterize an event, we need both space and time. These postulates

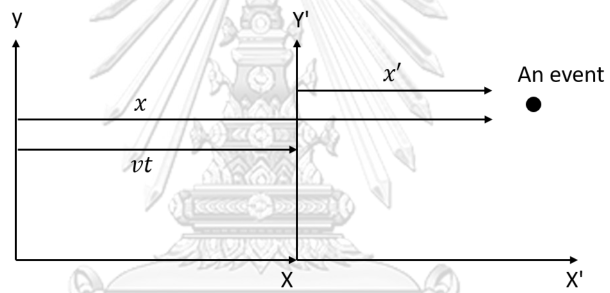

Figure 3.1: The relation between two obervsers in inertial frames

break down the idea of absolute space and time in Newtonian mechanics and Galilean transformation as shown in figure (3.1).

$$\begin{aligned} t' &= t, \\ x' &= x - vt, \\ y' &= y, \\ z' &= z. \end{aligned} \tag{3.1.2}$$

The generalization of Galilean transformation which obeys special relativity is called "Lorentz transformation" given by

$$\begin{aligned} t' &= \gamma(t - xv), \\ x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z \end{aligned} \tag{3.1.3}$$

where $\gamma = \sqrt{1 - \frac{v^2}{c^2}}$.

It is more useful at this point to introduce the idea of Minkowski spacetime, which was proposed by Hermann Minkowski by generalizing Euclidean space to Minkowski spacetime. The invariant distance between two points in Euclidean

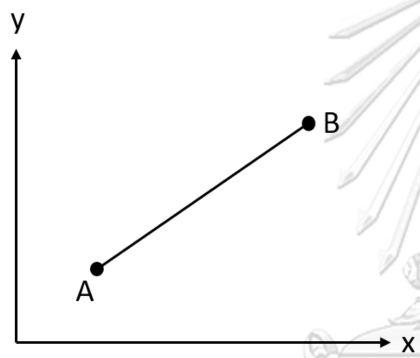

Figure 3.2: The image shows the distance in Euclidean space.

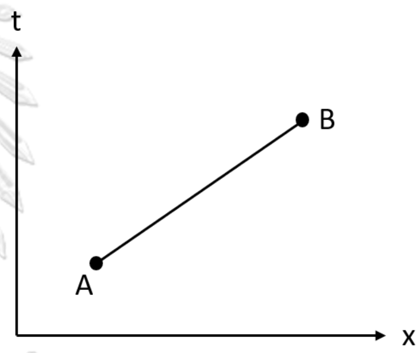

Figure 3.3: The image shows the distance in Minkowski spacetime.

space obeying Pythagoras' theorem is also generalized to spacetime interval given by

$$ds^2 = dx^2 + dy^2 \rightarrow ds^2 = -dt^2 + dx^2. \tag{3.1.4}$$

In spacetime, the more you move in space, the less you move in time. In four-dimensional spacetime, we can conveniently write the spacetime interval as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{3.1.5}$$

where $\eta_{\mu\nu}$ is Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.1.6}$$

Not only does the spacetime diagram help us to unify space and time, but also introduces the idea of the light cone diagram as shown in (3.4). The light cone is a physical structure of spacetime, unlike the coordinate which is just a convenient choice of frame. The light cone divide spacetime into three regions of spacetime as follows

$$ds^2 \begin{cases} < 0, & \text{timelike.} \\ > 0, & \text{spacelike.} \\ = 0, & \text{lightlike.} \end{cases}$$

A timelike path is a path on which massive particles move such as electrons, protons, neutrinos, etc. A lightlike interval is a track for massless particles such as photons, gravitons, etc. To move along a spacelike trajectory, particles must exceed the speed of light which violates special relativity. Therefore, points connected by spacelike paths cannot influence each other. In spacetime diagram, the Lorentz transformation can be shown in figure (3.5).

It is manifestly obvious that the speed of light is constant in both frames as shown in (3.5) because the light cone appears the same. Let us further review the mechanics satisfying special relativity or the machanics in Minkowski spacetime so-called “Relativistic Mechanics”. Let $x^\mu = x^\mu(\tau) = (t, \vec{x})$ be a path in spacetime of a particle, we can define four-velocity u^μ of the particle as

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma(1, \vec{v}) \tag{3.1.7}$$

where τ is time elapsed in a frame moving with particle, t is a time coordinate, and $\vec{v}$ is a velocity through space.

We can further define a four-momentum p^μ as

$$p^\mu \equiv mu^\mu = \gamma(m, m\vec{v}) \tag{3.1.8}$$

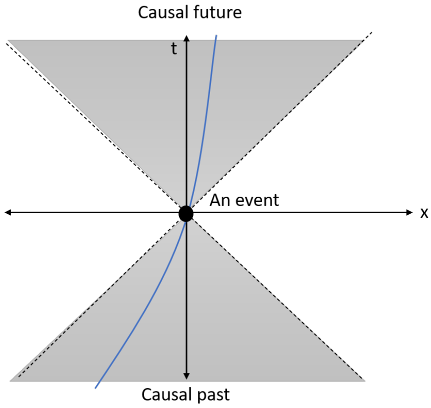

Figure 3.4: The image shows trajectory of massive particles must be in the light cone.

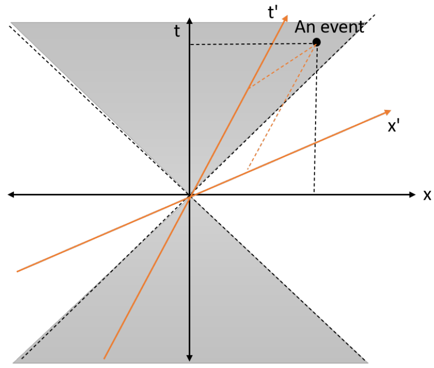

Figure 3.5: The image shows the relation between two obervsers in spacetime diagram via Lorentz transformation.

where γm is the energy of the particle and $\vec{p} = \gamma m\vec{v}$ is the momentum in space of the particle.

Then, we can write the famous mass-energy relation from the invariant product of $p^\mu p_\mu = p^{\mu'} p_{\mu'}$. Once we choose $p^{\mu'} = (m, 0)$ for the rest frame, and $p^\mu = (E, \vec{p})$ for an abitrary frame, we obtain the mass-energy relation

$$E^2 = p^2 + m^2. \tag{3.1.9}$$

Now, we can move to a curved spacetime which is dynamical and responding to matters and energies. Such a curved spacetime can be interpreted as the existence of gravity.

3.2 General Relativity

The dynamical variable of gravity can be represented by either the metric tensor $g_{\mu\nu}$ or vielbien e^a_μ. In the presence of only bosonic fields, there is no difference between the two. To find Einstein's field equation which governs the dynamics of gravity, we will use the principle of least action where the action of gravity has to be invariant under GCT. The Lagrangian must be also invariant under GCT and written in terms of either the metric tensor or vielbien.

Therefore, we write Lagrangian being a scalar function of Ricci scalar tensor, $f(R)$ as

$$S_{f(R)}[R] = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}f(R) \tag{3.2.1}$$

where κ is a constant. The action is called $f(R)$ gravity action where R is Ricci scalar tensor, and $g = \det g_{\mu\nu}$.

Einstein's field equation can be obtained by varying the action (3.2.1) with respect to the inverse of metric tensor $g^{\mu\nu}$

$$\begin{aligned}\delta_g S[R]_{f(R)} &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x((\delta\sqrt{-g})f(R) + \sqrt{-g}\delta f(R)) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x(-\frac{1}{2}g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}f(R) + \frac{\partial f(R)}{\partial R}\delta R) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x(-\frac{1}{2}g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}f(R) + \frac{\partial f(R)}{\partial R}(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}))\end{aligned} \tag{3.2.2}$$

where we use the following properties

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \\ g^{\mu\nu}\delta R_{\mu\nu} &= g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}\end{aligned} \tag{3.2.3}$$

with $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. By the vanishing of the surface integration after doing the integration by part, the variation of the action becomes

$$\delta_g S_{f(R)}[R] = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(F(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F(R)\right)\delta g^{\mu\nu} \tag{3.2.4}$$

where $F(R) = \frac{\partial f(R)}{\partial R}$.

Let $f(R) = R$ in which the action is called the Einstien-Hilbert action and $\delta_g S_{f(R)}[R] = 0$ for $\forall\delta g^{\mu\nu}$, we can finally obtain Einstein's field equations in vacuum as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{3.2.5}$$

where $G_{\mu\nu}$ is called Einstein tensor.

To couple gravity or spacetime with matter fields, we require the action of matter fields to be invariant under GCT. This can be done by introducing the minimal coupling to the action of matter fields where the ordinary derivative and volume element are replaced by the covariant derivative and the invariant volume element.

$$\partial \rightarrow \nabla \tag{3.2.6}$$

$$\int_{\mathcal{M}} d^4x \rightarrow \int_{\mathcal{M}} \sqrt{-g} d^4x \tag{3.2.7}$$

respectively. This is also known as the principle of general covariance.

The action of matter-coupled gravity can be written as

$$\begin{aligned} S &= S_{EH} + S_{\text{matter}} \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2}(R - 2\Lambda) + 2\mathcal{L}_{\text{matter}} \right) \end{aligned} \tag{3.2.8}$$

where we also add a cosmological constant Λ into the Einstein-Hilbert action.

Einstein field's equation with matter fields can be found by varying the Einstein-Hilbert action and a matter-field term with respect to the inverse of the metric tensor $g^{\mu\nu}$.

$$\delta_g S = \frac{\delta S_{EH}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \delta g^{\mu\nu}. \tag{3.2.9}$$

If we define the stress-energy tensor as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}, \tag{3.2.10}$$

we can write the variation of the matter field explitly as

$$\begin{aligned} \delta_g S_{\text{matter}} &= \int_{\mathcal{M}} d^4x \left(\delta(\sqrt{-g}) \mathcal{L}_{\text{matter}} + \frac{\delta \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} \sqrt{-g} \delta g^{\mu\nu} \right) \\ &= \int_{\mathcal{M}} d^4x \left(\sqrt{-g} (\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} - \frac{1}{2} \mathcal{L}_M g_{\mu\nu}) \delta g^{\mu\nu} \right) \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(-T_{\mu\nu} \right) \delta g^{\mu\nu}. \end{aligned} \tag{3.2.11}$$

After the variation, $\delta_g S$ can be written as

$$\delta_g S = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^4x \sqrt{-g} \big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} - \kappa^2 T_{\mu\nu} \big) \delta g^{\mu\nu}. \tag{3.2.12}$$

The constant $\kappa = \sqrt{\frac{8\pi G}{c^4}}$ which can be computed by using Newtonian limit of non-relativistic regime and weakly static gravitational field. We finally obtain the Einstein's field equations with the stress-energy tensor

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.2.13}$$

where G is the universal gravitational constant. We also notice that Einstein's field equation is a generalization to Passion's equation

$$\nabla^2\Phi = 4\pi G\rho \tag{3.2.14}$$

where the gravitational field Φ is replaced by the metric tensor $g_{\mu\nu}$ and the matter density ρ is generalized to the stress-energy tensor $T_{\mu\nu}$. We will consider $T_{\mu\nu}$ for a few examples of matter fields.

3.2.1 Scalar Fields

We first consider a matter field in terms of a real scalar field $\phi(x)$ given by

$$S_{\text{scalar}} = \int_{\mathcal{M}} \sqrt{-g}d^4x(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2). \tag{3.2.15}$$

In order to find stress-tensor energy for scalar fields, we need to do the variation with respect to the metric tensor field $g^{\mu\nu}$

$$\begin{aligned} \delta S_{\text{scalar}} &= -\frac{1}{2}\int_{\mathcal{M}} d^4x\big(\sqrt{-g}\partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu} + (\partial_\rho\phi\partial^\rho\phi - m^2\phi^2)\delta\sqrt{-g}\big) \\ &= -\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\rho\phi\partial^\rho - m^2\phi^2))\delta g^{\mu\nu} \end{aligned} \tag{3.2.16}$$

So, we get the stress-energy tensor of the scalar field

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\rho\phi\partial^\rho - m^2\phi^2). \tag{3.2.17}$$

3.2.2 Vector Fields

We now consider the situation where a vector field couples to the gravitational field. This can be done simply by considering the Maxwell field describing electromagnetic fields in a four-dimensional curved spacetime

$$S_{\text{vector}} = -\frac{1}{4}\int_{\mathcal{M}} d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \tag{3.2.18}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The variation is given by

$$\begin{aligned}
\delta_g S_{\text{vector}} &= -\frac{1}{4}\delta \int_{\mathcal{M}} d^4x \sqrt{-g} F_{\mu\nu}F^{\mu\nu} \\
&= -\frac{1}{4}\int_{\mathcal{M}} d^4x \big(\sqrt{-g}\,\delta(F_{\mu\nu}F^{\mu\nu}) + \delta\sqrt{-g} F_{\mu\nu}F^{\mu\nu}\big) \\
&= -\frac{1}{2}\int_{\mathcal{M}} d^4x \big(\sqrt{-g} F_{\mu\lambda}F^{\lambda}{}_{\nu}\delta g^{\mu\nu} + \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\delta\sqrt{-g}\big) \\
&= -\frac{1}{2}\int_{\mathcal{M}} d^4x \sqrt{-g}\big(F_{\mu\lambda}F^{\lambda}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\big)\delta g^{\mu\nu}.
\end{aligned} \tag{3.2.19}$$

Then, we have the stress-energy tensor on the curved spacetime

$$T_{\mu\nu} = F_{\mu\lambda}F^{\lambda}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{3.2.20}$$

One of the most useful features is that the trace of $T^{\mu\nu}$ is zero

$$g^{\mu\nu}T_{\mu\nu} = 0. \tag{3.2.21}$$

This is true only in four-dimensional spacetime.

3.3 First and Second-Order of Gravity Theory

We have used the metric tensor as a dynamical variable to describe gravity and the action is given by

$$S[g] = \int_{\mathcal{M}} d^4x \sqrt{-g}\big(\frac{1}{2\kappa^2}R(g) + \mathcal{L}_{\text{matter}}\big) \tag{3.3.1}$$

where the matter fields in $\mathcal{L}_{\text{matter}}$ are bosonic matter fields.

However, in order to construct an interacting theory of fermion fields and gravity, we must have expressed the metric tensor in terms of the vielbein e^a_μ to which the fermion fields couple. Therefore, we can use the relation between the metric tensor $g_{\mu\nu}$ and the vielbein $e^a{}_\mu$ given in (2.3.3), so that we can write the equivalent form of Einstein-Hilbert action

$$S[\omega(e)] = \int_{\mathcal{M}} d^4x e\big(\frac{1}{2\kappa^2}R(e) + \mathcal{L}_{\text{matter}}\big) \tag{3.3.2}$$

with only the presence of the boson fields, the field's equations are second-order in $g_{\mu\nu}$ or $e^a{}_\mu$ since Christoffel symbol $\Gamma^\rho{}_{\mu\nu}$ and the spin connection $\omega^a{}_b$ can be fully determined by the metric tensor and the vielbein, respectively. This is called second-order formalism.

However, we can also construct the alternative theory of gravity under the non-torsion free condition called "Palatini action" given by

$$S_P[e,\omega] = \int_{\mathcal{M}} d^4xe(\frac{1}{2\kappa^2}e^{\mu}{}_{a}e^{\nu}{}_{a}R^{ab}{}_{\mu\nu}((\omega,e)) + \mathcal{L}_{\text{matter}}). \tag{3.3.3}$$

In vacuum, $\mathcal{L}_{\text{matter}} = 0$. The field's equation for the Palatini action can be expressed in terms of two sets of the first-order field's equations of motion of the vielbein and the spin connection, respectively.

The former results from doing the variation of the action with respect to the vielbein $\delta S_P/\delta e^{\mu}_{a}$. The latter results from doing the variation of the action with respect to the spin connection $\delta S_P\delta/\omega_{\mu ab}$. This formalism is called first-order formalism. Then, we have the variation of the Palatini action as follows

$$\delta S_P[e,\omega] = -\frac{1}{2}e\left(R^{a}{}_{\mu} - \frac{1}{2}e^{a}{}_{\mu}R\right)\delta e^{\mu}{}_{a} - \frac{3}{2}e(D_{\mu}e^{a}{}_{\nu})e^{\mu}{}_{[a}e^{\nu}{}_{b}e^{\rho}{}_{c]}\delta\omega_{\rho}{}^{cb}. \tag{3.3.4}$$

However, the spacetime in physics appears torsion-free, so the second-order and the first-order result in the same physics. As a result, the Palatini formulation is exactly equivalent to the second-order formalism of the Einstein-Hilbert action because of the vanishing of the second term in the variation of the Palatini action.

However, if $\mathcal{L}_{\text{matter}}$ contains fermion fields, the two theories will not be the same, rather differ by the presence of ψ^4 terms resulting from the torsion of fermion fields. For example, the action of interacting fields between massless Dirac field ψ and gravity in the second-order formalism is given by

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\big[R(\omega(e)) - \kappa^2\frac{1}{2}\overline{\psi}\gamma^{\mu}\nabla_{\mu}\psi + \kappa^2\frac{1}{2}\overline{\psi}\overleftarrow{\nabla}_{\mu}\gamma^{\mu}\psi\big] \tag{3.3.5}$$

where $\overline{\psi} = \psi^{\dagger}i\gamma^0$ is the Dirac adjoint, and $\nabla_{\mu} = \mathcal{D}_{\mu}\psi_{\nu} - \Gamma_{\mu\nu}{}^{\rho}\psi_{\rho}$. In this case the spinor has no coordinate indices, so $\nabla \to \mathcal{D}$. Then, we can write $\nabla_{\mu} = \partial_{\mu} + \frac{1}{4}\omega_{\mu}{}^{ab}\gamma_{ab}$ and $\overleftarrow{\nabla} = \overleftarrow{\partial_{\mu}} - \frac{1}{4}\omega_{\mu}{}^{ab}\gamma_{ab}$.

We can add the term of ψ^4 by considering the first-order formalism. the first-order formalism, we can find the field equation of the spin connection and find the torsion from $\delta_{\omega}S_{\text{EH}} + \delta_{\omega}S_{\text{Dirac}} = 0$. We can relate the torsion and the contorsion terms to each other.

Now, we can substitute the contorsion consisting of ψ^4 via $\omega = \omega(e) + K$ through the Ricci scalar tensor $R(\omega)$ and covariant derivative $\mathcal{D}(\omega)$ in the first-order formalism. Therefore, we obtain the second-order formalism with the existence of ψ^4 in the action given by

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\big[R(\omega(e)) - \kappa^2\overline{\psi}\gamma^\mu\nabla_\mu\psi + \kappa^2\overline{\psi}\overleftarrow{\nabla}_\mu\gamma^\mu\psi + \kappa^4\frac{1}{16}(\overline{\psi}\gamma_{\mu\nu\rho}\psi)(\overline{\psi}\gamma^{\mu\nu\rho}\psi)\big]. \tag{3.3.6}$$

As a result, the physical effects of the existence of fermion fields with and without torsion will differ by the presence of ψ^4. However, the existence of the terms of ψ^4 plays no rule in general relativity of gravity, but plays a very important rule to prove the invariance of supersymmetry in supergravity.

Therefore, the first-order formalism would be easier to prove the local supersymmetric property of supergravity by the existence of quartic terms in the fermionic fields especially when we replace the massless Dirac field with the gravitino. We will discuss this process more carefully in the next chapter, once we construct gravity theory which is invariant under $N = 1$ local supersymmetry. On the other hand, the second formalism is very convenient for many applications in an ordinary theory of gravity where fermions are neglected.

3.4 Black Holes

Black holes are an object that even light cannot escape from a certain distance called the “event horizon”. They naturally arise in the solution of general relativity, and black holes have been named after their solutions. Here, we will recall some important solutions and discuss some crucial properties of the corresponding black hole, see also [67–69].

3.4.1 Schwarzschild Balck hole

The Schwarzschild solution is the first analytic solution of Einstein's equation solved by Karl Schwarzschild and the corresponding spacetime is also called Schwarzschild spacetime. Schwarzschild spacetime is an exterior solution of a spherically static massive object. There are four Killing vectors for Schwarzschild spacetime. Three of them generate an isometry group of $SO(3)$ on the space-like hypersurface Σ_t. The other is a stationary Killing vector $K^\mu = (\partial_t)^\mu$ implying the time-independent metric tensor elements namely, $\frac{\partial}{\partial t} g_{\mu\nu} = 0$.

Schwarzschild spacetime can be written as

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.4.1}$$

The exterior vacuum condition implies that $T_{\mu\nu} = 0$, so Einstein's equation becomes

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{3.4.2}$$

After the calculation for an asymptotically flat spacetime, we find $\beta = -\alpha$ and

$$e^{2\alpha(r)} = 1 + \frac{c}{r}. \tag{3.4.3}$$

The constant c can be found by using Newtonian limit and the asymptotically flat Schwarzschild spacetime, manely $g_{\mu\nu} \to \eta_{\mu\nu}$.

We can finally write the Schwarzschild metric as

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{1}{(1 - \frac{2GM}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.4.4}$$

where the metric tensor can be written as

$$g_{\mu\nu} = \begin{pmatrix} -(1 - \frac{2MG}{r}) & 0 & 0 & 0 \\ 0 & \frac{1}{(1 - \frac{2MG}{r})} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin\theta \end{pmatrix}. \tag{3.4.5}$$

At the boundary $r = 2MG$ is called "event horizon" and $r = 0$ is called singularity. We can consider the metric in the Eddington-Finkelstein coordinate

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + (dvdr + drdv) + r^2 d\Omega^2 \tag{3.4.6}$$

where $v = t + r^*$, $u = t - r^*$, $r^* = r + 2M\log(\frac{r}{2M} - 1)$. Eddington-Finkelstein coordinate states that there is no problem at Schwarzschild radius, it is just a point that the light cones tilt over from time-like becoming space-like and all future-directed paths point in the direction of decreasing r. Moreover, we can make a maximal extension of the Schwarzschild spacetime to describe the whole manifold by Kruskal-Szekeres diagram as shown in figure (3.6) or condense it into a finite region constructing its conformal diagram called Penrose diagram as shown in figure (3.7).

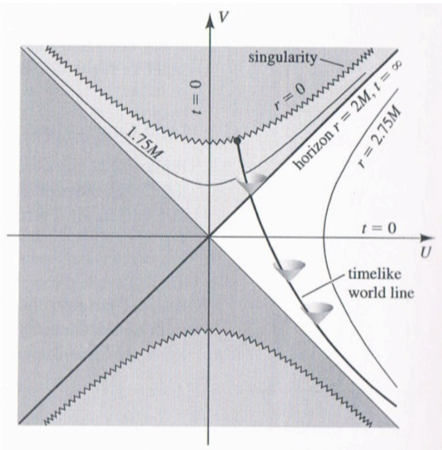

Figure 3.6: This images shows the Schwarzschild mretic in the Kruskal-Szekeres coordinate. Ask a Mathematician/Physicist.

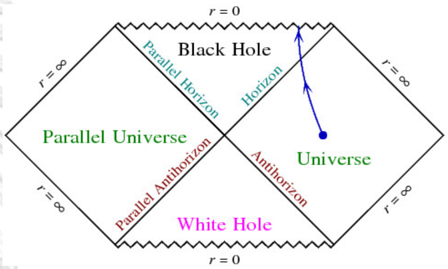

Figure 3.7: This image shows the Penrose diagram of extended Schwarzschild spacetime. jila.colorado.edu.

3.4.2 Reissner-Nordstrom Black holes

The Reissner-Nordstrom spacetime is an extension of Schwarzschild spacetime where the spherically non-rotating object has the electric charge q and the magnetic charge p. Even though in astrophysics, the Reissner-Nordstrom Black hole rarely exists because the evolution of a black hole would be quickly neutralized by inverse beta decay. Therefore, most of the matter would be neutral. However, Reissner-Nordstrom black hole represents several important features.

The action of the Reissner-Nordstrom spacetime can be written as

$$S_{EM} = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right). \tag{3.4.7}$$

This is called Einstein-Maxwell theory and the field equations are also called Einstein-Maxwell's equation.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{3.4.8}$$

where $T_{\mu\nu} = F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$ is the stress-energy tensor of electromagnetic fields.

The Maxwell's equations results in the electromagnetic stress tensor in spherical coordinate given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & qr^{-2} & 0 & 0 \\ qr^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & p\sin\theta \\ 0 & 0 & -p\sin\theta & 0 \end{pmatrix}. \tag{3.4.9}$$

Then, the Reissner-Nordstrom metric written in terms of spherical coordinate is given by

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{3.4.10}$$

where $d\Omega^2 = r^2 d\theta^2 + r^2\sin\theta^2 d\theta^2$. By solving Einstein's field equations, we can obtain

$$\begin{aligned} A(r) &= \left(1 - \frac{2m}{r} + \frac{(q^2+p^2)}{r^2}\right) \\ &= B(r)^{-1} \end{aligned} \tag{3.4.11}$$

where we set $G = 1$. Therefore, the Reissner-Nordstrom metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2m}{r} + \frac{(q^2+p^2)}{r^2}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2m}{r} + \frac{(q^2+p^2)}{r^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin\theta^2 \end{pmatrix}. \tag{3.4.12}$$

The Ricci scalar is

$$R = \frac{12(mr - (q^2+p^2))^2}{r^6} + \frac{2(q^2+p^2)^2}{r^8}. \tag{3.4.13}$$

As we have discussed in Schwarzschild black hole, there is only one real curvature singularity at $r = 0$. The difference between Schwarzschild black hole and Reissner–Nordstrom Black hole is that Reissner-Nordstrom Black hole has more than one possible event horizon where

$$A(r_\pm) = 0 = \big(1 - \frac{2m}{r_\pm} + \frac{(q^2+p^2)}{r_\pm^2}\big) = 0. \tag{3.4.14}$$

We obtain

$$r_\pm = m \pm \sqrt{m^2 - (q^2+p^2)} \tag{3.4.15}$$

with the following possibilities.

1. $m^2 < (q^2 + p^2)$ There is a naked singularity, just like the $M < 0$ in Schwarzschild spacetime. $A(r)$ is always positive. Timelike coordinate becomes space-like coordinate only at $r = 0$. This solution is unphysical.

2. $m^2 > (q^2 + p^2)$ This solution is realistic gravitational collapse and $A(r)$ is negative between r_+ and r_-. In region 1 outside r_+, a particle moves along a time-like coordinate until r reaches r_+, at which the time-like coordinate becomes a space-like coordinate. In region 2 between r_+ and r_-, the particle inevitably moves toward r_- where the space-like coordinate becomes a time-like coordinate. In region 3, the particle can choose either to move to the singularity or move in the direction of increasing r until the particle reaches r_- where again the time-like coordinate becomes a space-like coordinate. At this stage, the particle is forced to move out of the event horizon at r_+.

3. $m^2 = (q^2 + p^2)$ This solution is called extremal Reissner-Nordstrom solution which is very important in supersymmetric theories.

Let us consider the extremal Reissner-Nordstrom black hole. The extremal Reissner-Nordstrom black hole is a special case of the Reissner-Nordstrom black hole in which the mass of a black is equal to the charge of the black hole $m = q$ without the monopole.

The metric of the extreme Reissner-Nordstrom black hole can be written as

$$ds^2 = -(1 - \frac{m}{r})^2 dt^2 + (1 - \frac{m}{r})^{-2} dr^2 + r^2 d\Omega^2. \tag{3.4.16}$$

By defining the radial coordinate transformation

$$\bar{r} = r - m, \tag{3.4.17}$$

we find that there is an isotropic form of the metric.

$$ds^2 = -(1 - \frac{m}{\bar{r}})^{-2} dt^2 + (1 - \frac{m}{\bar{r}})^2 (d\bar{r}^2 + \bar{r}^2 d\Omega^2). \tag{3.4.18}$$

We can rewrite it as

$$ds^2 = -H_1^2(\bar{r}) dt^2 + H_2^2(\bar{r}) [d\bar{r}^2 + \bar{r}^2 d\Omega^2]. \tag{3.4.19}$$

where $H_1 = (1 - \frac{m}{\bar{r}})^{-1}$ and $H_2 = {H_1}^{-1}$.

We can look at the near-horizon limit of such a background, as $\bar{r} \to 0$

$$ds^2 = -\frac{\bar{r}^2}{(m)^2} dt^2 + (m)^2 \frac{d\bar{r}^2}{\bar{r}^2} + (m)^2 d\Omega_2^2. \tag{3.4.20}$$

Then, we define a new coordinate $\nu = \frac{\bar{r}^2}{(m)^2}$, and we rewrite the near-horizon solution again

$$ds^2 \approx \frac{(m)^2}{\nu^2}(-dt^2 + d\nu^2) + (m)^2 d\Omega_2^2. \tag{3.4.21}$$

We can see that the metric turns asymptotically into two two-dimensional spaces, which are characterized (t, r) and (θ, ϕ). The (θ, ϕ)-space is a two-sphere with radius m, and (t, r)-spacetime is the two-dimensional anti de-Sitter spacetime AdS_2, with radius m. They both are a maximally symmetric space and can be written as coset manifolds given by

$$S^2 = \frac{SO(3)}{SO(2)} \qquad \text{, and} \qquad AdS_2 = \frac{SO(2,1)}{SO(1,1)} \tag{3.4.22}$$

$AdS^2 \times S^2$ is the horizon geometry of AdS supersymmetric black hole solution that would be useful to study the AdS/CFT correspondence. It is also known as the Bertotti-Robinson solutions, which are the solution of the Einstein-Maxwell equation.

3.5 The Black Hole Mechanics

We look at the first realization of the holographic principle resulting from the information paradox of the black hole. The question arises from whether or not information of the falling object disappears. The study has shown that the area of the black hole is always increasing, namely $\delta A \geq 0$, which is equivalent to the fact that the changing of entropy of a system should also be greater than zero, namely $\delta S \geq 0$.

However, this leads to a problem because if the black hole has entropy, it must have a non-zero temperature. This means that the black hole must radiate. According to classical general relativity, nothing can escape from the black hole, so the temperature of the black hole must be zero. Nevertheless, Stephen Hawking shown that the black hole has radiation called "Hawking radiation".

Holographically, this indicates that the information somehow does not disappear, but it is encoded in the area of the event horizon. Therefore, the information of bulk spacetime can be related to the physics of the boundary of the bulk spacetime, which is called the holographic principle, in particular with the AdS/CFT correspondence being a particular example.

3.5.1 Killing horizons and Surface gravity

A Killing horizon is a null hypersurface Σ where Killing vector fields χ^μ become null and normal to the Killing horizon. This is also the regime that the timelike vector becomes spacelike at this boundary. Therefore, Stephen Hawking in 1972 shown that the killing horizon Σ, in stationary asymptotically flat spacetime, is not necessarily the Killing horizon of the stationary Killing vector $K^\mu = (\partial_t)^\mu$ rather some Killing vector fields of χ^μ.

For a trivial example, if the spacetime geometry is static (The Schwarzschild geometry), the corresponding Killing vector field χ^μ coincides with the stationary Killing vector $K^\mu = (\partial_t)^\mu$ which represents time translations. However, if the spacetime is stationary but not static (The Kerr geometry), the Killing vector field will be the linear combination between the stationary Killing vector K^μ and the rotational axisymmetric Killing vector field $R^\mu = (\partial_\phi)^\mu$ as $\chi^\mu = K^\mu + \Omega_H R^\mu$ for a constant Ω_H describing the angular velocity of the black hole.

One of the most important features of the Killing vector is the association with the surface gravity $\tilde{\kappa}$. Since the event horizon is a null hypersurface namely, $\chi^\mu\chi_\mu = 0$ everywhere on the horizon, the gradient is normal to null hypersurface Σ. In particular, it is parallel to itself

$$\nabla_\nu(\chi^\mu\chi_\mu) = -2\tilde{\kappa}\chi_\nu. \tag{3.5.1}$$

Regarding the LHS as the Killing's equation. We obtain

$$\chi^\mu\nabla_\mu\chi^\nu = -\tilde{\kappa}\chi^\nu \tag{3.5.2}$$

where $\tilde{\kappa}$ is the surface gravity which is some function of the coordinates and will be constant over the horizon. Finally, the surface gravity is given by

$$\tilde{\kappa}^2 = -\frac{1}{2}(\nabla^\mu\chi^\nu)(\nabla_\mu\chi_\nu). \tag{3.5.3}$$

In static black holes, the surface gravity can be interpreted as the acceleration of a static observer near the horizon or the exerted force to keep a test-particle stationary on the horizon observed by a static observer at infinity. The surface gravity of the Schwarzschild spacetime is

$$\tilde{\kappa}_{SH} = \frac{1}{4m}. \tag{3.5.4}$$

The surface gravity of the Schwarzschild solution decreases as the mass of the black hole increases. It means that the surface gravity of a supermassive black hole is smaller than a small black hole.

3.5.2 Thermodynamics and black holes

We have focused on surface gravity because it plays an important role in the connection between other physical quantities and physical properties of black holes. The first one is that the existence of the entropy of the black hole

$$S = \frac{A}{4} \tag{3.5.5}$$

where A is an area of the black hole horizon. The second one is the temperature of the black hole called "Hawking temperature"

$$T_{HB} = \frac{\tilde{\kappa}\hbar}{2\pi} \tag{3.5.6}$$

where $\tilde{\kappa}$ is a surface gravity.

The zeroth law of the black hole Mechanics. If the stress tensor $T_{\mu\nu}$ obeys the dominant energy condition, the surface gravity of a stationary black hole is constant over the horizon. This is analogous to the zeroth law of thermodynamics which states that the temperature is constant over a body in thermal equilibrium. Therefore, surface gravity is analogous to temperature.

The first law of the black hole Mechanics. The change of stationary black holes with (m, J, q) to another black holes with $(m+\delta m, J+\delta J, q+\delta q)$ with $(\tilde{\kappa}, \Omega_H, \Phi_H)$ is given by

$$dm = \frac{\tilde{\kappa}}{\kappa} dA + \Omega_H dJ + \Phi_H dq \tag{3.5.7}$$

where (m, J, q) are mass, angular momentum, and charge, respectively.$(\tilde{\kappa}, \Omega_H, \Phi_H)$ are surface gravity of the event horizon, angular velocity, and electric surface potential, respectively. Analogously, the first law of thermodynamics states the conservation of energy of the system.

The second law of the black hole Mechanics. The second law corresponds to Hawking and Bekenstein's area theorem that the change of the area of a black hole horizon is always greater than or equal to zero. This is analogous to the second law of thermodynamics stating that the change of the entropy in an isolated system is always greater than or equal zero.

$$\frac{dA}{dt} \geq 0 \tag{3.5.8}$$

Provided that $T_{\mu\nu}$ satisfies weak energy condition and the cosmic censorship hypothesis. It is a strong statement of a link between entropy and the area of a black hole horizon.

The third law of the black hole. It is impossible to make an extreme black hole from a normal one by setting $Q = M$ and $J = M$ in order to have the vanishing surface gravity and the temperature of the black hole.

A certain black hole obeys "the no-hair theorem". It states that no matter what the initial condition of a star used to have, at the end it is just described by three parameters which are mass, charge, and angular momentum.

We can use the same idea in thermodynamics of black hole that the entropy of the system has to be calculated by the microscopic degree of freedom via Boltzmann's equation

$$S_{stat} = \ln N(M, Q, J). \tag{3.5.9}$$

It must agree with the macroscopic parameters in classical gravity which is characterized by M, Q, and J as in equation (3.5.5). To account for the microstate of the black holes, we need a quantum theory of gravity. This is also the quest for studying black holes in string theory [112]. Moreover, the computation of AdS black hole entropy can be also holographically calculated by studying the holographic supersymmetric AdS black hole solution of gauged supergravity in various dimensions.

CHAPTER IV

Supergravity

4.1 Supersymmetry

Supersymmetry is a theoretical conjecture of the relation between bosons and fermions. They lead to a multiplet called supermultiplet. In this chapter, we will review some crucial points about supersymmetry, see also [89–93, 108, 111].

4.1.1 Superalgebra

The No-go theorem proposed in 1967 by Coleman-Mandula which states that the most general symmetries of the S-matrix based on conventional Lie algebra are the direct sum of the Poincare group with an internal symmetry group

$$\text{Ponicare} \otimes \text{Internal}. \tag{4.1.1}$$

So, there is no mix between Poincare Lie algebra and internal Lie algebra. In opposed to the No-go theorem, supersymmetry algebra is an extension of Poincare algebra where we can add fermion generators because we replace Lie algebra by the so-called graded Lie algebra. A similar idea was also applied to add fermions into string theory to develop superstring theory by Schwarz, Gervais, and Sakita. In 1974, Haag, Lopouszanski, and Sohnuis showed that fermion operators of supersymmetry are spinors of representation $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ called supercharges given by

$$Q^i_a \quad \text{, and} \quad \overline{Q}^i_{\dot{a}} \tag{4.1.2}$$

respectively where $i, j, = 1, ..., N$ and $a, \dot{a} = 1, 2$. These operators obey the graded Lie algebra of the form

$$[\mathcal{O}_a, \mathcal{O}_b\} = \mathcal{O}_a\mathcal{O}_b - (-1)^{\eta_a\eta_b}\mathcal{O}_b\mathcal{O}_a = f_{ab}{}^{c}T_c \tag{4.1.3}$$

and Super-Jacobi identity

$$(-1)^{\eta_a\eta_c}[[\mathcal{O}_a, \mathcal{O}_b\}, \mathcal{O}_c\} + (-1)^{\eta_a\eta_b}[[\mathcal{O}_b, \mathcal{O}_c\}, \mathcal{O}_a\} + (-1)^{\eta_b\eta_c}[[\mathcal{O}_c, \mathcal{O}_a\}, \mathcal{O}_b\} = 0 \tag{4.1.4}$$

where η_a defines the type of generators as

$$\eta_a = \begin{cases} 1, & \text{if} \quad \mathcal{O}_a \quad \text{is bosonic generators.} \\ 0, & \text{if} \quad \mathcal{O}_a \quad \text{is fermionic generators.} \end{cases}$$

These are called Lie superalgebra. Moreover, we can find the commutation relations of superalgebra given by

$$\begin{aligned}
[P^\rho, J^{\mu\nu}] &= i(\eta^{\mu\nu}P^\nu - \eta^{\nu\rho}P^\mu), \\
[P^\mu, P^\nu] &= 0, \\
[J^{\mu\nu}, J^{\rho\sigma}] &= -i(J^{\mu\sigma}\eta^{\nu\rho} - J^{\nu\sigma}\eta^{\mu\rho} + J^{\nu\rho}\eta^{\mu\sigma} - J^{\mu\rho}\eta^{\nu\sigma}), \\
[P^\mu, Q_{ai}] &= [P^\mu, \overline{Q}^i_{\dot{a}}] = 0, \\
[Q_{ai}, J^{\mu\nu}] &= i(\sigma^{\mu\nu})_a{}^b Q_{bi}, \\
[\overline{Q}^{\dot{a}}i, J^{\mu\nu}] &= i(\overline{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}}\overline{Q}^{bi}, \\
\{Q_{ai}, Q_{bj}\} &= -\frac{1}{2}\epsilon_{ab}Z_{ij}, \\
\{Q_{ai}, \overline{Q}^j_{\dot{a}}\} &= -\frac{1}{2}\delta^j_i\sigma_{\mu a\dot{a}}P^\mu, \\
\{\overline{Q}^i_{\dot{a}}, \overline{Q}^j_{\dot{b}}\} &= -\frac{1}{2}\epsilon_{\dot{a}\dot{b}}Z^{ij}
\end{aligned} \tag{4.1.5}$$

where P^μ is translation generators, $J^{\mu\nu}$ is Lorentz generators, and Z^{ij} is an antisymmetric matrix providing the abelian subalgebra of internal symmetries which commutes with the other generators

$$[Z^{ij}, P^\mu] = [Z^{ij}, J^{\mu\nu}] = [Z^{ij}, Q^k_\alpha] = [Z^{ij}, Z^{kl}] = [Z^{ij}, T_A] = 0 \tag{4.1.6}$$

where T_A are internal generators of supersymmetry with the algebra

$$\begin{aligned}
[T_A, T_B] &= f_{AB}{}^{C}T_C, \\
[Q_{ai}, T_A] &= (S_A)_i{}^j Q_{aj}, \\
[\bar{Q}^i_{\dot{a}}, T_A] &= -(S^{*A})^i{}_j\bar{Q}^j_{\dot{a}}
\end{aligned} \tag{4.1.7}$$

where $(S_A)_i{}^j$ is generators T_A in representation of supercharges and $f_{AB}{}^C$ is the structure constant. Moreover, supersymmetry is also equipped with its own internal symmetry called R-symmetry denoted by H_R.

4.1.2 Supermultiplet

Supermultiplet is a set of fermions and bosons that transform into each other under supersymmetry. In supergravity, we are interested in gravity multiplet comprising graviton and gravitino. The number of gravitino corresponds to the number of supersymmetry N which is constrained by the number supercharges $4N$. In four-dimensional spacetime, the number of gravitinos can vary from $0 \leq N \leq 8$. If $N = 0$, we get general relativity and if $N = 8$, the theory is called maximal supergravity. In supermultiplet, the number of boson (n_B) is equal to the number of fermion (n_F)

$$n_B = n_F. \tag{4.1.8}$$

This valids in any supermultiplet.

Massless Multiplet

Massless representation is very crucial in order to construct the theory of elementary particle physics. The category of massless gravity multiplet in four dimensions can be shown as in the table (4.1). Moreover, for $N \leq 4$, we can also couple gravity multiplet to other matter multiplets. For $N = 1, 2$, we can couple chiral, vector, and scalar multiples to gravity multiplet. For $3 \leq N \leq 4$, we can only couple vector multiplet to gravity multiplet. The massless representation of supersymmetry of $N = 1$ supermultiplet comprises of only two states given by

$$|p_\mu, \lambda\rangle \qquad \text{, and} \qquad |p_\mu, (\lambda + \frac{1}{2})\rangle \tag{4.1.9}$$

with λ and $\lambda + \frac{1}{2}$ being the helicity of a particle and the corresponding superpartner. To account for the discrete CPT symmetry, we usually start with the negative helicity of the particle and add the CPT conjecture helicity of the supermulitplet.

supermultiplet of gravity $\lambda = -2$					
The number of Supersymmetry	s=2	s=3/2	s=1	s=1/2	s=0
N=1	1	1			
N=2	1	2	1		
N=3	1	3	3	1	
N=4	1	4	6	4	1+1
N=5	1	5	10	10+1	5+5
N=6	1	6	15+1	20+6	15+15
N=7	1	7	21+7	35+21	35+35
N=8	1	8	28	56	70

Table 4.1: The table shows the field contents in supermultiplet of gravity.

We can write possible $N = 1$ supermultiplets as follows.

Chiral multiplet

We will set $\lambda = -\frac{1}{2}$, we therefore obtain

$$|p_\mu, \lambda = 0\rangle \qquad \text{, and} \qquad |p_\mu, \lambda = \pm\frac{1}{2}\rangle. \tag{4.1.10}$$

These states represent a pair of partcles of spin 0 and 1/2, respectively.

Gauge or Vector multiplet

We will set $\lambda = -1$, we therefore obtain

$$|p_\mu, \lambda = \pm\frac{1}{2}\rangle \qquad \text{, and} \qquad |p_\mu, \lambda = \pm 1\rangle. \tag{4.1.11}$$

This representation shows a pair of particles of spin 1/2 and 1, respectively.

Gravity multiplet

Finally, if we set $\lambda = -2$, we eventually get

$$|p_\mu, \lambda = \pm 2\rangle \qquad \text{, and} \qquad |p_\mu, \lambda = \pm\frac{3}{2}\rangle. \tag{4.1.12}$$

This representation shows the massless spin-2 particle with its massless superpartner gravitino of spin 3/2.

To find an extended supermultiplet, we can simply apply the same method of (4.1.9) for each of the resulting states until we reach the desired number of supersymmetry. In four dimensions, the maximum number of supersymmetry is $N = 8$, we then get the previous table of massless gravity multiplet $\lambda = -2$.

Rarita-Schwinger field

The action of massive the Rarita-Schwinger in D-dimensional flat Minkowski spacetime is given by

$$S_{RS} = -\int d^D x \overline{\psi}_\mu (\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho})\psi_\rho \tag{4.1.13}$$

where $\gamma^{\mu\nu\rho} \equiv \frac{1}{3!}\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]}$.

To write the action of gravitino which is the massless spin-3/2 field, we can generalize the action of massive the Rarita-Schwinger in D-dimensional flat to the action of massless the Rarita-Schwinger in a D-dimensional curved spacetime given by

$$S_{RS} = -\frac{1}{2\kappa^2}\int d^D x \overline{\psi}_\mu \gamma^{\mu\nu\rho}\mathcal{D}_\nu \psi_\rho \tag{4.1.14}$$

where $e =$det ${e_\mu}^a$, $\gamma^\mu = {e_a}^\mu \gamma^a$, and $\mathcal{D}_\mu = \partial_\mu \epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}$. The term $\frac{1}{2\kappa^2}$ is added to be consistent when we consider supergravity.

4.1.3 Manifolds in Supergravity

Supergravity is an extension of general relativity, so the geometry of manifold still plays an important role. From a geometrical physics point of view, there is the idea of isometry which means the equality of measure under a group of transformations called the isometry group G_{iso}. The elements of the isometry group are diffeomorphism. Besides, in order to complete the idea of isometry, we need to define a variable associated with the measurable quantities, such as area, volume, length, angle, etc. The variable is the metric tensor $g(x)$. As a result, we can characterize the geometry of spacetime based on the isometry group of transformations equipped with the metric tensor. The geometry is either called Riemannian or pseudo-Riemannian manifold denoted by $(M, g(x))$. [81, 95]

Coset Manifold

To relate Lie group G with the isometry of a group G_{iso}, we need to impose an additional constraint to the property of the isometry group called the transitive action on G_{iso}, which means that any two points can be related to each other by the isometry group. Moreover, the manifold is said to be homogeneous. This means that the isometry group can be described by a given Lie group G, $G_{iso} \to G$. Therefore, the homogenous manifold can be represented by the coset manifold $\mathcal{M}$.

$$\mathcal{M} = \frac{G}{H'} \tag{4.1.15}$$

G is a transitive action group on a generic point on $\mathcal{M}$ and act on bosonic fields, graviton e^a_μ, vector fields A^Λ_μ, and scalar fields ϕ^i. $H^{'}$ is a holonomy group that leaves a generic point on $\mathcal{M}$ invariant. As a result, the elements which describes the manifold $\mathcal{M}$ are the set of $g^{'} \sim g$ if $g = g^{'}h$ where $g^{'} \in G$ and $h \in H^{'}$ meaning that $g^{'}X_0 = X$ and $hX_0 = X_0$, then $gX_0 = g^{'}h = g^{'}X_0 = X$. Therefore, the dimension of coset manifold is dim(G)-dim($H^{'}$), which corresponds to the number of coordinate characterizing the coset manifold. Moreover, the manifold is said to be homogenous symmetric space meaning that a holonomy group $H^{'}$ becomes the holonomy group H of the compact maximal subgroup of non-compact group G, and G becomes a semi-simple group. The coset manifold can be written as

$$\mathcal{M} = \frac{G}{H}. \tag{4.1.16}$$

$H = H_R \times H_m$ is holonomy group acting on fermionic fields, gravitinos ψ, and chiral fields χ_i of spin 1/2, where H_R is the R-symmetry, automorphism group of supersymmetry, and H_m is a compact subgroup acting on matter fields. For $N \geq 5$, $H = H_R$ because there is no matter multiplets. In four dimensional spacetime with $1 \leq N \leq 8$, if $N < 8$, $H_R = U(N)$ and if $N = 8$, $H_R = SU(8)$.

4.2 Supergravity

The law of physics is usually based on symmetries, spacetime symmetries, and internal symmetries. Besides, there exists the biggest symmetry relating spacetime symmetries and internal symmetries together so-called supersymmetry. Field theories being invariant under supersymmetry is called supersymmetric field theory. Like other symmetries, supersymmetry has both global (rigid) and local (gauge) versions. The generalization of local supersymmetric field theory is called supergravity which allows us to have the interacting field theory between graviton with other spin-s particles. We will focus on supergravities in four-dimensional spacetime, and there are many excellent reviews on supergravity, see [79, 87, 89, 96, 97, 105–111]. In this section, we will discuss the general properties of supergravity and we will work on $N = 6$ supergravity in detail in the next chapter.

4.2.1 Minimal $N = 1$ Supergravity in four dimensions

We start with the "minimal $N = 1$ pure supergravity" where we will implement the idea of second and first-order formalism. The latter is used to conveniently include the higher order terms in gravitino ψ_μ. In second-order formalism, the action of minimal $N = 1$ pure supergravity can be written as the combination of the usual Einstien-Hilbert action of massless spin-2 graviton and Rarita-Schwinger action of massless spin-3/2 gravitino given by

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\left(R(\omega) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho\right) \tag{4.2.1}$$

where the supersymmetry variations are given by

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \tag{4.2.2}$$

$$\delta\psi_\mu = \mathcal{D}_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab} \tag{4.2.3}$$

with useful formulas resulting from the above ones

$$\delta e = \frac{1}{2}e(\bar{\epsilon}\gamma^\rho\psi_\rho) \tag{4.2.4}$$

$$\delta e^\mu_a = -\frac{1}{2}\bar{\epsilon}\gamma^\mu\psi_a \qquad \text{and} \qquad \delta e^{\mu a} = -\frac{1}{2}\bar{\epsilon}\gamma^\mu\psi^a. \tag{4.2.5}$$

We now consider the supersymmetry variation between graviton and gravitino.

We first consider the action of gravity S_2 expressed in terms of frame fields

$$\begin{aligned} S_2 &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xeR(\omega) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xeg^{\mu\nu}g^{\rho\lambda}e^a_\rho e^b_\nu R_{\lambda\mu ab}(\omega) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xee^{\mu a}e^{\nu b}R_{\mu\nu ab}(\omega) \end{aligned} \tag{4.2.6}$$

and its supersymmetry variation given by

$$\begin{aligned} \delta S_2 &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\left((\delta e)e^{\mu a}e^{\nu b}R_{\mu\nu ab} + 2e(\delta e^{\mu a})e^{\nu b}R_{\mu\nu ab} + ee^{\mu a}e^{\nu b}(\delta R_{\mu\nu ab})\right) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\left(\frac{1}{2}e(\bar{\epsilon}\gamma^\rho\psi_\rho R(\omega) + 2e(-\frac{1}{2}\bar{\epsilon}\gamma^\rho\psi^a e^{\nu b}R_{\mu\nu ab}(\omega)))\right) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\left(\frac{1}{2}\bar{\epsilon}\gamma^\rho\psi_\rho R(\omega) - \bar{\epsilon}\gamma^\rho\psi^\lambda e^a_\lambda g^{\nu\sigma}e^b_\sigma R_{\mu\nu ab}(\omega)\right) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\left(\frac{1}{2}\bar{\epsilon}\gamma^\rho\psi_\rho R(\omega) - \bar{\epsilon}\gamma^\rho\psi^\lambda R_{\mu\lambda}(\omega)\right) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)(-\bar{\epsilon}\gamma^\mu\psi^\nu)\,. \end{aligned} \tag{4.2.7}$$

The last term of the first line vanishes due to the surface terms.

We next look at $S_{3/2}$ given by

$$S_{3/2} = -\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho \tag{4.2.8}$$

and, consider the supersymmetry variation of the gravitino action $S_{3/2}$

$$\begin{aligned} \delta S_{3/2} &= -\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\left(\delta\bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho + \bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\delta\psi_\rho\right) \\ &= -\frac{1}{\kappa^2}\int_{\mathcal{M}} d^4xe\left(\delta\bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\mu\psi_\rho\right) \\ &= -\frac{1}{\kappa^2}\int_{\mathcal{M}} d^4xe\bar{\epsilon}\overleftarrow{\mathcal{D}_\mu}\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho \\ &= \frac{1}{\kappa^2}\int_{\mathcal{M}} d^4xe\left(\bar{\epsilon}\mathcal{D}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho + \bar{\epsilon}\gamma^{\mu\nu\rho}\mathcal{D}_\mu\mathcal{D}_\nu\psi_\rho\right) \\ &= \frac{1}{\kappa^2}\int_{\mathcal{M}} d^4xe\bar{\epsilon}\gamma^{\mu\nu\rho}\mathcal{D}_\mu\mathcal{D}_\nu\psi_\rho \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\gamma^{\mu\nu\rho}[\mathcal{D}_\mu,\mathcal{D}_\nu]\psi_\rho \\ &= \frac{1}{8\kappa^2}\int_{\mathcal{M}} d^4xe\bar{\epsilon}\gamma^{\mu\nu\rho}\gamma^{ab}R_{\mu\nu ab}\psi_\rho \end{aligned} \tag{4.2.9}$$

where the factor of 2 from the second line results from the identical contribution of the variation of ψ and $\bar{\psi}$, we use the integration by part on third line and neglect the surface term, the vanishing of first term of fourth line due to $\gamma^{\mu\nu\rho}\mathcal{D}\psi_\rho = \gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho$ and $\nabla_\mu\gamma_\nu = 0$, and the existence of Riemann tensor of the last line results from $[\mathcal{D}_\mu, \mathcal{D}_\nu]\psi = \frac{1}{4}R_{\mu\nu ab}\gamma^{ab}\psi$.

To continue, we use

$$\gamma^{\mu\nu\rho}\gamma_{\tau\sigma} = \gamma^{\mu\nu\rho}{}_{\tau\sigma} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]}. \tag{4.2.10}$$

Then, we have

$$\begin{aligned}\gamma^{\mu\nu\rho}\gamma^{ab}R_{\mu\nu ab} &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 6R_{\mu\nu}{}^{[\rho}{}_{b}\gamma^{\mu\nu]b} + 6\gamma^{[\mu}R_{\mu\nu}{}^{\rho\nu]} \\ &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 2R_{\mu\nu}{}^{\rho}{}_{b}\gamma^{\mu\nu b} + 4R_{\mu\nu}{}^{\mu}{}_{b}\gamma^{\nu\rho b} \\ &\quad + 4\gamma^{\mu}R^{\rho\nu}_{\mu\nu} + 2\gamma^{\rho}R_{\mu\nu}{}^{\nu\mu} \\ &= 4\gamma^{\mu}R_{\mu\nu}{}^{\rho\nu} + 2\gamma^{\rho}R_{\mu\nu}{}^{\nu\mu} \\ &= 4\gamma^{\mu}R_{\mu}{}^{\rho} + 2\gamma^{\rho}(-R)\end{aligned} \tag{4.2.11}$$

where the rank fifth gamma matrix $\gamma^{\mu\nu\rho ab}$ vanishes in four dimensions, the second term of the second line vanishes due to $R_{\mu[\nu\rho\sigma]} = 0$, and the third term of the second line vanishes by the contraction of symmetric $R_{\nu b}$ and anti-symmetric matrix $\gamma^{\mu\rho b}$.

We finally obtain

$$\begin{aligned}\delta_{3/2}S &= \frac{1}{8\kappa^2}\int_{\mathcal{M}} d^4xe\left(4\bar{\epsilon}\gamma^{\mu}\psi_\rho R_{\mu}{}^{\rho} - 2\bar{\epsilon}\gamma^{\rho}\psi_\rho R\right) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\left(\bar{\epsilon}\gamma^{\mu}\psi^{\nu}R_{\mu\nu} - \frac{1}{2}\bar{\epsilon}\gamma^{\mu}\psi^{\nu}g_{\mu\nu}R\right) \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\left(\bar{\epsilon}\gamma^{\mu}\psi^{\nu}\right).\end{aligned} \tag{4.2.12}$$

Therefore, the action is manifestly invariant $\delta S_2 + \delta S_{3/2} = 0$ under local supersymmetry to the first order in ψ_ν which also holds for any D-dimensional spacetime. To complete the local supersymmetry of the theory, we need to add ψ_μ^4 terms to the theory by using the first order formalism. In particular, we use $\omega = \omega(e) + K$.

K can be obtained by the field equation for the spin connection given by

$$\begin{aligned}\delta_\omega S_2 &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xee^{\mu a}e^{\nu b}\delta R_{\mu\nu ab}\\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xee^{\mu a}e^{\nu b}\left(\nabla_\mu\delta\omega_{\nu ab}-\nabla_\nu\delta\omega_{\mu ab}\right)\\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xee^{\mu a}e^{\nu b}\left(2\nabla_\mu\delta\omega_\nu{}^{ab}+T_{\mu\nu}{}^{\rho}\delta\omega_\rho{}^{ab}\right)\\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4e\left(T_{\rho a}{}^{\rho}e^\nu_b-T_{\rho b}{}^{\rho}e^\nu_a+T_{ab}{}^{\nu}\right)\delta\omega_\nu{}^{ab}\end{aligned} \tag{4.2.13}$$

where we use $\delta R_{\mu\nu ab}=\nabla_\mu\delta\omega_{\nu ab}-\nabla_\nu\delta\omega_{\mu ab}$ in the first line, $2\nabla_{[\mu}\psi_{\nu]}=2\mathcal{D}_{[\mu}\psi_{\nu]}-T^\rho_{\mu\nu}\psi_\rho$ and omitting the surface term.

We get

$$\begin{aligned}\delta_\omega S_{3/2} &= -\frac{1}{2\kappa^2}\delta_\omega\int_{\mathcal{M}} d^4xe\bar\psi_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho\\ &= -\frac{1}{8\kappa^2}\int_{\mathcal{M}} d^4xe\left(\bar\psi_\mu\gamma^{\mu\nu\rho}\gamma_{ab}\psi_\rho\right)\delta\omega_\nu{}^{ab}\\ &= -\frac{1}{8\kappa^2}\int_{\mathcal{M}} d^4xe\left(\bar\psi_\mu(6\gamma^{[\mu}e^\nu{}_{[b}e^{\rho]}{}_{a]})\psi_\rho\right)\end{aligned} \tag{4.2.14}$$

where we use $\mathcal{D}_\mu=\partial_\mu+\frac{1}{4}\omega^{ab}_\mu\gamma_{ab}$ in the first line, and $\gamma^{\mu\nu\rho}\gamma_{ab}=\gamma^{\mu\nu\rho}{}_{ab}+6\gamma^{[\mu}e^\nu{}_{[b}e^{\rho]}{}_{a]}$ with again the fifth rank gamma matrix $\gamma^{\mu\nu\rho}{}_{ab}=0$ in four dimensions for the second line.

By $\delta_\omega S_2+\delta_\omega S_{3/2}=0$, we obtain

$$T_{\rho a}{}^{\rho}e^\nu_b-T_{\rho b}{}^{\rho}e^\nu_a+T_{ab}{}^{\nu}=\frac{1}{2}(\bar\psi_a\gamma^\nu\psi_b+\bar\psi_\rho\gamma^\rho\psi_a e^\nu_b-\bar\psi_\rho\gamma^\rho\psi_b e^\nu_a) \tag{4.2.15}$$

and

$$T_{ab}{}^{\mu}=\frac{1}{2}\bar\psi_a\gamma^\mu\psi_b. \tag{4.2.16}$$

We then obtain the contorsion K given by

$$K_{\mu\nu\rho}=-\frac{1}{4}(\bar\psi_\mu\gamma_\rho\psi_\nu-\bar\psi_\nu\gamma_\mu\psi_\rho+\bar\psi_\rho\gamma_\nu\psi_\mu). \tag{4.2.17}$$

Finally, we can substitute the contorsion in $\omega=\omega(e)+K$ entering via Riemann tensor $R(\omega)$, and $\mathcal{D}(\omega)$ in the first order formalism action. Then, we obtain the equivalent second-order formalism of the minimal $N=1$ supergravity action with the extra term of ψ^4 written as

$$\begin{aligned}S &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4xe\big[R-\bar\psi_\mu\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho+\frac{1}{4}(\bar\psi_\mu\gamma^\nu\psi_\nu)(\bar\psi^\mu\gamma^\rho\psi_\rho)\\ &\quad -\frac{1}{16}(\bar\psi^\rho\gamma^\mu\psi^\nu)(\bar\psi_\rho\gamma_\mu\psi_\nu+2\bar\psi_\rho\gamma_\nu\psi_\mu)\big].\end{aligned} \tag{4.2.18}$$

This action is completely invariant under local supersymmetry to all order in ψ.

4.2.2 Extended Supergravity

Scalar Sector In supergravity, scalar fields $\phi^i, i = 1, 2, ..., n_s$ describe the coordinate of a manifold, so such a manifold is said to be the Riemannian scalar manifold $\mathcal{M}_{\text{scl}}$ with the definition of the metric tensor $\mathcal{G}_{st}(\phi)$. The behavior of scalar fields can be described by non-linear sigma model coupling to graviton written as

$$\mathcal{L}_s = -\frac{1}{2}\mathcal{G}_{st}(\phi)\partial_\mu\phi^s\partial^\mu\phi^t. \tag{4.2.19}$$

To couple scalar fields to vector and fermion fields for a certain N extended supersymmetry, we need extra constraints and additional structures on the scalar manifold $\mathcal{M}_{\text{scl}}$. The first extra structure is a flat symplectic bundle on $\mathcal{M}_{\text{scl}}$ and the symplectic electric-magnetic duality. The second structure results from the fact that fermion fields transform under the local Lorentz group which displays the properties of spacetime and they also transform in the representation of the holonomy group H of the scalar manifold which would correspond to the holonomy of the Levi-Civita connection. Therefore, these would determine the interaction between scalar fields and vector fields, and fermion fields respectively.

The coset gh, $g \in G$ where G is the global symmetry group and $h \in H$ of manifold $\mathcal{M}_{\text{scal}}$ can be characterized by the set of coordinates of dimensions $\dim(G)$-$\dim(H)$. Therefore, the representation of scalar coset manifold can be written in terms of scalar fields as

$$L(\phi^s(x)) = \frac{G}{H} \tag{4.2.20}$$

such that the action of h on $L(\phi^s(x))$ is fixed to be local right action $h \to h(x)$ on $L(\phi^s(x))$ while the action of g on $L(\phi^s(x))$ is said to be globally left action on $L(\phi^s(x))$.

Vector sector In supergravity, there exist A^Λ_μ, $\Lambda = 1, 2, 3..., n_v$ which define the electric field strength $F^\Lambda_{\mu\nu}$ given by

$$F^\Lambda_{\mu\nu} = \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu \tag{4.2.21}$$

and the magnetic dual tensor $G_{\Lambda\mu\nu}$ defined as

$$G_{\Lambda\mu\nu} = -\epsilon_{\mu\nu\rho\sigma}\frac{\partial \mathcal{L}}{\partial F^\Lambda_{\mu\nu}} = R_{\Lambda\Sigma}F^\Sigma_{\mu\nu} - I_{\Lambda\Sigma} * F^\Sigma_{\mu\nu} \tag{4.2.22}$$

where

$$*F^\Lambda_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\Lambda\rho\sigma} \tag{4.2.23}$$

This relation shows the duality between the electric field and the magnetic field. We can also express F^Λ and G_Λ as a $2n_v$ dimensional vector as

$$\mathcal{G} = (\frac{1}{2}G^M_{\mu\nu}dx^\mu dx^\nu) = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \frac{dx^\mu dx^\nu}{2} \tag{4.2.24}$$

we have

$$\mathcal{G}^M = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}. \tag{4.2.25}$$

$\mathcal{G}$ would describe the electric field and the magnetic dual tensor where the index $M = ({}^\Lambda, {}_\Lambda)$, so the field equations and Bianchi's identities can be written as

$$d\mathcal{G}^M = 0 \tag{4.2.26}$$

And the duality properties can be written as

$$*\mathcal{G} = -\mathbb{C}\mathbb{M}(\phi)\mathcal{G} \tag{4.2.27}$$

where $\mathbb{C}$ is a symplectic matrix

$$\mathbb{C} = (C^{MN}) = \begin{pmatrix} 0 & I_{n_v} \\ -I_{n_v} & 0 \end{pmatrix} \tag{4.2.28}$$

with $\mathbb{M}(\phi)$ also a symplectic matrix

$$\mathbb{M}(\phi)\mathbb{C}\mathbb{M}(\phi) = \mathbb{C}. \tag{4.2.29}$$

$M(\phi)$ would contain the matrix $I_{\Lambda\sigma}$ and $R_{\Lambda\Sigma}$ which characterize the non-minimal couplings of the scalars to the vector fields.

Global symmetry group For supergravity $N > 1$, G is the symmetry group of scalar action, now we need to promote G to be global symmetry group of the bosonic action, both vector field and scalar field.

The action of $g \in G$ on scalar and vector fields are non-linear and linear action written as

$$\begin{aligned} \phi^r &\to g \star \phi^r \qquad &&\text{, non-linear} \\ \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} &= \mathcal{R}_V[g] \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \qquad &&\text{, linear} \end{aligned} \tag{4.2.30}$$

where in general the symplectic representation of g can be written as

$$\mathcal{R}_V[g] = \begin{pmatrix} A[g]^\Lambda{}_\Sigma & B[g]^{\Lambda\Sigma} \\ C[g]_{\Lambda\Sigma} & D[g]_\Lambda{}^\Sigma \end{pmatrix}. \tag{4.2.31}$$

Both symplectic representation of $\mathcal{R}_V[g]$ and $\mathbb{M}[\phi]$ form a flat symplectic structure on scalar manifold $\mathcal{M}$. To preserve the field's equations, Bianchi's identity, and Lagrangian, so we need $B[g]^{\Lambda\Sigma}$=0. The corresponding group is the electric subgroup of the isometry group, $G_{el} \in G$.

$$\mathcal{R}_V[g]^M{}_N = \begin{pmatrix} A[g]^\Lambda{}_\Sigma & 0 \\ C[g]_{\Lambda\Sigma} & D[g]_\Lambda{}^\Sigma \end{pmatrix} \tag{4.2.32}$$

In $N = 6$ supergravity, $\mathcal{M}_{\text{scl}} = \frac{SO^*(12)}{U(6)}$, where $SO^*(12)$ is a special non-compact form of $SO(12)$ with maximal compact subgroup of $U(6) \sim SU(6) \times U(1)$ and the symplectic representation of $\mathcal{R}_V[g] = 32_c$.

Fermion sector The fermionic fields transform under the holonomy group of local H while the bosonic fields transform under the global symmetry group of G. The fermionic sector comprises gravitinos of spin-3/2 denoted by $\psi_{\mu A}$ and spin-$\frac{1}{2}$ particle denoted by χ_{ABC}, λ_{IA}, and λ_α with the corresponding charge conjugate spinors ψ_μ^A, χ^{ABC}, λ_I^A, and λ^α respectively. The fermionic fields can be represented by the Weyl spinors with positive chirality

$$\gamma^5 \begin{pmatrix} \psi_{\mu A} \\ \chi_{ABC} \\ \lambda_{IA} \\ \lambda_{\alpha} \end{pmatrix} = \begin{pmatrix} \Psi_{\mu A} \\ \chi_{ABC} \\ \lambda_{IA} \\ \lambda_{\alpha} \end{pmatrix} \tag{4.2.33}$$

while the chirality of the charge conjugate is negative

$$\gamma^5 \begin{pmatrix} \psi^{\mu A} \\ \chi^{ABC} \\ \lambda_I^A \\ \lambda^{\alpha} \end{pmatrix} = -\begin{pmatrix} \Psi^{\mu A} \\ \chi^{ABC} \\ \lambda_I^A \\ \lambda^{\alpha} \end{pmatrix} \tag{4.2.34}$$

To couple bosonic fields and fermionic fields, we need quantities that transform in both H and G. The coset representative $L(\phi)$ is such a quantity. Therefore, the scalar sector is the link of the interaction between bosonic fields and fermionic fields. Moreover, the transformation under G and the local group H is similar to general coordinate transformation (GCT) and local Lorentz transformation (LLT) in spacetime respectively.

Since, bosonic fields or tensors transform under GCT and fermionic fields or spinors transform under LLT, the Lagrangian density which is invariant under H must have the form of covariant derivative having Q as the composite connection since H is a gauge symmetry

$$\mathcal{D}_\mu \psi = \nabla_\mu \psi + Q_\mu \circ \psi. \tag{4.2.35}$$

$\mathcal{D}_\mu$ can be written in terms of spin connection ω and Christoeffel connection Γ.

4.2.3 Gaugings

In this section, we will promote a suitable subgroup G_{g} of the electric subgroup G_{el} belonging to the isometry group G to a gauge group gauged by vector fields of the theory with the non-Abelian structure of vector fields. This procedure is to obtain gauged supergravity from the ungauged one with the same number of supersymmetry by using embedding tensor formalism.

The theory is gauged by the vector fields $A^{\hat{\Lambda}}$ belonging to the adjoint representation of G_{g} of the theory so we have the condition

$$\mathrm{dim}G_{\mathrm{g}} \leq n_v. \tag{4.2.36}$$

We introduce the minimal coupling with the gauge connection defined by

$$\Omega_{g\mu} = gA^{\hat{\Lambda}}_{\mu}X_{\hat{\Lambda}} \tag{4.2.37}$$

where g is a gauged coupling constant, and $X_{\hat{\Lambda}}$ is gauge generators of the gauge group of G_{g}.

The gauge generators can be written in terms of symplectic representation of $R_v[X_{\hat{\Lambda}}]$ as

$$(X_{\hat{\Lambda}})^{\hat{M}}{}_{\hat{N}} = R_v[X_{\hat{\Lambda}}]^{\hat{M}}{}_{\hat{N}} = \begin{pmatrix} X^{\hat{\Sigma}}_{\hat{\Lambda}\ \hat{\Gamma}} & 0 \\ -X_{\hat{\Lambda}\hat{\Sigma}\hat{\Gamma}} & X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} \end{pmatrix} \tag{4.2.38}$$

We are interested in the case that $X_{\hat{\Lambda}\hat{\Sigma}\hat{\Gamma}} = 0$ with the quadratic constraint on gauge generators written as

$$[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = -X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}}X_{\hat{\Gamma}} \tag{4.2.39}$$

We also have the linear constraint

$$X_{(\hat{\Lambda}\hat{\Sigma}\hat{\Gamma})} = 0. \tag{4.2.40}$$

Under the infinitesimal gauge transformation of $g(x) \in G_{\mathrm{g}}$, and $g(x) = 1 + g\xi^{\hat{\Lambda}}(x)X_{\hat{\Lambda}}$, we define the covariant derivative as

$$\nabla_{\mu}\xi^{\hat{\Lambda}} \equiv \partial_{\mu}\xi^{\hat{\Lambda}} + gX_{\hat{\Sigma}\hat{\Gamma}}{}^{\hat{\Lambda}}A^{\hat{\Sigma}}_{\mu}\xi^{\hat{\Gamma}} \tag{4.2.41}$$

which is the total covariant derivative of general coordinate transformation and local Lorentz transformation including H and G_{g}. We will also redefine the curvature 2-form as

$$F^{\hat{\Lambda}}_{\mu\nu} = 2\partial_{[\mu}A^{\hat{\Lambda}}_{\nu]} + gX_{\hat{\Sigma}\hat{\Gamma}}{}^{\hat{\Lambda}}A^{\hat{\Sigma}}_{\mu}A^{\hat{\Sigma}_{\nu}} \tag{4.2.42}$$

and define $\hat{\Omega}_{\mu} = \hat{P}_{\mu} + \hat{Q}_{\mu}$ as

$$\hat{\Omega}_{\mu} = L^{-1}\nabla_{\mu}L = L^{-1}(\partial_{\mu} - gA^{\hat{\Lambda}}_{\mu}X_{\hat{\Lambda}})L \tag{4.2.43}$$

where

$$\begin{aligned}\hat{P}_\mu &= P_\mu - A_\mu^{\hat{\Lambda}} P_{\hat{\Lambda}} \\ \hat{Q}_\mu &= Q_\mu - A_\mu^{\hat{\Lambda}} Q_{\hat{\Lambda}}\end{aligned}. \tag{4.2.44}$$

$P_{\hat{\Lambda}}$ and $Q_{\hat{\Lambda}}$ represent projection of $L^{-1} X_{\hat{\Lambda}} L$ onto subspace of coset space t and h respectively, written as

$$\begin{aligned}P_{\hat{\Lambda}} &= L^{-1} X_{\hat{\Lambda}} L|_{\mathfrak{k}} \\ Q_{\hat{\Lambda}} &= L^{-1} X_{\hat{\Lambda}} L|_{\mathfrak{h}}\end{aligned} \tag{4.2.45}$$

where g is Lie algebra of G and $g = h \oplus k$ with $\mathfrak{k} \in k$ and $\mathfrak{h} \in h$ being generators of the coset space and Lie algebra of H, respectively.

Embedding Tensor The embedding tensor $\Theta_{\hat{\Lambda}}^{\Sigma}$ is the projection operator from Lie algebra of global symmetry group of G_{el} onto to Lie algebra of gauge group G_{g} as

$$X_{\hat{\Lambda}} = \Theta_{\hat{\Lambda}}^{\Sigma} t_\Sigma \tag{4.2.46}$$

$\Theta_{\hat{\Lambda}}^{\Sigma}$ is in the representation of $n_v \times \mathrm{adj}(G_e)$, $\hat{\Lambda} = 1, ..., n_v$ and $\Sigma = 1, ..., \dim(G_{\mathrm{el}})$. Moreover, the embedding tensor is independent of symplectic frame. We can write the locality constraint as

$$\mathbb{C}^{MN} \Theta_M^a \Theta_N^b = 0 \tag{4.2.47}$$

which implies that $\dim(G_g) = \mathrm{rank}(\Theta^a) \leq n_v$ and the embedding tensor is invariant under a gauge transformation.

these deformations with the introduction of minimal coupling, the Lagrangian of ungauged supergravity will not be invariant under supersymmetry transformations, and the extra terms are needed. Such a term can be written in terms of a tensor under the holonomy group H called T-tensor as

$$T_{\overline{M}} = \mathbb{L}_{\overline{M}}{}^{N} L^{-1} X_N L. \tag{4.2.48}$$

Then, we can write this in the complex basis as

$$T_{\overline{M},\overline{N}}{}^{\overline{P}} = \mathbb{L}_{\overline{M}}{}^{M} \mathbb{L}_{\overline{N}}{}^{N} X_{MN}{}^{P} (\mathbb{L}^{-1})_P{}^{\overline{P}} \tag{4.2.49}$$

We can also write the local, linear, quadratic constraint of T-tensor as

$$\begin{aligned} \mathbb{C}^{\overline{MN}} T_{\overline{M}}{}^{a} T_{\overline{N}}{}^{b} &= 0 \\ T_{(\overline{MNP})} &= 0 \\ [T_{\overline{M}}, T_{\overline{N}}] + T_{\overline{MN}}{}^{\overline{P}} T_{\overline{P}} &= 0 \end{aligned} \tag{4.2.50}$$

To preserve the supersymmetry of the original theory after applying minimal coupling, we need to add the extra terms of the order of g to the deformed action. Such a term is called "Yukawa term" written as

$$e^{-1}\mathcal{L}_{\text{Yukawa}} = g(-2\overline{\phi}_{\mu}^{A}\gamma^{\mu\nu}\psi_{\nu}^{B}S_{AB} + \overline{\lambda}^{I}\gamma^{\mu}\psi_{A\mu}N_{I}{}^{A} + \overline{\lambda}^{I}\lambda^{J}M_{IJ}) + \text{c.c.} \tag{4.2.51}$$

where S_{AB},$N_I{}^A$, M_{IJ} are called fermion-shift matrices. Moreover, the supersymmetric variations will also be deformed with the extra term at order g given by

$$\delta\psi_{A\mu} = \nabla_{\mu}\epsilon_{A} - gS_{AB}\gamma_{\mu}\epsilon^{B} + ... \tag{4.2.52}$$

$$\delta\lambda_{I} = \hat{P}_{\mu}^{A}{}_{I}\gamma^{\mu}\epsilon_{A} + gN_{I}{}^{A}\epsilon_{A} + ... \tag{4.2.53}$$

Besides, if we do the variation of Yukawa term with respect to $\psi_{A\mu}$ and λ_I, there exist the term of order g^2 because $\delta\psi_{A\mu}$ and $\delta\lambda_I$ contain g. Such an extra term of order g^2 can be canceled by adding the quadratic term of the fermion-shift matrices S_{AB} and $N_I{}^A$ into the action. Such an extra term is called "scalar potential" and can be written as

$$\delta_{A}^{B}V = g^{2}(N^{I}{}_{A}N_{I}{}^{B} - 12S_{AC}S^{CB}). \tag{4.2.54}$$

In summary, to promote ungauged to gauged supergravity, we can deform the original ungauged supergravity by introducing minimal coupling and the modified tensor field strength with the non-Abelian gauge field. We obtain the supergravity $\mathcal{L}^{(0)}_{\text{gauged}}$ which is invariant under gauge group G_{g}, but it is not invariant under supersymmetry.

To preserve supersymmetry of the original supergravity, we need to add the "Yukawa term" and "scalar potential term" into the theory and the extra terms into the supersymmetry transformation of fermion fields. The complete Largrangian is given by

$$e^{-1}\mathcal{L}_{\text{gauged}} = e^{-1}\mathcal{L}_{\text{ungauged}}(\partial \to \nabla, dA \to dA + A \wedge A) + \mathcal{L}_{\text{Yukawa}} + \text{c.c.} - V \quad (4.2.55)$$

CHAPTER V

N=6 gauged supergravity with $SO(6)$ gauge group

N=6 gauged supergravity with $SO(6)$ gauge group gets more attention due to the compactification from the type IIA supergravity on ten-dimensional $AdS_4 \times CP_3$ [10–14] and eleven-dimensional supergravity on $AdS_4 \times S^7$ [50–55]. In this section, we will review the structure of N=6 gauged supergravity with $SO(6)$ gauge group in the embedding tensor formalism as described in [4]. We also discuss the relation between the fermion-shift matrices and, T-tensor, coset representative, and gauge group generators.

5.1 Supermultiuplet of $N = 6$ supergravity

In N=6 supergravity, there exists only the gravity supermultiplet without matter multiplets due to the constraints of supersymmetry. Field contents of four-dimensional N=6 supergravity are

$$(e_\mu^{\hat{\mu}}, \psi_{\mu A}, A_\mu^{AB}, A_\mu^0, \chi_{ABC}, \chi_A, \phi_{AB}). \tag{5.1.1}$$

The bosonic sector consists of the single graviton $e_\mu^{\hat{\mu}}$, sixteen vectors $A_\mu^{AB} = -A_\mu^{BA}$ and and A_μ^0, fifteen complex scalars $\phi_{AB} = -\phi_{BA}$. Real and imaginary parts of ϕ_{AB} are usually called scalars and pseudo-scalars, respectively. The manifold of the $N = 6$ supergravity is the scalar manifold of the form

$$\mathcal{M}_{\text{scl}} = \frac{G}{H} = \frac{SO^*(12)}{U(6)} \tag{5.1.2}$$

with isometry symmetries forming the global symmetry group of $G = SO^*(12)$ with only R-symmetry sitting in the maximal compact subgroup $H = H_R = U(6)$ of $G = SO^*(12)$. The fermion sectors comprise six gravitini $\psi_{\mu A}$, , twenty-six spin-$\frac{1}{2}$ fields $\chi_{ABC} = \chi_{[ABC]}$ and χ_A.

In order to study N=6 gauged supergravity, we follow most of the convention of N=6 gauged supergravity truncated from maximal N=8 gauged supergravity [4] with the gauging procedure following [96]. In this thesis, we will follow the spacetime and tangent space indices denoted by $\mu, \nu, \ldots = 0, 1, 2, 3$ and $\hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2, 3$, respectively. Moreover, we also use the indices of $A, B, \ldots = 1, 2, \ldots, 6$ as the fundamental representation of $SU(6)$ being the subgroup of the R-symmetry $U(6) \sim SU(6) \times U(1)$. The 15 complex scalar denoted by ϕ_{AB} are the set of coordinates spanning the scalar manifold $SO^*(12)/U(6)$ written by the coset representative in representation **32** of $SO^*(12)$ of the form

$$\mathcal{V}_M{}^{\underline{M}} = \mathcal{A}^\dagger e^Y \tag{5.1.3}$$

with the Claley matrix

$$\mathcal{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_{16} & i\mathbb{I}_{16} \\ \mathbb{I}_{16} & -i\mathbb{I}_{16} \end{pmatrix} \tag{5.1.4}$$

and

$$Y = \begin{pmatrix} 0 & 0_{1\times 15} & 0 & \phi_{CD} \\ 0_{15\times 1} & 0_{15\times 15} & \phi_{AB} & \frac{1}{2}\epsilon_{ABCDEF}\bar{\phi}^{EF}\phi_{CD} \\ 0 & \bar{\phi}^{CD} & 0 & 0_{1\times 15} \\ \bar{\phi}^{AB} & \frac{1}{2}\epsilon^{ABCDEF}\phi_{EF} & 0_{15\times 1} & 0_{15\times 15} \end{pmatrix}. \tag{5.1.5}$$

where $\bar{\phi}^{AB} = (\phi_{AB})^*$.

In following analysis, it is more useful to define 16×16 submatrices of $\mathcal{V}_M{}^{\underline{M}}$ by the use of this identification

$$\mathcal{V}_M{}^{\underline{M}} = \begin{pmatrix} \bar{h}_\Lambda{}^{\underline{\Lambda}} & h_{\Lambda\underline{\Lambda}} \\ \bar{f}^{\Lambda\underline{\Lambda}} & f^\Lambda{}_{\underline{\Lambda}} \end{pmatrix} \tag{5.1.6}$$

where we can define $\mathbf{f}$, $\mathbf{h}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{h}}$ such that they satisfy the relations

$$
\begin{aligned}
&(\mathbf{f}\mathbf{f}^\dagger)^T = \mathbf{f}\mathbf{f}^\dagger, \qquad (\mathbf{h}\mathbf{h}^\dagger)^T = \mathbf{h}\mathbf{h}^\dagger, \qquad \mathbf{f}\mathbf{h}^\dagger - \bar{\mathbf{f}}\mathbf{h}^T = i\mathbb{I}_{16}, \\
&\mathbf{f}^\dagger\mathbf{h} - \mathbf{h}^\dagger\mathbf{f} = -i\mathbb{I}_{16}, \qquad \mathbf{f}^T\mathbf{h} - \mathbf{h}^T\mathbf{f} = 0\,.
\end{aligned} \tag{5.1.7}
$$

We can write the inverse of $\mathcal{V}_M{}^{\underline{M}}$ in terms of $\mathbf{f}$ and $\mathbf{h}$ as

$$
\mathcal{V}_{\underline{M}}{}^{M} = \begin{pmatrix} -if^{\Lambda}{}_{\underline{\Lambda}} & ih_{\Lambda\underline{\Lambda}} \\ i\bar{f}^{\Lambda\underline{\Lambda}} & -i\bar{h}_{\Lambda}{}^{\underline{\Lambda}} \end{pmatrix}. \tag{5.1.8}
$$

The sixteen electric gauge fields A^{AB} and A^0 can be combined into a single $A^\Lambda = (A^0, A^{AB})$ along with its magnetic dual A_Λ written as

$$
A^M = (A^\Lambda, A_\Lambda) \tag{5.1.9}
$$

where, the gauge fields transform as **32** representation of $SO^*(12)$

5.2 Gaugings

Gaugings can be efficiently described by the embedding tensor formalism in which we define the linear combination of the generators of global symmetry as gauge generators

$$
X_M = \theta_M{}^n t_n \tag{5.2.1}
$$

where t^n is the global $SO^*(12)$ generators and $\theta_M{}^n$ is called the embedding tensor. Therefore, we can define the covariant derivative introducing the minimal coupling of various fields written as $\theta_M{}^m$.

$$
D_\mu = \nabla_\mu - gA^M_\mu X_M\,. \tag{5.2.2}
$$

∇_μ is the usual spacetime covariant derivative including the local $U(6)$ composite connection if exist. g is the gauge coupling constant which we can absorb into the definition of $\theta_M{}^m$.

In **32** representation and $SO^*(12)$ generators $(t_n)_M{}^N$, the embedding tensor can be described by the generalized structure constants

$$
X_{MN}{}^P = \theta_M{}^n (t_n)_N{}^P\,. \tag{5.2.3}
$$

To preserve all of the original supersymmetry of the ungauged theory under a proper gauging procedure, the embedding tensor must satisfy linear and quadratic constraints written respectively as

$$X_{(MN}{}^{L}\Omega_{P)L} = 0 \qquad \text{and} \qquad \theta_M{}^m \theta_N{}^n f_{mn}{}^p + X_{MN}{}^P \theta_P{}^p = 0 \tag{5.2.4}$$

where $f_{mn}{}^p$ is the $SO^*(12)$ structure constants. The former implies that the embedding tensor $\theta_M{}^m$ belongs to the representation **351** of $SO^*(12)$ and the latter results in

$$[X_M, X_N] = -X_{MN}{}^P X_P\,. \tag{5.2.5}$$

As a result, the gauge generators form a closed sub-algebra and $X_{MN}{}^P$ act as the corresponding structure constants.

In gauging an ungauged supergravity theory, the invariant of supersymmetry requires some deformations of the ungauged theory and also supersymmetric transformations. Such deformations are of first and second order in the gauge coupling constant g, and can be encoded in terms of the T-tensor written as

$$T_{\underline{MN}}{}^{\underline{P}} = \mathcal{V}_{\underline{M}}{}^M \mathcal{V}_{\underline{N}}{}^N \mathcal{V}_P{}^{\underline{P}} X_{MN}{}^P\,. \tag{5.2.6}$$

In general, both of the electric and magnetic fields can play a role in the gaugings which leads to various possible gauge groups. Nevertheless, in this work, we focus only on $SO(6)$ gauge group embedded electrically in $U(6) \subset SO^*(12)$. Therefore, the gauging only associates with electric gauge fields A^{AB} where we have the gauge generators written as

$$X_{I_1J_1,I_2J_2}{}^{I_3J_3} = 4g\delta^{[I_3}_{[I_1}\delta_{I_2][J_2}\delta^{J_3]}_{J_2]} \qquad \text{and} \qquad X_{I_1J_1}{}^{I_3J_3}{}_{I_2J_2} = -X_{I_1J_1,I_2J_2}{}^{I_3J_3} \tag{5.2.7}$$

with all remaining components of gauge generators vanishing. In particular, the components of $X^{\Lambda}{}_M{}^N$ which couples to the magnetic gauge fields vanish.

5.2.1 T-Tensor and Fermion-Shift Matrices

In order to find the expression of T-tensor and fermion-shift matrices, we follow the truncation of N=8 gauged supergravity to N=6. The T-tensor of the N=8 theory can be written as

$$T_{ij,kl}{}^{pq} = -\frac{1}{2\sqrt{2}}\delta^{[p}_{[k}N^{q]}{}_{l]ij} - \sqrt{2}\delta^{[p}_{[k}S_{l][i}\delta^{q]}_{j]} \tag{5.2.8}$$

where there are only $S_{AB}, N^{A}{}_{B}, N_{AB}, N^{A}{}_{BCD}$ fermion-shift matrices survived in truncated N=6 theory along with the splitting of indices $\underline{\Lambda}, \underline{\Sigma}, \ldots$ as $(0, [AB])$.

Therefore, we can write the various component of T-tensor associated with the fermion-shift matrices as

$$\begin{aligned} S_{AB} &= \frac{\sqrt{2}}{5}T_{C(A,B)E}{}^{CE}, \\ N^{A}{}_{B} &- 2\sqrt{2}T_{\alpha\beta,BC}{}^{AC}, \\ N_{AB} &= -\frac{8\sqrt{2}}{2}T_{C[A,B]E}{}^{CE}, \\ N^{A}{}_{BCD} &= -2\sqrt{2}T_{[CD,B]E}{}^{AE} - \frac{1}{2}\delta^{A}_{[B}N_{CD]}. \end{aligned} \tag{5.2.9}$$

These upper and lower indices of the fermion-shift matrices are related to each other by the complex conjugate, for example, S_{AB} is a symmetric matrix, and $S_{AB} = (S^{AB})^{*}$.

Moreover, the S_{AB} tensor will play a crucial role in the association of superpotential. This implies how many the supersymmetry of a certain holographic solution will be preserved corresponding to the number of non-vanishing Killing spinors.

5.2.2 T-Tensor and Gauge Structure Constant

The non-vanishing structure constant of gauge group of $SO(6)$ can be written as

$$(X_{\Lambda})_{\Sigma}{}^{\Gamma} = \begin{pmatrix} X_{\Lambda\Sigma}{}^{\Gamma} & 0 \\ 0 & X_{\Lambda}{}^{\Sigma}{}_{\Gamma} \end{pmatrix} \tag{5.2.10}$$

where the indices split Λ, Σ, and Γ to $(0, [I_1J_1])$, $(0, [I_2J_2])$, and $(0, [I_3J_3])$ respectively under $SO(6)$ $[I, J] = 1, 2, ..., 6$.

$$\begin{aligned} X_{I_1J_1I_2J_2}{}^{I_3J_3} &= 4g\delta^{I_3}_{[I_1}\delta_{J_1][I_3}\delta^{J_3]}_{J_2]} \\ X_{I_1J_1}{}^{I_3J_3}{}_{I_2J_2} &= -X_{I_1J_1I_2J_2}{}^{I_3J_3} \end{aligned} \tag{5.2.11}$$

We can write down the expression of T-tensor with the structure constant of gauge group as

$$\begin{aligned}
T_{\underline{M},\underline{N}}{}^{\underline{P}} &= [L^{-1} \star X]_{\underline{M},\underline{N}}{}^{\underline{P}} \\
&= (L^{-1})_{\underline{M}}{}^{M}(L^{-1})_{\bar{N}}{}^{N} L_{P}{}^{\underline{P}} X_{MN}{}^{P} \\
&= (L^{-1})_{\underline{M}}{}^{\Lambda}(L^{-1})_{\underline{N}}{}^{\Sigma}(L)_{\Gamma}{}^{\underline{P}} X_{\Lambda\Sigma}{}^{\Gamma} + (L^{-1})_{\underline{M}}{}^{\Lambda}(L^{-1})_{\underline{N}\Gamma} L^{\Sigma\underline{P}} X_{\Lambda}{}^{\Gamma}{}_{\Sigma}.
\end{aligned} \tag{5.2.12}$$

Plugging in the above condition with the splitting of indices $\Lambda, \Sigma, \ldots = (0, [IJ])$ and the definition (5.2.6), we find the component of T-tensor associated with the coset representative as

$$\begin{aligned}
T_{EF,AB}{}^{CD} &= -\frac{g}{2} f^{I_1 J}{}_{EF}(f^{JJ_1}{}_{AB}\bar{h}_{I_1 J_1}{}^{CD} + h_{I_1 J_1 AB}\bar{f}^{JJ_1 CD}), \\
T_{\alpha\beta,AB}{}^{CD} &= -\frac{g}{2} f^{I_1 J}{}_{\alpha\beta}(f^{JJ_1}{}_{AB}\bar{h}_{I_1 J_1}{}^{CD} + h_{I_1 J_1 AB}\bar{f}^{JJ_1 CD}) \\
&= T_{AB}{}^{CD} \\
&= -\frac{g}{2} f^{I_1 J}(f^{JJ_1}{}_{AB}\bar{h}_{I_1 J_1}{}^{CD} + h_{I_1 J_1 AB}\bar{f}^{JJ_1 CD}).
\end{aligned} \tag{5.2.13}$$

It is straightforward to obtain all the fermion-shift matrices and the scalar potential, once we define the explicit parametrizations of scalar fields.

5.2.3 Scalar Potential

By considering N=8 Ward identity

$$\delta^i_j V^{N=8}(\phi) = g^2 \left(-12 S^{ik}_{jk} + \frac{1}{6} N_j{}^{klm} N^i{}_{klm} \right), \tag{5.2.14}$$

the truncated scalar potential of the N=6 theory can be expressed in the fermion-shift matrices as

$$V(\phi) = \left(-2 S^{AB} S_{AB} + \frac{1}{36} N_A{}^{BCD} N^A{}_{BCD} + \frac{1}{6} N_A{}^B N^A{}_B \right). \tag{5.2.15}$$

This scalar potential can be used to consider supersymmetric AdS_4 critical points if the critical point of the scalar potential coincides with the superpotential,. This plays an important role in considering holographic solutions.

5.3 Lagrangian and Supersymmetry transformation

To write down the field's equations of the $N=6$ gauged supergravity, we first write down the bosonic Lagrangian as

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{24}P_{\mu ABCD}P^{\mu ABCD} - \frac{i}{4}\left(\mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}_{\mu\nu}F^{+\Sigma\mu\nu} - \overline{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}_{\mu\nu}F^{-\Sigma\mu\nu}\right) - V\,. \tag{5.3.1}$$

The first term is the Einstein-Hilbert action of general relativity while the second and the third term are scalar and gauge kinetic terms, respectively. The scalar kinetic terms can be written in term of the vielbein $P^{ABCD}_{\mu} = (P_{\mu ABCD})^*$ on the scalar coset manifold of $SO^*(12)/U(6)$ as

$$\begin{aligned} P^{ABCD}_{\mu} &= \mathcal{V}^{ABM}D_{\mu}\mathcal{V}_{M}{}^{CD} \\ &= i\left(\bar{f}^{\Lambda AB}D_{\mu}\bar{h}_{\Lambda}{}^{CD} - \bar{h}_{\Lambda}{}^{AB}D_{\mu}\bar{f}^{\Lambda CD}\right). \end{aligned} \tag{5.3.2}$$

and the scalar matrix $\mathcal{N}$ appearing in the gauge kinetic terms of the Lagrangian is also given by

$$\mathcal{N}_{\Lambda\Sigma} = -\bar{h}_{\Lambda}{}^{\underline{\Lambda}}(f^{-1})_{\underline{\Lambda}\Sigma} \tag{5.3.3}$$

where $\overline{\mathcal{N}}_{\Lambda\Sigma}$ is the complex conjugate of $\mathcal{N}_{\Lambda\Sigma}$. Besides, the complex self-dual and anti-self-dual gauge field strengths can be defined as

$$F^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}\left(F^{\Lambda}_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\Lambda\rho\sigma}\right) \tag{5.3.4}$$

with $F^{\Lambda}_{\mu\nu}$ written as

$$F^{\Lambda}_{\mu\nu} = \partial_{\mu}A^{\Lambda}_{\nu} - \partial_{\nu}A^{\Lambda}_{\mu} + X_{\Gamma\Sigma}{}^{\Lambda}A^{\Gamma}_{\mu}A^{\Sigma}_{\nu}\,. \tag{5.3.5}$$

Moreover, the deformed supersymmetry transformations of the $N=6$ gauged supergravity, with all fermionic fields vanishing, are given by

$$\delta\psi_{\mu A} = D_{\mu}\epsilon_A - S_{AB}\gamma_{\mu}\epsilon^B - \frac{1}{4\sqrt{2}}\hat{F}^{+}_{\rho\sigma AB}\gamma^{\rho\sigma}\gamma_{\mu}\epsilon^B, \tag{5.3.6}$$

$$\delta\chi_A = -\frac{1}{4!}\epsilon_{ABCDEF}P^{BCDE}_{\mu}\gamma^{\mu}\epsilon^F + N^B{}_A\epsilon_B - \frac{1}{2\sqrt{2}}\hat{F}^{+}_{\mu\nu}\gamma^{\mu\nu}\epsilon_A, \tag{5.3.7}$$

$$\delta\chi_{ABC} = -P_{\mu ABCD}\gamma^{\mu}\epsilon^D + N^D{}_{ABC}\epsilon_D - \frac{3}{2\sqrt{2}}\hat{F}^{+}_{\mu\nu[AB}\epsilon_{C]}\,. \tag{5.3.8}$$

Despite, the vanishing of all fermionic fields, the variations of fermionic fields are not necessarily zero. Furthermore, we also note the chiralities of the fermionic fields

$$\gamma_5\psi_{\mu A} = -\psi_{\mu A}, \qquad \gamma_5\chi_{ABC} = -\chi_{ABC}, \qquad \gamma_5\chi_A = -\chi_A \tag{5.3.9}$$

with the opposite chiralities for ψ_μ^A, χ^{ABC} and χ^A. We can also write tensors $\hat{F}^+_{\mu\nu AB} = (\hat{F}^{-AB}_{\mu\nu})^*$ as

$$\hat{F}^{-AB}_{\mu\nu} = \mathcal{V}_M{}^{AB} G^{-M}_{\mu\nu} \tag{5.3.10}$$

where

$$G^M_{\mu\nu} = \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\Lambda\mu\nu} \end{pmatrix} \tag{5.3.11}$$

and $G_{\Lambda\mu\nu} = i\epsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}}{\partial F^\Lambda_{\rho\sigma}}$. We can also write

$$\hat{F}^+_{\mu\nu} = (\mathcal{V}_M{}^0 G^{-M}_{\mu\nu})^*. \tag{5.3.12}$$

In addition, we define the covariant derivative of ϵ_A as

$$D_\mu\epsilon_A = \partial_\mu\epsilon_A + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon_A + \frac{1}{2}Q_{\mu A}{}^B\epsilon_B \tag{5.3.13}$$

with the composite connection $Q_{\mu A}{}^B$ given by

$$Q_{\mu A}{}^B = \frac{2i}{3}\left(h_{\Lambda AC}\partial_\mu\bar{f}^{\Lambda AB} - f^\Lambda{}_{AC}\partial_\mu\bar{h}_\Lambda{}^{BC}\right) - gA^M_\mu Q_{MA}{}^B \tag{5.3.14}$$

where $Q_{MA}{}^B$ can be obtained from

$$Q_{MAB}{}^{CD} = \mathcal{V}_{AB}{}^P X_{MP}{}^N \mathcal{V}_N{}^{CD} \tag{5.3.15}$$

with the relation $Q_{MAB}{}^{CD} = 4\delta^{[C}_{[A}Q_{MB]}{}^{D]}$. This will play an important role in finding the supersymmetric AdS_4 black holes in which we preserve the supersymmetry by performing topological twists which we will discuss later.

CHAPTER VI

Holographic solutions of N=6, D=4 Gauged Supergravity

In this chapter, we will be working out various types of supersymmetric solutions to the $N = 6$ gauged supergravity with $SO(6)$ gauge group. The $N = 6$ theory has been studied in [4] and the theory admits supersymmetric $N = 6$ AdS_4 vacuum with the cosmological constant $V_0 = -48g^2$ and vanishing scalar fields. Thanks to the AdS/CFT correspondence, this is dual to a three-dimensional $N = 6$ SCFT. We will find asymptotically AdS_4 geometric solutions which can be interpreted as various types of deformation of the dual $N = 6$ SCFT [113].

6.1 Holographic RG flows

In low energy limit and large-N limit of the AdS/CFT correspodence, gauged supergravities theories in AdS_{d+1} spacetime dual to strongly-coupled superconformal field theories (SCFTs) at the d-dimensional boundary of anti-de Sitter spacetime. We can consider the flow in the radial coordinate r of the AdS spacetime as the energy scale of the operators in SCFTs, so the correspondence relates the IR regime of the SCFTs at the deep interior of AdS and the UV regime of SCFTs at the boundary. On the other hand, when gauged supergravities are only in asympotically anti-de Sitter space $AAdS_{n+1}$, the dual theories that appear at the boundary are just approximated SCFTs and they are superconformal field theory at only the conformal fixed points. As a result, the flow from the UV regime of the dual SCFT at the boundary to the IR regime of quantum theory at the deep interior could break conformal symmetry, or supersymmetry and the corresponding quantum theory could become non-conformal field theories. This is called "Renormalization group flow" or "holographic RG flow" or "RG flow". Different types of IR geometries horizon describe different types of RG flows, see also [98–100].

Therefore, the holographic RG flow is the study of supersymmetric solutions of gauged supergravity which characterizes the flows from SCFTs at the conformal fixed point in (UV) to other conformal fixed points or non-conformal phases of the deformed dual SCFTs at (IR). This means that we can study the behavior of superconformal field theory in UV related to conformal field theory or quantum field theory in IR. This helps us to understand the dynamics of strongly-coupled quantum field theory providing a non-perturbative technique of quantum mechanical system.

To preserve the Poincare symmetry on $AAdS_4$ of RG flows, supersymmetric solutions of gauged supergravities must take the form of the domain wall spacetime

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + dr^2. \tag{6.1.1}$$

Then, we can calculate the corresponding spin connection by using Cartan's equation

$$de^{\hat{\mu}} = e^{\hat{\nu}} \wedge \omega^{\hat{\mu}}{}_{\hat{\nu}} \tag{6.1.2}$$

where

$$\begin{aligned} e^{\hat{\mu}} &= e^{A(r)} dx^\mu, \\ e^{\hat{r}} &= dr. \end{aligned} \tag{6.1.3}$$

with $'$ denoting r derivative, and the non-vanishing component of spin connection

$$\begin{aligned} \omega^{\hat{\mu}\hat{r}} &= \omega^{\hat{\mu}}{}_{\hat{r}} \eta^{\hat{r}\hat{r}} \\ &= A'(r) e^{\hat{\mu}}. \end{aligned} \tag{6.1.4}$$

In supersymmetric domain wall solutions, the scalar fields and Killing spinors will only depend on the coordinate r denoted by $\Phi(r)$ and $\epsilon_A(r)$, respectively. We also switch off one-form and two-forms in our theory.

The supersymmetry variations for fermions must vanish. For the $\mu = 0, 1, 2$, the partial derivative of Killing spinors ϵ_A vanishes. The components of gravitino supersymmetry transformation of $N = 6$ gauged supergravity take the form

$$\begin{aligned}
\delta\psi_{A\mu} &= \nabla_\mu \epsilon_A - S_{AB}\gamma_\mu \epsilon_B \\
&= \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon_B - S_{AB}\gamma_\mu\epsilon_B \\
&= \frac{1}{2}A'\gamma^{\hat{\mu}}\gamma_{\hat{\mu}}\gamma_{\hat{r}}\epsilon_A - S_{AB}\gamma^{\hat{\mu}}\gamma_{\hat{\mu}}e^{\hat{\mu}}{}_\mu\epsilon^B \\
&= (\frac{1}{2}A'\delta_{AB}e^{i\Lambda} - S_{AB})\epsilon^B \\
&= 0.
\end{aligned} \tag{6.1.5}$$

To solve all the BPS conditions, we have used the projection condition

$$\gamma^{\hat{r}}\epsilon_A = e^{i\Lambda}\epsilon^A \tag{6.1.6}$$

for Λ being a real function of r. In this thesis, we will use the Majorana representation for which the gamma matrices are real, but γ_5 is purely imaginary. The projection condition relates the two chiralities of Killing spinors of the domain walls, ϵ_A, and ϵ^A. As a result, the flow solutions preserve only half of the original supersymmetry or 1/2-BPS solutions or 12 supercharges in this case.

Since S_{AB} is a symmetric matrix, one can diagonalize S_{AB} with its eigenvalues leading to the "superpotential" as

$$S_{AB} = \frac{1}{2}\mathcal{W}\delta_{AB}. \tag{6.1.7}$$

Then, we get

$$A'e^{i\Lambda} - \mathcal{W} = 0 \tag{6.1.8}$$

and obtain the equation

$$\begin{aligned}
A' &= \pm|\mathcal{W}|, \\
e^{i\Lambda} &= \pm\frac{\mathcal{W}}{|\mathcal{W}|}.
\end{aligned} \tag{6.1.9}$$

In the following, we define $W = |\mathcal{W}|$ and choose the upper sign to make the supersymmetric AdS_4 critical point locate at $r \to \infty$. Then, we can also look at the r component of the gravitino variation

$$\begin{aligned}
\delta\psi_{Ar} &= \nabla_\mu \epsilon_A - S_{AB}\gamma_\mu \epsilon_B \\
&= \partial_r \epsilon_A - \frac{1}{2}\mathcal{W}\gamma_{\hat{r}}\epsilon^A \\
&= \partial_r \epsilon_A - \frac{1}{2}A' \epsilon_A \\
&= 0.
\end{aligned} \tag{6.1.10}$$

This gives the solution of Killing spinors as

$$\epsilon_A(r) = e^{\frac{A}{2}}\tilde{\epsilon}_A \tag{6.1.11}$$

where $\tilde{\epsilon}_A$ is constant spinor satisfying the projection condition.

To find the full set of BPS equation, we also need to consider $\delta\chi_{ABC} = 0$, $\delta\chi_A = 0$

$$\begin{aligned}
\delta\chi_{ABC} &= -P_{\mu ABCD}\gamma^\mu \epsilon^D + N^D{}_{ABC}\epsilon_D \\
&= -P_{\hat{r}ABCD}\gamma^{\hat{r}}\epsilon^D + N^D{}_{ABC}\epsilon_D \\
&= (-P_{\hat{r}ABCD}e^{-i\Lambda} + N^D{}_{ABC})\epsilon_D
\end{aligned} \tag{6.1.12}$$

and

$$\begin{aligned}
\delta\chi_A &= -\frac{1}{4!}\epsilon_{ABCDEF}P_\mu{}^{BCDE}\gamma^\mu\epsilon^F + N^F{}_A\epsilon_F \\
&= -\frac{1}{4!}\epsilon_{ABCDEF}P_{\hat{r}}{}^{BCDE}\gamma^{\hat{r}}\epsilon^F + N^F{}_A\epsilon_F \\
&= (-\frac{1}{4!}\epsilon^{ABCDEF}P_{\hat{r}}{}^{BCDE}e^{-i\Lambda} + N^F{}_A)\epsilon_F.
\end{aligned} \tag{6.1.13}$$

Once the last two variations are satisfied, we get the BPS equations of four-dimensional gauged supergravity. However, it is very complicated to do the calculation of thirty scalar fields, so we will be working on some of the thirty scalar fields non-vansihing. In order for the solution of non-vanishing scalar fields to satisfy the field's equations of thirty scalar fields, we will choose a set of singlet scalar fields under a subgroup of gauge group of $SO(6)$.

6.1.1 Solutions with $SO(2) \times SO(4)$ symmetry

We first consider solutions with $SO(2) \times SO(4)$ symmetry. The embedding of $SO(6)$ implies that the scalar ϕ_{AB} transform as an adjoint representation of $SO(6)$.

The singlet of $SO(2) \times SO(4) \subset SO(6)$ can be explicitly written by

$$\phi_{AB} = \phi(\delta^1_A \delta^2_B - \delta^1_B \delta^2_A). \tag{6.1.14}$$

We can write ϕ as

$$\phi = \varphi e^{i\zeta} \tag{6.1.15}$$

where the r dependent φ and ζ are a real scalars.

By a straightforward computation using the equations given in the previous section, we find the scalar potential

$$V = -8\left(1 + 4e^{2\varphi} + e^{4\varphi}\right) g^2 \tag{6.1.16}$$

and the fermion-shift matrix of S_{AB}

$$S_{AB} = \frac{1}{2} \mathcal{W} \delta_{AB}\,. \tag{6.1.17}$$

The real superpotential is given by

$$\mathcal{W} = 4g \cosh\varphi. \tag{6.1.18}$$

As we mention in the previous section, S_{AB} implies that the solution of the singlet scalar $SO(2) \times SO(4)$ would either preserve the full $N = 6$ supersymmetry with all components Killing spinors ϵ_A non-vanishing or no supersymmetry at all if all the Killing spinors vanish. Moreover, in order that the critical point of the scalar potential to be a supersymmetric $N = 6$ AdS_4 vacuum, the scalar potential critical point must coincide with the superpotential's critical point, which is the case at $\varphi = 0$.

From the $\delta\psi_{A\mu}$, we have

$$A' = 4g\cosh\varphi \qquad , \text{and} \qquad e^{i\Lambda} = 1\,. \tag{6.1.19}$$

The variations of $\delta\chi_{ABC}$ and $\delta\chi_A$ result in the following BPS equations

$$\varphi' = -4g \sinh\varphi \qquad , \text{and} \qquad \zeta' = 0\,. \tag{6.1.20}$$

As a result, we finally obtain the set of BPS equations solving all the supersymmetry conditions. These equations also imply the second-order equations from the

Lagrangian

$$e^{-1}\mathcal{L} = \frac{1}{16}e^{-4\varphi}\left(-(-1+e^{4\varphi})\zeta'^2 - 16e^{4\varphi}\varphi'^2\right) + 8\left(1+4e^{2\varphi}+e^{4\varphi}\right)g^2. \quad (6.1.21)$$

Therefore, we can analytically solve these BPS equations with the solutions given by

$$4gr = \ln(1+e^{\varphi}) - \ln(1-e^{\varphi}), \quad (6.1.22)$$

$$A = \varphi - \ln(1-e^{2\varphi}). \quad (6.1.23)$$

where we have neglected the integration constants in these equations by shifting the radial coordinate and scaling the $x^{0,1,2}$ coordinates, respectively. As $r \to \infty$ corresponding to AdS critical point, we find that

$$\varphi \sim e^{-4gr} \sim e^{-\frac{r}{L}} \qquad \text{and} \qquad A \sim 4gr \sim \frac{r}{L} \quad (6.1.24)$$

with L being the AdS_4 radius related to the cosmological constant by

$$L = \sqrt{-\frac{3}{V_0}} = \frac{1}{4g}. \quad (6.1.25)$$

For the choice of convenience, we have defined $g > 0$.

According to

$$m^2L^2 = \Delta(\Delta - d) \quad (6.1.26)$$

with $d = 3$, and $m^2L^2 = -1$, we can see that the behavior of φ is dual to a relevant operator of dimensions $\Delta = 1, 2$ in the dual SCFT. Besides, there also exists the singularity at $r \to 0$ with

$$\varphi \sim \pm\ln(4gr) \qquad \text{and} \qquad A \sim \ln(4gr)\,. \quad (6.1.27)$$

We find that near the singularity$\varphi \to \pm\infty$, the scalar potential

$$V \sim -8g^2e^{\pm 2\varphi} \to -\infty\,. \quad (6.1.28)$$

By the criterion given in [57], we find that the singularity of the solution with $SO(2) \times SO(4)$ symmetry is physical. Therefore, we have the solution that describes the RG flows from the UV $N = 6$ SCFT to a non-conformal phase in the

IR. Such a flow is governed by an operator of dimensions $\Delta = 1, 2$ corresponding to scalar or fermion mass terms in three dimensions. Moreover, the flow will break superconformal symmetry but preserves all of the $N = 6$ Poincare supersymmetry. The R-symmetry $SO(6)$ is broken to $SO(2) \times SO(4)$ subgroup. This precisely agrees with the field theory result given in [16]. Therefore, the solution would describe mass deformations of $N = 6$ SCFT in three dimensions.

6.1.2 Solutions with $U(3)$ symmetry

We continue with solutions of a residual symmetry $U(3) \sim SU(3) \times U(1) \subset SO(6)$. The $U(3)$ generators can be written in $SO(6)$ fundamental representation as

$$X = \begin{pmatrix} A_{3\times 3} & S_{3\times 3} \\ -S_{3\times 3} & A_{3\times 3} \end{pmatrix} \tag{6.1.29}$$

where the $A_{3\times 3}$ and $S_{3\times 3}$ matrices are anti-symmetric and symmetric, respectively. The matrices $A_{3\times 3}$ generate an $SO(3) \subset SU(3)$ resulting in a diagonal subgroup of $SO(3) \times SO(3) \subset SO(6)$. Moreover, the $U(1)$ factor corresponds to the matrices of $S_{3\times 3} = \mathbb{I}_3$. Therefore, we have only one singlet scalar given by

$$\phi_{AB} = \begin{pmatrix} 0_{3\times 3} & \phi \mathbb{I}_3 \\ -\phi \mathbb{I}_3 & 0_{3\times 3} \end{pmatrix} = \phi J_{AB} \, . \tag{6.1.30}$$

where the matrix J_{AB} is identified with the Kahler form of CP^3 on which the ten-dimensional type IIA theory compactifies [4].

By the definition of $\phi = \varphi e^{i\zeta}$, we find the scalar potential

$$V = -24 g^2 e^{-2\varphi}(1 + e^{4\varphi}) \tag{6.1.31}$$

which coincides with the potential given in [4] which admits AdS_4 critical point at $\varphi = 0$ dual to an $N = 6$ SCFT in three dimensions. We also find the fermion-shift matrix S_{AB}

$$S_{AB} = \frac{1}{2} \mathcal{W} \delta_{AB} \tag{6.1.32}$$

along with the complex superpotential

$$\mathcal{W} = \frac{1}{2} e^{-3\varphi - i\zeta} \left[(e^{6\varphi} + 3e^{2\varphi})(1 + e^{i\zeta}) + (1 + e^{4\varphi})(e^{i\zeta} - 1) \right] . \tag{6.1.33}$$

Besides, the supersymmetry variations of $\delta\chi_A$ and $\delta\chi_{ABC}$ result in

$$e^{-i\Lambda}(2\varphi' \pm i\sinh(2\varphi)\zeta') = -ge^{-3\varphi}(e^{4\varphi}-1)[1-e^{i\zeta}+e^{2\varphi}(1+e^{i\zeta})] \qquad (6.1.34)$$

implying $\zeta' = 0$. In addition, the field's equations obtained from Lagrangian

$$e^{-1}\mathcal{L} = -\frac{3}{16}e^{-4\varphi}\left((-1+e^{4\varphi})\zeta'^2+16e^{4\varphi}\varphi'^2\right)+24g^2e^{-2\varphi}(1+e^{4\varphi}) \qquad (6.1.35)$$

also requires $\zeta' = 0$ resulting in $\zeta = \zeta_0$ which are compatible with all the BPS equations. In the following task, we set $\zeta_0 = 0$, so the BPS equations can be written as

$$\varphi' = -e^{-\varphi}(e^{4\varphi}-1) \qquad \text{and} \qquad A' = ge^{-\varphi}(3+e^{4\varphi}). \qquad (6.1.36)$$

The analytic solutions can also found as

$$A = 3\varphi - \ln(1-e^{4\varphi}), \qquad (6.1.37)$$

$$4gr = 2\tan^{-1}e^{\varphi} - \ln(1-e^{\varphi}) + \ln(1+e^{\varphi}). \qquad (6.1.38)$$

Similar to the $SO(2)\times SO(4)$ case, the solution is asymptotic to the supersymmetric AdS_4 with φ dual to an operator of dimensions $\Delta = 1, 2$. At the singularity $r = 0$, the solutions become

$$\varphi \sim \ln(gr) \qquad \text{and} \qquad A \sim 3\varphi \sim 3\ln(gr) \qquad (6.1.39)$$

and

$$\varphi \sim -\ln(gr) \qquad \text{and} \qquad A \sim -\varphi \sim \ln(gr). \qquad (6.1.40)$$

Both of these give

$$V \sim -24g^2e^{\pm 2\varphi} \to -\infty, \qquad (6.1.41)$$

so, both singularities are physical which results in the interpretation of a holographic dual of RG flows from the $N = 6$ SCFT to non-conformal phases in the IR. The flow solution will preserve $N = 6$ Poincare supersymmetry in three dimensions similar to the $SO(2)\times SO(4)$ case. However, the flow breaks the $SO(6)$ R-symmetry to $U(3)$ by the mass deformation of the dual $N = 6$ SCFT similar to the $SO(2)\times SO(4)$ case.

6.1.3 Solutions with $SO(2) \times SO(2) \times SO(2)$ symmetry

We move to a more interesting solution by considering a smaller symmetry of $SO(2) \times SO(2) \times SO(2) \subset SO(6)$ symmetry. The parameterization of the singlet complex scalars can be explicitly given by

$$\phi_{AB} = \begin{pmatrix} \phi_1 i\sigma_2 & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \phi_2 i\sigma_2 & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & \phi_3 i\sigma_2 \end{pmatrix}. \tag{6.1.42}$$

with the complex scalars defined similarly as in the previous solutions

$$\phi_\alpha = \varphi_\alpha e^{i\zeta_\alpha}, \qquad \alpha = 1, 2, 3. \tag{6.1.43}$$

We find the scalar potential

$$V = -16g^2[\cosh(2\varphi_1) + \cosh(2\varphi_2) + \cosh(2\varphi_3)]. \tag{6.1.44}$$

It is obvious that the potential admits the critical point at $\varphi_1 = \varphi_2 = \varphi_3 = 0$ which is the supersymmetric $N = 6$ AdS_4 vacuum along with the fermion-shift matrix S_{AB}

$$S_{AB} = \frac{1}{2}\begin{pmatrix} \mathcal{W}_1\mathbb{I}_2 & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \mathcal{W}_2\mathbb{I}_2 & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & \mathcal{W}_3\mathbb{I}_2 \end{pmatrix} \tag{6.1.45}$$

where

$$\begin{aligned} \mathcal{W}_1 \;\; = \;\; & \frac{1}{2} g e^{-\varphi_1-\varphi_2-\varphi_3}\left[e^{i(\zeta_1-\zeta_2-\zeta_3)}(e^{2\varphi_1}-1)(e^{2\varphi_2}-1)(e^{2\varphi_3}-1)\right. \\ & \left. -(1+e^{2\varphi_1})(1+e^{2\varphi_2})(1+e^{2\varphi_3})\right]. \end{aligned} \tag{6.1.46}$$

$\mathcal{W}_2$ and $\mathcal{W}_3$ are similar to $\mathcal{W}_1$ with only the phase $e^{i(\zeta_1-\zeta_2-\zeta_3)}$ replaced by $e^{i(\zeta_2-\zeta_1-\zeta_3)}$ and $e^{i(\zeta_3-\zeta_1-\zeta_2)}$, respectively.

To be able to write these eigenvalues $\mathcal{W}_a$ as the superpotential in term of which the scalar potential (6.1.44) can be written, we need the condition that $\zeta_1 = \zeta_2 = \zeta_3 = 0$. This is also implied by the consistency of the field's equations given from the Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L} = \frac{1}{16}e^{-4(\varphi_1+\varphi_2+\varphi_3)}(&-e^{-4(\varphi_2+\varphi_3)}(-1+e^{4\varphi_1})^2\zeta_1'^2 - e^{4\varphi_1}(e^{4\varphi_3}(-1+e^{4\varphi_2})^2\zeta_2'^2 + \\ & e^{4\varphi_2}((-1+e^{3\varphi_3})^2\zeta_3'^2 + 16e^{4\varphi_3}(\varphi_1'^2+\varphi_2'^2+\varphi_3'^2)))) \end{aligned} \tag{6.1.47}$$

In the following, we set $\zeta_1 = \zeta_2 = \zeta_3 = 0$, which results in $e^{i\Lambda} = \pm 1$, and the set of BPS equations can be written as

$$\varphi_1' = -ge^{-\varphi_1-\varphi_2-\varphi_3}[e^{2(\varphi_1+\varphi_2)} + e^{2(\varphi_1+\varphi_3)} - e^{2(\varphi_2+\varphi_3)} - 1], \quad (6.1.48)$$

$$\varphi_2' = -ge^{-\varphi_1-\varphi_2-\varphi_3}[e^{2(\varphi_1+\varphi_2)} + e^{2(\varphi_2+\varphi_3)} - e^{2(\varphi_1+\varphi_3)} - 1], \quad (6.1.49)$$

$$\varphi_3' = -ge^{-\varphi_1-\varphi_2-\varphi_3}[e^{2(\varphi_1+\varphi_3)} + e^{2(\varphi_2+\varphi_3)} - e^{2(\varphi_1+\varphi_2)} - 1], \quad (6.1.50)$$

$$A' = ge^{-\varphi_1-\varphi_2-\varphi_3}[e^{2(\varphi_1+\varphi_2)} + e^{2(\varphi_1+\varphi_3)} + e^{2(\varphi_3+\varphi_3)} + 1]. \quad (6.1.51)$$

Nevertheless, we find the analytic solutions by writing the linear combination of these equations as

$$\varphi_1' + \varphi_2' = -2ge^{-\varphi_1-\varphi_2-\varphi_3}(e^{2(\varphi_1+\varphi_2)} - 1). \quad (6.1.52)$$

we further transform to a new radial coordinate ρ given by

$$\frac{d\rho}{dr} = e^{-\varphi_1-\varphi_2-\varphi_3} \quad (6.1.53)$$

and obtain a first solution as

$$\varphi_2 = 2g\rho - \varphi_1 - \frac{1}{2}\ln(e^{4g\rho} + C_2) \quad (6.1.54)$$

with C_2 being a constant for φ_2 solution.

We follow the same strategy to find other solutions by substituting the φ_2 solution, in the combination $\varphi_1' + \varphi_3'$. We get

$$\varphi_3 = 2g\rho - \varphi_1 - \frac{1}{2}\ln(e^{4g\rho} + C_3). \quad (6.1.55)$$

After that, the φ_1 solution can be obtain by inserting φ_1 and φ_2 in (6.1.48)

$$\varphi_1 = \frac{1}{4}\ln\left[\frac{e^{4g\rho}(e^{4g\rho} + C_1)}{(e^{4g\rho} + C_2)(e^{4g\rho} + C_3)}\right]. \quad (6.1.56)$$

Finally, we are able to evaluate the solution for A given by

$$A = g\rho + \frac{1}{4}\ln(e^{4g\rho} + C_1) + \frac{1}{4}\ln(e^{4g\rho} + C_2) + \frac{1}{4}\ln(e^{4g\rho} + C_3). \quad (6.1.57)$$

We therefore are able to consider the behavior of the solutions as $\varphi_\alpha \sim 0$ resulting in $\rho \sim r$ and

$$\varphi_1 \sim \frac{1}{4}(C_1 - C_2 - C_3)e^{-4g\rho}, \qquad \varphi_{2,3} \sim -\frac{1}{4}(C_1 - C_{3,2})e^{-4g\rho}, \qquad A \sim 4g\rho\,. \quad (6.1.58)$$

This is the kind of what we expect because the solutions are asymptotic to the supersymmetric AdS_4 vacuum. Similar to the previous cases, the singularity of the solutions occurs as $4g\rho \to \ln(-C_\alpha)$ which can be categorized as the follows. For $C_1 \neq C_2 \neq C_3$, there are three possibilities:

- For $C_1 > C_{2,3}$, the singularity occurs as $4g\rho \to \ln(-C_1)$ with

$$\begin{aligned} \varphi_1 &\sim \frac{1}{4}\ln(4g\rho - \tilde{C}_1), \qquad \tilde{C}_1 = \ln(-C_1), \\ \varphi_{2,3} &\sim -\varphi_1, \qquad A \sim \varphi_1\,. \end{aligned} \tag{6.1.59}$$

- For $C_2 > C_{1,3}$ or $C_3 > C_{1,2}$, we find that

$$\begin{aligned} \varphi_1 &\sim -\frac{1}{4}\ln(4g\rho - \tilde{C}_{2,3}), \qquad \tilde{C}_{2,3} = \ln(-C_{2,3}), \\ \varphi_{2,3} &\sim \varphi_1, \qquad A \sim -\varphi_1\,. \end{aligned} \tag{6.1.60}$$

In the first scenario, we find $\varphi_1 \to -\infty$ and $\varphi_{2,3} \to \infty$ while in the second scenario, the solution gives $\varphi_{1,2,3} \to \infty$. All of these behaviors lead to $V \to -\infty$. Therefore, the singularities are physically acceptable, so the set of fully preserved $N = 6$ Poincare supersymmetric solution describes different types of mass deformations within the dual $N = 6$ SCFT to non-conformal phases with $SO(2)\times SO(2)\times SO(2)$ symmetry.

6.1.4 Solutions with $SO(3)$ symmetry

As a final RG flow solution, we consider solutions with a residual symmetry of $SO(3) \subset SO(3) \times SO(3) \subset SO(6)$ generated by the $A_{3\times 3}$ antisymmetric matrices in the upper-left block of (6.1.29). There are three singlet scalar written as

$$\phi_{AB} = \begin{pmatrix} 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \hat{A}_{3\times 3} \end{pmatrix} \tag{6.1.61}$$

with

$$\hat{A} = \begin{pmatrix} 0 & \tilde{\phi}_1 & \tilde{\phi}_2 \\ -\tilde{\phi}_1 & 0 & \tilde{\phi}_3 \\ \tilde{\phi}_2 & -\tilde{\phi}_3 & 0 \end{pmatrix}. \tag{6.1.62}$$

For convenience, we set the three singlet scalars $\tilde{\phi}_\alpha = \varphi_\alpha e^{i\zeta_\alpha}$ to the form

$$\varphi_1 = \Phi\cos\theta, \qquad \varphi_2 = \Phi\sin\theta\cos\vartheta, \qquad \varphi_3 = \Phi\sin\theta\sin\vartheta \tag{6.1.63}$$

and

$$\zeta_1 = \zeta, \qquad \zeta_2 = \zeta+\eta, \qquad \zeta_3 = \zeta+\xi. \tag{6.1.64}$$

We find the scalar potential

$$\begin{aligned} V &= -g^2\big[16\cos^4\theta(2+\cosh 2\Phi) + 16\cosh^4\theta\sin^4\theta(2+\cosh 2\Phi) \\ &\quad +16\sin^4\theta\sin^4\vartheta(2+\cosh 2\Phi) - \cos^2\theta\sin^2\theta\cos^2\vartheta\times \\ &\quad (\cosh 4\Phi - 8\cos 2\eta\sinh^4\Phi - 36\cosh 2\Phi - 61) + \sin^2\theta\sin^2\vartheta\times \\ &\quad \big[8\sinh^4\Phi(\cos^2\theta\cos 2\xi + \cos^2\vartheta\sin^2\theta\cos[2(\eta-\xi)]) \\ &\quad +(61+36\cosh 2\Phi - \cosh 4\Phi)(\cos^2\theta + \cos^2\vartheta\sin^2\theta)\big]\big]. \end{aligned} \tag{6.1.65}$$

In the present case, the scalar potential explicitly depends on the phases of the complex scalars, so the analysis would be more complicated. In order that the calculation is more practical, we will truncate away a singlet scalar $\tilde{\phi}_3 = 0$ which is equivalent to setting $\vartheta = 0$ and $\xi = -\zeta$. We find the eigenvalues of the diagonalized fermion-shift matrix S_{AB}

$$S^{\text{diag}}_{AB} = \text{diag}(-2g\cosh\Phi_{\times 4}, \frac{1}{2}\mathcal{W}_+, \frac{1}{2}\mathcal{W}_-) \tag{6.1.66}$$

where we write $\mathcal{W}_\pm$ for

$$\begin{aligned} \mathcal{W}_\pm &= 2g(\cos 2\eta + 2\sin\eta)\sinh^4\frac{\Phi}{2}(\cos 4\theta\sin\eta \pm i\sin 2\theta) \\ &\quad -\frac{1}{4}g(3+12\cosh\Phi+\cosh 2\Phi) \end{aligned} \tag{6.1.67}$$

and the corresponding eigenvectors are

$$\hat{\epsilon}_\pm = -\frac{1}{2}\sec 2\theta\left(2\cos\eta\sin 2\theta \mp \sqrt{3+\cos 2\eta + 2\cos 4\theta\sin^2\eta}\right). \tag{6.1.68}$$

To further do the calculation, we write the Lagrangian of scalar kinetic term as

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= -\frac{1}{2}G_{\alpha\beta}\phi^{\alpha\prime}\phi^{\beta\prime}\\ &= -\Phi'^2 - \sinh^2\Phi\theta'^2 - \frac{1}{4}\sinh^2 2\Phi\zeta'^2 - \frac{1}{2}\sin^2\theta\sinh^2 2\Phi\zeta'\eta'\\ &\quad -\frac{1}{4}\sin^2\theta\sinh^2\Phi(3+\cosh 2\Phi - 2\cos 2\theta\sinh^2\Phi)\eta'^2 \end{aligned} \tag{6.1.69}$$

with $\phi^\alpha = (\Phi, \theta, \zeta, \eta)$. It is very useful to state the inverse of $G_{\alpha\beta}$

$$G^{\alpha\beta} = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \text{csch}^2\Phi & 0 & 0\\ 0 & 0 & -\text{sech}^2\Phi + \text{csch}^2\Phi\sec^2\theta & -\text{csch}^2\Phi\sec^2\theta\\ 0 & 0 & -\text{csch}^2\Phi\sec^2\theta & 4\text{csc}^2 2\theta\text{csch}\Phi \end{pmatrix}. \tag{6.1.70}$$

The scalar potential can be obtained from the real superpotential $W = |\mathcal{W}_+| = |\mathcal{W}_-|$ as

$$\begin{aligned} V &= -2G^{\alpha\beta}\frac{\partial W}{\partial\phi^\alpha}\frac{\partial W}{\partial\phi^\beta} - 3W^2\\ &= g^2\left[\cos^2\theta\sin^2\theta\left(\cosh 4\Phi - 8\cos 2\eta\sinh^4\Phi - 36\cosh 2\Phi - 61\right)\right.\\ &\quad \left. -4(3+\cos 4\theta)(2+\cosh 2\Phi)\right]. \end{aligned} \tag{6.1.71}$$

By setting $\epsilon^{1,2,3,4} = 0$ along with the projection conditions

$$\gamma_{\hat{r}}\epsilon_\pm = e^{\pm i\Lambda}\epsilon^\pm \quad \text{with} \quad e^{\pm i\Lambda} = \frac{\mathcal{W}_\pm}{W}, \tag{6.1.72}$$

we find the BPS equation as follows

$$\Phi' = \frac{1}{16W}g^2\left[8\sinh^3\Phi\cosh\Phi(\cos 2\eta + 2\cos 4\theta\sin^3\eta) - 30\sin 2\Phi - \sinh 4\Phi\right], \tag{6.1.73}$$

$$\theta' = -\frac{1}{W}g^2\sin^2\eta\sin 4\theta\sinh^2\Phi, \tag{6.1.74}$$

$$\zeta' = \frac{2}{W}g^2\sin 2\eta\sin^2\theta\sinh^2\Phi, \tag{6.1.75}$$

$$\eta' = -\frac{2}{W}g^2\sin 2\eta\sinh^2\Phi, \tag{6.1.76}$$

$$A' = W. \tag{6.1.77}$$

These equations can be written in a more compact form as

$$\phi^{\alpha\prime} = 2G^{\alpha\beta}\frac{\partial W}{\partial \phi^{\beta}}\,. \tag{6.1.78}$$

It is straightforward to show that the BPS equations satisfy the second-order equations resulting from the Lagrangian given above. We can see from these equations that there exists only one supersymmetric critical point, at $\Phi' = \theta' = \zeta' = \eta' = 0$, with $\phi^{\alpha} = 0$.

We also mention that even though the superpotential and the scalar potential do not depend on ζ, ζ' does not vanish anyway because of the existence of the mixed terms between ζ and η in the matrix $G_{\alpha\beta}$. Furthermore, we can further truncate away some scalars. For example, once we set $\eta = 0$ or $\theta = 0$, we find the BPS equations in the case of $N = 6$ supersymmetry with all the six eigenvalues of S_{AB} leading to

$$\mathcal{W} = 4g\cosh\Phi\,. \tag{6.1.79}$$

This is similar to the case of $N = 5$ gauged supergravity [31] in which the existence of the differences in the phases of the scalars play an important role in the supersymmetry breaking to lower supersymmetry.

We now look at the BPS solutions to the equations by combining η' and θ', so we get

$$\frac{d\theta}{d\eta} = \frac{1}{4}\sin 4\theta\tan\eta \tag{6.1.80}$$

The corresponding solution is given by

$$\cot 2\theta = C_1\cos\eta\,. \tag{6.1.81}$$

By doing similar strategy, combining ζ' and η', we get

$$\frac{d\zeta}{d\eta} = -\sin^2\theta\,. \tag{6.1.82}$$

Along with the previous solution of θ, we find the solution for ζ as

$$\zeta = \zeta_0 - \frac{\eta}{2} + \frac{\sqrt{C_1^2 + \sec^2\eta}\cos\eta\tan^{1}\frac{\sqrt{2}C_1\sin\eta}{\sqrt{2+C_1^2(1+\cos 2\eta)}}}{\sqrt{4+2C_1^2(1+\cos 2\eta)}} \tag{6.1.83}$$

where ζ_0 is constant.

We then combine Φ' and η' equations by using all the previous solutions with the change of variable

$$\tilde{\Phi} = \sinh\Phi \tag{6.1.84}$$

and find

$$\frac{d\tilde{\Phi}}{d\eta} = \csc 2\eta(1+\tilde{\Phi}^2)\left(\frac{\tilde{\Phi}\tan^2\eta}{C_1^2+\sec^2\eta} + \frac{2}{\tilde{\Phi}}\right). \tag{6.1.85}$$

The solution is written as

$$\frac{\tilde{\Phi}^2}{4} = -\frac{1 + C_1^2\cos^2\eta - C_2\sqrt{(1+\cos 2\eta)(2+C_1^2(1+\cos 2\eta))}}{3+4C_1^2\cos^2\eta + \cos 2\eta - 4C_2\sqrt{(1+\cos 2\eta)(2+C_1^2(1+\cos 2\eta))}} \tag{6.1.86}$$

By taking into an account all the previous results in η', we eventually find the solution for $\eta(r)$ written implicitly as

$$\begin{aligned} 8gr &= \sinh^{-1}\left[2C_2\sqrt{\frac{\Xi-1}{(1+C_1^2)[2+(C_1^2-4C_2^2)(1+\Xi)]}}\right] \\ &\quad - \tanh^{-1}\sqrt{\frac{(1+C_1^2-4C_2^2)(1+\Xi)}{\Xi-1}} \end{aligned} \tag{6.1.87}$$

where we write

$$\Xi = \cos 2\eta\,. \tag{6.1.88}$$

The final solution of $A(\Xi)$ can be found as

$$\begin{aligned} A &= \frac{1}{4}(\tanh^{-1}\alpha_+ - \tanh^{-1}\alpha_-) - \frac{1}{2}\tanh^{-1}\left[2C_2\sqrt{\frac{\Xi+1}{2+C_1^2(1+\Xi)}}\right] \\ &\quad -\frac{1}{8}\ln\left[4C_1^4(1+\Xi)^2+(3+\Xi)^2-4(1+\Xi)[8C_2^2+C_1^2(4C_2^2(1+\Xi)-3-\Xi)]\right] \\ &\quad +\frac{1}{4}\ln[2+(1+\Xi)(C_1^2-4C_2^2)] \end{aligned} \tag{6.1.89}$$

where $\alpha_\pm$ given by

$$\alpha_\pm = \sqrt{-\frac{2+C_1^2(1+\Xi)}{(1+\Xi)\left[1+C_1^2-4C_2(2C_2+\pm\sqrt{4C_2^2-C_1^2-1})\right]}}\,. \tag{6.1.90}$$

The solution preserves $N = 2$ supersymmetry and breaks $SO(6)$ R-symmetry to $SO(3)$. The singularity of the solution occurs when

$$\cos^2\eta = -\frac{1}{1 + 2C_1^2 - 8C_2^2 \pm 4C_2\sqrt{4C_2^2 - C_1^2 - 1}} \tag{6.1.91}$$

This results in $\tilde{\Phi} \to \pm\infty$ or $\Phi \to \pm\infty$ leading to

$$V \to g^2 e^{4|\Phi|} \cos^2\theta \sin^2\eta \sin^2\theta\,. \tag{6.1.92}$$

It is obvious that the scalar potential is unbounded as by $V \to +\infty$, so the IR singularities of the $N = 2$ solutions are unphysical. However, if $\theta = 0$ or $\eta = 0$, the solution becomes the $N = 6$ solution as mentioned above and are physical.

6.2 Supersymmetric Janus solutions

Supersymmetric Janus solution is another form of the solution of gauged supergravity where the metric takes the form of AdS_3-sliced domain wall

$$ds^2 = e^{2A(r)}\{e^{\frac{2\xi}{\ell}}(-dt^2 + dx^2) + d\xi^2\} + dr^2. \tag{6.2.1}$$

We can use the supersymmetric Janus solution to explain the defect in conformal field theory which allows us to study the structure and the dynamics of conformal field theory. In the limit $\ell \to \pm\infty$, we would get the original domain wall spacetime of the form

$$\begin{aligned} ds^2 &= e^{2A(r)}(-dt^2 + dx^2 + d\xi^2) + dr^2 \\ &= e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2. \end{aligned} \tag{6.2.2}$$

The vielbeins read

$$\begin{aligned} e^{\hat{\mu}} &= e^{A(r)+\frac{\xi}{\ell}}dx^\mu, \\ e^{\hat{\xi}} &= e^{A(r)}d\xi, \\ e^{\hat{r}} &= dr \end{aligned} \tag{6.2.3}$$

with the non-vanishing spin connection

$$\begin{aligned}\omega^{\hat{\mu}\hat{r}} &= A'(r)e^{\hat{\mu}},\\ \omega^{\hat{\mu}\hat{\xi}} &= \frac{1}{\ell}e^{-A}e^{\hat{\mu}},\\ \omega^{\hat{r}\hat{\xi}} &= A'e^{\hat{\xi}}.\end{aligned} \tag{6.2.4}$$

We can consider the variation of the gravitino $\delta\psi_{A\mu} = 0$ along the direction of $\mu = 0, 1$

$$\begin{aligned}\delta\psi_{\mu A} &= \mathcal{D}_\mu\epsilon_A - S_{AB}\gamma_\mu\epsilon^B\\ &= \partial_\mu\epsilon_A + \frac{1}{4}\omega_\mu{}^{ab}\gamma_a\gamma_b\epsilon_A - S_{AB}\gamma_\mu\epsilon^B\\ &= \frac{1}{2}\left(\omega_{\hat{\mu}}{}^{\hat{\mu}\hat{r}}\gamma_{\hat{\mu}}\gamma_{\hat{r}} + \omega_{\hat{\mu}}{}^{\hat{\mu}\hat{\xi}}\gamma_{\hat{\mu}}\gamma_{\hat{\xi}}\right) - S_{AB}\gamma_{\hat{r}}\epsilon^B\end{aligned} \tag{6.2.5}$$

where we use $\{\gamma_{\hat{a}}, \gamma_{\hat{b}}\} = 0$, for $a \neq b$. We obtain

$$A'\gamma_{\hat{r}}\epsilon_A + \frac{1}{\ell}e^{-A}\gamma_{\hat{\xi}}\epsilon_A - \mathcal{W}\epsilon^A = 0. \tag{6.2.6}$$

This equation leads to

$$A'^2 + \frac{1}{\ell^2}e^{-2A} = W^2 \tag{6.2.7}$$

where $W = |\mathcal{W}|$ is the absolute value of eigenvalue of fermion-shift matrix S_{AB}. Moreover, we also impose an additional projection condition as

$$\gamma_\xi\epsilon_A = i\kappa e^{i\Lambda}\epsilon^A \tag{6.2.8}$$

where we have $\kappa^2 = 1$ implying $\kappa = \pm 1$ which defines the chirality of the Killing spinors on the two-dimensional defects. Therefore, the two projection conditions in supersymmetric Janus solutions read

$$\begin{aligned}\gamma_{\hat{r}}\epsilon_A &= e^{i\Lambda}\epsilon^A,\\ \gamma_\xi\epsilon_A &= i\kappa e^{i\Lambda}\epsilon^A.\end{aligned}$$

These projection conditions also lead to the phase factor

$$e^{i\Lambda} = \frac{A'}{W} + \frac{i\kappa}{\ell}\frac{e^{-A}}{W}. \tag{6.2.9}$$

If the eigenvalue of $\mathcal{W}$ is real, then

$$e^{i\Lambda} = \frac{\mathcal{W}}{A' + \frac{i\kappa}{\ell}e^{-A}}. \tag{6.2.10}$$

We can also consider the variation of gravitino along the direction of ξ which reads

$$
\begin{aligned}
\delta\psi_{\hat{\xi}A} &= \partial_{\hat{\xi}}\epsilon_A + \frac{1}{2}\omega_{\hat{\xi}}{}^{\hat{r}\hat{\xi}}\gamma_{\hat{r}}\gamma_{\hat{\xi}} - S_{AB}\gamma_{\hat{\xi}}\epsilon^B \\
&= e^{-A}\partial_\xi\epsilon_A + \frac{1}{2}\left(A'\gamma_{\hat{r}}\gamma_{\hat{\xi}}\epsilon_A - \mathcal{W}\gamma_{\hat{\xi}}\epsilon^A\right) \\
&= e^{-A}\partial_\xi\epsilon_A - \frac{1}{2l}e^{-A}\epsilon_A \\
&= 0.
\end{aligned}
\tag{6.2.11}
$$

We then obtain the Killing spinors as

$$
\epsilon_A = e^{\frac{\xi}{2l}}\varepsilon_A(r) \tag{6.2.12}
$$

where ε_A depends only on r. In the supersymmetric Janus case, we can finally look at the variation of the gravitino in the radial direction

$$
\begin{aligned}
\delta\psi_{Ar} &= 0 \\
&= \nabla_\mu\epsilon_A - S_{AB}\gamma_\mu\epsilon_B \\
&= \partial_r\epsilon_A - \frac{1}{2}\mathcal{W}\gamma_{\hat{r}}\epsilon^A \\
&= 2\partial_r\epsilon_A - A'\gamma_{\hat{r}}\epsilon^A \\
&= 2\partial_r\varepsilon_A - A'\varepsilon_A - i\frac{\kappa}{l}e^{-A}\varepsilon_A
\end{aligned}
\tag{6.2.13}
$$

where we have used (6.2.6), (6.2.12), and the projection conditions. Then, we obtain the solution of Killing spinors in case of supersymmetric Janus solution as

$$
\epsilon_A = e^{\frac{A}{2}+\frac{\xi}{2l}+i\frac{\Lambda}{2}}\epsilon_A^{(0)} \tag{6.2.14}
$$

where $\epsilon_A^{(0)}$ is constant Killing spinor satisfying the projection conditions and we can see that ϵ_A is in different representations can have different phase factor e^{Λ}. If we take $\ell \to \infty$, we would obtain the domain wall scenario. Finally, one can also look at $\delta\chi_A$, and $\delta\chi_{ABC}$, and it turns out that they are the same as the case of the supersymmetric domain wall. As a result, one could obtain the same form of Killing spinors equations and, we can proceed by a similar analysis with the redefinition of phase factor in (6.2.13).

It turns out that in the previous singlet scalars there are only $SO(2) \times$

$SO(4)$ and $SO(3)$ case which can possess supersymmetric Janus solutions because of the non-vanishing pseudoscalars. For $SO(3)$ case, the analysis is extremely complicated as we have already seen in the RG flow solution. Therefore, we will only consider supersymmetric Janus solution with $SO(2)\times SO(4)$ symmetry which is more traceable and can be analytically calculated.

6.2.1 Janus solutions with $SO(2) \times SO(4)$ symmetry

In this case, we have a real superpotential

$$\mathcal{W} = 4g\cosh\varphi, \tag{6.2.15}$$

so we will use the definition of the phase $e^{i\Lambda}$ of the form (6.2.10). We note that $\epsilon^{1,2}$ and ϵ^{A} with $A = 3, 4, 5, 6$ transform differently under $SO(2) \times SO(4)$ as $(\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{4})$, so they satisfy different projection conditions. We further find that the consistency of BPS equations require $\kappa = 1$ and $\kappa = -1$ for $\epsilon^{1,2}$ and $\epsilon^{3,\ldots,6}$, respectively. Therefore, the surface detect preserves $N = (2, 4)$ or $N = (4, 2)$ superconformal symmetry.

The BPS equations are given by

$$\begin{aligned}
\varphi' &= -\frac{8g^2\ell^2 A' e^{2A}\sinh(2\varphi)}{1+\ell^2 A'^2 e^{2A}}, && (6.2.16)\\
\zeta' &= -\frac{16g^2\kappa\ell e^{A}}{1+\ell^2A'^2e^{2A}}, && (6.2.17)\\
A'^2 + \frac{e^{-2A}}{\ell^2} &= 16g^2\ell^2\cosh^2\varphi\,. && (6.2.18)
\end{aligned}$$

We also note that for $\ell \to \infty$, these equations reduce to the equations of RG flow studied in the case of domain wall solutions.

To find the solution, we take φ as an independent variable, so we can evaluate $A(\varphi)$ and $\zeta(\varphi)$ given by

$$\begin{aligned}
A &= C - \ln\sinh\varphi, && (6.2.19)\\
\cosh(2\varphi) &= \frac{32g^2\ell^2\tanh^2[4g(r-r_0)]}{16g^2\ell^2-1}, && (6.2.20)\\
\kappa\tan\zeta &= -\sqrt{1-16g^2\ell^2}\sinh[4g(r-r_0)] && (6.2.21)
\end{aligned}$$

where C and r_0 are constant. We note that these solutions are similar to those in $N = 8, 5, 3$ gauged supergravities, see [32], [33] and [31] .

We also mention that the unbroken supersymmetries on the conformal defects in these cases are $N = (4, 4)$, $N = (4, 1)$ and $N = (2, 1)$. All of these solutions should be related by the consistent truncations of the maximal $N = 8$ gauged supergravity to $N = 3$ and $N = 5, 6$ theories. Thanks to the AdS/CFT correspondence, we expect that the dual $N = 3, 5, 6$ SCFTs possess the same two-dimensional conformal defect as in the $N = 8$ SCFT.

Finally, we also comment on the possible $SO(3)$ symmetric Janus solution due to the complicated analysis. By doing partial analysis, it turns out that the BPS equations for Janus solutions being very similar to the case of $N = 5$ theory in [31]. Therefore, we expect that $N = 6$ gauged supergravity should also give a supersymmetric $N = 2$ Janus solution with $SO(3)$ symmetry.

6.3 Supersymmetric AdS black holes

In this section, we will study the supersymmetric AdS_4 black hole by considering solutions interpolating between AdS_4 and $AdS_2 \times \Sigma^2$ geometries. The AdS_4 describes the asymptotic spacetime at a large distance from the black hole and the $AdS_2 \times \Sigma^2$ governs the near horizon geometries with Σ^2 being a two-dimensional Riemann surface.

We can write down the metric ansatz as

$$ds^2 = -e^{2f(r)}dt^2 + dr^2 + e^{2h(r)}(d\theta^2 + F(\theta)^2 d\phi^2). \tag{6.3.1}$$

The metric on Σ^2 takes the form of two-sphere S^2 and hyperbolic H^2, with

$$F(\theta) = \begin{cases} \sin\theta, & \text{if } \Sigma^2 = S^2. \\ \sinh\theta, & \text{if } \Sigma^2 = H^2. \end{cases}$$

As mentioned above, we are interested in the asymptotic anti-de Sitter space, so a result one should expect at large distance limit $r \to \infty$ is given by

$$f = h \to \frac{r}{L} \tag{6.3.2}$$

with L being AdS_4 radius. Moreover, in order that the solution approaches an supersymmetric $AdS_2 \times \Sigma^2$ fixed point at near horizon limit $r \to -\infty$, we require the boundary condition as

$$\varphi_I' \to 0, \qquad h' \to 0, \quad \text{and} \quad f' \to \frac{1}{L_{AdS_2}}. \tag{6.3.3}$$

with I indicates the number of scalars.

We can also write down the vielbein as

$$e^{\hat{t}} \;= e^{f(r)}dt, \qquad e^{\hat{r}} \;= dr, \tag{6.3.4}$$

$$e^{\hat{\theta}} \;= e^{h(r)}d\theta, \qquad e^{\hat{\phi}} \;= e^{h(r)}F(\theta)d\phi. \tag{6.3.5}$$

The non-vanishing components of spin connection are

$$\omega^{\hat{t}\hat{r}} \;= f^{'}(r)e^{\hat{t}}, \qquad \omega^{\hat{\theta}\hat{r}} \;= h^{'}(r)e^{\hat{\theta}}, \tag{6.3.6}$$

$$\omega^{\hat{\phi}\hat{r}} \;= h^{'}(r)e^{\hat{\phi}}, \qquad \omega^{\hat{\theta}\hat{\phi}} \;= \frac{F^{'}(\theta)}{F}e^{-h}e^{\hat{\phi}}. \tag{6.3.7}$$

To further consider the BPS equations of supersymmetric AdS_4 black hole solutions, we first consider the supersymmetry variation of the $\delta\psi_{\mu A}$ given by

$$\delta\psi_{\mu A} = \mathcal{D}_\mu \epsilon_A - S_{AB}\gamma_\mu \epsilon^B - \frac{1}{4\sqrt{2}}\hat{F}^+_{\rho\sigma AB}\gamma^{\rho\sigma}\epsilon^B \tag{6.3.8}$$

along the direction of $\hat{\phi}$

$$\begin{aligned}
\delta\psi_{\hat{\phi}A} &= \mathcal{D}_{\hat{\phi}}\epsilon_A - \mathcal{W}\gamma_{\hat{\phi}}\delta_{AB}\epsilon^B - \frac{1}{4\sqrt{2}}\hat{F}^+_{\rho\sigma AB}\gamma^{\rho\sigma}\epsilon^B \\
&= \frac{1}{4}\omega_{\hat{\phi}}{}^{\hat{\phi}\hat{r}}\gamma_{\hat{\phi}\hat{r}}\epsilon_A + \frac{1}{2}\omega_{\hat{\phi}}{}^{\hat{\theta}\hat{\phi}}\gamma_{\hat{\theta}\hat{\phi}}\epsilon_A + \frac{1}{2}Q_{\hat{\phi}A}{}^B\epsilon_B - \frac{1}{2}\mathcal{W}\gamma_{\hat{\phi}}\epsilon^A - \frac{1}{4\sqrt{2}}\hat{F}^+_{\rho\sigma AB}\gamma^{\rho\sigma}\gamma_{\hat{\phi}}\epsilon^B \\
&= \frac{1}{2}\frac{F'}{F}e^{-h}\gamma_{\hat{\theta}\hat{\phi}} + \frac{1}{2}h'\gamma_{\hat{\phi}\hat{r}}\epsilon_A + \frac{1}{2}Q_{\hat{\phi}A}{}^B\epsilon_B - \frac{1}{2}\mathcal{W}\gamma_{\hat{\phi}}\epsilon^A - \frac{1}{4\sqrt{2}}\hat{F}^+_{\rho\sigma AB}\gamma^{\rho\sigma}\gamma_{\hat{\phi}}\epsilon^B.
\end{aligned} \tag{6.3.9}$$

In general, the existence of Riemannian surface Σ^2 on the world-volume of the domain wall will break supersymmetries. However, there is a method called the topological twist that will preserve some amount of supersymmetry. To preserve such supersymmetries, we need to turn on gauge fields along Σ^2 written in term of gauge connection, which enters the covariant derivative of ϵ_A through the composite connection Q_A^B. To cancel the spin connection $\omega^{\hat{\theta}\hat{\phi}}$ with the gauge connection in $\delta\psi_{\hat{\phi}A}$, we write the four-dimensions gauge field ansatz as

$$\begin{aligned} A^{m+} &= A_t^m dt - p^m F'(\theta) d\phi \\ A^{m-} &= \tilde{A}_t^m dt - e_m F'(\theta) d\phi \end{aligned} \tag{6.3.10}$$

where p^m and e_m denote magnetic and electric charges of the black holes. If magnetic and electric charges are present the solution is called dyonic solution, however, we are interested in only the presence of magnetic charge corresponding to the absence of e_m, A_t^m, and $\tilde{A}_t$. We can also write the gauge fields in term of gauge generators as

$$A^{IJ} = (-p^m F'(\theta))(T_m)^{IJ} \tag{6.3.11}$$

$$= (-p^m \frac{F'}{F} e^{-h} e^{\hat{\phi}})(T_m)^{IJ}. \tag{6.3.12}$$

In this work, we focus on only solutions with $SO(2) \times SO(2) \times SO(2)$ and a truncated $SO(2)$ of $SO(2) \times SO(2) \times SO(2)$ twists. We first consider $SO(2) \times SO(2) \times SO(2)$ with the following gauge field ansatz

$$A^{12} = -p_1 F'(\theta) d\phi, \qquad A^{34} = -p_2 F'(\theta) d\phi, \qquad A^{56} = -p_3 F'(\theta) d\phi\,. \tag{6.3.13}$$

where p_i, $i = 1, 2, 3$, are magnetic charges with the corresponding field strengths given by

$$F^{12} = \kappa p_1 F(\theta) d\theta \wedge d\phi, \qquad F^{34} = \kappa p_2 F(\theta) d\theta \wedge d\phi, \qquad F^{56} = \kappa p_3 F(\theta) d\theta \wedge d\phi\,. \tag{6.3.14}$$

We define a parameter κ with $\kappa = 1, -1$ for Σ^2 being S^2 or H^2, respectively. Note that$F''(\theta) = -\kappa F(\theta)$.

For $N = 6$, with $SO(6)$ gauging, we have

$$Q_{IJAB}{}^{CD} = -4i \left(f^{AB}{}_{JK} \overline{h}_{IK}{}^{CD} + h_{ABIK} \overline{f}^{JKCD} \right) \tag{6.3.15}$$

and

$$Q_{IJA}{}^{B} = \frac{1}{4} Q_{IJAC}{}^{BC} - \frac{\delta_B{}^A}{40} Q_{IJCD}{}^{DC} \tag{6.3.16}$$

for $SO(2)\times SO(2)\times SO(2)$ subgroup of gauge group of $SO(6)$, the composite connection are given by

$$\begin{aligned} Q_{\hat{\phi}A}{}^{B} &= = -gA_{\mu}{}^{\Lambda}Q_{\Lambda A}{}^{B} \\ &= -\frac{1}{2}gA_{\mu}{}^{IJ}Q_{IJA}{}^{B} \\ &= 2gi\sigma_2\otimes\begin{pmatrix} A^{12}_{\hat{\phi}} & & \\ & A^{34}_{\hat{\phi}} & \\ & & A^{56}_{\hat{\phi}} \end{pmatrix}. \end{aligned} \tag{6.3.17}$$

We find that the spin connection $\omega^{\hat{\theta}\hat{\phi}}$ can be cancelled by the following topological twist

$$0 = \frac{1}{2}\frac{F'}{F}e^{-h}\gamma_{\hat{\theta}\hat{\phi}}\epsilon_A + \frac{1}{2}Q_{\hat{\phi}A}{}^{B}\epsilon_B. \tag{6.3.18}$$

As a result, we find the following projector

$$\gamma_{\hat{\theta}\hat{\phi}}\epsilon_A = (i\sigma_2\otimes\mathbb{I}_3)_A{}^{B}\epsilon_B \tag{6.3.19}$$

with the twist condition

$$2gp_1 = 2gp_2 = 2gp_3 = 1\,. \tag{6.3.20}$$

We have $p_1 = p_2 = p_3$. The twist can be obtained from the diagonal subgroup $SO(2)_{\text{diag}}\subset SO(2)\times SO(2)\times SO(2)$ similar to pure $N=4$ and $N=5$ gauged supergravities investigated in [58] and [31], respectively. Moreover, it appears that in the case of $SO(2)\times SO(2)\times SO(2)$, we need to turn on the $SO(2)\sim U(1)$ gauge field of $U(6)\sim SU(6)\times U(1)$ to find the consistent BPS equation. Similarly, we define the ansatz of the $U(1)$ gauge field as

$$A^0 = -p_0F(\theta)d\phi \qquad \text{and} \qquad F^0 = +\kappa p_0 d\theta\wedge\phi\,. \tag{6.3.21}$$

We note that both A^0 and A^{IJ} appear in the BPS equation because of the off-diagonal element of the scalar coset representative. In particular, we can write the relations

$$\hat{F}^{+}_{AB} = h_{\Lambda AB}F^{+\Lambda} = h_{0AB}F^{+0} + \frac{1}{2}h_{IJ,AB}F^{+IJ}, \tag{6.3.22}$$

$$\hat{F}^{+} = h_{\Lambda 0}F^{+\Lambda} = h_{00}F^{+0} + \frac{1}{2}h_{IJ0}F^{+IJ}\,. \tag{6.3.23}$$

We now consider the relation among F^{IJ}, F^{+IJ} and $\hat{F}^{+IJ}$ by considering the field strengths

$$\begin{aligned} F^{IJ} &= \kappa p F(\theta) d\theta \wedge d\phi \\ &= \kappa p F(\theta) \left(\frac{e^{\hat{\theta}}}{e^h} \wedge \frac{e^{\hat{\phi}}}{F(\theta) e^h} \right) \\ &= \kappa p e^{-2h} e^{\hat{\theta}} \wedge e^{\hat{\phi}} \end{aligned} \tag{6.3.24}$$

so, the components read

$$F^{12}_{\hat{\theta}\hat{\phi}} = F^{34}_{\hat{\theta}\hat{\phi}} = F^{56}_{\hat{\theta}\hat{\phi}} = \kappa p e^{-2h}. \tag{6.3.25}$$

By using the definition of the complex self-dual and anti-self-dual gauge field strengths

$$F^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}\left(F^{\Lambda}_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma} F^{\Lambda\rho\sigma} \right) \tag{6.3.26}$$

we find that there are two non-vanishing tangent spacetime components of

$$\begin{aligned} F^{+IJ}_{\hat{\theta}\hat{\phi}} &= \frac{1}{2}\left(F^{+IJ}_{\hat{\theta}\hat{\phi}} + \frac{i}{2}\epsilon_{\hat{\theta}\hat{\phi}\hat{\rho}\hat{\sigma}} F^{+IJ\rho\sigma} \right) \\ &= \frac{1}{2} F^{+IJ}_{\hat{\theta}\hat{\phi}} \\ &= \frac{1}{2}\kappa p e^{-2h} \end{aligned} \tag{6.3.27}$$

and

$$\begin{aligned} F^{+IJ}_{\hat{t}\hat{r}} &= \frac{1}{2}\left(F^{+IJ}_{\hat{t}\hat{r}} + \frac{i}{2}\epsilon_{\hat{t}\hat{r}\hat{\theta}\hat{\phi}} F^{+IJ\theta\phi} \right) \\ &= \frac{i}{4}\left(F^{+IJ\theta\phi} - F^{+IJ\phi\theta} \right) \\ &= \frac{i}{2}\kappa p e^{-2h} \end{aligned} \tag{6.3.28}$$

so, we obtain

$$\begin{aligned} F^{+12}_{\hat{\theta}\hat{\phi}} &= F^{+34}_{\hat{\theta}\hat{\phi}} = F^{+56}_{\hat{\theta}\hat{\phi}} = \frac{1}{2}\kappa p e^{-2h}, \\ F^{+12}_{\hat{t}\hat{r}} &= F^{+34}_{\hat{t}\hat{r}} = F^{+56}_{\hat{t}\hat{r}} = \frac{i}{2}\kappa p e^{-2h}. \end{aligned} \tag{6.3.29}$$

Return to the BPS equation resulting from $\delta\psi_{\hat{\phi}A}=0$, we now have

$$\begin{aligned}
\delta\psi_{\hat{\phi}A} &= \frac{1}{2}\frac{F'}{F}e^{-h}\gamma_{\hat{\theta}\hat{\phi}}+\frac{1}{2}h'\gamma_{\hat{\phi}\hat{r}}\epsilon_A+\frac{1}{2}Q_{\hat{\phi}A}{}^{B}\epsilon_B-\frac{1}{2}\mathcal{W}\gamma_{\hat{\phi}}\epsilon^A-\frac{1}{4\sqrt{2}}\hat{F}^{+}_{\rho\sigma AB}\gamma^{\rho\sigma}\gamma_{\hat{\phi}}\epsilon^B\gamma_{\hat{\phi}}\epsilon^A \\
&= \frac{1}{2}h'\gamma_{\hat{\phi}}\gamma_{\hat{r}}\epsilon_A-\frac{1}{2}\mathcal{W}\gamma_{\hat{\phi}}\epsilon^A-\frac{1}{4\sqrt{2}}\hat{F}^{+}_{\rho\sigma AB}\gamma^{\rho\sigma}\gamma_{\hat{\phi}}\epsilon^B \\
&= \frac{1}{2}h'\gamma_{\hat{\phi}}\gamma_{\hat{r}}\epsilon_A-\frac{1}{2}\mathcal{W}\gamma_{\hat{\phi}}\epsilon^A-\frac{1}{2\sqrt{2}}\left(\hat{F}^{+}_{\hat{t}\hat{r}AB}\gamma^{\hat{t}\hat{r}}\gamma_{\hat{\phi}}+\hat{F}^{+}_{\hat{\theta}\hat{\phi}AB}\gamma^{\hat{\theta}\hat{\phi}}\gamma_{\hat{\phi}}\right)\epsilon^B \\
&= \frac{1}{2}h'\gamma_{\hat{\phi}}\gamma_{\hat{r}}\epsilon_A-\frac{1}{2}\mathcal{W}\gamma_{\hat{\phi}}\epsilon^A+\frac{1}{2\sqrt{2}}\gamma_{\hat{\phi}}\left(\hat{F}^{+}_{\hat{\theta}\hat{\phi}AC}-i\hat{F}^{+}_{\hat{t}\hat{r}AC}\right)(i\sigma_2\otimes\mathbb{I}_3)^{C}{}_{B}\epsilon^B \\
&= \frac{1}{2}h'\gamma_{\hat{\phi}}\gamma_{\hat{r}}\epsilon_A-\frac{1}{2}\mathcal{W}\gamma_{\hat{\phi}}\epsilon^A-\frac{1}{2}\mathcal{Z}\gamma_{\hat{\phi}}\epsilon_A \\
&= h'\gamma_{\hat{r}}\epsilon_A-\mathcal{W}\epsilon^A-\mathcal{Z}\epsilon_A \\
&= \left(h'e^{i\Lambda}-\mathcal{W}-\mathcal{Z}\right)\epsilon^A.
\end{aligned} \tag{6.3.30}$$

We also note that we have the definition of $\epsilon_{\hat{0}\hat{r}\hat{\theta}\hat{\phi}}=1$ and $\gamma_5\epsilon_A=-\epsilon_A$ and the projection conditions, defined in(6.1.6), and the twist condition implying

$$\gamma^{\hat{0}\hat{r}}\epsilon_A=-i\gamma^{\hat{\theta}\hat{\phi}}\epsilon_A=(\sigma_2\otimes\mathbb{I}_3)_A{}^{B}\epsilon_B\,. \tag{6.3.31}$$

We also define the "central charge" matrix as

$$\mathcal{Z}_{AB}=-\frac{1}{\sqrt{2}}(\hat{F}^{+}_{\hat{\theta}\hat{\phi}AC}-i\hat{F}^{+}_{\hat{0}\hat{r}AC})(i\sigma_2\otimes\mathbb{I}_3)^{C}{}_{B}\,. \tag{6.3.32}$$

In the present case, the central charge $\mathcal{Z}_{AB}$ is proportional to the identity matrix as

$$\mathcal{Z}_{AB}=\mathcal{Z}\delta_{AB}. \tag{6.3.33}$$

We find a BPS equation resulting from $\delta\psi_{\hat{\theta}A}$ as

$$h'e^{i\Lambda}-\mathcal{W}-\mathcal{Z}=0 \tag{6.3.34}$$

which gives

$$h'=\pm|\mathcal{W}+\mathcal{Z}| \qquad \text{and} \qquad e^{i\Lambda}=\pm\frac{\mathcal{W}+\mathcal{Z}}{|\mathcal{W}+\mathcal{Z}|}\,. \tag{6.3.35}$$

Then, we consider $\delta\psi_{\hat{\theta}A}=0$

$$\begin{aligned}
\delta\psi_{\hat{\theta}A} &= \frac{1}{4}\omega_{\hat{\theta}}{}^{\mu\nu}\gamma_{\mu\nu}\gamma_{\hat{\theta}}\epsilon_A-\frac{1}{2}\mathcal{W}\epsilon^A-\frac{1}{2}\mathcal{Z}\gamma_{\hat{\theta}}\epsilon^A \\
&= (h'e^{i\Lambda}-\mathcal{Z}-\mathcal{W})\epsilon^A
\end{aligned} \tag{6.3.36}$$

which gives the same BPS equation as $\delta\psi_{\hat{\phi}A}$. Moreover, we can also look at $\delta\psi_{\hat{t}A}=0$ as follows

$$\begin{aligned}\delta\psi_{\hat{t}A} &= \frac{1}{2}f'\gamma_{\hat{t}\hat{r}}+\frac{1}{2}Q_{\hat{t}A}{}^{B}\epsilon_B-\frac{1}{2}\gamma_{\hat{t}}\epsilon^A-\frac{1}{2\sqrt{2}}\gamma_{\hat{t}}\left(\hat{F}^{+}_{\hat{\theta}\hat{\phi}AC}-i\hat{F}^{+}_{\hat{t}\hat{r}AC}\right)(i\sigma_2\otimes\mathbb{I}_3)^{C}{}_{B}\epsilon^B\\ &= \left(f'e^{i\Lambda}+2igA_1e^{i\Lambda}-\mathcal{W}+\mathcal{Z}\right)\epsilon^B\end{aligned} \tag{6.3.37}$$

With the previous BPS equations, we obtain

$$f'=\mathrm{Re}[e^{-i\Lambda}(\mathcal{W}-\mathcal{Z})] \qquad \text{and} \qquad 2gA_1=\mathrm{Im}[e^{-i\Lambda}(\mathcal{W}-\mathcal{Z})]. \tag{6.3.38}$$

The latter fixes the time component of the gauge fields. Finally, we can look at $\delta\psi_{\hat{r}A}=0$ given as

$$\begin{aligned}\delta\psi_{\hat{r}A} &= \partial_r\epsilon_A-\frac{1}{2}\mathcal{W}e^{-i\Lambda}\epsilon_A+\frac{1}{2}\mathcal{Z}e^{-i\Lambda}\epsilon_A\\ &= \partial_r\epsilon_A-\frac{1}{2}\left(f'e^{i\Lambda}\right)e^{-i\Lambda}\epsilon_A\\ &= \partial_r\epsilon_A-\frac{1}{2}f'\epsilon_A\end{aligned} \tag{6.3.39}$$

We then get r-dependence of the Killing spinors

$$\epsilon_A=e^{\frac{f}{2}}\epsilon_{A(0)} \tag{6.3.40}$$

which is similar to the case of domain walls and Janus solutions. To complete the consistent set of the BPS equations, we will consider $\delta\chi_{ABC}=0$ and $\delta\chi_A=0$

$$\begin{aligned}\delta\chi_{ABC} &= -P_{\mu ABCD}\gamma^{\mu}\epsilon^D+N^{D}{}_{ABC}\epsilon_D-\frac{3}{2\sqrt{2}}\gamma^{\mu\nu}\hat{F}^{+}_{\mu\nu[AB}\epsilon_{C]}\\ &= -P_{\hat{r}ABCD}\gamma^{\hat{r}}\epsilon^D+N^{D}{}_{ABC}\epsilon_D-\frac{3}{\sqrt{2}}(-i\hat{F}^{+}_{\hat{t}\hat{r}}+\hat{F}^{+}_{\hat{\theta}\hat{\phi}})_{[AB}(i\sigma_2\otimes\mathbb{I}_3)_{C]}{}^{D}\epsilon_D\\ &= \left(-P_{\hat{r}ABCD}e^{i\Lambda}+N^{D}{}_{ABC}-\frac{3}{\sqrt{2}}(-i\hat{F}^{+}_{\hat{t}\hat{r}}+\hat{F}^{+}_{\hat{\theta}\hat{\phi}})_{[AB}(i\sigma_2\otimes\mathbb{I}_3)_{C]}{}^{D}\right)\epsilon_D\end{aligned} \tag{6.3.41}$$

and

$$\begin{aligned}\delta\chi_A &= -\frac{1}{4!}\epsilon_{ABCDEF}P_{\mu}{}^{BCDE}\gamma^{\mu}\epsilon^F+N^{F}{}_{A}\epsilon_F-\frac{1}{2\sqrt{2}}\hat{F}^{+}_{\mu\nu}\gamma^{\mu\nu}\epsilon_A\\ &= -\frac{1}{4!}\epsilon_{ABCDEF}P_{\hat{r}}{}^{BCDE}\gamma^{\hat{r}}\epsilon^F+N^{F}{}_{A}\epsilon_F-\frac{1}{\sqrt{2}}(-i\hat{F}^{+}_{\hat{t}\hat{r}}+\hat{F}^{+}_{\hat{\theta}\hat{\phi}})(i\sigma_2\otimes\mathbb{I}_3)_{A}{}^{F}\epsilon_F\\ &= \left(-\frac{1}{4!}\epsilon_{ABCDEF}P_{\hat{r}}{}^{BCDE}e^{i\Lambda}+N^{F}{}_{A}-\frac{1}{\sqrt{2}}(-i\hat{F}^{+}_{\hat{t}\hat{r}}+\hat{F}^{+}_{\hat{\theta}\hat{\phi}})(i\sigma_2\otimes\mathbb{I}_3)_{A}{}^{F}\right)\epsilon_F.\end{aligned} \tag{6.3.42}$$

With these equations, we can find the BPS equations of scalar fields.

6.3.1 Solutions with $SO(2) \times SO(2) \times SO(2)$ twist

We start the supersymmetric black hole solution with the case of non-vanishing p_i. In this case, the topological twists allow the full $N = 6$ supersymmetry corresponding to non-vanishing of ϵ^A, $A = 1, 2, \ldots, 6$. Similar to the RG flow analysis, the consistency of the BPS equations coming from $\delta\psi_{\hat{\mu}A} = 0$ and $\delta\chi_A$, $\delta\chi_{ABC}$ require

$$\zeta_1 = \zeta_2 = \zeta_3 = 0 \qquad (6.3.43)$$

$$p_0 = \kappa p_1 \qquad (6.3.44)$$

respectively, and the former result in real $\mathcal{W}$ and $\mathcal{Z}$ giving $e^{i\Lambda} = \pm 1$.

We note that, in this scenario, the condition of $p_0 = 0$ will break all supersymmetry. This indicates that the $SO(2) \times SO(2) \times SO(2)$ twist goes together with the $U(1)$ gauge field A^0_μ.

With all of these, we find the set of consistent BPS equations written as

$$
\begin{aligned}
\varphi_1' &= -\frac{\partial|\mathcal{W}+\mathcal{Z}|}{\partial\varphi_1}, \\
&= \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\left[2g(1+e^{2(\varphi_2+\varphi_3)}-e^{2(\varphi_1+\varphi_2)}-e^{2(\varphi_1+\varphi_3)})\right. \\
&\quad \left.-p_1\kappa e^{-2h+2\varphi_1+2\varphi_2+\varphi_3}\right], \qquad (6.3.45) \\
\varphi_2' &= -\frac{\partial|\mathcal{W}+\mathcal{Z}|}{\partial\varphi_2}, \\
&= \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\left[2g(1-e^{2(\varphi_2+\varphi_3)}-e^{2(\varphi_1+\varphi_2)}+e^{2(\varphi_1+\varphi_3)})\right. \\
&\quad \left.-p_1\kappa e^{-2h+2\varphi_1+2\varphi_2+\varphi_3}\right], \qquad (6.3.46) \\
\varphi_3' &= -\frac{\partial|\mathcal{W}+\mathcal{Z}|}{\partial\varphi_3}, \\
&= \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\left[2g(1-e^{2(\varphi_2+\varphi_3)}+e^{2(\varphi_1+\varphi_2)}-e^{2(\varphi_1+\varphi_3)})\right. \\
&\quad \left.-p_1\kappa e^{-2h+2\varphi_1+2\varphi_2+\varphi_3}\right], \qquad (6.3.47) \\
h' &= |\mathcal{W}+\mathcal{Z}| \\
&= \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\left[2g(1+e^{2(\varphi_2+\varphi_3)}+e^{2(\varphi_1+\varphi_2)}+e^{2(\varphi_1+\varphi_3)})\right. \\
&\quad \left.+p_1\kappa e^{-2h+2\varphi_1+2\varphi_2+\varphi_3}\right], \qquad (6.3.48) \\
f' &= |\mathcal{W}-\mathcal{Z}| \\
&= \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\left[2g(1+e^{2(\varphi_2+\varphi_3)}+e^{2(\varphi_1+\varphi_2)}+e^{2(\varphi_1+\varphi_3)})\right. \\
&\quad \left.-p_1\kappa e^{-2h+2\varphi_1+2\varphi_2+\varphi_3}\right]. \qquad (6.3.49)
\end{aligned}
$$

For the existence of an $AdS_2\times\Sigma^2$ fixed point, we need $\varphi_1'=\varphi_2'=\varphi_3'=h'=0$ and $f'\sim\frac{1}{L_{AdS_2}}$ as $r\to-\infty$. However, the BPS equations given above do not admit any$AdS_2\times\Sigma^2$ fixed points.

Despite of the lack of supersymmetric $AdS_2\times\Sigma^2$ fixed point, we still can find the analytic solutions to these BPS equations which may be useful for some holographic studies. By the change of the variable of ρ using $\frac{d\rho}{dr}=e^{\varphi_3}$, we find the linear combinations

$$\frac{d}{d\rho}(\varphi_1-\varphi_2) = 2g(e^{\varphi_2-\varphi_1}-e^{\varphi_1-\varphi_2}) \qquad (6.3.50)$$

and

$$\frac{d}{d\rho}(\varphi_2-\varphi_3) = 2g(e^{\varphi_1-\varphi_2}-e^{\varphi_1+\varphi_2-2\varphi_3}). \qquad (6.3.51)$$

The former equation can be solved analytically with the solution given by

$$\varphi_1 = \ln\left[\frac{e^{\varphi_2}(e^{4g\rho}+e^{4g\rho_0})}{e^{4g\rho}-e^{4g\rho_0}}\right] \tag{6.3.52}$$

with ρ_0 being constant. By using this solution in the latter, we find

$$\varphi_2 = \ln\left[\frac{e^{\varphi_3-2g\rho}(e^{4g\rho}-e^{4g\rho_0})}{\sqrt{e^{4g\rho}+e^{8g\rho_0}+8gC}}\right] \tag{6.3.53}$$

with again C being constant.

Assume that f and h are the function of φ_3, we find the solutions for f and h as

$$\begin{aligned} f &= -\frac{1}{2}\ln\left[e^{4g\rho_0}\left(256g^3C\tilde{C}-16g\tilde{C}e^{8g\rho_0}+\kappa p_1\ln\left[\frac{1+e^{8g(\rho-\rho_0)}+8gCe^{4g\rho-8g\rho_0}}{e^{8g(\rho-\rho_0)}-1}\right]\right)\right. \\ &\quad \left. -8gC\kappa p_1\tanh^{-1}e^{4g(\rho-\rho_0)}\right]+h, \qquad (6.3.54) \\ h &= \frac{1}{2}\ln\left[e^{4g\rho_0}\left(16g\tilde{C}(e^{8g\rho_0}-16g^2C^2)-\kappa p_1\ln\left[\frac{1+e^{8g(\rho-\rho_0)}+8gCe^{4g\rho-8g\rho_0}}{e^{8g(\rho-\rho_0)}-1}\right]\right)\right. \\ &\quad \left.+8gC\kappa p_1\tanh^{-1}e^{4g(\rho-\rho_0)}\right]+\frac{1}{2}\ln\left[\frac{e^{12g\rho_0}(1-e^{8g(\rho-\rho_0)})^2}{8g(e^{8g\rho_0}-16g^2C^2)}\right]+\varphi_3-4g\rho \qquad (6.3.55) \end{aligned}$$

Finally, the solution of $\varphi_3(\rho)$ can be written as

$$\begin{aligned} 4C_0e^{4g\rho}(e^{8g\rho_0}+e^{8g\rho}+8gCe^{4g\rho}) &= \beta_0+\beta_1\ln\left[\frac{e^{4g(\rho_0-\rho)}+1}{e^{4g(\rho_0-\rho)}-1}\right] \\ &\quad +\beta_2\ln\left[\frac{e^{8g(\rho-\rho_0)}-1}{1+e^{8g(\rho-\rho_0)}+8gCe^{4g\rho}}\right] \end{aligned} \tag{6.3.56}$$

with C_0 being a constant, and the coefficients β_0, β_1 and β_2 are defined in term of $\varphi_3(\rho)$ by

$$\begin{aligned} \beta_0 &= -16g\tilde{C}e^{4\rho_0}(16Cg^2-e^{8g\rho_0})\left[2e^{4\varphi_3+8g(\rho+\rho_0)}+8gCe^{4g\rho}(e^{8g\rho}+e^{8g\rho_0})\right. \\ &\quad \left.+(e^{16g\rho_0}+e^{16g\rho})(1-e^{4\varphi_3})+2e^{8g(\rho+\rho_0)}\right], \qquad (6.3.57) \\ \beta_1 &= \left[\frac{\kappa p_1}{2(e^{4g\rho}+e^{4g(3\rho-2\rho_0)}+8Cge^{8g(\rho-\rho_0)})}\right]\left[e^{12g\rho}+e^{4g(\rho+2\rho_0)}+4Cge^{8g\rho}(3+e^{4\varphi_3})\right. \\ &\quad \left.+16g^2C^2e^{4g\rho}(1+e^{8g(\rho-\rho_0)})-2Cg(e^{4\varphi_3}-1)(e^{8g\rho_0}+e^{8g(2\rho-\rho_0)})\right], \qquad (6.3.58) \\ \beta_2 &= \frac{\kappa p_1[(e^{8g\rho_0}+e^{8g\rho}+8Cge^{4g\rho})^2-e^{4\varphi_3}(e^{8g\rho_0}-e^{8g\rho})^2]}{4e^{4g(\rho-\rho_0)}(e^{8g\rho_0}+e^{8g\rho}+8Cge^{4g\rho})}. \qquad (6.3.59) \end{aligned}$$

Because the solution does not permit any $AdS_2\times\Sigma^2$ fixed point in the IR, the solution describes a flow from the locally supersymmetric AdS_4 vacuum to a curved domain wall with world-volume $\mathbb{R}\times\Sigma^2$. Thanks to the AdS/CFT correspondence, the solution is thought to govern an RG flow from the $N = 6$ SCFT in three dimensions to a supersymmetric quantum mechanics in the IR resulting from a twisted compactification on Σ^2.

6.3.2 Solutions with $SO(2)$ twist

We now look at a possible truncation of the previous result so that we can have a $AdS_2 \times \Sigma^2$ fixed point. The strategy is to set $p_2 = p_3 = 0$ and $\varphi_2 = \varphi_3 = 0$. Therefore, we obtain a solution preserving $SO(2)\times SO(4)$ symmetry with the twist performed along the $SO(2)$ factor, and the supersymmetry is unbroken with only $\epsilon^{1,2}$. With the condition given above, and $\epsilon^{3,4,5,6} = 0$, we find the BPS equations

$$\varphi' = \frac{1}{4}e^{-2h-\varphi}[8ge^{2h} - p_0 + \kappa p_1 - e^{2\varphi}(8ge^{2h} + p_0 + \kappa p_1)], \qquad (6.3.60)$$

$$h' = \frac{1}{4}e^{-2h-\varphi}[8ge^{2h} - p_0 + \kappa p_1 + e^{2\varphi}(8ge^{2h} + p_0 + \kappa p_1)], \qquad (6.3.61)$$

$$f' = \frac{1}{4}e^{-2h-\varphi}[8ge^{2h} + p_0 - \kappa p_1 + e^{2\varphi}(8ge^{2h} - p_0 - \kappa p_1)] \qquad (6.3.62)$$

where $\varphi_1 = \varphi$.

We note that with only the $SO(2)$ twist, there is no need to set $p_0 = \kappa p_1$. Nevertheless, to obtain an $AdS_2 \times \Sigma^2$ fixed point, we require the vanishing of p_0. The corresponding $AdS_2 \times \Sigma^2$ fixed point give

$$\varphi = \varphi_0, \qquad h = \frac{1}{2}\ln\left[-\frac{\kappa p_1}{8g}\right], \quad L_{AdS_2} = \frac{1}{8g\cosh 2\varphi_0}\,. \qquad (6.3.63)$$

where φ_0 is a constant. Moreover, for the possible value of h, it requires that $\kappa = -1$ which means that this fixed point is an $AdS_2 \times H^2$ fixed point.

Finally, we give the flow solution by the similar method as in the previous calculation as

$$h = \varphi - \ln(1 - e^{2\varphi}) + C, \qquad (6.3.64)$$

$$f = h - 2\varphi + \ln[\kappa p_1(1 + e^{4\varphi}) + 2e^{2\varphi}(4g - \kappa p_1)], \qquad (6.3.65)$$

$$8g(\rho - \rho_0) = 2\sqrt{\frac{2g}{\kappa p_1 - 2g}}\tan^{-1}\left[\frac{4g + \kappa p_1(e^{2\varphi} - 1)}{2\sqrt{2g(\kappa p_1 - 2g)}}\right] + \ln\left[\frac{\kappa p_1(1 + e^{4\varphi}) + 2e^{2\varphi}(4g - \kappa p_1)}{(1 - e^{2\varphi})^2}\right] \qquad (6.3.66)$$

with a new radial coordinate ρ defined by$\frac{d\rho}{dr} = e^{\varphi}$. Besides, we omit the integral constant of f by rescaling time coordinate t to absorb such a constant.

At large distance limit of $r \sim \rho \to \infty$, we find

$$\varphi \sim e^{-4gr}, \qquad h \sim f \sim 4gr \tag{6.3.67}$$

being an asymptotically locally AdS_4 critical point. Moreover, with near horizon limit, where $\varphi_0 = \frac{1}{2}\ln\left(1 - 2\sqrt{-\frac{2g}{\kappa p_1}}\right)$ and $C = -\varphi_0$, we find $\varphi \to \varphi_0$ and the following result

$$h \sim \frac{1}{2}\ln\left[-\frac{\kappa p_1}{8g}\right] \qquad \text{and} \qquad f \sim 8gr\frac{1-\sqrt{-\frac{2g}{\kappa p_1}}}{\sqrt{1-2\sqrt{-\frac{2g}{\kappa p_1}}}} \tag{6.3.68}$$

These solutions reinforce that the solution admits $AdS_2 \times H^2$ fixed point.

6.3.3 $U(3)$ symmetric solutions

Finally, we study $U(3)$ symmetric solutions with a twist performed along the $SO(2) \sim U(1)$ factor. The gauge generator of the $U(1)$ factor can be written as $X_{14} + X_{25} + X_{36}$. As a result, we just turn on the following gauge fields

$$\mathcal{A} = A^{14} = A^{25} = A^{36} = A(r)dt - \kappa p F'(\theta)d\phi. \tag{6.3.69}$$

where the singlet scalar of $U(3)$ given in (6.1.29), we find the composite connection as

$$Q_A{}^B = 2gi\mathcal{A}(\mathbb{I}_3 \otimes i\sigma_2). \tag{6.3.70}$$

The twist condition results in

$$\gamma_{\hat{\theta}\hat{\phi}}\epsilon_A = (i\sigma_2 \otimes \mathbb{I}_3)_A{}^B\epsilon_B \qquad \text{and} \qquad 2gp = 1 \tag{6.3.71}$$

We note that, similar to the case of $SO(2) \times SO(2) \times SO(2)$ twist, all of the Killing spinor ϵ_A are non-vanishing. Besides, we also need non-vanishing A^0 using the ansatz (6.3.21). In this case, consistency requires $p_0 = -\kappa p$.

Similar to the RG flows, in order to have the consistency between the BPS equations and the field equations, we set $\zeta = 0$. As a result, we obtain $A(r) = 0$, and the BPS equations given by

$$\varphi' = g(e^{-\varphi} - e^{3\varphi}) + \frac{1}{2}\kappa p e^{-2h-3\varphi}, \tag{6.3.72}$$

$$h' = g(3e^{-\varphi} + e^{3\varphi}) + \frac{1}{2}\kappa p e^{-2h-3\varphi}, \tag{6.3.73}$$

$$\varphi' = g(3e^{-\varphi} + e^{3\varphi}) - \frac{1}{2}\kappa p e^{-2h-3\varphi}\,. \tag{6.3.74}$$

It is obviously that there is no $AdS_2 \times \Sigma^2$ fixed point in this case. Moreover, we also cannot obtain the analytic flow solution.

CHAPTER VII

Conclusions and comments

In this thesis, we have studied four-dimensional $N = 6$ gauged supergravity with $SO(6)$ gauge group resulting from a consistent truncation of the maximal $N = 8$ theory with $SO(8)$ gauge group. The $N = 6$ gauged theory admits a unique $N = 6$ supersymmetric AdS_4 vacuum preserving the full $SO(6)$ gauge symmetry, which can be explicitly identified with $AdS_4 \times CP^3$ geometry in type IIA theory dual to a three-dimensional $N = 6$ SCFT. We have found several holographic solutions with various symmetries governing the RG flows from this $N = 6$ SCFT to possible non-conformal phases in the IR. In particular, there is the solution with $SO(2) \times SO(4)$ symmetry in which the $SO(6)$ R-symmetry are broken with unbroken $N = 6$ Poincare supersymmetry. This precisely coincides with the field theory result on mass deformations of $N = 6$ SCFTs given in [16]. We also found other solutions, breaking the $SO(6)$ R-symmetry to $U(3)$, $SO(3)$ and $SO(2) \times SO(2) \times SO(2)$ symmetries. While most of the solutions preserve $N = 6$ supersymmetry, the solution with $SO(3)$ symmetry possibly beaks $N = 6$ to $N = 2$ supersymmetry. We found all analytic solutions, however, the $N = 2$ solution gives non-physical IR singularities by the criterion given in [57].

We have also generalized the flat domain walls to the curved ones, AdS_3-sliced domain walls. We have found a supersymmetric Janus solution, which describes a two-dimensional conformal defect within the $N = 6$ SCFT, with $SO(2) \times SO(4)$ symmetry and $N = (2, 4)$ supersymmetry on the defect. The solution has the similar form as those in $N = 8$, $N = 5$ and $N = 3$ gauged supergravities, [32], [31] and [33] respectively. Therefore, we argue that these solutions can be related to the $N = 8$ solution by consistent truncations. For Janus solutions to exist, it is indispensable that there must exist non-vanishing pseudoscalars pointed out in [32]. As a result, among the remaining cases, only the $SO(3)$ invariant sector possibly admits supersymmetric Janus solutions.

Moreover, we have also studied supersymmetric solutions of the form $AdS_2 \times \Sigma^2$ together with solutions interpolating between these geometries and the $N = 6$ AdS_4 vacuum. We found one $AdS_2 \times H^2$ fixed point with $SO(2) \times SO(4)$ symmetry from $SO(2)$ twist. The solution interpolating between this fixed point and the AdS_4 vacuum preserves two supercharges while the IR fixed point $AdS_2 \times H^2$ has four supercharges. Holographically, this solution describes an RG flow from the $N = 6$ SCFT to superconformal quantum mechanics which is useful in the entropy computation of black hole [47–49].

For $SO(2) \times SO(2) \times SO(2)$ twist, the BPS equations are more complicated but admit no $AdS_2 \times \Sigma^2$ fixed point. However, we could find the flow solution between the AdS_4 critical point to a curved domain wall with world-volume $\mathbb{R} \times \Sigma^2$ in the IR. The solution preserves $N = 6$ supersymmetry in three dimensions, or twelve supercharges, and $SO(2) \times SO(2) \times SO(2)$ symmetry, which should be dual to a twisted compactification on Σ^2 of the UV $N = 6$ SCFT to a supersymmetric quantum mechanics in the IR. Besides, we have also considered an $SO(2) \sim U(1)$ twist in the case of $U(3)$ symmetric solutions, and the corresponding $AdS_2 \times \Sigma^2$ geometries, however there exist no $AdS_2 \times \Sigma^2$ fixed points.

We hope that these analytic solutions could be useful in the study of gauge/ gravity holography and other related aspects. Since this is the only first-step in classifying supersymmetric solutions of $N = 6$ gauged supergravity, there are some directions we can further investigate. It would be very interesting to uplift the RG flow solutions to M-theory via the embedding in $N = 8$ gauged supergravity which can be obtained from a consistent truncation of M-theory on S^7. This would lead to a complete holographic description of mass deformations of $N = 6$ CSM theory and possible related M-brane configurations.

In this work, we have only considered gauged supergravity with $SO(6)$ gauge group which is electrically embedded in the global $SO^*(12)$ symmetry. It would be interesting to study magnetic and dyonic gaugings involving also magnetic gauge fields.

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. **2** (1998) 231-252, arXiv: hep-th/9711200.

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge Theory Correlators from Non-Critical String Theory”, Phys. Lett. **B428** (1998) 105-114, arXiv: hep-th/9802109.

[3] E. Witten, “Anti De Sitter Space and holography”, Adv. Theor. Math. Phys. **2** (1998) 253-291, arXiv: 9802150.

[4] L. Andrianopoli, R. D’Auria, S. Ferrara, P. A. Grassi, M. Trigiante, “Exceptional $N = 6$ and $N = 2$ AdS_4 Supergravity, and Zero-Center Modules”, JHEP 04 (2009) **074**, arXiv: 0810.1214.

[5] B. de Wit, H. Samtleben and M. Trigiante, “The maximal $D = 4$ supergravities”, JHEP 06 (2007) **049**, arXiv:0705.2101.

[6] B. de Wit and H. Nicolai, “$N = 8$ Supergravity”, Nucl. Phys. **B208** (1982) 323.

[7] B. de Wit, H. Samtleben and M. Trigiante, “On Lagrangians and gaugings of maximal supergravities”, Nucl. Phys. **B655** (2003) 93, arXiv hep-th/0212239.

[8] B. de Wit, H. Samtleben and M. Trigiante, “Magnetic charges in local field theory,” JHEP 0509 (2005) **016**, arXiv: hep-th/0507289.

[9] S. Lust, P. Ruter and J. Louis, “Maximally Supersymmetric AdS Solutions and their Moduli Spaces”, JHEP 03 (2018) **019**, arXiv: 1711.06180.

[10] B. E. W. Nilsson and C. N. Pope, "Hopf fibration of eleven-dimensional supergravity", Class. Quant. Grav. **1** (1984) 499-515.

[11] G. Arutyunov and S. Frolov, "Superstrings on $AdS_4 \times CP^3$ as a Coset Sigma-model', JHEP 09 (2008) **129**, arXiv: 0806.4940.

[12] B. Stefanski jr, "Green-Schwarz action for Type IIA strings on $AdS_4 \times CP^3$", Nucl. Phys. **B808** (2009) 80-87, arXiv: 0806.4948.

[13] R. D' Auria, P. Fre, P. A. Grassi and M. Trigiante, "Superstrings on $AdS_4 \times CP^3$ from Supergravity", Phys. Rev. **D79** (2009) 086001, arXiv: 0808.1282.

[14] G. Bonelli, P. A. Grassi and H. Safaai, "Exploring Pure Spinor String Theory on $AdS_4 \times CP^3$", JHEP 10 (2008) **085**, arXiv: 0808.1051.

[15] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals", JHEP 10 (2008) **091**, arXiv: 0806.1218.

[16] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, "$N = 5, 6$ Superconformal Chern-Simons Theories and M2-branes on Orbifolds", JHEP 09 (2008) **002**, arXiv: 0806.4977.

[17] R. Corrado, K. Pilch and N. P. Warner, "An $N = 2$ supersymmetric membrane flow", Nucl. Phys. **B629** (2002) 74-96, arXiv: hep-th/0107220.

[18] C. N. Gowdigere and N. P. Warner, "Flowing with Eight Supersymmetries in M-Theory and F-theory", JHEP 12 (2003) **048**, arXiv: hep-th/0212190.

[19] K. Pilch, A. Tyukov and N. P. Warner, "Flowing to Higher Dimensions: A New Strongly-Coupled Phase on M2 Branes", JHEP 11 (2015) **170**, arXiv: 1506.01045.

[20] C. Ahn and K. Woo, "Supersymmetric Domain Wall and RG Flow from 4-Dimensional Gauged $N = 8$ Supergravity", Nucl. Phys. **B599** (2001) 83-118, arXiv: hep-th/0011121.

[21] C. Ahn and T. Itoh, "An $N = 1$ Supersymmetric G_2-invariant Flow in M-theory", Nucl. Phys. **B627** (2002) 45-65, arXiv: hep-th/0112010.

[22] N. Bobev, N. Halmagyi, K. Pilch and N. P. Warner, "Holographic, $N = 1$ Supersymmetric RG Flows on M2 Branes", JHEP 09 (2009) **043**, arXiv: 0901.2376.

[23] A. Guarino, "On new maximal supergravity and its BPS domain-walls", JHEP 02 (2014) **026**, arXiv: 1311.0785.

[24] J. Tarrio and O. Varela, "Electric/ magnetic duality and RG flows in AdS_4/CFT_3", JHEP 01 (2014) **071**, arXiv: 1311.2933.

[25] Y. Pang, C. N. Pope and J. Rong, "Holographic RG Flow in a New $SO(3) \times SO(3)$ Sector of ω-Deformed $SO(8)$ Gauged $N = 8$ Supergravity", JHEP 08 (2015) **122**, arXiv: 1506.04270.

[26] P. Karndumri, "Holographic RG flows in $N = 3$ Chern-Simons-Matter theory from $N = 3$ 4D gauged supergravity", Phys. Rev. **D94** (2016) 045006, arXiv: 1601.05703.

[27] P. Karndumri and K. Upathambhakul, "Gaugings of four-dimensional $N = 3$ supergravity and AdS_4/CFT_3 holography", Phys. Rev. **D93** (2016) 125017 arXiv: 1602.02254.

[28] P. Karndumri, "Supersymmetric deformations of 3D SCFTs from tri-sasakian truncation", Eur. Phys. J. C (2017) **77**, 130, arXiv: 1610.07983.

[29] P. Karndumri and K. Upathambhakul, "Supersymmetric RG flows and Janus from type II orbifold compactification", Eur. Phys. J. C (2017) **77**, 455, arXiv: 1704.00538.

[30] P. Karndumri and K. Upathambhakul, "Holographic RG flows in N = 4 SCFTs from half-maximal gauged supergravity", Eur. Phys. J. **C78** (2018) 626, arXiv:hep-th/1806.01819.

[31] P. Karndumri and C. Maneerat, "Supersymmetric solutions from $N = 5$ gauged supergravity", Phys. Rev. **D101** (2020) 126015, arXiv: 2003.05889.

[32] N. Bobev, K. Pilchand N. P. Warner, "Supersymmetric Janus Solutions in Four Dimensions", JHEP 1406 (2014) **058**, arXiv: 1311.4883.

[33] P. Karndumri, "Supersymmetric Janus solutions in four-dimensional $N = 3$ gauged supergravity", Phys. Rev. **D93** (2016) 125012, arXiv: 1604.06007.

[34] M. Suh, "Supersymmetric Janus solutions of dyonic $ISO(7)$-gauged $N = 8$ supergravity", JHEP 04 (2018) **109**, arXiv: 1803.00041.

[35] N. Kim and S. J. Kim, "Re-visiting Supersymmetric Janus Solutions: A Perturbative Construction", arXiv: 2001.06789.

[36] M. Cvetic, M. Duff, P. Hoxha, J. T. Liu, H. Lu, et al., "Embedding AdS black holes in ten-dimensions and eleven-dimensions", Nucl. Phys. **B558** (1999) 96–126, arXiv: hep-th/9903214.

[37] M. J. Duff and J. T. Liu, "Anti-de sitter black holes in gauged $N = 8$ supergravity", Nucl. Phys. **B554** (1999) 237–253, arXiv: hep-th/9901149.

[38] J. P. Gauntlett, N. Kim, S. Pakis and D. Waldram, "Membranes Wrapped on Holomorphic Curves", Phys. Rev. **D65** (2002) 026003, arXiv: 0105250.

[39] S. L. Cacciatori and D. Klemm, "Supersymmetric AdS(4) black holes and attractors", JHEP 01 (2010) **085**, arXiv: 0911.4926.

[40] G. Dall'Agata and A. Gnecchi, "Flow equations and attractors for black holes in $N = 2$ $U(1)$ gauged supergravity, JHEP 03 (2011) **037**, arXiv: 1012.3756.

[41] K. Hristov and S. Vandoren, "Static supersymmetric black holes in AdS_4 with spherical symmetry", JHEP 04 (2011) **047**, arXiv: 1012.4314.

[42] N. Halmagyi, “BPS Black Hole Horizons in $N = 2$ Gauged Supergravity”, JHEP 02 (2014) **051**, arXiv: 1308.1439.

[43] N. Halmagyi, M. Petrini, and A. Zaffaroni, “BPS black holes in AdS_4 from M-theory”, JHEP 08 (2013) **124**, arXiv: 1305.0730.

[44] A. Guarino and J. Tarrio, “BPS black holes from massive IIA on S^6”, JHEP 09 (2017) **141**, arXiv: 1703.10833.

[45] A. Guarino, “BPS black hole horizons from massive IIA”, JHEP 08 (2017) **100**, arXiv: 1706.01823.

[46] P. Karndumri, “Supersymmetric $AdS_2 \times \Sigma_2$ solutions from tri-sasakian truncation”, Eur. Phys. J. C (2017) **77**, 689, arXiv: 1707.09633.

[47] F. Azzurli, N. Bobev, P. M. Crichigno, V. S. Min, and A. Zaffaroni, “A Universal Counting of Black Hole Microstates in AdS_4”, arXiv: 1707.04257.

[48] F. Benini, H. Khachatryan and P. Milan, “Black hole entropy in massive type IIA”, Class. Quant. Grav. **35** (2018) 035004, arXiv: 1707.06886.

[49] S. M. Hosseini, K. Hristov and A. Passias, “Holographic microstate counting for AdS_4 black holes in massive IIA supergravity”, JHEP 10 (2017) **190**, arXiv: 1707.06884.

[50] B. de Wit and H. Nicolai, “The consistency of the S^7 truncation in $D = 11$ supergravity”, Nucl. Phys. **B281** (1987) 211.

[51] B. de Wit, H. Nicolai, and N. P. Warner, “The embedding of gauged $N = 8$ supergravity into $d = 11$ supergravity”, Nucl. Phys. **B255** (1985) 29.

[52] H. Nicolai and K. Pilch, “Consistent truncation of $d = 11$ supergravity on $AdS_4 \times S^7$”, JHEP 03 (2012) **099**, arXiv: 1112.6131.

[53] B. de Wit and H. Nicolai, “Deformations of gauged $SO(8)$ supergravity and supergravity in eleven dimensions”, JHEP 05 (2013) **077**, arXiv: 1302.6219.

[54] H. Godazgar, M. Godazgar, Hermann Nicolai, “Testing the non-linear flux ansatz for maximal supergravity”, Phys. Rev. **D87** (2013) 085038, arXiv: 1303.1013.

[55] H. Godazgar, M. Godazgar, O. Krueger, H. Nicolai, “Consistent 4-form fluxes for maximal supergravity”, JHEP 10 (2015) **169**, arXiv: 1507.07684.

[56] N. P. Warner, “Some properties of the scalar potential in gauged supergravity theories”, Nucl. Phys. **B231** (1984) 250-268.

[57] S. S. Gubser, “Curvature singularities: the good, the bad and the naked”, Adv. Theor. Math. Phys. **4** (2000) 679-745, arXiv: hep-th/0002160.

[58] N. Bobev and P. M. Crichigno, “Universal RG Flows Across Dimensions and Holography”, JHEP 12 (2017) **065**, arXiv: 1708.05052.

[59] A. Borghese, Y. Pang, C. N. Pope and E. Sezgin, “Correlation Functions in ω-Deformed N= 6 Supergravity”, JHEP 02 (2015) **112**, arXiv: 1411.6020.

[60] G. Dall’Agata, G. Inverso and M. Trigiante, “Evidence for a family of $SO(8)$ gauged supergravity theories”, Phys. Rev. Lett. **109** (2012) 201301, arXiv: 1209.0760.

[61] G. Dall’ Agata, G. Inverso and Al, Marrani, “Symplectic Deformations of Gauged Maximal Supergravity”, JHEP 07 (2014) **133**, arXiv: 1405.2437.

[62] A. Borghese, G. Dibitetto, A. Guarino, D. Roest and O. Varela, “The $SU(3)$-invariant sector of new maximal supergravity”, JHEP 03 (2013) **082**, arXiv: 1211.5335.

[63] A. Gallerati, H. Samtleben, M. Trigiante, “The $N > 2$ supersymmetric AdS vacua in maximal supergravity”, JHEP 12 (2014) **174**, arXiv: 1410.0711.

[64] A. Guarino, “On new maximal supergravity and its BPS domain-walls”, JHEP 02 (2014) **026**, arXiv: 1311.0785.

[65] J. Tarrio and O. Varela, “Electric/magnetic duality and RG flows in AdS4/ CFT3”, JHEP 10 (2014) **071**, arXiv: 1311.2933.

[66] D. Roest and H. Samtleben, “Twin Supergravities”, Class. Quant. Grav. **26** (2009) 155001.

[67] Bambi, Cosimo. ”Astrophysical Black Holes: A Compact Pedagogical Review.” Annalen der Physik 530.6 (2018): 1700430.

[68] Reall, Harvey. ”Part 3 Black Holes.” Lecture notes given as part of the Cambridge University Mathematical Tripos (2014).

[69] Townsend, Paul K. ”Black holes.” arXiv preprint gr-qc/9707012 (1997).

[70] Abbott, Benjamin P., et al. ”Observation of gravitational waves from a binary black hole merger.” Physical review letters 116.6 (2016): 061102.

[71] Carlip, Steven. ”Quantum gravity: a progress report.” Reports on Progress in Physics 64.8 (2001): 885.

[72] Schulz, Benjamin. ”Review on the quantization of gravity.” arXiv preprint arXiv:1409.7977 (2014).

[73] Hartnoll, Sean A., Andrew Lucas, and Subir Sachdev. Holographic quantum matter. MIT press, 2018.

[74] Michael B. Green, John H. Schwarz, Edward Witten. ”Superstring theory. Volume 1- Introduction((Book)).” Cambridge and New York, Cambridge University Press, 1987, 479 (1987).

[75] Green, Michael B., John H. Schwarz, and Edward Witten. Superstring theory: volume 2, loop amplitudes, anomalies and phenomenology. Cambridge university press, 2012.

[76] Polchinski, Joseph. String theory: Volume 1, an introduction to the bosonic string. Cambridge university press, 1998.

[77] Polchinski, Joseph. String theory: Volume 2, superstring theory and beyond. Cambridge university press, 1998.

[78] Zwiebach, Barton. A first course in string theory. Cambridge university press, 2004.

[79] Ortín, Tomás. Gravity and strings. Cambridge University Press, 2004.

[80] Fré, Pietro Giuseppe, and Alexander Fedotov. Groups and Manifolds: Lectures for Physicists with Examples in Mathematica. Walter de Gruyter GmbH & Co KG, 2017.

[81] Frè, Pietro Giuseppe. Advances in Geometry and Lie Algebras from Supergravity. Springer, 2018.

[82] Nakahara, Mikio. Geometry, topology and physics. CRC Press, 2003.

[83] Schutz, Bernard. A first course in general relativity. Cambridge university press, 2009.

[84] Carroll, Sean M. ”An introduction to general relativity: spacetime and geometry.” Addison Wesley 101 (2004): 102.

[85] Misner, Charles W., Kip S. Thorne, and John Archibald Wheeler. Gravitation. Macmillan, 1973.

[86] Wald, Robert M. General relativity. University of Chicago press, 2010.

[87] Freedman, Daniel Z., and Antoine Van Proeyen. Supergravity. Cambridge university press, 2012.

[88] Yepez, Jeffrey. ”Einstein’s vierbein field theory of curved space.” arXiv preprint arXiv:1106.2037 (2011).

[89] Wess, Julius, and Jonathan Bagger. Supersymmetry and supergravity. Princeton university press, 1992.

[90] Weinberg, Steven. The quantum theory of fields: volume 3, supersymmetry. Cambridge university press, 2005.

[91] Martin, Stephen P. "A supersymmetry primer." Perspectives on supersymmetry II. 2010. 1-153.

[92] Bilal, Adel. "Introduction to supersymmetry." arXiv preprint hep-th/0101055 (2001).

[93] Tanedo, Flip. "Supersymmetry and Extra Dimensions."

[94] Rarita, William, and Julian Schwinger. "On a theory of particles with half-integral spin." Physical Review 60.1 (1941): 61.

[95] Castellani, Leonardo. "On G/H geometry and its use in M-theory compactifications." Annals of Physics 287.1 (2001): 1-13.

[96] Trigiante, Mario. "Gauged supergravities." Physics Reports 680 (2017): 1-175.

[97] Samtleben, Henning. "Lectures on gauged supergravity and flux compactifications." Classical and Quantum Gravity 25.21 (2008): 214002.

[98] Zaffaroni, Alberto. "Introduction to the AdS-CFT correspondence." Classical and Quantum Gravity 17.17 (2000): 3571.

[99] Skenderis, Kostas. "Lecture notes on holographic renormalization." Classical and Quantum Gravity 19.22 (2002): 5849.

[100] Fukuma, Masafumi, So Matsuura, and Tadakatsu Sakai. "Holographic renormalization group." Progress of theoretical physics 109.4 (2003): 489-562.

[101] RobinsonMatthew. Symmetry and the Standard Model: Mathematics and Particle Physics. Springer, 2011.

[102] Zee, Anthony. Group theory in a nutshell for physicists. Vol. 17. Princeton University Press, 2016.

[103] Weinberg, Steven, and O. W. Greenberg. "The quantum theory of fields, vol. 1: Foundations." PhT 48.11 (1995): 78.

[104] Gilmore, Robert. Lie groups, Lie algebras, and some of their applications. Courier Corporation, 2012.

[105] Samtleben, Henning. "Introduction to supergravity." Lectures presented at the 13th Saalburg School on Fundamentals and New Methods in Theoretical Physics (2007): 3-14.

[106] Tani, Yoshiaki. "Introduction to supergravities in diverse dimensions." Soryushiron Kenkyu Electronics 96.6 (1998): 315-351.

[107] West, Peter C. Introduction to supersymmetry and supergravity. World scientific, 1990.

[108] Louis, Jan. "Introduction to Supersymmetry and Supergravity." Fulltext on Jan Louis's personal page (2014).

[109] Brandt, Friedemann. "Lectures on supergravity." Fortschritte der Physik: Progress of Physics 50.10-11 (2002): 1126-1172.

[110] Taylor, Washington. "TASI lectures on supergravity and string vacua in various dimensions." arXiv preprint arXiv:1104.2051 (2011).

[111] Boels, R., and F. Brümmer. "Introduction to Supersymmetry and Supergravity."

[112] Mohaupt, Thomas. "Black holes in supergravity and string theory." Classical and Quantum Gravity 17.17 (2000): 3429.

[113] Karndumri, Parinya, and Jakkapat Seeyangnok. "Supersymmetric solutions from N= 6 gauged supergravity." Physical Review D 103.6 (2021): 066023.

APPENDIX

APPENDIX A

Group

Symmetry is a transformation that leaves physical quantities invariant. Symmetries play a very crucial role in physics because a certain physical theory is usually based on a certain set of symmetry called a symmetry group. The symmetry group is the study of the structure between elements of the symmetry group and the operation among themselves denoted by $(G, \circ)$. A certain group represents a particular structure, [101–104]. Symmetry group is a group, so it must satisfy group axioms.

1. Closure : The operation between group elements are trapped in the group, so $g_1 \circ g_2 \in G$, if g_1, $g_2 \in G$.

2. Associativity : The order of operation does not matter, so $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

3. Identity : There exists e so that there exist the operation of doing nothing, so $e \circ g_i = g_i \circ e = g_i$ where $g_i \in G$.

4. Inverse : No matter where we go, we are always be able to go back to the same point, so there exists g_i^{-1} such that $g_i^{-1} \circ g_i = e$.

The number of elements of the group is called the order of the group. H is a subgroup of G, if H satisfies the group axiom and $h_i \in H$ are the elements in G. We can construct a bigger group by a group product. If G and H are a group, the product can be classified by

1. Direct product : If G and K are commute, the product can be written by $G \times K$.

2. Semi-direct product : If G and K are not commute, the product can be written by $G \ltimes K$.

Representation We can represent the elements of groups by a set of numbers called a group representation $D(G)$. The representation of the group is said to be homomorphism if it preserves the group namely

$$D(g_i) \circ D(g_j) = D(g_i \circ g_j). \tag{A.0.1}$$

The elements of G become $D(e), D(g_1), D(g_2), ..., D(g_n)$. It does not matter how to choose to represent the elements. However, one representation may be more useful than another representation for a certain application. To see the set of number which represents a elements of a group, we can use a vector space as the representation of the group where the number of basis vector space corresponds to the group elements

$$e = g_1 \to |e\rangle, g_2 \to |g_2\rangle, ..., g_n \to |g_n\rangle. \tag{A.0.2}$$

The matrix representation of the group elements g_k are $n \times n$ matrices of dimensions n denoted by

$$D(g_k)_{ij} = \langle g_i | D(g_k) | g_j \rangle \tag{A.0.3}$$

where $i, j = 1, 2, ..., n$. The $n \times n$ matrices are called a regular representation. Moreover, the representation is said to be a faithful representation if $D(G)$ is an isomorphism. However, one may find representations that are smaller than the regular representation, and the smallest one is called the fundamental representation corresponding to the number of Cartan generators. If a regular representation can be reduced to smaller dimensions, there are subspaces. Let V be n-dimensional space spanned by n basis vectors. U and W are subspaces of V if for every vector v such that $v = u + w$ where $v \in V$, $w \in W$, and $u \in U$. We write

$$V = U \oplus W. \tag{A.0.4}$$

Let X be an operator in an n-dimensional space of V which is composed of two subspaces U and W with corresponding operators A_m and B_k, respectively. X is a block diagonal matrix of A_m and B_k written as

$$X_{n\times n} = \begin{pmatrix} A_m & 0 \\ 0 & B_k \end{pmatrix} \tag{A.0.5}$$

with $m+k=n$. This means that there exists a similarity transformation of $D(g)$ with an invertible matrix $S_{n\times n}$ such that

$$D^{'}(g) = SD(g)S^{-1} = \begin{pmatrix} A_{n_1}(g) & & & & \\ & B_{n_2}(g) & & & \\ & & C_{n_3}(g) & & \\ & & & \ddots & \\ & & & & D_{n_n}(g) \end{pmatrix} \tag{A.0.6}$$

Each subspace, A_{n_1}, $B_{n_2}(g)$, $C_{n_3}(g)$, ..., $D_{n_n}(g)$ is called an invariant subspace.

A.1 Lie Group

Lie group is a continuous group with infinite group elements, however such group elements can be parameterized by a finite set of continuous variables $\alpha_i(\theta)$, where $i = 1, 2, ..., n$. n is called the dimension of Lie group corresponding to the number of generators of the Lie group itself.

A.1.1 Generators

The elements of the group can be defined by $g(\alpha_i)$, and there exists the element corresponding to the identity as

$$g(\alpha_i)|_{\alpha_i=0} = e \tag{A.1.1}$$

The corresponding representation can be written as

$$D_n(g(\alpha_I))|_{\alpha_i=0} = I_{n\times n} \tag{A.1.2}$$

Due to the continuous parameterization, we can look at the infinitesimal elements of D_n around the identity by Taylor's series

$$D_n(g(\alpha_i)) = 1 + i\alpha^a T_a \tag{A.1.3}$$

where i indicates that T_a is a hermitian operator and $D_n(g(\alpha_i))$ is unitary operator. T_a is called generators written as

$$T_a = -i\frac{\partial g}{\partial \alpha^a}|_{\alpha^a=0}. \tag{A.1.4}$$

There is a generator for each parameter corresponding to the element of the group, and the entire representation can be completely defined by the generators T_a. These generators form the basis vector of the parameter space near the identity, and each point in the parameter space would correspond to a particular element of the group. So, the number of generators is equal to the dimensions of the Lie group which turns out to be the number of dimensions on a tangent space on a manifold. Therefore, one can characterize the manifold as Lie group called a coset manifold.

A.1.2 Lie Algebra

The element of Lie group is defined by the values of the parameters in the parameter space spanned by the generators, so the generators themselves form the algebra called Lie algebra as the consequence of the group axiom composed of Lie bracket resulting from the closure given by

$$[T_a, T_b] = if_{ab}{}^{c}T_c \tag{A.1.5}$$

and Jacobi identity resulting from the associativity given by

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0. \tag{A.1.6}$$

Representation of Lie group

A particular representation of any group element can be written by

$$D_n(\alpha_i) = e^{i\alpha^a T_a}. \tag{A.1.7}$$

In this representation of a vector space, there exists a set of eigenvectors and the corresponding eigenvalues denoted by $|j : m\rangle$,on which Lie group act, where j are the eigenvectors and m, are the eigenvalues. The basis of the vector space is the set of simultaneously diagonalized generators called Cartan generator which forms a closed algebra, an algebra being commute among themselves but not outside the algebra. The number of Cartan generators is called the rank of the group. The list of eigenvalues is called the weight vector. The remaining generators can be formed as linear combinations which transform one vector into another and the resulting vector space is a physical space called root space.

A.2 Lorentz Group

Lorentz group is a symmetry underlying special relativity. We will review some important features of Lorentz group and Lorentz algebras to discuss representations of Lorentz group which are objects representing fields of elementary particles in physics.

Lorentz Algebra

Lorentz group $SO(1,3)$ is a group which preserves spacetime interval

$$s^2 = -t^2 + \vec{x}^2 \tag{A.2.1}$$

so, an action which is invariant under $SO(1,3)$ is called Lorentz invariant theory and all physical theories of our universe should be invariant under this symmetry.

Lie algebra can be written as

$$[J^{\mu\nu}, J^{\rho\lambda}] = i\left(\eta^{\lambda\mu}J^{\rho\nu} - \eta^{\nu\lambda}J^{\rho\mu} - \eta^{\rho\mu}J^{\lambda\nu} + \eta^{\nu\rho}J^{\lambda\mu}\right). \tag{A.2.2}$$

There are six generators of Lorentz group which consist of three rotations denoted by $[J_i]^{\mu}{}_{\nu}$ and three boosts denoted by $[K_i]^{\mu}{}_{\nu}$ where

$$\begin{aligned} J^i &= \frac{1}{2}\epsilon^{ijk}J^{jk}, \\ K^i &= J^{0i}. \end{aligned} \tag{A.2.3}$$

We can write the Lorentz algebra as

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk}J_k, \\
[J_i, K_j] &= i\epsilon_{ijk}K_k, \\
[K_i, K_j] &= -i\epsilon_{ijk}J_k.
\end{aligned} \tag{A.2.4}$$

We can rewrite such generators as the combination as

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) \tag{A.2.5}$$

where the commutation relation of new generators lead to

$$\begin{aligned}
[N_i^+, N_j^+] &= i\epsilon_{ijk}N_k^+, \\
[N_i^-, N_j^-] &= i\epsilon_{ijk}N_k^-, \\
[N_i^+, N_j^-] &= 0.
\end{aligned} \tag{A.2.6}$$

Here, we can obviously see that $SO(1,3)$ Lorentz algebra is made of two pieces of $SU(2)$ Lie algebra. This is true in four-dimensional spacetime.

A.2.1 Lorentz Representation

The representation of $SO(1,3)$ can be denoted by doublet (j, j') of $(2j+1)(2j+1) \times (2j'+1)(2j'+1)$ matrices.

$(0,0)$ **Representation** This representation is 1×1 matrix or scalar which is a trivial representation.

$(1/2, 0)$ **Representation** By setting $N_i^- = 0$, so

$$\begin{aligned}
N_i^- &= \frac{1}{2}(J_i - iK_i) = 0 \rightarrow J_i = iK_i, \\
N_i^+ &= \frac{1}{2}(J_i + iK_i) \rightarrow iK_i = \frac{1}{2}\sigma_i
\end{aligned} \tag{A.2.7}$$

and we get

$$\begin{aligned}
R(\theta) &= e^{i\vec{\theta}\cdot\vec{J}} \rightarrow R(\theta) = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}, \\
B(\phi) &= e^{i\vec{\phi}\cdot\vec{J}} \rightarrow B(\phi) = e^{i\vec{\phi}\cdot\frac{\vec{\sigma}}{2}}
\end{aligned} \tag{A.2.8}$$

where $R(\theta)$ and $B(\phi)$ correspond to rotation and boost transformation the $(1/2, 0)$ representation, respectively.

$(0, 1/2)$ **Representation** By setting $N_i^+ = 0$

$$\begin{aligned} N_i^- &= \frac{1}{2}(J_i - iK_i) \to -ik_i = \frac{1}{2}\sigma_i, \\ N_i^+ &= \frac{1}{2}(J_i + iK_i) = 0 \to J_i = -ik_i \end{aligned} \tag{A.2.9}$$

and we obtain

$$\begin{aligned} R(\theta) &= e^{i\vec{\theta}\cdot\vec{J}} \to R(\theta) = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}, \\ B(\phi) &= e^{i\vec{\phi}\cdot\vec{J}} \to B(\phi) = e^{-i\vec{\phi}\cdot\frac{\vec{\sigma}}{2}} \end{aligned} \tag{A.2.10}$$

where $R(\theta)$ and $B(\phi)$ correspond to rotation and boost transformation the $(0, 1/2)$ representation, respectively.

Therefore, the two $(1/2, 0)$ and $(0, 1/2)$ are identical under rotation, but slightly different under boost. Therefore, the $(1/2, 0)$ and $(0, 1/2)$ representation is called left-handed spinor and right-handed spinor respectively, and they are conjugate of each other.

$(1/2, 1/2)$ **Representation** This representation is called the vector representation which comprises of two non-overlapping copies of $SU(2)$.

A.3 Clifford Algebra

Clifford algebra is a general idea for looking at the spinor representation in various n-dimensional spacetime, that corresponds to a set of n matrices of γ^μ, $\mu = 0, ..., n-1$ with components $(\gamma^\mu)^{ab}$ satisfying anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{A.3.1}$$

We can also show that Clifford algebra can be related to Lorentz algebra by defined

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{A.3.2}$$

This leads to the algebra given by

$$[S^{\mu\nu}, S^{\rho\lambda}] = i(\eta^{\lambda\mu}S^{\rho\nu} - \eta^{\nu\lambda}S^{\rho\mu} - \eta^{\rho\mu}S^{\lambda\nu} + \eta^{\nu\rho}S^{\lambda\mu}) \tag{A.3.3}$$

which coincides exactly with Lorentz algebra (A.2.2).

Moreover, Clifford algebra can also naturally arise from Dirac's equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{A.3.4}$$

once, we try to obtain Klein-Gordon equation

$$(i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\mu - m)\psi = (\gamma^\nu \gamma^\mu \partial_\mu + m^2)\psi = 0. \tag{A.3.5}$$

We then obtain the Clifford algebra (A.3.1). Let's solve Clifford algebra in four dimensions by using a trick given by

$$\begin{aligned} A \otimes B &= \begin{pmatrix} Ab_{11} & Ab_{12} \\ Ab_{21} & Ab_{22} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} b11 & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} b_{12} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} b_{21} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{12} & a_{12}b_{12} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12} \\ a_{11}b_{21} & a_{12}b_{21} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{21} & a_{22}b_{21} & a_{21}b_{22} & a_{22}b_{22} \end{pmatrix} \end{aligned} \tag{A.3.6}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

We also note that $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. Let us now mention few crucial solutions to the Clifford algebra.

Chiral representation In this representation, we have

$$\begin{aligned} \gamma^0 = (\sigma^0 \otimes \sigma^1) = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \gamma^i = i(\sigma^i \otimes \sigma^2) = i \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \end{aligned} \tag{A.3.7}$$

It is also called Weyl representation.

Dirac representation In this representation, we give

$$\begin{aligned} \gamma^0 &= (\sigma^0 \otimes \sigma^3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \gamma^i &= i(\sigma^i \otimes \sigma^2) = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \end{aligned} \tag{A.3.8}$$

Majorana representation Lastly, Majorana representation can be obtained by giving

$$\begin{aligned} \gamma^0 &= (\sigma^2 \otimes \sigma^1) = \begin{pmatrix} 0\sigma^2 & \\ \sigma^2 & 0 \end{pmatrix}, \\ \gamma^1 &= (\sigma^3 \otimes \sigma^0) = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \\ \gamma^2 &= i(\sigma^2 \otimes \sigma^2) = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \\ \gamma^3 &= i(\sigma^1 \otimes \sigma^0) = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}. \end{aligned} \tag{A.3.9}$$

This results in every non-vanishing components to be imaginary number, so we have real spinor in Majorana representation. Moreover, we also note that one can generally identify the chirality of spinor by using projection operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5) \tag{A.3.10}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ being invariant under Lorentz transformation denoted by $[S^{\mu\nu,\gamma^5}] = 0$. This holds in any representation of the Clifford algebra of γ^μ matrices. Furthermore, we can also consider a higher rank of gamma matrices as

$$\gamma^{\mu_1\mu_2\cdots\mu_n} \equiv \gamma^{[\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n]} \tag{A.3.11}$$

Moreover, one can also have the Clifford algebra on local Lorentz coordinate can be mapped to the manifold on a tangent space as

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{A.3.12}$$

APPENDIX B

Field Theories

Fields area mathematical objects that eat a spacetime location and split out a value in the field space, and the type of the value depends on the type fields. In this appendix, we will review scalar fields, spinor fields, and vector fields describing particle with spin-0, spin-1/2, and spin-1 ,respectively. We will also discuss the the global and local symmetry called gauge symmetry of Lagrangian.

B.1 Scalar Fields

Scalar fields $\Phi_i(\vec{x}, t)$ are used to represent spinless particles such as the Higgs boson. In Minkowski spacetime, the dynamics of an real scalar field are captured by the action

$$S = \int d^4x(-\frac{1}{2}\partial_\mu \Phi(x)\partial^\mu \Phi(x) - V(x)) \tag{B.1.1}$$

where the first term is the kinetic energy of scalar field and the second term is potential term. Let $V(x) = \frac{1}{2}m^2\Phi^2$, the action becomes

$$S = \int d^4x(-\frac{1}{2}\partial_\mu \Phi(x)\partial^\mu \Phi(x) - \frac{1}{2}m^2\Phi^2). \tag{B.1.2}$$

We can evaluate the equation of motion by doing the variation of the action with respect to the scalar field itself, $\delta_\Phi S = 0$. We get

$$\partial_\mu \partial^\mu \Phi(x) - m^2\Phi = 0. \tag{B.1.3}$$

The field's equation of motion of scalar field is called the "Klein-Gordon equation". Moreover, we can generalize a real scalar field to a complex scalar field of $\Phi(\vec{x}, t) =$

$(\Phi_1 + i\Phi_2)/\sqrt{2}$ where Φ_1, and Φ_2 are real scalar fields. The action can be written as

$$S = \int d^4x(-\frac{1}{2}\partial_\mu \Phi(x)\partial^\mu \Phi(x)^\dagger - \frac{1}{2}m^2\Phi\Phi^\dagger). \tag{B.1.4}$$

The field's equations of motion for $\Phi(\vec{x}, t)$ and its complex conjugate $\Phi^\dagger$ can be written as

$$\begin{aligned} \partial_\mu\partial^\mu\Phi(x) - m^2\Phi = 0, \\ \partial_\mu\partial^\mu\Phi^\dagger - m^2\Phi^\dagger = 0. \end{aligned} \tag{B.1.5}$$

B.2 Vector Fields

Vector fields can be used to described spin-1 particles which play a very crucial role in elementary particle physics. They represents gauge bosons such as photon, ZW bosons, and gluon which are the gauge mediators of electromagnetic, weak, and strong interactions, respectively. Let us consider a massless vector field $A_\mu(\vec{x}, t) = (\phi(x), \vec{A}(x))$, where $\phi(x)$ and $\vec{A}(x)$ are called potential and vector potential, respectively. The dynamics of the vector field can be described by the action

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu) \tag{B.2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$ is electromagnetic field tensor and $j^\mu = (\rho, \vec{J})$ is four current.

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B1 & 0 \end{pmatrix} \tag{B.2.2}$$

and its Hodge dual tensor defined as $G_{\mu\nu} = *F^{\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$

$$G_{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E1 & 0 \end{pmatrix} \tag{B.2.3}$$

One can find the equation of motion of vector field by doing the variation of the action with respect to A_μ, then one obtain

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{B.2.4}$$

with the Bianchi's identity

$$\partial_{[\mu} F_{\nu\rho]} = 0. \tag{B.2.5}$$

These equations form Maxwell's equations

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{B.2.6}$$

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0, \tag{B.2.7}$$

$$\vec{\nabla} \cdot \vec{E} = \rho, \tag{B.2.8}$$

$$\frac{\partial \vec{E}}{\partial t} - \vec{\nabla} \times \vec{E} = -\vec{j}. \tag{B.2.9}$$

B.3 Spinor field

Spinor fields are used to describe half-integer spin particles. They are very important to describe fermionic particles such as leptons and quarks. Here, we will review the representation of spin-1/2 field.

B.3.1 Dirac Spinor

A spinor $\Psi(x)$ is used to describe spin half-integer particles. There are three interesting types of spinors, Dirac spinor, Weyl Spinor, and Majorana spinor. Let us firstly start reviewing Dirac spinor describing spin 1/2 particle. The dynamics of Dirac spinor field can be written as

$$S = \int d^4x (\overline{\Psi} \gamma^\mu \partial_\mu \Psi - m \overline{\Psi} \Psi). \tag{B.3.1}$$

By doing the variation of the action with respect to Ψ and $\bar{\Psi}$.

One obtain the equations of motion as

$$\gamma^\mu \partial_\mu \Psi - m\Psi = 0 \tag{B.3.2}$$

$$\partial_\mu \overline{\Psi} \gamma^\mu - m\overline{\Psi} = 0 \tag{B.3.3}$$

which is the set of first order equation of motion, one can also find the corresponding second order equation of motion as

$$\partial^\mu \partial_\mu \Psi - m^2 \Psi = 0 \tag{B.3.4}$$

This shows that the Dirac's equation satisfies the Klein-Gordon equation.

B.3.2 Weyl Spinor

Dirac spinor can written in terms of two Weyl spinor written as

$$\Psi = \begin{pmatrix} \psi \\ \overline{\chi} \end{pmatrix} \tag{B.3.5}$$

In Chiral representation, one can write Dirac equation as

$$\overline{\sigma}^\mu \partial_\mu \psi = m\overline{\chi} \tag{B.3.6}$$

$$\sigma^\mu \partial_\mu \overline{\chi} = m\psi \tag{B.3.7}$$

and the action in term of this representation can be written as

$$S = i \int d^4x (\overline{\chi}^\dagger \sigma^\mu \partial_\mu \overline{\chi} - \psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi + m\psi^\dagger \overline{\chi} - m\overline{\chi}^\dagger \psi) \tag{B.3.8}$$

If $m = 0$, we can obviously see that Dirac spinor is a reducible representation of Weyl spinors. However if $m \neq 0$, Dirac spinor is irreducible, but Dirac spinor will be equivalent to two Weyl spinors. The action of massless spinor can be written as

$$S = -i \int d^4x \psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi \tag{B.3.9}$$

with the equation of motion as

$$\overline{\sigma}^\mu \partial_\mu \psi = 0 \tag{B.3.10}$$

B.3.3 Majorana Spinor

A real Majorana spinor is a direct result from the Majorana representation of Clifford algebra with reality condition called Majorana constraint given by

$$\psi = \psi^{(c)} = C\bar{\psi}^T \tag{B.3.11}$$

where C being charge conjugate. The corresponding Majorana Lagrangian is given by

$$\mathcal{L}_M = i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda - \frac{i}{2} m \left(\lambda^* \sigma^2 \lambda^* - \lambda^T \sigma^2 \lambda\right) \tag{B.3.12}$$

where we define the Majorana field to be

$$\psi_M = \begin{pmatrix} \lambda \\ (i\sigma^2)\lambda^* \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{B.3.13}$$

where we express the right and left component into each other and $\lambda = \psi_L = (i\sigma^2_R)$. The physical interpretation of charge conjugate can be obvioulsy seen by taking the complex conjugate of gauge theory of Dirac's equation given (D.1.10)

$$\begin{aligned} ((i\gamma^\mu(\partial_\mu + iqA_\mu) - m)\psi)^* &= 0, \\ \left(i\gamma^\mu(\partial_\mu - iqA_\mu)\psi^{(c)}\right) &= 0. \end{aligned} \tag{B.3.14}$$

We see that if ψ does satisfy the Dirac's equation for a particle with charge q, $\psi^{(c)}$ need to also satisfy Dirac's equation with negative charge $-q$. As a result, C is the charge conjugate in the sense that it does swap the charge of the field.

APPENDIX C

Symmetry

Symmetry is a set of transformation which leaves physical system invariant and the physical system can be described by the functional action. If the action is invariant under generic field transformation without imposing field's equations and boundary conditions, the symmetry is called "off-shell symmetry". If the action is only invariant by imposing field's equations and boundary conditions, the symmetry is called "on-shell symmetry". The mathematical tool which is used to study symmetry is group theory. According to group theory, symmetry can be classified in many different ways whether or not the symmetry is discrete or continuous, spacetime or internal, global or local (gauged), abelian or non-abelian. Here, we mostly focus on continuous symmetry. One of the most important roles of symmetry is the applications of understanding interactions between fields or particles.

C.1 Spacetime and Internal Symmetry

Spacetime symmetry is a symmetry which act on spacetime, so it does transform spacetime such as Poincare symmetry

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\nu. \tag{C.1.1}$$

Moreover, there is also the symmetry which does not act on spacetime, but it acts on hidden degree of freedom of the field

$$\phi^i(x) \to U^i{}_j \phi^j(x). \tag{C.1.2}$$

This is called "internal symmetry".

C.2 Global and Gauged Symmetry

Both of the spacetime and internal symmetry can be either global symmetry or guage symmetry, where the parameter of transformation is independent and dependent of spacetime, respectively. Global symmetry is a symmetry which acts on the field at all points on spacetime in the same way at once such as for the spacetime symmetry of Poincare symmetry

$$x^{\mu} \to \Lambda^{\mu}{}_{\nu}(x)x^{\nu} + a^{\nu} \tag{C.2.1}$$

or the internal symmetry of $SU(N)$ symmetry

$$\phi^{i}(x) \to U^{i}{}_{j}\phi^{j}(x) \tag{C.2.2}$$

The gauge symmetry is a symmetry which depends on the point in spacetime.

$$U^{i}{}_{j} = U^{i}{}_{j}(x). \tag{C.2.3}$$

C.3 Symmetry and Noether Current

For a certain continuous symmetry, there exists a corresponding conserved quantity according to Noether's theorem and the conversed charges correspond with the Noether currents.

For the action

$$S[\phi^{i}] = \int d^{4}x\mathcal{L}(\phi^{i}, \partial\phi^{i}) \tag{C.3.1}$$

which is invariant under a certain transformation,

$$\phi(x) \to \phi^{'}(x^{'}) \tag{C.3.2}$$

namely,

$$S[\phi^{i}] = S[\phi^{i'}]. \tag{C.3.3}$$

The variation of the field can be expressed as

$$\delta\phi^i(x) = \epsilon^a \Delta_a \phi^i(x) \tag{C.3.4}$$

and the variation of the Lagrangian can written as

$$\delta\mathcal{L} = \big[\frac{\delta\mathcal{L}}{\delta\phi^i} - \partial_\mu\big(\frac{\delta\mathcal{L}}{\partial\partial_\mu\phi^i}\big)\big]\delta\phi^i + \partial_\mu\big(\frac{\delta\mathcal{L}}{\partial\partial_\mu\phi^i}\delta\phi^i\big). \tag{C.3.5}$$

The variation of Lagrangian may not vanish under the transformation, however, it can be vanished up to the total derivative of the action given by

$$\delta\mathcal{L} = \epsilon^a \partial_\mu K^\mu{}_a. \tag{C.3.6}$$

We can write

$$\partial_\mu\big(K^\mu{}_a - \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\Delta_a\phi^i\big) = \big(\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\big(\frac{\partial\mathcal{L}}{\partial\partial_{\mu\phi^i}}\big)\big)\Delta_a\phi^i \tag{C.3.7}$$

where we have assumed that ϵ is spacetime independent. And we impose Euler equation, we then obtain the Noether current as

$$j^\mu{}_a = K^\mu{}_a - \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\Delta_a\phi^i \tag{C.3.8}$$

with the conservation law of

$$\partial_\mu j^\mu{}_a = 0. \tag{C.3.9}$$

APPENDIX D

Gauge Theories

To consider the interaction between fields or particles, we need an interaction term in our Lagrangian $\mathcal{L}_{\text{int}}$. This can be done by coupling Noether current of a field and then multiply it by the field to which we need to couple together with a coupling constant. However, this process can be done with a more fundamental process called "Gauge Theory". Gauge theory is a way to include an interaction by promoting a certain global symmetry group of Lagrangian to gauge symmetry group and force the Lagrangian to be invariant under the gauge symmetry. This process forces us to define a covariant derivative which automatically includes gauge fields with a coupling constant into the theory. As a result, we have an interaction term in the Lagrangian. Therefore, gauge theory plays a central role in modern physics especially in the theory of fundamental interaction of elementary particle physics. Here, we will review some crucial features of gauge theory.

D.1 Abelian Gauge Theories

It is obvious that Maxwell's equations of the action

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{D.1.1}$$

are invariant under gauge transformation of

$$A_\mu \to A_\mu^{'} = A_\mu + \partial_\mu \alpha(x) \tag{D.1.2}$$

where $\alpha(x)$ is the infinitesimal gauge parameter which depends on spacetime. As a result, one can have the action of electromagnetic fields which is invariant under local $U(1)$. Moreover, the existence of mass term of gauge field A_μ will violate the local symmetry $U(1)$.

For the Dirac action which govern the dynamics of spin 1/2 particle

$$\mathcal{L}_{\mathrm{D}} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \tag{D.1.3}$$

One can promote the global symmetry $U(1)$ of the Dirac Lagrangian

$$\psi \to e^{-iq\alpha}\psi \tag{D.1.4}$$

to a gauge symmetry by setting $\alpha = \alpha(x)$ and under local $U(1)$ transformation we obtain

$$\mathcal{L}_{\mathrm{D}} \to \overline{\psi}(i\gamma^\mu\partial_\mu - m - \gamma^\mu\partial_\mu\alpha(x))\psi. \tag{D.1.5}$$

We need to way a way to cancel the extra term. One way is to define a new gauge field which transforms under $U(1)$ as

$$A_\mu \to A_\mu - \frac{1}{q}\partial_\mu\alpha(x) \tag{D.1.6}$$

so, we have just introduced a covariant derivative written as

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu. \tag{D.1.7}$$

The covariant derivative indicate that the field is charged under the gauge symmetry, and the Lagrangian can be rewritten as

$$\mathcal{L}_{\mathrm{D}} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{D.1.8}$$

which is invariant under both global and gauged $U(1)$ symmetry. Here, we have just added the gauge field into our theory, however there is no dynamics of gauge field due to the lack of kinematic term of gauge field. One can add the extra term which contribute the dynamics of the gauge field given by

$$\mathcal{L}_{\mathrm{kin,A}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{D.1.9}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ due to the abelian property of $U(1)$ gauge symmetry. Therefore, the interacting theory of spin 1/2 field cam be written as

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{D.1.10}$$

if one quantize this Lagrangian, one would get the theory of quantum electrodynamics or QED.

D.2 Non-Abelian Gauge Theories

Now, we will generalize our gauging procedure of arbitrary compact Lie group of dimensions G, which transforms the field by

$$\phi_i \to U_{ij}\phi_j \tag{D.2.1}$$

and the structure of the group can be represented by Lie algebra

$$[T^a, T^b] = if_{abc}T^c \tag{D.2.2}$$

and the corresponding gauge field transformation can be written as

$$A^\mu \to U(x)A^\mu U^\dagger(x) + \frac{i}{g}U(x)\partial^\mu U(x)^\dagger. \tag{D.2.3}$$

We can introduce the covariant derivative in the similar way, which indicate the introduction of gauge fields and the field of theory would be charged by the gauged fields under the covariant derivative given by

$$D_\mu = \mathbb{I}_{N\times N}\partial_\mu - igA_\mu \tag{D.2.4}$$

where A_μ is the $N \times N$ matrix, and it acts on the fields as

$$(D_\mu\phi)_j = \partial_\mu\phi_j(x) - ig[A_\mu(x)]_{jk}\phi_k(x). \tag{D.2.5}$$

The field strength can be defined as

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu(x) - \partial_\mu A_\mu - ig[A_\mu(x), A_\nu(x)]. \tag{D.2.6}$$

The invariant form of field strength under gauge group is the trace of field strength given by

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{D.2.7}$$

Here, we have just promoted a non-interacting field theory with a global symmetry to interacting field theory with gauge symmetry gauged by vector fields. For a certain Lie group, there exist particular gauge fields with a particular interaction.

Vitae

My name is Jakkapat Seeyangnok. I am currently a graduate student at Chulalongkorn University, Thailand. I was born in Khon Kaen on 5th July, 1994. I graduated from Seekan(Wattananunupathum) school. I received my bachelor's degree in physics with first-class honors from Kasetsart University, Thailand. After finishing my bachelor's degree, I was in the military as a private soldier for a year. As I went to graduate school, I had an opportunity to join summer school at Deutsches Elektronen-Synchrotron (DESY), Hamburg, Germany.

Publications

1. Karndumri, Parinya, and Jakkapat Seeyangnok. "Supersymmetric solutions from N= 6 gauged supergravity." Physical Review D 103.6 (2021): 066023.

Presentations, Talks

1. Talked "Charge particles spectra in deep inelastic ep scattering", Summer Student DESY 2019 Hamburg, Germany.

International Schools

1. Attended, Summer Student DESY 2019 Hamburg, Germany.

2. Attended, National Astronomical Research Institute of Thailand, international school, workshop, and conference in data analysis on high energy physics, particle physics, and cosmology 2014, Thailand.

3. Attended, Tah Poe The Institute for Fundamental Study for Theoretical Physics, international summer school 2014, and workshop in cosmology and astroparticle physics, Thailand.