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ADS_4/CFT_3 HOLOGRAPHY FROM FOUR-DIMENSIONAL GAUGED SUPERGRAVITY

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เราได้ศึกษาอาร์จีโฟลว์แบบโฮโลกราฟีจากเกจซูเปอร์กราวิตีที่มีจำนวนซูเปอร์ซิมเมทรีเท่ากับสามและเท่ากับสี่ในสี่มิติ มานิโฟลด์สเกลาร์ของเกจซูเปอร์กราวิตีที่มีจำนวนซูเปอร์ซิมเมทรีเท่ากับสามจะอยู่ในรูปของ $G/H = SU(3,n)/SU(3) \times SU(n) \times U(1)$ เกจกรุปที่เป็นไปได้สำหรับกรณีนี้คือ $SO(3) \times SO(3)$, $SO(3,1)$, $SO(2,2)$, $SO(2,1) \times SO(2,2)$ และ $SL(3,R)$ เราได้ศึกษา เกจซูเปอร์กราวิตีที่มีจำนวนซูเปอร์ซิมเมทรีเท่ากับสี่จากการยุบขนาดมิติของทฤษฎีสตริงแบบ II บน $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ และเกจกรุปแบบเซมิซิมเปิล มานิโฟลด์สเกลาร์ของเกจซูเปอร์กราวิตีที่มีจำนวนซูเปอร์ซิมเมทรีเท่ากับสี่จะอยู่ในรูปของ $SL(2,R)/SO(2) \times SO(6,n)/SO(6) \times SO(n)$ เกจกรุปที่ได้จากการยุบขนาดมิติเชิงเรขาคณิตนอกแบบของทฤษฎีสตริงแบบ IIB คือ $ISO(3) \times ISO(3)$ ซึ่งฝังตัวใน $SO(6,6)$ ผ่านสับกรุป $SO(3,3) \times SO(3,3)$ เรายังได้พิจารณาเกจซูเปอร์กราวิตีที่มีจำนวนซูเปอร์ซิมเมทรีเท่ากับสี่จากการยุบขนาดมิติแบบจีเคพีของทฤษฎีสตริงแบบ IIB เกจกรุปที่ได้จากการยุบขนาดมิติเชิงเรขาคณิตของทฤษฎีสตริงแบบ IIA เป็นเกจกรุปแบบนอนเซมิซิมเปิล $ISO(3) \ltimes U(1)^6$ สำหรับเกจกรุปแบบเซมิซิมเปิลเราได้พิจารณาเกจกรุปดังต่อไปนี้ $SO(4) \times SO(4)$, $SO(3,1) \times SO(3,1)$, $SO(2,2) \times SO(2,2)$, $SO(4) \times SO(3,1)$, $SO(4) \times SO(2,2)$ และ $SO(3,1) \times SO(2,2)$ เราพบจุดวิกฤตแบบ AdS_4 ที่มีสมมาตรยิ่งยวดสำหรับแต่ละเกจกรุป เราได้หาคำตอบอาร์จีโฟลว์ที่เชื่อมโยงระหว่างจุดวิกฤตเหล่านี้รวมทั้งเชื่อมโยงจุดวิกฤตไปยังทฤษฎีสนามแบบนอนคอนฟอร์มอลในแต่ละเกจกรุป เรายังได้ให้ตัวอย่างคำตอบเจนัสสำหรับเกจซูเปอร์กราวิตีที่ได้จากการยุบขนาดมิติเชิงเรขาคณิตนอกแบบของทฤษฎีสตริงแบบ IIB อีกด้วย

ภาควิชา ฟิสิกส์........................ ลายมือชื่อนิสิต..

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We study holographic RG flows from $N = 3$ and $N = 4$ gauged supergravities in four dimensions. The scalar manifold of $N = 3$ gauged supergravity is in the form of the coset space $G/H = SU(3, n)/SU(3) \times SU(n) \times U(1)$. Possible gauge groups, in this study, are given by $SO(3) \times SO(3)$, $SO(3, 1)$, $SO(2, 2)$, $SO(2, 1) \times SO(2, 2)$, and $SL(3, R)$. We then study $N = 4$ gauged supergravity from type II compactification on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with non-semisimple gaugings. The scalar manifold of $N = 4$ gauged supergravity is in the form of $SL(2, R)/SO(2) \times SO(6, n)/SO(6) \times SO(n)$ coset. The gauge group arising from non-geometric compactification of type IIB is $ISO(3) \times ISO(3)$, which is embedded in $SO(6, 6)$ via the $SO(3, 3) \times SO(3, 3)$ subgroup. We also consider $N = 4$ gauged supergravity from type IIB GKP (Giddings-Kachru-Polchinski) compactification. We similarly study non-semisimple $ISO(3) \ltimes U(1)^6$ gauge group arising from type IIA geometric compactification. For semisimple gaugings, we consider $SO(4) \times SO(4)$, $SO(3, 1) \times SO(3, 1)$, $SO(2, 2) \times SO(2, 2)$, $SO(4) \times SO(3, 1)$, $SO(4) \times SO(2, 2)$, and $SO(3, 1) \times SO(2, 2)$ gauge groups. A number of supersymmetric AdS_4 critical points for each gauge group are found. We give RG flow solutions interpolating between these critical points together with possible flows to non-conformal theories, in each gauge group. We also give examples of Janus solutions for $N = 4$ gauged supergravity obtained from type IIB non-geometric compactification.

Department : Physics .... Student's Signature ....

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CHAPTER I

Introduction

In this chapter, we give an introduction to the background materials used in this dissertation. We start with ideas and the motivations for quantum gravity. We then move to the topic of the AdS/CFT correspondence, holographic renormalization group (RG) flows and gauged supergravities. The aim of this dissertation is to study holographic RG flows from four-dimensional gauged supergravities with $N = 3$ and $N = 4$.

1.1 Quantum gravity

Quantum gravity is one of the most prevailing research in high energy physics. It is an attempt to give a quantum description of gravity. Three of the four fundamental forces are described in the framework of quantum theory. The standard model successfully gives a quantum description of the electromagnetic, weak, and strong interactions, along with the existence of elementary particles. The standard model also successes in providing experimental predictions. By including a quantum description of gravity, it is a hope to have a theory with a unification of the four fundamental forces.

Besides quantum effects, general relativity provides a wonderful classical framework for gravity. It describes gravitation as a geometry of spacetime. General relativity also gives precise predictions, including gravitational time dilation, gravitational lensing, gravitational redshift. Recently, the gravitational wave from merging of a binary black hole system, which is predicted in the theory was directly observed using the Laser Interferometer Gravitational-Wave Observatory (LIGO) [1].

There are no less than 16 major approaches proposed to give a quantum description to general relativity [2]:

1. Canonical quantum gravity
2. Manifestly covariant quantization
3. Euclidean quantum gravity
4. R-squared gravity
5. Supergravity
6. String and brane theory
7. Renormalization group and Weinberg's asymptotic safety
8. Non-commutative geometry
9. Twistor theory
10. Asymptotic quantization
11. Lattice formulation
12. Loop space representation
13. Quantum topology, motivated by Wheeler's quantum geometrodynamics
14. Simplicial quantum gravity and null-strut calculus
15. Condensed-matter view: the universe in a helium droplet
16. Affine quantum gravity.

The first eight approaches are based on the Lagrangian/Hamiltonian framework which uses the action principle. String theory is the only one among the first eight approaches that are not a field theory of conventional point particles and spacetime is is required to have extended structures. The later eight approaches use different mathematical structures of conventional pictures.

However, there are problems that obstruct the attempt to quantize the theory of gravity. We often encounter divergences or some inconsistency of quantum gravity theories. For example, field quantization in quantum field theory neglects modes of fields that possess zero-point energy. The number of these mode is infinite. The vacuum energy is then infinite implying infinite gravity which is coupled to this energy. From observations, a cosmological constant which is corresponding to the vacuum energy is small. This is a problem. Also from the viewpoint of field theory in the unit with $\hbar = c = 1$, the gravitational coupling constant has a unit of $energy^{-2}$. Theories with coupling constant of positive dimensions usually turn out to be finite, while theories with a dimensionless coupling constant are candidates to be renormalizable. Theories that have coupling constants of negative dimensions, usually have divergences and they are not renormalizable. Quantum description of general relativity falls into the last category.

In string theory, as the candidate for a quantum theory of gravity, the problem of non-renormalizability has been cured. One-loop diagrams in this theory are finite and free of any ultraviolet divergences [3]. A solution to the cosmological constant problem is possible [4]. String theory also provides a potentially powerful tool, the AdS/CFT correspondence, to solve complex problems in various areas of theoretical physics.

1.2 AdS/CFT correspondence

Recently, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence (also known as gauge/gravity duality), which is proposed by Juan Maldacena in 1997 [5], is one of the most outstanding developments in string theory with exceed 14,700 citations [6]. It is a duality between conformal field theories, which are the gauge theories, and string theory on anti-de Sitter spaces. With this duality, one can study strongly coupled quantum field theories, which cannot be described by perturbative expansion, via the gravitational theories which are weakly interacting.

The AdS/CFT correspondence is not only interesting in the field of high energy physics and string theories, but also used to study other areas, i.e. condensed matter physics and nuclear physics. Condensed matter physics also uses the formalism of quantum field theory to describe exotic states of matter. However, some phenomena are described by strongly coupled field theories. Some condensed matter physicists believe that we can study these phenomena by using the AdS/CFT correspondence [7].

Recently, there has been a lot of studies using the AdS/CFT correspondence and string theories in the fields of hadron physics [8] and condensed matter, including superconductors [9] [10] [11] [12] [13] [14] and superfluids [15]. As conventional superconductors are well-described by Cooper pair fluid in BCS theory, the understanding of pairing mechanism for unconventional superconductors are still incomplete since a normal state of some materials is not well-described by the standard Fermi liquid theory. Recently, the AdS/CFT correspondence technique is used to study unconventional superconductors by introducing a $3 + 1$ dimensional black hole with a charged scalar field dual to a strongly coupled gauge theory on a layered unconventional superconductor. This is the well-known AdS_4/CFT_3 correspondence. With this technique, we can obtain some results for conductivity, phase transition, and energy gap from Einstein-Maxwell scalar theory.

This correspondence can also be used to study the other side of the coin, as it could give a microscopic description of black hole thermodynamics. The AdS/CFT correspondence provides a wonderful framework to study a non-perturbative definition in quantum gravity of asymptotically-AdS black hole, in the context of a conformal field theory living on the boundary of the AdS space. The Bekenstein-Hawking entropy of a class of BPS black holes could be reproduced in the dual

CFT [16] [17] [18] [19] [20] [21].

AdS/CFT correspondence provides an excellent framework in the study of the renormalization group flows in theories with a strongly interacting system, which is a non-perturbative system. We can study a deformation in a CFT by constructing a dual geometry in a gravity theory with an asymptotically AdS background. The conjecture gives a one-to-one map between fields in the gravity theory and operators in the conformal field theory. A map between the radial coordinate of the AdS geometry to the energy scale in the CFT allows us to holographically study a non-perturbative RG flows from the UV to IR fixed points.

1.3 Four-dimensional gauged supergravity

To study holographic RG flows, working in lower-dimensional gauged supergravities, as the effective theories of superstring theories or M-theory, has proven to be useful and effective. There are geometries in the form of the $AdS_4 \times X^7$, identified with AdS_4 vacua of the scalar potential of four-dimensional gauged supergravity. The isometry of the internal manifold corresponds to gauge symmetry at the AdS_4 vacua. The effective four-dimensional $N = 8$ $SO(8)$ gauged supergravity constructed in [22] results in the $AdS_4 \times S^7$ geometry preserving maximal supersymmetry. Holographic study within this gauged supergravity has been investigated in [23] [24] [25] [26] [27]. These results give holographic descriptions of the deformations leading to various types of RG flows in the superconformal field theories in three dimensions.

In this dissertation, we study holographic RG flows from four-dimensional gauged supergravities in the context of AdS_4/CFT_3 correspondence. Moreover, we focus on gauged supergravities with $N = 3, 4$. We will study supersymmetric solutions with only the metric and scalar fields non-vanishing. Supergravity theories with $N > 2$ have enough supersymmetries to determine the geometry of the scalar manifold in the form of the coset space G/H. We are also not interested in supergravities with $N > 4$ since they have no matter multiplet. Thus, we only interested in gauged supergravities with $N = 3$ and $N = 4$.

In four-dimensional $N = 3$ gauged supergravity, there is a unique non-maximal AdS_4 solutions from a compactification of the eleven-dimensional supergravity [28]. The internal manifold is the tri-sasakian N^{010} with $SU(2) \times SU(3)$ isometry. The corresponding Kaluza-Klein spectrum has been given in [29], and the structure of $N = 3$ multiplets is investigated in [30]. A possible $N = 3$ SCFT dual to M-theory compactified on $AdS_4 \times N^{010}$ is studied in [31]. The gravity dual to $N = 3$ SCFT is also studied in many aspects [32] [33] [34] [35] [36] [37]. These result in a significant match between $N = 3$ SCFT and the AdS_4 solution from the compactification of eleven-dimensional configurations in M-theory. The eleven-dimensional supergravity compactified on the $AdS_4 \times N^{010}$ can be described by an $N = 3$, $SU(3) \times SO(3)$ gauged supergravity as an effective theory [29] [30]. The theory with eight vector multiplets is constructed in [38] [39] [40]. Various deformations and supersymmetric vacua have been identified in [41]. The eleven-dimensional configurations to these solutions might be obtained through a consistent reduction ansatz if exists.

Four-dimensional $N = 4$ Gauged supergravity has been studies for a long time [42] [43] [44]. The embedding tensor formalism, which included all the deformations, has been given in [45]. There are also $N = 4$ gauged supergravity obtained from the compactifications of type IIA and type IIB superstring theories with various fluxes [46] [47] [48] [49].

1.4 Outline

This dissertation focuses on studying holographic RG flow solutions from $N = 3$ and $N = 4$ gauged supergravities in the context of AdS_4/CFT_3 correspondence. In chapter 2, we give an introduction to AdS_5/CFT_4 correspondence. This includes reviews of $N = 4$ super Yang-Mills theory and type IIB superstring theory on $AdS_5 \times S^5$. We will later consider a generalization to the AdS_4/CFT_3 correspondence and give holographic solutions of interest in chapter 4 and chapter 5.

In chapter 3, we review some general structures of gauged supergravities in four dimensions. The specific structures for $N = 3$ and $N = 4$ gauged supergravities will be given in each corresponding chapter.

In chapter 4, we study holographic RG flows from $N = 3$ gauged supergravity in four dimensions. In chapter 5, we study holographic RG flows from $N = 4$ gauged supergravity obtained from the compactifications of type II string theories with non-semisimple gaugings and also consider solutions from semisimple gauge groups. We conclude the dissertation in chapter 6.

CHAPTER II

AdS/CFT Correspondence

Anti-de Sitter/conformal field theory correspondence or AdS/CFT correspondence, sometimes called Maldecena's conjecture, is a relationship between $D+1$-dimensional quantum gravity theories with AdS background and quantum field theories with conformal symmetry, living on D-dimensional boundary of the gravity theories. As the boundary theories are similar to the Yang-Mills theories which are gauge theories that describe elementary particles, the correspondence is sometimes called gauge/gravity duality.

The correspondence was proposed by Juan Maldacena in 1997 [5], as a relation between type IIB string theory on $AdS_5 \times S^5$ space and $\mathcal{N} = 4$ super Yang-Mills theory on $3+1$-dimensional Minkowski space. It was showed later that the conformal field theory lives on the boundary of the corresponding anti-de Sitter space [50] [51]. This made the conjecture more precise and has more physical aspects.

There are many ways the correspondence can be extended and generalized. Generally, it is a duality between a gravitational theory on AdS_{D+1} and a D-dimensional conformal field theory. There are also extensions in which nonconformal quantum field theories correspond to quantum gravity theories on the domain wall backgrounds, called domain wall/QFT correspondence [52].

The AdS/CFT correspondence is now one of the largest areas of study in string theory. Maldacena's paper had become the most cited papers in high energy physics by the year 2019, with more than 14,700 citations [6]. Although the correspondence has not been concretely proved, the consecutive researches provide considerable evidences of the correspondence.

In this chapter, we review Maldacena's conjecture which is a relation between $\mathcal{N}=4$ super Yang-Mills theory on $3+1$-dimensional Minkowski space and type IIB string theory on $AdS_5 \times S^5$ space. We first review $\mathcal{N}=4$ super Yang-Mills theory, a system of D3-branes in type IIB string theory, and the conjecture.

We will later assume that the AdS_4/CFT_3 correspondence, which will be used to discuss holographic RG flows, works in a similar way. Although there is no formal mathematical proof for general cases, a large number of researches from both AdS and CFT sides are in agreement to make it evident that the correspondence could be generalized to various dimensions.

2.1 $\mathcal{N}=4$ super Yang-Mills theory

For a gauge group $SU(N)$ with coupling constant g_{YM}, each $N=4$ vector supermultiplet consists of a gauge fields ${A_\mu}^A$, four Weyl spinors $\lambda^A_{\alpha a}$ and six real scalar fields X^{Ai}, with $SU(N)$ index $A=1,\ldots,N^2-1=dim(SU(N))$, spacetime index $\mu=0,\ldots,3$ with signature $\eta=diag(-1,+1,+1,+1)$, Weyl index $\alpha=1,2$, R-symmetry $SU(4)\simeq SO(6)$ indices $i=1,\ldots,6$, $a=1,\ldots,4$. $SU(N)$ generators T^A satisfy the algebra

$$[T^A,T^B]=i{f^{AB}}_C T^C. \tag{2.1.1}$$

All fields in vector multiplet can be written into a form of $N\times N$ matrices,

$$A_\mu \equiv {A_\mu}^A T^A, \quad \lambda_{\alpha a}\equiv \lambda^A_{\alpha a}T^A, \quad X^i \equiv X^{Ai}T^A. \tag{2.1.2}$$

Lagrangian for $\mathcal{N}=4$ vector multiplet is

$$\mathcal{L}_{\mathcal{N}=4}=\frac{1}{g^2_{YM}}tr\left(-\frac{1}{2}F^2_{\mu\nu}-i\bar{\lambda}^a \not{D}\lambda_a-(D_\mu X^i)^2\right. \tag{2.1.3}$$

$$\left.+C^{ab}_i\lambda_a[X^i,\lambda_b]+\text{h.c}+\frac{1}{2}[X^i,X^j]^2\right), \tag{2.1.4}$$

where the field strength tensor and covariant derivative are defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{2.1.5}$$

$$D_\mu X^i = \partial_\mu X^i + i[A_\mu, X^i]. \tag{2.1.6}$$

Note that there are a possibly θ-term $\frac{\theta}{16\pi^2} tr \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$ with constant θ, and a gauge fixing and ghost terms for quantum computations. The coefficients C_i^{ab} are needed to make a singlet form λ_a, λ_b and X^i transforming respectively as **4**, **4**, and **6** of the global $SU(4)$ R-symmetry. These coefficients can be derived by dimensional reduction of $\mathcal{N} = 1$ super Yang-Mills theory in $9 + 1$ dimensions, containing a ten-dimensional gauge field $A_M = (A_\mu, X^i)$ and a Majorana-Weyl spinor $\Psi_\sigma = (\lambda_{\alpha a})$. The 10-dimensional Lagrangian is

$$\mathcal{L}_{10} = -\frac{1}{g_{YM}^2} tr \left(-\frac{1}{2} F_{MN}^2 + i\bar{\Psi}\Gamma^M D_M \Psi \right), \tag{2.1.7}$$

with 10-dimensional gamma matrices

$$\Gamma^M \equiv (\Gamma^\mu, \Gamma^i), \tag{2.1.8}$$

where

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}, \quad \Gamma^i = \gamma^{(5)} \otimes \tilde{\gamma}^i. \tag{2.1.9}$$

The matrices γ^μ are the usual 4 Dirac matrices, $\gamma^{(5)} = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\tilde{\gamma}^i$ are 8×8 Dirac matrices of $SO(6)$. In the Weyl representation, these matrices can be written in the form of Pauli matrices σ^μ and the coefficients C^i_{ab}

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \tilde{\gamma}^i = \begin{pmatrix} 0 & C^i \\ \bar{C}^i & 0 \end{pmatrix}. \tag{2.1.10}$$

These gamma matrices satisfy ten-dimensional Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}. \tag{2.1.11}$$

Since C^i_{ab} are constructed to be invariant under $SU(4)$, the lagrangian is manifestly invariant under R-symmetry. The other manifest symmetries are gauge $SU(N)$ and Poincaré symmetries. However, there are also other symmetries. It is called the $\mathcal{N} = 4$ super Yang-Mills theory because there are 4 supersymmetries, by the transformations

$$Q^a_\alpha X^i \to C^{iab}\lambda_{\alpha b} \tag{2.1.12}$$

$$Q^a_\alpha \lambda_{\beta b} \to f_{\alpha\beta}\delta^a_b + [X^i, X^j]\epsilon_{\alpha\beta}C_{ij}{}^a{}_b \tag{2.1.13}$$

$$Q^a_\alpha \bar{\lambda}^b_{\dot{\beta}} \to C^{ab}_i \bar{\sigma}^\mu_{\alpha\dot{\beta}} D_\mu X^i \tag{2.1.14}$$

$$Q^a_\alpha A_\mu \to \sigma_{\mu\alpha}{}^{\dot{\beta}}\bar{\lambda}^a_{\dot{\beta}}, \tag{2.1.15}$$

which can be also derived from 10 dimensions. The $f_{\alpha\beta}$ is the self-dual component of $F_{\mu\nu}$ written as $f_{\alpha\beta} = F_{\mu\nu}\sigma^{\mu\nu}{}_{\alpha\beta}$. Note that Q^a_α has the R-symmetry index a. This is the general definition of an R-symmetry, a symmetry that acts on the supercharges. The fundamental relations among supercharges are

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b, \tag{2.1.16}$$

$$\{Q^a_\alpha, Q^b_\beta\} = [P_\mu, Q^a_\alpha] = 0. \tag{2.1.17}$$

It can be shown that the β-function of $\mathcal{N} = 4$ super Yang-Mills theory vanishes at all loops. For example, the β-function to one loop is given by

$$\beta(g_{YM}) = -\frac{1}{16\pi^2}\left(\frac{11}{3}C(A) - \frac{2}{3}\sum_\lambda C(\lambda) - \frac{1}{6}\sum_X C(X)\right) g^3_{YM} \tag{2.1.18}$$

where the second term on the right-hand side is a sum over all Weyl fermion and the third term is a sum over all real scalars. For $C(A) = C(\lambda) = C(X) = N$, the right hand side can be factored out and gives zero. In a theory with $\beta = 0$, there is no dynamical scale generated. Hence there are no ”particles”, and, strictly speaking, no S-matrix, although one can talk about perturbative S-matrix for a scattering of (gauge variant) gluons, gluinos, etc.. For theory with $\beta = 0$ the Poincaré group has a larger bosonic extension known as a conformal group.

In any quantum field theory, the 2-point function of an operator $\Theta(x)$, let it be scalar for simplicity, can be defined as

$$\langle 0| T\left(\Theta(x)\Theta(0)\right)|0\rangle \equiv G(x^2, \mu, g) \tag{2.1.19}$$

where μ is a renormalization scale, g is a renormalized coupling and $G(x^2, \mu, g)$ is the renormalized (finite, well-defined) correlation function depending only on $x^2 = x_\mu x^\mu$ to preserve Lorentz invariant. We have indicated that there are no dependences on all other parameters.

Let Θ have engineering dimension D_Θ, which is the mass dimension in $\hbar = c = 1$ units, e.g.

Θ	D_Θ
Φ	1
Ψ	3/2
A_μ	1
$F_{\mu\nu}$	2
Φ^3	3
F^2	4

Therefore, we can redefine $G(x^2, \mu, g)$ as

$$G(x^2, \mu, g) = (x^2)^{-D_\Theta} \hat{G}(t, g) \tag{2.1.20}$$

where the function $\hat{G}(t, g)$ depends on the dimensionless parameter $t = \frac{1}{2} log(x^2\mu^2)$. The renormalization group (RG) equation tells us that

$$\left[\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + 2\gamma(g)\right] \hat{G}(t, g) = 0. \tag{2.1.21}$$

General solution: First find a solution to the auxiliary equation,

$$\frac{d}{dt}\bar{g}(t, g) = \beta(\bar{g}(t, g)) \tag{2.1.22}$$

with initial condition of the running coupling $\bar{g}(0, g) \equiv g$. Then we can solve the RG equation for

$$\hat{G}(t, g) \equiv \xi(\bar{g}(t, g))e^{-2\int_0^t dt' \gamma(\bar{g}(t', g))}. \tag{2.1.23}$$

A critical point is given by $\beta(g^*) = 0$ for some g^* then

$$\hat{G}(t, g^*) = \xi(g^*)e^{-2\gamma(g^*)t} \tag{2.1.24}$$

$$= \xi(g^*)(\mu^2 x^2)^{-\gamma(g^*)}. \tag{2.1.25}$$

Note that the second line in the above equation came from dimensional analysis. Hence

$$G(x^2, \mu, g) = \frac{\text{constant}}{(x^2)^{D_\Theta + \gamma(g^*)}} \tag{2.1.26}$$

where $\gamma(g^*)$ acts as an anomalous dimension. Such power-law behavior is typical of conformally invariant theories.

For example, let's consider a single bosonic field $\phi(x)$ in 4 dimensions with a free action,

$$S = \frac{1}{2}\int d^4x \partial_\mu \phi \partial^\mu \phi = \frac{1}{2}\int d^4x \partial_\mu \phi \partial_\nu \phi \eta^{\mu\nu} \tag{2.1.27}$$

The form of the action, including the explicit from of the metric $\eta = diag(-1, +1, +1, +1)$, is invariant under the spacetime translations $x^\mu = x'^\mu + a^\mu$ and Lorentz transformation $x^\mu = \Lambda^\mu{}_\nu x'^\mu$ as long as $\phi(x)$ transforms as a scalar, $\phi(x) = \phi'(x')$.

If we take a scale transformation into account, let $x^\mu = Ax'^\mu$ where A is a constant. This is clearly not a Lorentz transformation. The metric is then changes

$$\eta'_{\mu\nu} = A^2 \eta_{\mu\nu}, \quad \eta_{\mu\nu} = \frac{1}{A^2}\eta'_{\mu\nu}. \tag{2.1.28}$$

Since the change is proportional to the metric itself, we also could try to compensate by rescaling the fields,

$$\frac{1}{2}\int d^4x \frac{\partial \phi(x)}{\partial x^\mu}\frac{\partial \phi(x)}{\partial x^\nu}\eta^{\mu\nu} = \frac{1}{2}\int d^4x' A^4 \frac{1}{A}\frac{\partial}{\partial x'^\mu}\phi(Ax')\frac{1}{A}\frac{\partial}{\partial x'^\nu}\phi(Ax')\eta^{\mu\nu}. \tag{2.1.29}$$

The previous equation can be brought to the same form by letting

$$A\phi(Ax') = \phi'(x'), \quad A\phi(x) = \phi'(\frac{1}{A}x). \tag{2.1.30}$$

The 2-point function is also the same,

$$\langle \phi'(x')\phi'(0)\rangle = \frac{\text{constant}}{|x'|^2} = \frac{A^2\text{constant}}{|x|^2} \tag{2.1.31}$$

$$= \langle A\phi(x)A\phi(0)\rangle\,. \tag{2.1.32}$$

This is an example where the mass dimension or scaling dimension equal to one for a scalar field.

In general, a conformal invariant condition applies if

$$\Theta'(x') = A^{\Delta_\Theta}\Theta(x) \tag{2.1.33}$$

with the scaling dimension or mass dimension $\Delta_\Theta = D_\Theta + \gamma_\Theta$. In this case, one can show

$$\langle \Theta(x)\Theta(0)\rangle = \frac{\text{constant}}{|x|^{2\Delta_\Theta}}. \tag{2.1.34}$$

This can be proved by letting

$$\langle \Theta(x)\Theta(0)\rangle = f(|x|) = f(A|x'|). \tag{2.1.35}$$

The 2-point function is then

$$\langle \Theta(x)\Theta(0)\rangle = \frac{1}{A^{2\Delta_\Theta}}\langle \Theta'(x')\Theta'(0)\rangle = \frac{1}{A^{2\Delta_\Theta}} f(|x'|) \tag{2.1.36}$$

By setting $|x'| = 1$, the scaling A give a relation for function $f(|x|)$,

$$f(A) = \frac{\text{const}}{A^{2\Delta_\Theta}} = \frac{f(1)}{A^{2\Delta_\Theta}}. \tag{2.1.37}$$

We got a lot of information by just considering a scale transformation, $x^\mu = Ax'^\mu$ where $A =$ constant. In fact, there is another change of coordinates, special conformal transformation, which is within a conformal group,

$$x^\mu = \frac{x'^\mu + a^\mu x'^2}{1 + 2x'^\nu a_\nu + a^2 x'^2}. \tag{2.1.38}$$

One can show that under the special conformal transformation, the metric transforms as $\eta_{\mu\nu} \to e^{2\omega(x)}\eta_{\mu\nu}$.

It is possible to rescale away $\omega(x)$. However, one can use the covariant expression for the action

$$S = \int d^D x\sqrt{g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{D-2}{8(D-1)}R\phi^2\right). \tag{2.1.39}$$

Let us see how it works in general. Suppose there is a coordinate transformation $x^\mu = x^\mu(x')$, under which, by covariance,

$$g'_{\mu\nu}(x') \equiv g_{\rho\sigma}(x(x'))\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} = e^{2\omega(x')}g_{\mu\nu}(x'). \tag{2.1.40}$$

In order to do this, it is convenient to think of the transformation of ϕ as a scalar $\phi(x) = \tilde{\phi}(x')$ and Poincaré rescaling $\tilde{\phi}(x') = e^{-\frac{D-2}{2}\omega(x')}\phi'(x')$. Note that for $D = 4$, $e^{\omega} = A$ is a constant, and we get $\phi'(x') = A\phi(x)$ as before.

Be general covariance, the action transforms as

$$\begin{aligned}&\int d^Dx\sqrt{g(x)}\left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \frac{D-2}{8(D-1)}R(x)\phi^2(x)\right)\\&\qquad = \int d^Dx'\sqrt{g'(x')}\left(\frac{1}{2}\partial'_\mu\tilde{\phi}(x')\partial'^\mu\tilde{\phi}(x') + \frac{D-2}{8(D-1)}R'(x')\tilde{\phi}^2(x')\right)\end{aligned} \tag{2.1.41}$$

Now if the coordinate transformation is such that $g'_{\mu\nu}(x') = e^{2\omega(x')}g_{\mu\nu}(x')$, we have

$$R'(x') = e^{-2\omega(x')}\left(R(x') - 2(D-1)\nabla'^2\omega - (D-1)(D-2)\partial'_\mu\omega\partial'^\mu\omega\right). \tag{2.1.42}$$

Letting $\tilde{\phi}(x') = e^{-\frac{D-2}{2}\omega(x')}\phi'(x')$, we get

$$\begin{aligned}&\int d^Dx'\sqrt{g}e^{(D-2)\omega}\left(\frac{1}{2}\partial'_\mu(e^{-\frac{D-2}{2}\omega}\phi')\partial'^\mu(e^{-\frac{D-2}{2}\omega}\phi')\right.\\&\quad\left.+\frac{D-2}{8(D-1)}\left(R - 2(D-1)\nabla'^2\omega - (D-1)(D-2)\partial'_\mu\omega\partial'^\mu\omega\right)\phi'^2\right)\\&= \int d^Dx'\sqrt{g}\left(\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' + \frac{D-2}{8(D-1)}R\phi'^2\right) + \int d^Dx'\sqrt{g}\left(-\frac{D-2}{2}\partial'_\mu\omega\phi'\partial'^\mu\phi' +\right.\\&\quad\left.+\frac{1}{2}(\frac{D-2}{2})^2\partial'_\mu\omega\partial'^\mu\omega\phi'^2 - \frac{D-2}{4}\nabla'^2\omega\phi'^2 - \frac{(D-2)^2}{8}(\partial'^\mu\omega)^2\phi'^2\right).\end{aligned} \tag{2.1.43}$$

Note that everything in the above equation is at x'. The second and the fourth terms in the second integral of the right-hand side cancel each other. The first and third terms also cancel by integration by part.

Now we see what the connection with $\beta = 0$ is. In theories where $\beta \neq 0$ a scale parameter will be generated in the quantum theory spoiling conformal invariance even if it was present in the classical action (e.g. in QCD with massless quarks [53].) In $\mathcal{N} = 4$ SYM conformal invariance is exact. Let's look at the generators of the conformal group. Let $A \cong 1 + \epsilon$ with $\epsilon \ll 1$. An infinitesimal scale transformation is then

$$\begin{aligned}\delta\phi(x) &= \phi'(x) - \phi(x) = (1+\epsilon)\phi(x+\epsilon x) - \phi(x)\\&\simeq \epsilon(1 + x^\mu\partial_\mu)\phi(x)\end{aligned} \tag{2.1.44}$$

which can be defined as a scale generator $D = i(1 + x^\mu \partial_\mu)$. Note that this can be generalized to $D = i(\Delta + x^\mu \partial_\mu)$. Similarly, we will get a set of conformal generators (without spin),

$$P_\mu = i\partial_\mu \tag{2.1.45}$$

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2.1.46}$$

$$D = i(\Delta + x^\mu \partial_\mu) \tag{2.1.47}$$

$$K_\mu = i(x^2 \partial_\mu - 2x_\mu x^\rho \partial_\rho - 2\Delta x_\mu) \tag{2.1.48}$$

where P_μ and $M_{\mu\nu}$ are Poincaré generators and K_μ denotes a special conformal generator. These generators make a conformal group $SO(4,2) \simeq SU(2,2)$.

The last thing to notice is that K_μ and Q^a_α do not commute. Thus we need to introduce 4 extra fermionic generators $\bar{S}^a_{\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}[K_\mu, Q^{a\alpha}]$ to close the superalgebra. Hence the global symmetry of $\mathcal{N} = 4$ super Yang-Mills theory is $PSU(2,2|4)$, whose bosonic part is a product $SU(2,2) \times SU(4)$ of a conformal group and R-symmetry group.

2.2 Type IIB String Theory on $AdS_5 \times S^5$

In flat Minkowski $9+1$-dimensional space, type II string theory, after fixing the world sheet metric, has an action

$$S_{WS} = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left(\partial_\alpha X_M \partial^\alpha X^M - i\bar{\Psi}^M \gamma^\alpha \partial_\alpha \Psi_M \right) \tag{2.2.1}$$

where $X^M(\tau, \sigma)$ and $\Psi^M(\tau, \sigma)$ denote bosonic worldsheet coordinates in $9+1$ dimensional Minkowski space and ten 2-components Majorana fermions world sheet fields respectively. The index $\alpha = 0, 1$ is now the Lorentz index on the world sheet, which can be raised/lowered with $\eta_{\alpha\beta} = diag(-1, +1)$ $(\sigma^0 \equiv \tau, \sigma^1 \equiv \sigma.)$ 2-dimensional Dirac matrices are

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.2.2}$$

Equations of motion of (2.2.1),

$$\partial_\alpha \partial^\alpha X^M = 0 \tag{2.2.3}$$

$$i\gamma^\alpha \partial_\alpha \Psi^M = 0, \tag{2.2.4}$$

have a solution

$$X^M = X_+^M(\tau+\sigma) + X_-^M(\tau-\sigma) \tag{2.2.5}$$

$$\Psi^M = \begin{pmatrix} \Psi_-^M(\tau-\sigma) \\ \Psi_+^M(\tau+\sigma) \end{pmatrix}. \tag{2.2.6}$$

If we only look at closed strings, the left/right movers $(\tau \pm \sigma)$ are completely decoupled. For a string to be closed we need $X_\pm^M$ to be periodic, with period equal to π conventionally. However, fermionic fields $\Psi_\pm^M$ can be either periodic (Ramond) or antiperiodic (Neveu-Schwarz). We can express them as mode expansions, for bosonic string,

$$X_-^M = \frac{1}{2}x^M + \alpha' p^M(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0} \alpha_n^M e^{-2in(\tau-\sigma)} \tag{2.2.7}$$

$$X_+^M = \frac{1}{2}x^M + \alpha' p^M(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0} \tilde{\alpha}_n^M e^{-2in(\tau+\sigma)}. \tag{2.2.8}$$

For fermionic strings, there are mode expansions for Ramond (R) sector,

$$\Psi_-^M = \sum_{n\in\mathbb{Z}} d_n^M e^{-2in(\tau-\sigma)} \tag{2.2.9}$$

$$\Psi_+^M = \sum_{n\in\mathbb{Z}} \tilde{d}_n^M e^{-2in(\tau+\sigma)} \tag{2.2.10}$$

and for Neveu-Schwarz (NS) sector

$$\Psi_-^M = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_n^M e^{-2in(\tau-\sigma)} \tag{2.2.11}$$

$$\Psi_+^M = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_n^M e^{-2in(\tau+\sigma)}. \tag{2.2.12}$$

Since there are two (anti)periodic conditions for each endpoint, we have four sectors, NS-NS, R-R, NS-R, and R-NS. The first two sectors are spacetime bosons and the other two are spacetime fermions. With the similar quantization, $\alpha, \tilde{\alpha}, d, \tilde{d}, b, \tilde{b}$ become creation/annihilation operators generating the quanta of spacetime fields.

Let's look at the left movers (X_-^μ and Ψ_-^μ). The ground state of the NS sector $|0\rangle_{NS}$ has mass $m^2 = -\frac{1}{2\alpha'}$, which is a tachyon and removed by GSO projection. Excited states $b^M_{-\frac{1}{2}} |0\rangle_{NS}$ has $m^2 = 0$ corresponding to massless vector fields in $\mathbf{8}_V$ representation of $SO(8)$ massless little group. The ground state of R sector $|0\rangle_R$ is massless. Other excitations of both sectors are massive.

Since elements of Clifford algebra $\{d_0^M, d_0^N\} = 2\eta^{MN}$ acting as Dirac matrices, the R ground state carries a representation of $SO(8)$, i.e. it transforms like a spinor $\mathbf{8}_S \oplus \mathbf{8}_C$, where $\mathbf{8}_S$ has a positive chirality and $\mathbf{8}_C$ is a negative one. The GSO projection removes one of these imposing a chirality condition for left/right movers. Type IIA superstring theory results if we remove the opposite chiralities from left/right movers (non-chiral theory) while type IIB superstring is obtained by removing the same chiralities (chiral theory).

Let's consider bosonic massless spectra of both theories. For type IIA theories, the massless bosonic spectra of NS-NS sector come from $\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ of $SO(8)$ representation. The field content contains a dilaton ϕ, B-fields $B_{ij} = -B_{ji}$ and metric $G_{ij} = G_{ji}$ corresponding to the trace, antisymmetric and symmetric traceless parts of the state $b^i_{-\frac{1}{2}} \tilde{b}^j_{-\frac{1}{2}} |0\rangle \otimes |0\rangle$. The R-R sector massless spectra have 1-form C_μ and 3-form $C_{\mu\nu\rho}$, corresponding to the decomposition $\mathbf{8}_S \otimes \mathbf{8}_C = \mathbf{8}_v \oplus \mathbf{56}_v$ of $SO(8)$.

NS-NS massless bosonic spectra of type IIB theories are the same as type IIA. However, the bosonic R-R state $|0\rangle_\alpha \otimes |0\rangle_\beta$ decompositions are $\mathbf{8}_C \otimes \mathbf{8}_C = \mathbf{1}' \oplus \mathbf{28}' \oplus \mathbf{35}'$ of $SO(8)$, which corresponds to scalar C, 2-form C_{ij}, and 4^+-forms C^+_{ijkl}. In particular, C^+_{ijkl} are the physical components of a 4-form $C^+_{\mu\nu\rho\lambda}$ which has a self dual field strength $dC_4^+ = *dC_4^+$, i.e. $\partial_{[\mu_1} C^+_{\mu_2 \dots \mu_5]} = \frac{1}{5!} \epsilon_{\mu_1 \dots \mu_{10}} \partial^{[\mu_6} C^{+\mu_7 \dots \mu_{10}]}$.

The fermionic degree of freedom comes from two gravitinos Ψ^i_M of positive chirality and two dilatinos λ^i of negative chirality ($i = 1, 2$). The theory is chiral but all anomalies cancel.

String theory also contains an infinite number of massive fields, but in the low energy limit, we can restrict ourselves to the massless fields with only leading interactions. Thus we are considering type IIB supergravity. For $\kappa \sim \alpha'^2 g_s, g_s = e^{\phi_\infty}$, the bosonic action is

$$\begin{aligned} S =& \frac{1}{2\kappa^2} \int \left[e^{-2\phi} \left(*R + 4d\phi \wedge *d\phi - \frac{1}{2} H_3 \wedge *H_3 \right) \right. \\ & \left. -\frac{1}{2} F_1 \wedge *F_1 - \frac{1}{2} \tilde{F}_3 \wedge *\tilde{F}_3 - \frac{1}{4} \tilde{F}_5 \wedge *\tilde{F}_5 - \frac{1}{2} C_4^+ \wedge H_3 \wedge F_3 \right] \end{aligned} \tag{2.2.13}$$

where

$$F_1 = dC_0, \quad H_3 = dB, \quad F_3 = dC_2, \tag{2.2.14}$$

$$F_5 = dC_4^+, \quad \tilde{F}_3 = F_3 - C_0 \wedge H_3, \tag{2.2.15}$$

$$\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B \wedge F_3 \tag{2.2.16}$$

and, strictly speaking, it is $\tilde{F}_5$ which is self-dual, $\tilde{F}_5 = *\tilde{F}_5$. Note that this makes the kinetic term $*\tilde{F}_5 \wedge \tilde{F}_5$ vanished. Strictly speaking, we cannot write an action but we will do it anyway and impose $\tilde{F}_5 = *\tilde{F}_5$ at the level of the equation of motions.

The action 2.2.13 is in the "string frame." One can go to the "Einstein frame" by setting $\hat{G}_{MN} = e^{-\phi/2} G_{MN}$, with

$$S = \frac{1}{2\kappa^2} \int \sqrt{\hat{G}} \hat{R} + \dots \tag{2.2.17}$$

For all our purposes this will not make a difference since the solutions we are interested in have $\phi = 0$. Note that ϕ is only the deviation from the asymptotic value ϕ_∞ that we have already used to define g_s and absorbed into κ.

Certainly, flat spacetime $G_{MN} = \eta_{MN}$ with all other fields vanishing is a solution that preserves all supersymmetry, 32 supercharges for ten-dimensional $\mathcal{N} = 2$ theories. Recall that from supersymmetric quantum field theory, for a chiral superfield $\Phi = \varphi + \sqrt{2}\psi\theta + F\theta^2$, supersymmetry transformations are

$$\delta_\epsilon \varphi = [\epsilon Q, \varphi] = \epsilon\psi, \quad \delta_{\epsilon^\dagger} \varphi = [\epsilon^\dagger Q^\dagger, \varphi] = 0 \tag{2.2.18}$$

$$\delta_\epsilon \psi = [\epsilon Q, \psi] = \epsilon F, \quad \delta_{\epsilon^\dagger} \psi = [\epsilon^\dagger Q^\dagger, \psi] = -i\sigma^\mu \epsilon^\dagger \partial_\mu \varphi \tag{2.2.19}$$

$$\delta_\epsilon F = [\epsilon Q, F] = 0, \quad \delta_{\epsilon^\dagger} F = [\epsilon^\dagger Q^\dagger, F] = i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi \tag{2.2.20}$$

where ϵ is 2-component constant Grassmann Weyl spinor. If any of these things get a vacuum expectation value (VEV) then the supersymmetry is broken. Since unbroken supersymmetry implies $Q\left|0\right\rangle = Q^\dagger\left|0\right\rangle = 0$, to preserve Lorentz invariance, the only one that can possibly get a VEV is F. In this case, $F \neq 0$ for broken supersymmetric theory. So we must always check fermion supersymmetry transformations.

Let η be a 16 component Weyl constant Grassmann spinor, fermion transformations are

$$\delta_\eta \lambda = i\Gamma^M \eta^* P_M - \frac{i}{24}\Gamma^{MNP}\eta G_{MNP} + \text{fermions} \tag{2.2.21}$$

$$\begin{aligned}\delta_\eta \Psi_M &= D_M\eta + \frac{i}{480}\Gamma^{NPQRS}\Gamma_M F_{NPQRS} \\ &\quad + \frac{1}{96}({\Gamma_M}^{PQR}G_{PQR} - 9\Gamma^{NP}G_{MNP})\eta^* + \text{fermions}.\end{aligned} \tag{2.2.22}$$

We will only be interested in solutions only with $\tilde{F}_5 \neq 0$, $F_5 \equiv \tilde{F}_5$. In this case $\delta\lambda^i = 0$ is trivially satisfied since it does not depend on F_5 whereas $\delta\Psi^i_M = 0$ reduces to

$$D_M\eta + \frac{i}{480}\Gamma^{NPQRS}\Gamma_M F_{NPQRS}\eta = 0 \tag{2.2.23}$$

with

$$D_M\eta = (\partial_M + \frac{1}{4}{\omega_M}^{AB}\Gamma_{AB})\eta \tag{2.2.24}$$

$${\omega_M}^{AB} = e^{NA}\partial_{[M}e^B_{N]} - e^{NB}\partial_{[M}e^A_{N]} - e^{RA}e^{SB}{e^C}_M\partial_{[R}e_{S]C}. \tag{2.2.25}$$

Note that ${\omega_M}^{AB}$ and ${e^C}_M$ are spin connection and frame (zehnbein) in ten dimensions.

Obviously flat $9+1$ Minkowski space with $F_5 = 0$ gives $\partial_M\eta = 0$ for 32 real supersymmetries. However, there are other solutions. The one of interest here is

$$ds^2 = ds^2_{AdS_5} + ds^2_{S^5} \tag{2.2.26}$$

$$F_5 = \frac{1}{L}\text{Vol}_{AdS_5} + \frac{1}{L}\text{Vol}_{S^5} \tag{2.2.27}$$

where $ds^2_{AdS_5}$ is the metric of AdS_5 with radius L, Vol_{AdS_5} is its volume 5-form and similarly for S^5. More explicitly, we can write in Poincaré coordinates,

$$ds^2_{AdS_5} = \frac{r^2}{L^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu\right) + \frac{L^2}{r^2}dr^2 \tag{2.2.28}$$

$$ds^2_{S^5} = L^2 d\Omega_5 = L^2 \hat{g}_{\alpha\beta}d\theta^\alpha d\theta^\beta \tag{2.2.29}$$

$$F_{[0123r]} = \frac{r^3}{L^4}, \quad F_{\alpha_1\ldots\alpha_5} = L^4\sqrt{\hat{g}}\epsilon_{\alpha_1\ldots\alpha_5}. \tag{2.2.30}$$

Note that $F_5 = *_{10}F_5$. With this explicit form, we are able to investigate the numbers of solutions of the equation $\tilde{D}_M\eta = 0$ where $\tilde{D}_M = D_M + \theta(F)$, not just $\partial_M\eta = 0$. However, we should compute the integrability condition

$$[\tilde{D}_M, \tilde{D}_N]\eta \equiv \Xi_{MN}\eta \tag{2.2.31}$$

where an antisymmetric matrix $\Xi_{MN} = -\Xi_{NM}$ is a matrix acting on η similar to $[D_M, D_N] = \frac{1}{4}R_{MNPQ}\Gamma^{PQ}\eta$. The number of independent solutions to $\tilde{D}_M\eta = 0$ is the same as the number of zero eigenvalues of $\Xi_{MN}\eta = 0$. For our case, the maximal supersymmetry implies that $\Xi_{MN} = 0$ and all $32_{\mathbb{R}}$ of η are allowed.

2.2.1 D3-brane

The existence of RR potentials, $C_0, C_2, C_4^\dagger$ in type IIB and C_1, C_3 in type IIA, has been known since the beginning of superstring theory. However, the fundamental string is neutral under these fields. It was not clear how to describe the states charged under such fields until 1995 [54]. In the same way as electron couples to a 1-form electromagnetic potential via

$$e\int_{\mathbb{R}} d\tau A_\mu(x(\tau))\dot{x}^\mu(\tau). \tag{2.2.32}$$

An n-form potential $C_{\mu_1\ldots\mu_n}$ couples to a $p = n-1$ dimensional extended object spanning a $p+1 = n$ dimensional world-volume

$$T_p\int_{\Sigma_p\times\mathbb{R}} d\tau d^p\sigma C_{\mu_1\ldots\mu_{p+1}}(X(\tau,\vec{\sigma}))\partial_\tau X^{[\mu_1}\ldots\partial_{\sigma^p}X^{\mu_{p+1}]} \tag{2.2.33}$$

Note that p is even in type IIA and odd in type IIB. The objects coupled to the RR fields are known as Dp-branes.

There are two types of descriptions for Dp-brane. First, they are solutions to the supergravity equation [55],

$$ds^2 = H_p^{-1/2}(r)\left(-(dx^0)^2 + \sum_{i=1}^{p}(dx^i)^2\right) + H_p^{1/2}(r)\left(dr^2 + r^2 d\Omega_{8-p}\right) \quad (2.2.34)$$

$$F_{p+2} = dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p \wedge dH_p^{-1}(r), \quad e^{\phi} = H_p^{\frac{3-p}{4}}(r) \quad (2.2.35)$$

where

$$H_p(r) = 1 + k\frac{g_s N \alpha'^{\frac{7-p}{2}}}{r^{7-p}} \quad (2.2.36)$$

with k being a numerical factor. Note that this solution preserves half of the supersymmetry (16 supercharges).

Another description is Dp-brane being hypersurfaces on which open strings are allowed to end. To explain this, let $X^0, X^1, \ldots, X^p$ lie on Dp-branes, as in Figure 2.1. They will have usual Neumann-Neumann boundary conditions. In other words, they can move freely on Dp-branes. However, the open string ends are fixed to the branes. The directions $X^{p+1}, \ldots, X^9$ have Dirichlet-Dirichlet boundary conditions. This makes the left and right movers coupled into standing waves. The type of modding, integer for all X^M and integer/half-integer for Ψ^M in the R/NS sector respectively, remains the same. However, their mode expansions are different since there is no p-dependence in the direction perpendicular to Dp-brane. For example, mode expansions of a bosonic string are,

$$X^\mu = \frac{1}{2}x^\mu + \alpha' p^\mu \tau + \text{oscillators}; \mu = 0, 1, \ldots, p \quad (2.2.37)$$

$$X^i = \frac{1}{2}x^i - d^i \sigma + \text{oscillators}; i = p+1, \ldots, 9. \quad (2.2.38)$$

Note that $d^i = 0$ if all Dp-branes are on top of each other. Moreover, GSO projection works as before, removing the tachyon, and so on. Hence, the world-volume theory is that of maximally super Yang-Mills theory in $p+1$ dimensions, as we can map

$$b^\mu_{-\frac{1}{2}}\left|p^0 \ldots p^p; IJ\right\rangle_{NS} \to {A^\mu}_I{}^J \quad (2.2.39)$$

$$b^i_{-\frac{1}{2}}\left|p^0 \ldots p^p; IJ\right\rangle_{NS} \to {X^i}_I{}^J \quad (2.2.40)$$

$$\left|p^0 \ldots p^p; IJ\right\rangle_R \to {\lambda_{\alpha I}}^J \quad (2.2.41)$$

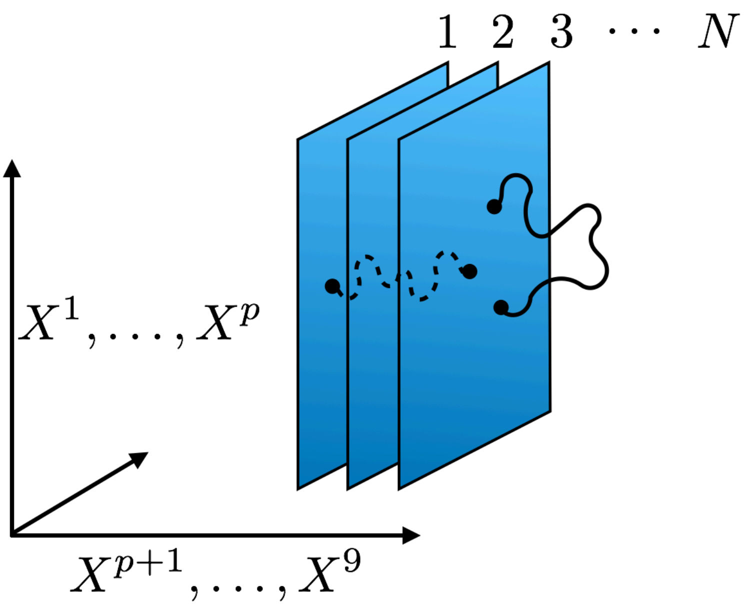


Figure 2.1: The stringy point of view of the Dp-branes with ends of open strings fixed on brane directions

with $g_{YM}^2 \propto g_S \alpha'^{\frac{p-3}{4}}$.

Now we are interested in the case when $p = 3$ or the IIB $D3$-brane. From the string point of view, we have directly $3+1$ dimensional $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ and $g_{YM}^2 = g_S$. From the supergravity point of view, we have $e^\phi = 1$ which implies $g_s = e^{\phi_\infty}$ being a constant everywhere. The metric of N-stack of $D3$-branes is

$$ds^2 = \left(1 + 4\pi \frac{g_S N \alpha'^2}{r^4}\right)^{-1/2} (dx^\mu)^2 + \left(1 + 4\pi \frac{g_S N \alpha'^2}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2). \tag{2.2.42}$$

The illustration of this metric is shown in Figure 2.2. For the region $\frac{g_s N \alpha'^2}{r^4} \ll 1$, the metric become flat and reduces to $9+1$ dimensional Minkowski space

$$ds^2 = (dx^\mu)^2 + dr^2 + r^2 d\Omega_5^2, \tag{2.2.43}$$

where the RHS represents $\mathbb{R}^{3,1}$ and $\mathbb{R}^6$ in spherical coordinates, respectively. For

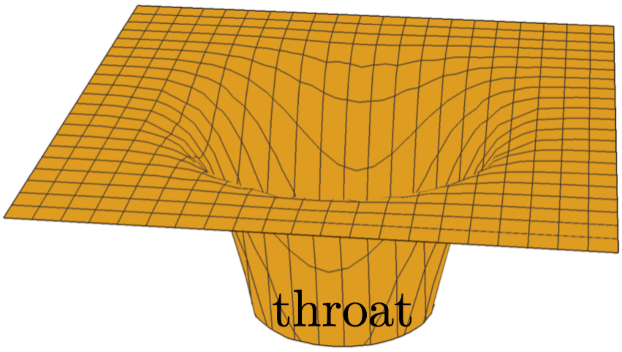


Figure 2.2: Illustration of the metric of N-stack of D3-brane (2.2.42) where the region $\frac{g_sN\alpha'^2}{r^4} \ll 1$ reduces to $9+1$-dimensional Minkowski space, and the region $\frac{g_sN\alpha'^2}{r^4} \gg 1$ gives $AdS_5 \times S^5$ throat

the region $\frac{g_sN\alpha'^2}{r^4} \gg 1$, the metric is much more interesting,

$$\begin{aligned} ds^2 &= \frac{r^2}{\sqrt{4\pi g_s N\alpha'^2}}(dx^\mu)^2 + \frac{\sqrt{4\pi g_s N\alpha'^2}}{r^2}(dr^2 + r^2 d\Omega_5^2) \\ &= \frac{r^2}{\sqrt{4\pi g_s N\alpha'^2}}(dx^\mu) + \frac{\sqrt{4\pi g_s N\alpha'^2}}{r^2}dr^2 + \sqrt{4\pi g_s N\alpha'^2}d\Omega_5^2 \end{aligned} \tag{2.2.44}$$

which is clearly a metric of $AdS_5 \times S^5$ with both of their radius $L^2 = \sqrt{4\pi g_s N\alpha'^2}$. Hence the stack of $D3$-branes has made its appearance as $AdS_5 \times S^5$ and the $\mathbb{R}^{3,1}$ is acting as its boundary.

Note that we can rescale $r \to L^2 r$ so that the metric (2.2.44) be

$$\begin{aligned} ds^2 = L^2 d\hat{s}^2 &= L^2\left(r^2(dx^\mu)^2 + \frac{dr^2}{r^2} + d\Omega_5^2\right) \\ &= L^2\left(\frac{(dx^\mu)^2 + dz^2}{z^2} + d\Omega_5^2\right) \end{aligned} \tag{2.2.45}$$

where recall that $z = 1/r$.

Let's discuss on a naïve argument for the correspondence. In the string picture, the total action consists of $D3$-brane action, interactions, and a bulk action,

$$S_{tot} = S_{brane} + S_{int} + S_{bulk}. \tag{2.2.46}$$

$D3$-brane action contains gauge fields of $SU(N)$ in $\mathcal{N}=4$ theory due to charges they carried by the open string ends. S_{int} is an interaction term between graviton and gauge fields. The bulk action contains the low energy effective action (supergravity) and other higher-order terms. To summarize,

$$S_{brane} = S_{\mathcal{N}=4} + S_{\text{higher order}} \\ \left(\sim \int d^4xF^2 + \dots\right) + \left(\sim \alpha'^2 \int d^4xF^4 + \dots\right) \tag{2.2.47}$$

$$S_{int} \sim \kappa d^4xhF^2 + \dots \tag{2.2.48}$$

$$S_{bulk} = S_{SUGRA} + S_{higherorder/massive} \\ \left(\sim \int (\partial h)^2 + \kappa h(\partial)^2 + \dots\right) + \left(\sim \kappa^6 \int (\partial h)^8) + \dots\right) \tag{2.2.49}$$

In low energy limits, the brane action is approximated by $S_{\mathcal{N}=4}$, no interaction and the bulk action will be approximated by free supergravity action.

On the other hand, in the brane picture at low energy. As in the previous discussion, there are two regions which strings can live, at the boundary $\mathbb{R}^{3,1}$ where $\frac{g_sN\alpha'^2}{r^4} \ll 1$ and in the $AdS_5 \times S^5$ throat where $\frac{g_sN\alpha'^2}{r^4} \gg 1$. The string at the boundary is decoupled from the brane since its cross-section on the brane is vanishing $\sigma \sim \omega^3L^8 \sim 0$ [56], where ω is the incident energy. Strings in $AdS_5 \times S^5$ are infinitely red shifted ($g_{tt} = 1/\sqrt{1+L^4/r^4} \to 0$ as $r \to 0$) and have no enough energy to come out. Thus, these two regions are decoupled form each other.

Hence, by comparing both pictures,

$$S_{\mathcal{N}=4} + S_{\text{free SUGRA}} \leftrightarrow S_{AdS_5\times S^5} + S_{\text{free SUGRA}}, \tag{2.2.50}$$

we can naïvely map $\mathcal{N}=4$ super Yang-Mills theory on $\mathbb{R}^{3,1}$ to type IIB gravity theory on $AdS_5 \times S^5$.

2.3 The Conjecture

In this section, we will discuss on a one-to-one map between two physical theories, $\mathcal{N}=4$ super Yang-Mills theory with gauge group $SU(N)$ and coupling constant

$g_{YM}(\theta)$ and type IIB string theory on $AdS_5 \times S^5$ with $\int_{S^5} F_5 = N$ and $g_s = g_{YM}^2$ $(C_0 \equiv \theta)$.

One might object that there is no one who knows how to formulate the AdS side, so this whole thing seems empty. However, there are various highly non-trivial limits of the above conjecture where many things can be tested. If nobody ever succeeds in formulating it or it turns out that there are ambiguities in the formulation, we might turn the conjecture around and use the super Yang-Mills side to define the AdS side.

To see how it all comes together, we must understand where the various theories live. First of all, let's take a stack of $D3$-branes and go deep into the throat. The five-dimensional metric then looks like

$$ds^2 \simeq \frac{r^2}{L^2}(dx^\mu)^2 + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \tag{2.3.1}$$

$$= L^2 \left(\frac{1}{z^2}(dx^2 + (dx^\mu)^2) + d\Omega_5{}^2 \right) \tag{2.3.2}$$

Thus, to the question where are the branes, one could answer that we are inside it so it is everywhere. Remember that when we illustrated the picture of a brane as a hyperplane this is an approximation. In fact, the brane curves spacetime at $r \neq 0$ as well. This is the space where type IIB closed strings live. They move inside $AdS_5 \times S^5$.

The next question is "where is $\mathcal{N} = 4$ super Yang-Mills?" One might say that the open strings are attached to the branes so it would be the region $r = 0$, but this is wrong. The $\mathcal{N} = 4$ super Yang-Mills theory lives on the conformal boundary of AdS, $r \to \infty$ or $z \to 0$. This seems like we are going away from the brane, but it is not really. In a sense, it is a matter of the order of limits. Once we are in the AdS limit, we cannot move away from the brane. We did throw away the Minkowski region.

We will argue that there is no other place $\mathcal{N} = 4$ could live. The conformal boundary of AdS_5 is four-dimensional Minkowski space $Mink_4$. The isometry of AdS_5, $SO(4,2)$, acts as a conformal group on $Mink_4$. As in A.1, we will write

AdS_5 as hyperboloid,

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = L^2 \tag{2.3.3}$$

Let rescale $Y_i = \Lambda y_i$, 2.3.3 is then

$$-y_{-1}^2 - y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 = \frac{L^2}{\Lambda^2}. \tag{2.3.4}$$

Let $\Lambda \to \infty$ as we move to the conformal boundary, the RHS of 2.3.4 is going to zero. Note that we can still write the identification $y_i \sim s y_i$ for $s > 0$. If we set $u_\pm = y_{-1} \pm y_4$, our embedded space is then

$$u_+ u_- - y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0. \tag{2.3.5}$$

We can use the equivalence to set $u_- = 1$, or in other words, we can chose a representative

$$\left(\frac{u_+}{u_-}, 1, \frac{y_0}{u_-}, \frac{y_1}{u_-}, \frac{y_2}{u_-}, \frac{y_3}{u_-}\right) \equiv \left(\frac{u_+}{u_-}, 1, x_0, x_1, x_2, x_3\right) \tag{2.3.6}$$

with $\frac{u_+}{u_-} = x_0^2 - x_1^2 - x_2^2 - x_3^2$ and x_μ independent from each other. The metric is simply $ds^2 = (dx^\mu)^2$. Going to the previous coordinates,

$$Y_{-1} = L \cosh\rho \cos\tau, \ldots \tag{2.3.7}$$

we see the $Y_i \to \infty$ limit corresponds to $\rho \to \infty$ or $\xi \to \frac{\pi}{2}$. Thus the type IIB theory lives in the bulk and $\mathcal{N} = 4$ lives on the boundary.

To refine the conjecture and make some tests, let's make a Wick rotation in Y_0 or x^0. An AdS_5 is turned into a hyperbolic plane or topologically a ball $H_5 \simeq B^5$. Four-dimensional Minkowski space $\mathbb{R}^{1,3}$ changes into $\mathbb{R}^4$, and $\mathbb{R} \times S^3$ changes into $S^4 = \partial B^5$. The AdS_5 metric is now

$$ds^2 = L^2 \frac{(dx^\mu)^2 + dz^2}{z^2} \tag{2.3.8}$$

which is in a form of Euclidean space. The metric is invariant under the isometry

$$\begin{pmatrix} z \\ x^\mu \end{pmatrix} = A \begin{pmatrix} z' \\ x^{\mu'} \end{pmatrix}. \tag{2.3.9}$$

The metric cannot take the boundary form by simply taking the limit $z \to 0$ since it will blow up. However, if we rescale the metric by a conformal transformation z^2, we can take the boundary limit,

$$\lim_{z\to 0} z^2 ds^2 = (dx^\mu)^2 = ds^2_{boundary}. \tag{2.3.10}$$

Note that $ds^2_{boundary}$ is not invariant under 2.3.9. It transforms as

$$ds^2_{boundary} = A^2 ds'^2_{boundary}. \tag{2.3.11}$$

2.3.1 Scalar Field in AdS Space

Let's consider a scalar field in AdS_5. Its action is

$$S = \frac{1}{2}\int dz d^4x \sqrt{g}\left(g^{MN}\partial_M\phi\partial_N\phi + m^2\phi^2\right). \tag{2.3.12}$$

By the definition of a scalar, it transforms as

$$\phi'(z', x') = \phi(z, x), \tag{2.3.13}$$

$$\phi'(Az, Ax) = \phi(z, x). \tag{2.3.14}$$

The equation of motion obtained from the action 2.3.12 is Klein-Gordon equation,

$$(-\Box + m^2)\phi = 0. \tag{2.3.15}$$

By using

$$g_{MN} = \frac{1}{z^2}\delta_{MN}, \quad g^{MN} = z^2\delta^{MN}, \quad \sqrt{g} = \sqrt{\left(\frac{1}{z^2}\right)^5} = \frac{1}{z^5}, \tag{2.3.16}$$

we expand the 4-gradient up to a subleading term

$$\Box = \frac{1}{\sqrt{g}}\partial_M\sqrt{g}g^{MN}\partial_N = z^5\partial_z z^3\partial_z + z^2\partial_{mu}\partial^{mu}. \tag{2.3.17}$$

At $z \approx 0$, the equation 2.3.15 can be approximated to

$$-z^5\partial_z z^{-3}\partial_z\phi + m^2\phi = 0. \tag{2.3.18}$$

We cannot hope to fix a scalar at boundary $\phi(z=0,x)$ because $\phi(z,x)$ is not a constant. Instead, let $\phi \sim z^{\alpha}$ and set $L=1$, then from 2.3.18 we get

$$-\alpha(\alpha-4)z^{\alpha}+m^2z^{\alpha}=0 \tag{2.3.19}$$

so that

$$-\alpha(\alpha-4)+m^2=0. \tag{2.3.20}$$

Hence, the solutions are $\alpha_{\pm}=2\pm\sqrt{4+m^2}$. Note that in general, we will get an equation $-\alpha(\alpha-d)+m^2=0$ for AdS_{d+1}.

Thus there are two linearly independent solutions, which depend on $z^{\alpha_{\pm}}$, at $z\to 0$. One linear combination is smooth in the bulk, so generically we can write $\phi(z,x)\sim z^{\alpha_-}+$subleading term. Therefore the best we can do is to define

$$\phi_{boundary}(x)=\lim_{z\to 0} z^{-\alpha_-}\phi(z,x). \tag{2.3.21}$$

By a scalar transformation $\phi'(z',x')=\phi(z,x)$, the transformation $(z,x)=(Az',Ax')$ gives

$$\begin{aligned}\phi_{boundary}(x) &= \lim_{z\to 0}(Az')^{-\alpha_-}\phi(Az',Ax')\\ &= \lim_{z'\to 0}(Az')^{-\alpha_-}\phi'(z',x')\\ &= A^{-\alpha_-}\lim_{z'\to 0} z'^{-\alpha_-}\phi'(z',x')\\ &= A^{-\alpha_-}\phi'_{boundary}(x').\end{aligned} \tag{2.3.22}$$

Thus $\phi'_{boundary}(x')=A^{\alpha_-}\phi_{boundary}(x)$. $\phi_{boundary}$ behaves as an object of scaling dimension α_-. We interpret it as a source on the boundary coupling to a gauge-invariant local operator $\Theta(x)$ by

$$\int dx\phi_{boundary}(x)\Theta(x). \tag{2.3.23}$$

A scale invariance of the interaction requires the scaling transformation of the operator to be $\Theta'(x')=A^{4-\alpha_-}\Theta(x)$,

$$\begin{aligned}\int dx\phi_{boundary}(x)\Theta(x) &= \int dx' A^4A^{-\alpha_-}\phi'_{boundary}(x')A^{-(4-\alpha_-}\Theta'(x')\\ &= \int dx'\phi'_{boundary}(x')\Theta'(x').\end{aligned} \tag{2.3.24}$$

Thus a scalar field at the boundary $\phi_{boundary}$ is dual to an operator Θ of mass dimension $\Delta = 4 - \alpha_- = \alpha_+ = 2 + \sqrt{4+m^2}$.

Given a specific $\phi_{boundary}(x)$ we can actually find the bulk field $\phi(z,x)$ by noticing that for an appropriate range of α,

$$\lim_{z\to 0} \frac{z^{4-2\alpha}}{(z^2+|x|^2)^{4-\alpha}} = C_\alpha \delta^4(x). \tag{2.3.25}$$

A constant C_α is obtained by integration

$$\int d^4x \frac{z^{4-2\alpha}}{(z^2+|x|^2)^{4-\alpha}} = \int d^4y \frac{z^{8-2\alpha}}{z^{2(4-\alpha)}(1+|y|^2)^{4-\alpha}} = \Omega_4 \int_0^\infty dr \frac{r^3}{(1+r^2)^{4-\alpha}} \tag{2.3.26}$$

where we use $x^\mu = zy^\mu$. Therefore

$$C_\alpha = \frac{\pi^2}{6 - 5\alpha + \alpha^2}. \tag{2.3.27}$$

Let's consider

$$\phi(z,x) = \frac{1}{C_\alpha} \int dx' \frac{z^{4-\alpha_-}\phi_{boundary}(x')}{(z^2+|x-x'|^2)^{4-\alpha_-}} \tag{2.3.28}$$

$$= z^{\alpha_-}\phi_{boundary}(x) \tag{2.3.29}$$

and check that it obeys the equation of motion

$$\begin{aligned}
&\left(-z^5\partial_z z^{-3}\partial_z - z^2\nabla^2 + m^2\right) \frac{z^{4-\alpha_-}}{(z^2+|x|^2)^{4-\alpha_-}} \\
&= \left(4(4-\alpha_-) - (4-\alpha_-)^2 + m^2\right) \frac{z^{4-\alpha_-}}{(z^2+|x|^2)^{4-\alpha_-}} \\
&= \left(4\alpha_+ - \alpha_+^2 + m^2\right) \frac{z^{4-\alpha_-}}{(z^2+|x|^2)^{4-\alpha_-}} \\
&= 0
\end{aligned} \tag{2.3.30}$$

at $x \neq 0$. Note that the last step is equal to zero by definitions of $\alpha_\pm$.

At this point, we get

$$\phi(z,x) \simeq \int dx' \frac{z^\Delta \phi_{boundary}(x')}{(z^2+|x-x'|^2)^\Delta}. \tag{2.3.31}$$

If we put it back into the action 2.3.12 and integrate by parts, we find

$$\begin{aligned} S &= \frac{1}{2}\int_{|z|\geq\epsilon} dzd^4x\partial_M\left(\phi\sqrt{g}g^{MN}\partial_N\phi\right) + \text{surface term} \\ &\simeq \int d^4x\phi(\epsilon,x)\frac{1}{\epsilon^5}\epsilon^2\,\partial_z\phi(z,x)|_{z=\epsilon}\,. \end{aligned} \tag{2.3.32}$$

To calculate the last step, we use $\phi(\epsilon,x)\simeq\epsilon^{4-\Delta}\phi_{boundary}(x)$,

$$\begin{aligned} \partial_z\phi(z,x)|_{z=\epsilon} &\cong \int dx'\frac{\Delta z^{\Delta-1}\phi_{boundary}(x')}{(z^2+|x-x'|^2)^{\Delta}} + \text{higher order}|_{z=\epsilon} \\ &\cong \Delta\epsilon^{\Delta-1}\int dx'\frac{\phi_{boundary}(x')}{|x-x'|^{2\Delta}} + \text{higher order}. \end{aligned} \tag{2.3.33}$$

All together, the action is then

$$S\simeq\int d^4xd^4x'\frac{\phi_{boundary}(x)\phi_{boundary}(x')}{|x-x'|^{2\Delta}}. \tag{2.3.34}$$

Note that we can compute a 2-point function

$$\frac{\delta}{\delta\phi_{boundary}(x)}\frac{\delta}{\delta\phi_{boundary}(x')}\,|_{\phi_{boundary}=0}\,e^{-S}\simeq\frac{1}{|x-x'|^{2\Delta}} \tag{2.3.35}$$

$$\simeq\left\langle\Theta_\Delta(x)\Theta_\Delta(x')\right\rangle. \tag{2.3.36}$$

We can extend this to an idea of the semiclassical partition function (on-shell) of ϕ as

$$Z[\phi_0(x),\epsilon]=e^{-S[\phi]}\,|_{\phi\to\epsilon^{\alpha_-}\phi_0} \tag{2.3.37}$$

where ϵ^{α_-} acts as IR regulator. This is equal to the generating function of Θ

$$\left\langle e^{\int dx\phi_0(x)\Theta(x)}\right\rangle_\epsilon \tag{2.3.38}$$

with $\Delta=2+\sqrt{4+m^2}$ for a scalar. Note that ϵ is a UV regulator.

2.4 AdS_5/CFT_4 Summary

We have review the AdS/CFT correspondence for the type IIB supergravity from coincident $D3$-branes and $\mathcal{N}=4$ supersymmetric Yang-Mills theory in flat space. The background created by a stack of N $D3$-branes can be written as

$$ds^2=h(r)^{-1/2}(dx^\mu)^2+h(r)^{1/2}(dr^2+r^2d\Omega_5^2) \tag{2.4.1}$$

$$g_sF_5=(1+*)d^4x\wedge dh^{-1}(r) \tag{2.4.2}$$

where

$$h(r) = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N \alpha'^2. \tag{2.4.3}$$

In the near horizon limit, $r \ll L$, we can approximate $h(r) \approx L^4/r^4$. The space-time geometry becomes $AdS_5 \times S^5$. Thus, we can describe a system of N-stack $D3$-branes in the near horizon limit by type IIB string theory on $AdS_5 \times S^5$ space-time.

In the world volume point of view, let's consider one $D3$-brane in flat space. It is a free gauge theory in $3+1$ dimensions in a low energy limit. The Lagrangian describing the gauge theory is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\sum_{i=1}^{6}(\partial_\mu \phi^i)^2 + \text{fermionic terms}. \tag{2.4.4}$$

This corresponds to $\mathcal{N} = 4$ supersymmetric $U(1)$ gauge theory. Since it is a free theory, the moduli space is simply $\mathbb{R}^6$.

The stack of N parallel $D3$-branes, in the low energy limit, gives the theory with gauge group $U(N)$. The lagrangian is

$$\mathcal{L} = Tr\left[-\frac{1}{4}F_{\mu\nu}^2 - (D_\mu \phi^i)^2 + \frac{g^2}{4}[\phi^i, \phi^j]^2\right] + \text{fermionic terms}. \tag{2.4.5}$$

The scalar fields ϕ^i transform in the adjoint representation of the gauge group $U(N)$. The moduli space is $(\mathbb{R}^6)^N/S_N$ where S_N is the permutation group of N elements.

The gauge group $U(N)$ is the same as $U(1) \times SU(N)$. One might think of $U(1)$ subgroup as a center of mass of a stack of N $D3$-brane. In a low energy limit, when all fields are neutral, the remaining gauge group is $SU(N)$ gauge group. Therefore this system is basically an $SU(N)$ supersymmetric Yang-Mills theory.

At this point, a stack of N $D3$-branes has two points of view. In a string theory point of view, the near horizon limit, it is a type IIB string theory on $AdS_5 \times S^5$ with N units of 5-form Ramond-Ramond flux. From the gauge theory point of view, the low energy limit gives an $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with $SU(N)$ gauge group. Thus, there are two descriptions of the same theory, 4-dimensional gauge field, and the 10-dimensional strings.

2.4.1 Extension to $AdS_5 \times Y^5$

We can generalize the AdS_5/CFT_4 correspondence to more general geometry $AdS_5 \times X^5$, where the X^5 is an internal compact 5-dimensional manifold. One of the interesting cases is when the $d\Omega_5$ in 2.4.2 is the base of the cone, as we place a stack of $D3$-branes at the tip of a Ricci-flat cone. This is an Einstein manifold Y^5, which gives the geometry $AdS_5 \times Y^5$ in the $r \to 0$ limit. If the cone is a Calabi-Yau space, the base of the cone is a Sasaki-Einstein manifold which preserves $\mathcal{N} = 1$, supersymmetry of $AdS_5 \times S^5$ [57].

Let's repeat the same procedure for a $D3$-brane at the tip of the cone. Consider the Calabi-Yau cone is a conifold C^4 which is a complex manifold described by a quadratic equation of complex variable u_i,

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0. \tag{2.4.6}$$

With this condition, it has 3 complex dimensions or 6 real dimensions. This is a cone with $S^2 \times S^3$ being its base [58]. A Ricci flat, Kähler metric of this conifold can be written as

$$ds^2 = dr^2 + r^2 ds^2_{T^{11}} \tag{2.4.7}$$

where $ds^2_{T^{11}}$ denotes the metric of Einstein space T^{11} which has the topology $S^2 \times S^3$,

$$ds^2_{T^{11}} = \frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin\theta_i^2 d\phi_i^2) + \frac{1}{9}(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i)^2. \tag{2.4.8}$$

The tip of the cone is singular. If we put a $D3$-brane at the singularity of the conifold, gauge theory in the low energy limit is an $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $U(1)_1 \times U(1)_2$. There are four chiral fields, z_1, z_2, w_1, w_2, in the theory transforming under gauge group $U(1)_1 \times U(1)_2$. The fields z_1, z_2 have charge $+1$ under $U(1)_1$ and -1 under $U(1)_2$, and in the opposite way for w_1, w_2.

There is no superpotential for one $D3$-brane case, there are no F-term equations. Thus we have to solve only the D-term equation to determine the moduli space. The moduli space of vacua can be obtained by imposing the vanishing of D-terms and dividing by the gauge group. The D-term of one $D3$-brane on the tip of this cone is in the form of auxiliary field D,

$$\mathcal{L}_D = \frac{D^2}{g^2} + D(|z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2)^2. \tag{2.4.9}$$

As usual, integrating out the D-term results in the scalar potential

$$V_D = g^2(|z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2)^2 \tag{2.4.10}$$

which gives the condition

$$|z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2 = 0, \tag{2.4.11}$$

and the gauge invariance gives $z_i \sim z_i e^{i\alpha}$ and $w_i \sim w_i e^{-i\alpha}$ with α being a gauge parameter. Thus this conifold is basically $(SU(2) \times SU(2))/U(1)$, which is often referred to as T^{11}. Thus there are two flavor symmetries transforming the chiral fields z_i, w_i as doublets and charged under $U(1)$ which is the anti-diagonal subgroup of $U(1)_1 \times U(1)_2$. Note that the matter chiral fields are neutral under the diagonal subgroup.

In the same way, this can be generalized to a stack of N $D3$-branes at the tip of the cone. It is an $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $U(N)_1 \times U(N)_2$. The chiral fields z_i, w_i transform in the fundamental representation of $U(N)_1$ and $U(N)_2$, respectively. As before, z_i, w_i transform in the anti-fundamental representation of $U(N)_2$ and $U(N)_1$, respectively.

Since in this case there are multiple $D3$-branes, we have to modify the superpotential. The only quartic function in the superfields and invariant under $SU(2) \times SU(2)$ is

$$Tr(\epsilon^{AC}\epsilon^{BD} z_A w_B z_C w_D) = Tr(z_1 w_1 z_2 w_2 - z_1 w_2 z_2 w_1). \tag{2.4.12}$$

There are two $U(1)$ factors from the gauge group $U(N)_1 \times U(N)_2$. The diagonal subgroup decouples and the anti-diagonal subgroup becomes a global symmetry in low energy limit since the gauge coupling flows to zero. Thus we have an $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with gauge group $SU(N)_1 \times SU(N)_2$. This is corresponding to its stringy picture, the type IIB string theory on $AdS_5 \times T^{1,1}$ with N Ramond-Ramond flux on $T^{1,1}$.

In the following chapters, we will generalize this principle to AdS_4/CFT_3 correspondence to study solutions from supergravity with the AdS_4 vacuum interpreted as renormalization group flows in CFT_3. Similar to the AdS_5/CFT_4 description with a stack of $D3$-branes in the previous discussion, $M2$-branes play an important role in describing the AdS_4/CFT_3 correspondence [57].

We will study supergravity solutions in the form of

$$\begin{aligned} ds^2 &= e^{2A(r)}(-dt^2 + dx_\mu^2) + dr^2 \\ \phi_i &= \phi_i(r), \end{aligned} \tag{2.4.13}$$

which we have assumed $A(r) \sim r/L$, and $\phi_i(r) \sim$ constant at large r. The geometries in the form (2.4.13) arise in the gauged supergravity descriptions of certain vacuum states of $N = 4$ super-Yang-Mills theory, and presume to be the gravity duals of the renormalization group flows emerging from relevant deformations in the CFT [59] [60] [61] [62] [63] [64] [65] [66] [67] [68] [69] [70] [71] [72] [73] [74] [75] [76].

The scalar potential $V(\phi_i)$ plays an important role in the study of holographic renormalization group flows. It should satisfy some conditions in order to have solutions with AdS vacua of radius L. First, the $V(\phi_i)$ must have a stationary point, without loss of generality, at $\phi_i = 0$. For gravity theory with AdS_4 background, we get

$$V(\phi_i = 0) = -\frac{3}{L^2}. \tag{2.4.14}$$

For $\phi_i \neq 0$, the $V(\phi_i)$ takes the form near $r = 0$

$$V(\phi_i) = -\frac{3}{L^2} + \frac{1}{2}m^2\phi_i^2 + \dots. \tag{2.4.15}$$

Note that the scalar masses obtained from (2.4.15) should satisfy the BF bound, $m^2L^2 \geq -\frac{9}{4}$ in order to maintain the stability of the solution.

It should be noted here that solving for supergravity solutions is sometimes lead to singular solutions. However, these singularities do not necessarily imply that the solutions are wrong, they mean the supergravity description fails. We have to look into the dual description instead. The criterion to determine if a singularity in the IR is allowed or not is proposed in [77], which states that large curvature in geometries are allowed only if the scalar potential is bounded above in the solution. The motivation for this criterion comes from a necessary condition for a family of black hole solutions, with horizons hiding the singularity. The Hawking temperature of the horizon is identified with a finite temperature in the dual field theory. A naked singularity may indicate the absence of a well-defined dual field theory or a field theory in an unphysical vacuum state. The criterion for singularities in the IR also discussed in [78] with the condition that the g_{tt} is bounded such that it gives a proper energy excitation in the dual field theory. For example, a Schwarzschild-like solution with a negative mass should not be allowed.

CHAPTER III

Gauged Supergravity Basic

In this chapter, we review the construction of gauged supergravity in four dimensions. Note that we will only focus on the structure that can be used with $2 < N \leq 4$ supersymmetry. This chapter mainly follows [79]. We begin with a discussion of bosonic Lagrangian of an ungauged four-dimensional supergravity coupled to a non-linear σ-model together with its supersymmetry transformations. We then discuss a procedure of gaugings some of the global symmetries of the ungauged Lagrangian in the notion of an embedding tensor.

3.1 Ungauged supergravity

Pure supergravity in four dimensions allows only for $N \leq 8$ number of supersymmetry. Theories with more supersymmetries must contain a massless particle with spin higher than 2. The bound $N \leq 8$ also imposes a condition on higher dimensional theories, as the $N = 8$ supergravity can be found from a dimensional reduction of eleven-dimensional supergravity. This restricts the number of supercharge not to be greater than 32.

Scalar fields in extended supergravity can be described by a non-linear sigma model. Supergravities with $N > 2$ have enough supersymmetries to determine the geometry of the scalar manifold. The scalar fields in these theories are described by a G/H coset space sigma-model. The group G is the global symmetry group of the theory, which is a non-compact group, generally. The group $H \subset G$ is its

N	G/H
3	$\frac{SU(3,n)}{SU(3)\times SU(n)}$
4	$\frac{SL(2,R)}{SO(2)} \times \frac{SO(6,n)}{SO(6)\times SO(n)}$
5	$\frac{SU(1,5)}{SU(1)\times SU(5)}$
6	$\frac{SO^*(12)}{U(1)\times SU(6)}$
8	$\frac{E_{7(7)}}{SU(8)}$

Table 3.1: Scalar manifolds of four-dimensional $N > 2$ extended supergravities

maximal compact subgroup, which is in the form

$$H = H_R \times H_m, \tag{3.1.1}$$

where H_R is the supersymmetry automorphism group or R-symmetry group, $H_R = U(N)$ for $N < 8$, and H_m is a compact group acting on the matter fields. The theories with $N > 4$ have no matter multiplet, thus $H = H_R$. The scalar manifolds of four-dimensional $N > 2$ extended supergravities are shown in Table 3.1.

The bosonic field content of four-dimensional supergravity theories with $N > 2$ consists of the metric $g_{\mu\nu}$, scalar field ϕ^i, and abelian vector fields A^M_μ. The dynamics of these fields is described by the bosonic Lagrangian, which is given by

$$e^{-1}\mathcal{L}_{\mathrm{B}} = \frac{1}{2}R - \frac{1}{2}G_{st}(\phi)\partial_\mu\phi^s\partial^\mu\phi^t - \frac{1}{4}I_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}F^{\mu\nu\Sigma} + \frac{1}{8}e^{-1}R_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}, \tag{3.1.2}$$

where $e = \sqrt{|\det g_{\mu\nu}|}$. The kinetic term of the graviton is described by the Einstein-Hilbert term where R is defined by contracting the Riemann tensor, $R = e^\mu_a e^\nu_b R_{\mu\nu}{}^{ab}$. The abelian field strength is defined by

$$F^M_{\mu\nu} = \partial_\mu A^M_\nu - \partial_\nu A^M_\mu. \tag{3.1.3}$$

The matrices $R_{\Lambda\Sigma}$ and $I_{\Lambda\Sigma}$ are real and imaginary parts of the matrix $\mathcal{N}_{\Lambda\Sigma}$, $\mathcal{N}_{\Lambda\Sigma} = R_{\Lambda\Sigma} + iI_{\Lambda\Sigma}$. The indices $s, t = 1, \ldots, n_s$ indicate all scalar fields from both supergravity and vector multiplets. The indices $\Lambda, \Sigma = 1, \ldots, n_v$ indicate all vector fields from both multiplets. Note that there is no scalar potential in an ungauged supergravity with $N > 1$. A non-trivial scalar potential can be introduced without breaking supersymmetry through the gauging procedure.

The scalar fields ϕ^s are described by a non-linear σ-model. The positive definite metric $G_{st}(\phi)$ describes the space whose coordinates are the scalar fields, called target space. It encodes the geometry of the scalar field space. For $N > 2$ supersymmetry, consistent couplings of the scalar fields to the vectors and fermions impose constraints on $G_{st}(\phi)$ such that the scalar manifold is in the form of coset manifold $\mathcal{M}_{\text{scalar}} = G/H$ in which G is a semisimple group and H is a maximal compact subgroup of G.

Fermionic fields transform under the holonomy group H, which contains the R-symmetry group H_R, of $\mathcal{M}_{\text{scalar}}$. Since the action of H is local on the scalar manifold, the covariant derivatives of the fermion fields require a composite connection Q_μ, which is defined in terms of the scalar fields ϕ^s and their derivatives $\partial_\mu \phi^s$. Consistency of the transformation property of the fermionic fields gives an additional structure on the scalar manifold.

The coset space can be parametrized by a coset representative $L(\phi)$, which transforms under left and right multiplications of $g \in G$ and $h(x) \in H$, respectively,

$$L(\phi) \to L'(x) = gL(x)h(x). \tag{3.1.4}$$

Note that the coset representative $L(\phi)$ can be parametrized by $n_s = dim(G) - dim(H)$ scalar fields. This defines $dim(G) - dim(H)$ coordinates of the coset space. It should be noted here that we can make use of the left-invariant current

$$J_\mu = L^{-1}\partial_\mu L \in \mathfrak{g} \equiv LieG, \tag{3.1.5}$$

to see the coset structure, as it can be decomposed to $Q_\mu \in \mathfrak{h}$ and $P_\mu \in \mathfrak{k}$,

$$J_\mu = Q_\mu + P_\mu, \tag{3.1.6}$$

where $\mathfrak{h} \equiv LieH$ and $\mathfrak{k}$ denotes its complement. This implies that the Lie algebra $\mathfrak{g}$ can be decomposed into $\mathfrak{h}$ and $\mathfrak{k}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}. \tag{3.1.7}$$

The Lie algebra $\mathfrak{g}$ is then described by Lie brackets,

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{h}. \tag{3.1.8}$$

This implies that the coset space $\mathfrak{k}$ represents the subgroup H, and the generators in the $\mathfrak{h}$ and $\mathfrak{k}$ are compact and non-compact generators, accordingly.

The geometry of the scalar manifold can be described by a Maurer-Cartan 1-form $\Omega \in \mathfrak{g}$,

$$\Omega = L^{-1}dL, \tag{3.1.9}$$

which satisfies the Maurer-Cartan equation,

$$d\Omega + \Omega \wedge \Omega = 0. \tag{3.1.10}$$

The Ω can be decomposed into a connection $Q \in \mathfrak{h}$ and a vielbein $P \in \mathfrak{k}$,

$$\Omega = Q + P. \tag{3.1.11}$$

We can define a covariant derivative of $L(\phi)$ by including the connection Q of the internal symmetry H,

$$DL = dL - LQ = LP. \tag{3.1.12}$$

The covariant derivative of the vielbein P satisfies

$$DP = dP + Q \wedge P + P \wedge Q = 0. \tag{3.1.13}$$

Covariant derivative for any field $\Phi(x)$, which transforms under H, is given by

$$D_r\Phi = \partial_r\Phi + Q_r \circ \Phi, \tag{3.1.14}$$

where $Q_r \circ \Phi$ is the connection Q acts on the field Φ. The scalar Lagrangian can be written in terms of P_μ,

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}eTr(P_\mu P^\mu), \tag{3.1.15}$$

which is invariant under global G and local H transformations,

$$\delta L = gL - Lh(x). \tag{3.1.16}$$

The currents Q_μ and P_μ transform as

$$\delta Q_\mu = -\partial_\mu h + [h, Q_\mu], \quad \delta P_\mu = [h, P_\mu], \tag{3.1.17}$$

which implies that the connection Q_μ acts as a gauge field under H. The current P_μ transforms in a linear representation of H, thus it can be used to construct H-invariant kinetic terms (3.1.15), and may be used to construct H-invariant fermionic interaction terms.

There are several ways to parametrize the coset representative $L(\phi)$. In this dissertation, we use unitary parametrization, which the matrix $L(\phi)$ is taken of the form

$$L = exp\{\phi^s Y_s\}, \tag{3.1.18}$$

where the non-compact generators Y_s span the space $\mathfrak{k}$, called coset generators. In this parametrization, the scalars ϕ^s transform in a linear representation of $H \subset G$. The global H-invariant of the Lagrangian is then manifest.

Fermionic fields in supergravities are depending on the numbers of supersymmetries. In $2 < N \leq 4$ supergravity theories, there are fermionic fields from supergravity and vector multiplets. For $N = 3$, there are $\psi_{\mu A}$ and χ_{ABC} from the supergravity multiplet, and λ_{Ai}, λ_{ABCi} from vector multiplets. For $N = 4$, there are $\psi_{\mu A}$ and χ_{ABC} from the supergravity multiplet and λ_{Ai} from vector multiplets. Note that the indices $A, B = 1, \ldots, N$ indicate the group $H_R = U(N)$, for $3 \leq N \leq 6$. The indices $i, j = 1, \ldots, n$ indicate the fundamental indices of $H_m = SU(n)$ for $N = 3$, and $H_m = SO(n)$ for $N = 4$.

Since fermionic fields transform under the group H, but not transform under the group G, the fermionic Lagrangian, which invariant under H, should be in the form of covariant derivatives with the connection Q, as the group H acts as a gauge symmetry. The covariant derivative for arbitrary fermions is given by

$$\mathscr{D}_\mu \psi = D_\mu \psi + Q_\mu \circ \psi, \tag{3.1.19}$$

where D_μ is the spacetime covariant derivative involving spin connection and Christoffel connection, and $Q_\mu = Q_s \partial_\mu \phi^s$. The term $Q_\mu \circ \psi$ is the connection Q

acts on the ψ. The fermionic kinetic term is then given by

$$\begin{aligned}
\mathcal{L}_{\text{f-kinetic}} =& i\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}^A_\mu \gamma_\nu \mathscr{D}_\rho \psi_{A\sigma} - \bar{\psi}_{A\mu}\gamma_\nu \mathscr{D}_\rho \psi^A_\sigma) \\
&- \frac{1}{12}e(\bar{\chi}^{ABC}\gamma^\mu \mathscr{D}_\mu \chi_{ABC} + \bar{\chi}_{ABC}\gamma^\mu \mathscr{D}_\mu \chi^{ABC}) \\
&+ \frac{1}{2}e(\bar{\lambda}^{Ai}\gamma^\mu \mathscr{D}_\mu \lambda_{Ai} + \bar{\lambda}_{Ai}\gamma^\mu \mathscr{D}_\mu \lambda^{Ai}). \qquad (3.1.20)
\end{aligned}$$

The kinetic terms for fermionic fields $\lambda_{ABCi} = \epsilon_{ABC}\lambda_i$, χ in the $N = 3$ case can be obtained similarly.

The supersymmetry transformations for bosonic and fermionic fields can be schematically described in the form

$$\begin{aligned}
\delta\text{Boson} &= \sum_{\text{Fermions}} \text{Fermion} \cdot \epsilon, \\
\delta\text{Fermion} &= \sum_{\text{Bosons}} \partial\text{Boson} \cdot \epsilon. \qquad (3.1.21)
\end{aligned}$$

The general form of the supersymmetry transformations for the bosonic and fermionic fields in the ungauged theory are given by

$$\delta e^a_\mu = \bar{\epsilon}^A \gamma^a \psi_{A\mu} + \bar{\epsilon}_A \gamma^a \psi^A_\mu, \qquad (3.1.22)$$

$$\delta A^\Lambda_\mu = \frac{1}{2} f^\Lambda{}_{AB} O^{AB}_\mu + f^\Lambda{}_i O^i_\mu + \text{h.c.}, \qquad (3.1.23)$$

$$P^{ABCD}_s \delta\phi^s = \Sigma^{ABCD}, \qquad (3.1.24)$$

$$P^{iAB}_s \delta\phi^s = \Sigma^{IAB}, \qquad (3.1.25)$$

$$\delta\psi_{A\mu} = \mathscr{D}_\mu \epsilon_A + \frac{i}{8} F^-_{\rho\sigma AB}\gamma^{\rho\sigma}\gamma_\mu \epsilon^B, \qquad (3.1.26)$$

$$\delta\chi_{ABC} = P_{sABCD}\partial_\mu \phi^s \gamma^\mu \epsilon^D + \frac{3}{4} i F^-_{\mu\nu[AB}\gamma^{\mu\nu}\epsilon_{C]}, \qquad (3.1.27)$$

$$\delta\lambda_{Ai} = P_{siAB}\partial_\mu \phi^s \gamma^\mu \epsilon + \frac{1}{4} i F^-_{\mu\nu i}\gamma^{\mu\nu}\epsilon_A. \qquad (3.1.28)$$

Note that for $N = 3$, there is an additional supersymmetry transformation for λ_i,

$$\delta\lambda_i = \frac{1}{2} P_{siAB}\partial_\mu \phi^s \gamma^\mu \epsilon_C \epsilon^{ABC}. \qquad (3.1.29)$$

The composite matrices P^{ABCD}_s and P^{iAB}_s describe the spin-0 states in the gravitational and vector multiplets, respectively. They are components of the $\mathfrak{k}$-valued vielbein one-form P_s. The tensors Σ^{ABCD}, Σ^{IAB}, O^{AB}_μ, and O^i_μ are the components of the $\mathfrak{k}$-generator Σ^c. Note that the explicit forms of the fermionic supersymmetry transformations for $N = 3$ and $N = 4$ will be given in the corresponding chapters.

3.2 Gauged supergravity

We now discuss the gauging of the supergravity theory, which is obtained by promoting a subgroup $G_0 \subset G$ to a local symmetry group gauged by the vector fields. The Lagrangian and the supersymmetry transformations are changed in order to have the same supersymmetries as the original ungauged theory. In this section, we will illustrate the construction of a gauged supergravity in an electric frame. We will discuss a covariant formalism in which the possible gaugings are encoded into an object call embedding tensor.

The gauging procedure starts by choosing a subgroup G_0 of the global symmetry G of the ungauged Lagrangian. The condition for gauging is that the dimension of a gauge group G_0 should not greater than the number of vector fields in the theory, $dimG_0 \leq n_v$. A subset $\{A^{\hat{\Lambda}}\}$ of the vector fields become the gauge vectors corresponding to the generators $X_{\hat{\Lambda}}$ of G_0. Let Ω_g be a gauge connection,

$$\Omega_{g\mu} = gA^{\hat{\Lambda}}_{\mu} X_{\hat{\Lambda}}, \tag{3.2.1}$$

where g is the gauge coupling. The $X_{\hat{\Lambda}}$ are generators of gauge group G_0, which satisfy the algebra

$$[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = f_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} X_{\hat{\Gamma}}. \tag{3.2.2}$$

The structure constant $f_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}}$ must obey the Jacobi identity,

$$f_{[\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} f_{\hat{\Delta}]\hat{\Gamma}}{}^{\hat{\Pi}} = 0, \tag{3.2.3}$$

in order to make the gauge group G_0 a closed subgroup.

In order to get the local invariance of the Lagrangian under G_0, ordinary derivatives are replaced by gauge-covariant derivatives,

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu - \Omega_{g\mu} = \partial_\mu - gA^{\hat{\Lambda}}_{\mu} X_{\hat{\Lambda}}. \tag{3.2.4}$$

This is where we introduce the minimal coupling of the vectors to the other fields. For homogeneous scalar manifold, the Maurer-Cartan 1-form Ω introduced in (3.1.9) is redefined by replacing the derivative,

$$\Omega_\mu = L^{-1}\partial_\mu L \to \hat{\Omega}_\mu \equiv L^{-1}\mathcal{D}_\mu L = \hat{P}_\mu + \hat{Q}_\mu. \tag{3.2.5}$$

The gauged vielbein and connection are given by

$$\hat{P}_\mu = P_\mu - gA_\mu^{\hat{\Lambda}} P_{\hat{\Lambda}}, \quad \hat{Q}_\mu = Q_\mu - gA_\mu^{\hat{\Lambda}} Q_{\hat{\Lambda}}, \tag{3.2.6}$$

where $P_{\hat{\Lambda}}$ and $Q_{\hat{\Lambda}}$ are the projections of $L^{-1}X_{\hat{\Lambda}}L$ onto $\mathfrak{k}$ and $\mathfrak{h}$, respectively. The gauge-covariant derivative is acting on a generic fermionic field ψ as

$$\mathscr{D}_\mu\psi = D_\mu\psi + \hat{Q}_\mu \circ \psi. \tag{3.2.7}$$

Generally, the gauging procedure can be done by replacing all P_μ and Q_μ with their corresponding gauged forms,

$$P_\mu \to \hat{P}_\mu, \quad Q_\mu \to \hat{Q}_\mu. \tag{3.2.8}$$

3.2.1 Embedding Tensor

In gauging some subgroup $G_0 \subset G$, a G-invariant object Θ is introduced by defining gauge generators as linear combinations of the global symmetry generators t_a of G,

$$X_{\hat{\Lambda}} = \Theta_{\hat{\Lambda}}{}^a t_a. \tag{3.2.9}$$

Note that the indices $\hat{\Lambda} = 1, \ldots, n_v$ and $a = 1, \ldots, dimG$ are adjoint indices of the gauge group and the global symmetry group, respectively. The dimension of the gauge group is given by the rank of $\Theta_{\hat{\Lambda}}{}^a$. We expect that rank of the gauge group should be less than the number of physical vector fields of the ungauged theory.

However, the gauging is not generally valid for an arbitrary choice of $\Theta_{\hat{\Lambda}}^a$. It needs to satisfy a set of constraints for consistency. We will not work out a complete set of constraints, however, it is useful to have some examples. One of the constraints is obtained by the requirement that the embedding tensor must be invariant under the action of the subgroup G_0. This implies a quadratic constraint in $\Theta_{\hat{\Lambda}}{}^{a}$,

$$\delta_{\hat{\Sigma}}\Theta_{\hat{\Lambda}}{}^{a} = \Theta_{\hat{\Sigma}}{}^{b}\delta_b\Theta_{\hat{\Lambda}}{}^{b} = \Theta_{\hat{\Sigma}}\hat{(t)}_{\hat{\Lambda}}{}^{\hat{\Gamma}}\Theta_{\hat{\Gamma}}{}^{a} + \Theta_{\hat{\Sigma}}{}^{b} f_{bc}{}^{a}\Theta_{\hat{\Lambda}}{}^{c} = 0, \tag{3.2.10}$$

which is equivalent to $[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = -X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}}X_{\hat{\Gamma}}$, where $X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} = \Theta_{\hat{\Lambda}}{}^{a}(t_a)_{\hat{\Sigma}}{}^{\hat{\Gamma}}$, due to the fact that generators in the adjoint representation can be written in terms of the structure constants. This also implies that $f_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} = -X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}}$ satisfy the Jacobi identity,

$$f_{[\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Pi}} f_{\hat{\Gamma}]\hat{\Pi}}{}^{\hat{\Delta}} = 0. \tag{3.2.11}$$

In summary, generators of the global symmetry satisfy a Lie algebra,

$$[t_a, t_b] = f_{ab}{}^{c} t_c, \tag{3.2.12}$$

where $f_{ab}{}^{c}$ is the structure constant of the global symmetry. The gauge generators also satisfy a Lie algebra $[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = f_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} X_{\hat{\Gamma}}$ since it forms a closed subgroup.

3.2.2 Lagrangian and supersymmetry transformations

In gauged supergravity, we modify the ungauged Lagrangian by promoting derivatives to covariant derivatives. However, the modified action is not invariant under the original supersymmetry transformation since there are additional gauge fields from the covariant derivatives. In order to restore the supersymmetry of the theory, we need to add coupling terms between scalars and fermions at first order in g, called Yukawa term,

$$e^{-1}\mathcal{L}_{\text{Yukawa}} = g(-2\bar{\psi}^{A}_{\mu}\gamma^{\mu\nu}\psi^{B}_{\nu}S_A B + \bar{\lambda}^{I}\gamma^{\mu}\psi_{A\mu}N_I{}^{A} + \bar{\lambda}^{I}\lambda^{J}M_{IJ}) + \text{h.c.}. \tag{3.2.13}$$

This term is also called a fermion mass-like term. The tensors S_{AB}, $N_I{}^{A}$, and M_{IJ} can be identified in term of T-tensor. By adding the Yukawa term, the fermionic supersymmetry transformations are modified to the form,

$$\delta\psi_{\mu A} = \nabla_\mu \epsilon_A - g S_{AB} \gamma_\mu \epsilon^B + \ldots, \tag{3.2.14}$$

$$\delta\lambda_I = \hat{P}^A_{\mu I} \gamma^\mu \epsilon_A + g N_I{}^A \epsilon_A + \ldots. \tag{3.2.15}$$

The tensors S_{AB}, $N_I{}^A$, and M_{IJ} are sometimes called fermion-shift matrices. Note that the bosonic supersymmetry transformations are the same as in the ungauged theory.

Besides promoting derivatives to covariant derivatives and adding Yukawa term, there are some terms left in the supersymmetry transformation of the Lagrangian. These terms can be canceled by adding a scalar potential term,

$$V(\phi) = \frac{1}{N} g^2 (N_I{}^A N^I{}_A - 12 S^{AB} S_{AB}). \tag{3.2.16}$$

This scalar potential plays an important role as its value at the given critical point will be identified as the cosmological constant, which is crucial to the study of holographic RG flows in the following chapters.

At this point, we are able to study supergravity solutions involving only the metrics and scalars non-vanishing. This leads to non-trivial scalar potential and fermionic supersymmetry transformations. The bosonic supersymmetry transformations are satisfied automatically with vanishing fermions. We can obtain a set of differential equations from the fermionic supersymmetry transformations as functions of scalar fields. For a given gauge group and a group of preserved symmetry, the differential equations can be solved for solutions interpolating between critical points of the scalar potential. In the following chapters, we study solutions interpolating between AdS_4 vacua, $V(\phi) < 0$, which are later interpreted as RG flows between CFTs.

CHAPTER IV

RG Flows from Four-dimensional $N = 3$ Gauged Supergravity

In this chapter, we first review four-dimensional $N = 3$ gauged supergravity. We will then discuss possible semisimple gauge groups allowed by supersymmetry. In each possible gauge group, the scalar potential and its possible supersymmetric AdS_4 vacua will be identified along with their possible holographic RG flows.

Among $N > 2$ supersymmetry, it has been found that there is a unique non-maximal AdS_4 solution with unbroken $N = 3$ supersymmetry from compactifying eleven-dimensional supergravity [28]. The internal manifold of this is a tri-sasakian N^{010} with $SU(2) \times SU(3)$ isometry. The corresponding Kaluza-Klein spectrum has been given in [29], and the structure of $N = 3$ multiplets is investigated in [30]. The possible $N = 3$ SCFT dual to M-theory compactified on $AdS_4 \times N^{010}$ is studied in [31]. The gravity dual to $N = 3$ SCFT is also studied in many aspects [32] [33] [34] [35] [36] [37]. These result in a significant match between $N = 3$ SCFT and the AdS_4 solution from the compactification of eleven-dimensional configurations in M-theory.

The eleven-dimensional supergravity compactified on the $AdS_4 \times N^{010}$ can be described by an $N = 3$, $SU(3) \times SO(3)$ gauged supergravity as an effective theory [29] [30]. The theory with eight vector multiplets is constructed in [38] [39] [40]. Various deformations and supersymmetric vacua have been identified in [41]. The eleven-dimensional configurations to these solutions might be obtained through a consistent reduction ansatz if exists.

4.1 $N = 3$ Gauged Supergravity

We first review four-dimensional $N = 3$ gauged supergravity. We follow most notations used in [38]. However, we use the mostly plus signature $(- + ++)$.

Four-dimensional $N = 3$ supersymmetry contains twelve supercharges. The field content of the supergravity multiplet is

$$(e^a_\mu, \psi_{\mu A}, A_{\mu A}, \chi) \tag{4.1.1}$$

which are given by a graviton e^a_μ, three gravitini ψ^i_μ, three vectors A^i_μ, and a spinor field χ. We denote indices $\mu, \nu = 0, \ldots, 3$ as spacetime indices, $a, b = 0, \ldots, 3$ as tangent space indices and $A, B = 1, 2, 3$ as $SU(3)_R$ R-symmetry triplets.

The supergravity multiplet can couple to n vector multiplets. Each of vector multiplet field content

$$(A_\mu, \lambda_A, \lambda, z_A)^i \tag{4.1.2}$$

contains a vector field A_μ, four spinor fields (λ_A, λ), which are a triplet and a singlet of $SU(3)_R$, and three complex scalar fields z_A. The indices $i, j = 1, \ldots, n$ denote each of the vector multiplets. Spinors in the theory are subject to chirality projection,

$$\begin{aligned}
\gamma_5 \psi_{\mu A} &= \psi_{\mu A}, \qquad \gamma_5 \chi = \chi, \qquad \gamma_5 \lambda_A = \lambda_A, \qquad \gamma_5 \lambda = -\lambda, \\
\gamma_5 \psi^A_\mu &= -\psi^A_\mu, \qquad \gamma_5 \lambda^A = -\lambda^A.
\end{aligned} \tag{4.1.3}$$

When coupled to n number of vector multiplet, $N = 3$ gauged supergravity has $3n$ complex or $6n$ real scalar fields. The scalar fields ${z_A}^i$ parametrized the coset space $SU(3, n)/SU(3) \times SU(n) \times U(1)$. The coset can be parametrized by the coset representative ${L(z)_\Lambda}^\Sigma$, which transforms under the global $SU(3.n)$ and the local $SU(3) \times SU(n) \times U(1)$ by the left and right multiplication, respectively. The $SU(3) \times SU(n)$ indices $\Lambda, \Sigma = 1, \ldots, n+3$ can split into (A, i), $A = 1, ;3$, $i = i, \ldots, n$, which are the fundamental $SU(3) \times SU(n)$. Accordingly, the coset representative ${L_\Lambda}^\Sigma$ can also be split into $({L_\Lambda}^A, {L_\Lambda}^i)$. The coset representative and its inverse are related by

$$(L^{-1})_\Lambda{}^\Sigma = J_{\Lambda\Pi} J^{\Sigma\Delta} (L_\Pi{}^\Delta)^* \qquad (4.1.4)$$

where $J_{\Lambda\Sigma}$ is an $SU(3,n)$ invariant tensor defined by

$$J_{\Lambda\Sigma} = J^{\Lambda\Sigma} = (\delta_{AB}, -\delta_{ij}). \qquad (4.1.5)$$

There are $n+3$ vector fields, n from vector multiplet, and three from supergravity multiplet. Together with their $n+3$ magnetic dual, they transform under the fundamental representation of $SU(3,n)$. We can also write the vector fields of the fundamental $SU(3,n)$ in the form of the fundamental $SU(3)$ and $SU(n)$, $A_\Lambda = (A_A, A_i)$. Note that he Lagrangian, which contains $n+3$ of vector fields, is invariant only under the $SO(3,n)$ subgroup of the duality symmetry $SU(3,n)$. There is an argument in [38] that the possible gauge groups are a subgroup of $SO(3,n)$, which transform the vector fields A_Λ to themselves.

The bosonic Lagrangian of the $N=3$ gauged supergravity, with only scalars and the metric non-vanishing, is given by

$$e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{1}{2}P_\mu{}^{iA} P^\mu{}_{Ai} - V, \qquad (4.1.6)$$

where $P_i{}^A$ is the vielbein of the $SU(3,n)/SU(3)\times SU(n)\times U(1)$ coset, and given by the (A,i)-components of the Mourer-Cartan one-form $\Omega_i{}^A = (\Omega_A{}^i)^*$. Note that the Mourer-Cartan one-form, in the presence of gaugings, is defined by

$$\Omega_\Lambda{}^\Pi = (L^{-1})_\Lambda{}^\Sigma dL_\Sigma{}^\Pi + (L^{-1})_\Lambda{}^\Sigma f_\Sigma{}^{\Omega\Gamma} A_\Omega L_\Gamma{}^\Pi. \qquad (4.1.7)$$

The scalar potential is given by

$$\begin{aligned} V &= -2S_{AC}S^{CA} + \frac{2}{3}\mathcal{U}_A\mathcal{U}^A + \frac{1}{6}\mathcal{N}_{iA}\mathcal{N}^{iA} + \frac{1}{6}\mathcal{M}^{iB}{}_A\mathcal{M}_{iB}{}^A \\ &= \frac{1}{8}|C_{iA}{}^B|^2 + \frac{1}{8}|C_i{}^{PQ}|^2 - \frac{1}{4}\left(|C_A{}^{PQ}|^2 - |C_P|^2\right) \qquad (4.1.8)\end{aligned}$$

where the fermion shift functions $S_{AB}, \mathcal{U}^A, \mathcal{N}_{iA}$ and $\mathcal{M}_{iA}{}^B$ are defined by

$$\begin{aligned}
S_{AB} &= \frac{1}{4}\left(\epsilon_{BPQ}C_A{}^{PQ} + \epsilon_{ABC}C_M{}^{MC}\right) \\
&= \frac{1}{8}\left(C_A{}^{PQ}\epsilon_{BPQ} + C_B{}^{PQ}\epsilon_{APQ}\right), \\
\mathcal{U}^A &= -\frac{1}{4}C_M{}^{MA}, \qquad \mathcal{N}_{iA} = -\frac{1}{2}\epsilon_{APQ}C_i{}^{PQ}, \\
\mathcal{M}_{iA}{}^B &= \frac{1}{2}(\delta_A^B C_{iM}{}^M - 2C_{iA}{}^B).
\end{aligned} \tag{4.1.9}$$

Note that these functions are written in terms of "boosted structure constants",

$$C^\Lambda{}_{\Pi\Gamma} = L_{\Lambda'}{}^\Lambda (L^{-1})_\Pi{}^{\Pi'}(L^{-1})_\Gamma{}^{\Gamma'} f_{\Pi'\Gamma'}{}^{\Lambda'} \qquad \text{and} \qquad C_\Lambda{}^{\Pi\Gamma} = J_{\Lambda\Lambda'}J^{\Pi\Pi'}J^{\Gamma\Gamma'}(C^{\Lambda'}{}_{\Pi'\Gamma'})^*, \tag{4.1.10}$$

where $C_P = -C_{PM}{}^M$.

In order to find supersymmetric solutions, we need fermionic supersymmetry transformations,

$$\delta\psi_{\mu A} = D_\mu \epsilon_A + S_{AB}\gamma_\mu \epsilon^B, \tag{4.1.11}$$

$$\delta\chi = \mathcal{U}^A \epsilon_A, \tag{4.1.12}$$

$$\delta\lambda_i = -P_{i\mu}^{\ A}\gamma^\mu \epsilon_A + \mathcal{N}_{iA}\epsilon^A, \tag{4.1.13}$$

$$\delta\lambda_{iA} = -P_{i\mu}^{\ B}\gamma^\mu \epsilon_{ABC}\epsilon^C + \mathcal{M}_{iA}{}^B \epsilon_B\,. \tag{4.1.14}$$

Note that we are considering the case where only scalars and the metric non-vanishing. By letting all fermions vanish, the bosonic supersymmetry transformations are automatically satisfied. The covariant derivative on the supersymmetry parameter ϵ_A is defined by

$$D\epsilon_A = d\epsilon_A + \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon_A + Q_A{}^B\epsilon_B + \frac{1}{2}nQ\epsilon_A, \tag{4.1.15}$$

where $Q_A{}^B$ and Q are the $SU(3)\times U(1)$ composite connections. These connections can be obtained from components of the Mourer-Cartan one-form together with the $SU(n)$ composite connections $Q_i{}^j$,

$$\Omega_A{}^B = Q_A{}^B - n\delta_A^B Q, \quad \Omega_i{}^j = Q_i{}^j + 3\delta_i^j Q. \tag{4.1.16}$$

Note that the connections have properties $Q_A{}^A = Q_i{}^i = 0$.

4.1.1 Possible Gauge Groups

The idea of possible gaugings when we are dealing with σ-model is that we can gauge the symmetries inside the isometry of the scalar manifold. In this case, the gauge group G must be a subgroup of the isometry group $\mathcal{G} = SU(3,n)$. However, supergravity gives the restriction that $\mathcal{G}$ is also a group of the duality transformations on the vector field strengths and their duals. The available gauge fields (and their duals) are then an irreducible representation of $\mathcal{G}$. In this case, it is a fundamental $\mathbf{3}+\mathbf{n}$ representation of $SU(3,n)$.

To find possible gauge groups of $SU(3,n)$, let $D(T)_\Lambda{}^\Sigma$ be a matrix representation of the $SU(3,n)$ generator T. By (4.1.5) being the $SU(3,n)$ invariant tensor, it yields

$$D^\dagger(T)J + JD(T) = 0. \tag{4.1.17}$$

By decomposing $D(T)$ into real and imaginary parts, $D(T) = X(T) + Y(T)$, we have

$$X^T J + JX(T) = 0 \tag{4.1.18}$$

$$-Y^T J + JY(T) = 0. \tag{4.1.19}$$

The equation (4.1.5) means there is an $SO(3,n) \subset SU(3,n)$ subgroup that preserve the invariant tensor J, which $Y(T) = 0$. By $\mathcal{G}$ being a duality group of the vector fields, the electric potential A_Λ and the magnetic potential B_Λ are transformed by

$$\delta H_\Lambda = D(T)_\Lambda{}^\Sigma H_\Sigma \tag{4.1.20}$$

where

$$H_\Lambda = A_\Lambda + B_\Lambda. \tag{4.1.21}$$

If we gauge only the electric potential, which is the real part of the complex vector $\mathbf{3}+\mathbf{n}$, the duality is broken into $SO(3,n)$, therefore G is a subgroup of $SO(3,n)$. Another requirement is that when restricted to gauge group G, the $\mathbf{3}+\mathbf{n}$ complex representation D must split into $adj \oplus adj$,

$$D \underset{G}{\rightarrow} adj \oplus adj. \tag{4.1.22}$$

In this case, $\mathbf{3+n}$ complex representation of $SU(3,n)$ split into two fundamental, real representations of $SO(3,n)$ which become an adjoint representation of the gauge group,

$$(\mathbf{3+n})_{\mathbb{C}} \to (\mathbf{3+n})_{\mathbb{R}} + (\mathbf{3+n})_{\mathbb{R}}. \tag{4.1.23}$$

When a particular gauge group $G \subset SO(3,n) \subset SU(3,n)$ is gauged, the global symmetry of the Lagrangian is broken down to G. The gauge field strength becomes non-abelian

$$F_\Lambda = dA_\Lambda + f_\Lambda{}^{\Sigma\Gamma} A_\Sigma \wedge A_\Gamma \tag{4.1.24}$$

where $f_\Lambda{}^{\Sigma\Gamma}$ are the structure constants of the gauge group. Gauge generators T_Λ of the gauge group G satisfy the Lie algebra,

$$[T_\Lambda, T_\Sigma] = f_{\Lambda\Sigma}{}^{\Gamma} T_\Gamma. \tag{4.1.25}$$

Note that we can also raise/lower $SU(3,n)$ indices of the structure constants by contract with the invariant matrix $J_{\Lambda\Sigma}$ and its inverse. In the present case, we are interested in semisimple gauge group with different couplings for each simple factor. We should note that the Mourer-Cartan one form on the scalar manifold is also modified in the presence of gaugings in the form of covariant derivative,

$$\Omega_\Lambda{}^\Sigma = (L^{-1})_\Lambda{}^\Sigma dL_\Sigma{}^\Pi + (L^{-1})_\Lambda{}^\Sigma f_\Sigma{}^{\Omega\Gamma} A_\Omega L_\Gamma{}^\Pi. \tag{4.1.26}$$

However, we will drop all of the gauge fields in the following since we are only interested in supersymmetric solutions with only scalars and the metric non-vanishing.

Supersymmetry constraints the gauge group G by requiring the structure constants to be totally antisymmetric

$$f_{\Lambda\Sigma\Gamma} = f_{\Lambda\Sigma}{}^{\Gamma'} J_{\Gamma'\Gamma} = f_{[\Lambda\Sigma\Gamma]}. \tag{4.1.27}$$

This can be satisfied by taking $J_{\Lambda\Gamma}$ to be the Killing form of the $(n+3)$-dimensional gauge group G. Since $J_{\Lambda\Sigma}$ has indefinite signs of the eigenvalues, the gauge group G can be either compact or non-compact types. In this case, $J_{\Lambda\Sigma}$ has three positive eigenvalues and an arbitrary number of negative eigenvalues. Thus we can have at most three compact or three non-compact directions. This restricts possible gauge groups G to be either $G = SO(3) \times H_n$, $G = SO(3,1) \times H_{n-3}$, or $SO(2,2) \times H_{n-3}$, where H_n is a compact group of dimension n, as pointed out in [38] [80]. We should mention that the consistency condition of the $SO(3,n)$ global symmetry is similar to the half-maximal gauged supergravity in seven dimensions constructed in [81], with possible gauge groups listed in [82]. The possible gauge groups of $N = 3$ gauged supergravity are expected to follow the same manner.

As mentioned in [82], all possible semisimple gauge groups take the form of $G_0 \times H$, where H is a compact group of dimension $n + 3 - dim(G_0)$ while the possible group G_0 is one of the following,

$$\begin{aligned} &SO(3), \quad SO(3,1), \quad SO(2,2), \quad SO(2,1), \\ &\qquad\qquad SO(2,1) \times SO(2,2), \quad SL(3,\mathbb{R}). \end{aligned} \tag{4.1.28}$$

All of G_0 above give structure constants which satisfy (4.1.27); hence they are suited to be gauge groups of $N = 3$ gauged supergravity coupled to vector multiplets.

4.2 AdS_4 Vacua, Masses, RG Flow Solutions

We now have enough ingredients to study the scalar potential and the BPS equations. In each gauge group, we will identify AdS_4 vacua from its scalar potential. We then find solutions interpolating between each critical point by solving the BPS equations.

There are $6n$ scalars parametrized the $SU(3,n)/SU(3) \times SU(n) \times U(1)$. In order to parametrize the coset, we introduce the notation for $6n$ non-compact generators of the general $SU(3,n)/SU(3) \times SU(n) \times U(1)$ coset,

$$\hat{Y}_{iA} \;=\; e_{i+3,A} + e_{A,i+3} \qquad \text{and} \qquad \tilde{Y}_{iA} = -ie_{i+3,A} + ie_{A,i+3} \tag{4.2.1}$$

where $i = 1, \dots, n$ and $(e_{\Lambda\Sigma})_{\Gamma\Delta} = \delta_{\Lambda\Gamma}\delta_{\Sigma\Delta}$.

To check unbroken supersymmetry and set up BPS equations to study domain wall solutions, we will consider fermionic supersymmetry transformations

(4.1.11)-(4.1.14). The AdS_4 metric ansatz is taken to be

$$ds^2 = e^{2A(r)}dx_{1,2}^2 + dr^2. \tag{4.2.2}$$

All scalars are depended only on radial coordinate r, for simplicity. We will use Majorana representation for gamma matrices, which all gamma matrices γ^a are real. The chirality projection $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is then purely imaginary. We will use the projection condition for parameters ϵ_A,

$$\gamma^{\hat{r}}\epsilon_A = e^{i\Lambda}\epsilon^A \tag{4.2.3}$$

where $e^{i\Lambda}$ is a phase factor.

We are now analyzing various possible gauge groups, namely $SO(3)\times SO(3)$, $SO(3,1)$, $SO(2,2)$, $SO(2,1)\times SO(2,2)$, and $SL(3,\mathbb{R})$. For each gauge group, we will consider various unbroken symmetries. The corresponding scalar potentials are computed, and we find their critical points. Note that we are only interested in AdS_4 vacua. Masses and the (dual) dimensions of the scalars will be given in each subsection. We then move to set up the BPS equations from the supersymmetry transformations. Solutions from these equations are interpreted as RG flows driven by the corresponding operators.

4.2.1 $SO(3)\times SO(3)$ Gauge Group

This gauge group can be obtained from $N = 3$ supergravity coupled to three vector multiplets. The structure constants are given by

$$f_{\Lambda\Sigma}{}^{\Gamma} = (g_1\epsilon_{ABC}, g_2\epsilon_{i+3,j+3,k+3}), \qquad i,j = 1,2,3\,. \tag{4.2.4}$$

In this case, there are 18 scalars parametrized by $SU(3,3)/SU(3)\times SU(3)\times U(1)$ coset manifold.

AdS_4 Vacua and RG Flows with $SO(3)$ Symmetry

We are considering solutions preserving $SO(3)_{\text{diag}} \subset SO(3)\times SO(3)$ symmetry. The 18 scalars transform in representations $(\mathbf{3},\bar{\mathbf{3}})_{-2} + (\bar{\mathbf{3}},\mathbf{3})_2$ of the local $SU(3)\times$

$SU(3)\times U(1)$. Note that we will drop the $U(1)$ charges from now on for simplicity. By embedding $SO(3)$ in $SU(3)$, as $\mathbf{3}\to\mathbf{3}$ and $\bar{\mathbf{3}}\to\mathbf{3}$, the 18 scalars transform in the representations

$$\mathbf{3}\times\mathbf{3}+\mathbf{3}\times\mathbf{3}=(\mathbf{1}+\mathbf{3}+\mathbf{5})+(\mathbf{1}+\mathbf{3}+\mathbf{5}). \tag{4.2.5}$$

There are two singlets corresponding to $SU(3,3)$ non-compact generators

$$Y_1=\hat{Y}_{11}+\hat{Y}_{22}+\hat{Y}_{33},\qquad Y_2=\tilde{Y}_{11}+\tilde{Y}_{22}+\tilde{Y}_{33}\,. \tag{4.2.6}$$

The coset representative can be parametrized by

$$L=e^{\Phi_1 Y_1}e^{\Phi_2 Y_2}\,. \tag{4.2.7}$$

The scalar potential is computed using (4.1.8),

$$\begin{aligned} V \;=\; & -\frac{3}{32}\cosh(2\Phi_2)\left[4\cosh(2\Phi_1)[1+\cosh(2\Phi_1)\cosh(2\Phi_2)]^2g_1^2\right.\\ & +2\sinh(2\Phi_1)\left[\cosh(4\Phi_1)-3+2\cosh^2(2\Phi_1)\cosh(4\Phi_2)\right]g_1g_2\\ & \left.+4\cosh(2\Phi_1)[\cosh(2\Phi_1)\cosh(2\Phi_2)-1]^2g_2^2\right]. \end{aligned} \tag{4.2.8}$$

We find two supersymmetric AdS_4 critical points from this scalar potential. The first AdS_4 critical point occurs at $\Phi_1=\Phi_2=0$. The cosmological constant V_0 and the AdS_4 radius L are given by

$$V_0=-\frac{3}{2}g_1^2,\qquad L^2=-\frac{3}{2V_0}=\frac{1}{g_1^2}\,. \tag{4.2.9}$$

The second AdS_4 critical point is given by

$$\begin{aligned} \Phi_1 &= \frac{1}{2}\ln\left[\frac{g_2-g_1}{g_2+g_1}\right],\qquad \Phi_2=0,\\ V_0 &= -\frac{3g_1^2g_2^2}{2(g_2^2-g_1^2)},\qquad L^2=\frac{g_2^2-g_1^2}{g_1^2g_2^2}\,. \end{aligned} \tag{4.2.10}$$

Note that reality of Φ_1 requires $g_2^2-g_1^2>0$ to give AdS_4 with $V_0<0$.

At the trivial critical point with all scalars vanishing, the $SO(3)\times SO(3)$ symmetry is unbroken, and all scalars have the same masses with $m^2L^2=-2$. By the relation $\Delta(\Delta-3)=m^2L^2$, these masses correspond to dual operators with dimensions $\Delta=1,2$ in the dual $N=3$ SCFT.

At the $SO(3)_{\text{diag}}$ critical point, scalar masses and their corresponding mass dimensions are shown in Table 4.1. All of the scalar masses satisfy the BF bound, which is expected for a supersymmetric critical point. Note that three massless Goldstone bosons indicates $SO(3) \times SO(3) \to SO(3)$ symmetry breaking.

$SO(3)_{\text{diag}}$ representations	m^2L^2	Δ
1	$4,\ -2$	$4,\ (1,2)$
3	$0_{(\times 3)},\ -2_{(\times 3)}$	$3,\ (1,2)$
5	$-2_{(\times 10)}$	$(1,2)$

Table 4.1: Scalar Masses at the $N = 3$ supersymmetric AdS_4 critical point in the $SO(3) \times SO(3)$ gauge group and their corresponding dimensions of the dual operators in $N = 3$ SCFT

For the two singlet scalars Φ_1, Φ_2 depending only on the radial coordinate r, the equation $\delta\chi = 0$ is already satisfied since $C_M{}^{MA} = 0$ also implies that $\mathcal{U}^A = 0$. The equations for $\delta\lambda_i = 0$ and $\delta\lambda_{iA} = 0$ reduced to two equations

$$\begin{aligned} e^{i\Lambda}[\cosh(2\Phi_2)\Phi_1' \pm i\Phi_2'] &= -\frac{1}{2}[\sinh(2\Phi_1) + i\cosh(2\Phi_1 \sinh(2\Phi_2))] \times \\ &\quad [\cosh\Phi_2(g_1\cosh\Phi_1 + g_2\cosh\Phi_1) \\ &\quad -i\sinh\Phi_2(g_1\sinh\Phi_1 + g_2\cosh\Phi_1)] \end{aligned} \tag{4.2.11}$$

In the case with only two $SO(3)$ singlets, the function S_{AB} is diagonal and in the form,

$$S_{AB} = \mathcal{W}\delta_{AB} \tag{4.2.12}$$

where the superpotential $\mathcal{W}$ which is given by

$$\begin{aligned} \mathcal{W} \;=\; & -[\cosh\Phi_1\cosh\Phi_2 - i\sinh\Phi_1\sinh\Phi_2][\cosh\Phi_1\cosh\Phi_2 + i\sinh\Phi_1\sinh\Phi_2]^2 g_1 \\ & +[\sinh\Phi_1\cosh\Phi_2 - i\cosh\Phi_1\sinh\Phi_2][\sinh\Phi_1\cosh\Phi_2 + i\cosh\Phi_1\sinh\Phi_2]^2 g_2\,. \end{aligned} \tag{4.2.13}$$

The equation $\delta\psi_{\mu A} = 0$, for $\mu = 0, 1, 2$, is then yield

$$\frac{1}{2}A' e^{i\Lambda} + \mathcal{W} = 0\,. \tag{4.2.14}$$

By writing $\mathcal{W} = |\mathcal{W}|e^{i\omega}$, the equation (4.2.14) can be separated into real and imaginary parts,

$$\frac{1}{2}A' + \frac{1}{2}|\mathcal{W}|(e^{i\omega - i\Lambda} + e^{-i\omega + i\Lambda}) = 0, \qquad (4.2.15)$$

$$\frac{1}{2}|\mathcal{W}|(e^{i\omega - i\Lambda} - e^{-i\omega + i\Lambda}) = 0, \qquad (4.2.16)$$

respectively. The second equation gives $e^{i\Lambda} = \pm e^{i\omega}$.

Equation (4.2.11) implies that $\Phi_2' = 0$ which is consistent with the field equation requirement $\Phi_2 = 0$. We will set $\Phi_2 = 0$ in the remaining analysis. This also implies that $\mathcal{W}$ is real and $\omega = 0$. The phase factor is then $e^{i\Lambda} = \pm 1$ and equations (4.2.11) and (4.2.14) become

$$\Phi_1' = \mp \sinh\Phi_1 \cosh\Phi_1 (g_1 \cosh\Phi_1 + g_2 \sinh\Phi_1), \qquad (4.2.17)$$

$$A' = \pm(g_1 \cosh^3\Phi_1 + g_2 \sinh^3\Phi_1). \qquad (4.2.18)$$

These equations admit two AdS_4 critical points with $N = 3$ supersymmetry, which are coincided with those identified previously. The corresponding Killing spinors could be obtained from $\delta\psi_{iA} = 0$, which gives $\epsilon_A = e^{\frac{A}{2}}\epsilon_A^{(0)}$. The constant spinors $\epsilon_A^{(0)}$ satisfy the condition $\gamma^r \epsilon_A^{(0)} = \pm\epsilon^{(0)A}$.

The equations (4.2.14) and (4.2.18) are similar to those studied in [41]. The solutions interpolating between the two supersymmetric AdS_4 critical points can be solved similarly. We choose the upper signs of these equations to identify UV critical point at $\Phi_1 = 0$ with $r \to \infty$. The solutions to these equations are then given by

$$g_1 g_2 r = 2g_1 \tan^{-1} e^{\Phi_1} + g_2 \ln\left[\frac{e^{\Phi_1}+1}{e^{\Phi_1}-1}\right] - 2\sqrt{g_2^2 - g_1^2}\tanh^{-1}\left[e^{\Phi_1}\sqrt{\frac{g_2+g_1}{g_2-g_1}}\right], \qquad (4.2.19)$$

$$A = \Phi_1 - \ln\left(1 - e^{4\Phi_1}\right) + \ln\left[(e^{2\Phi_1}+1)g_1 + (e^{\Phi_1}-1)g_2\right]. \qquad (4.2.20)$$

Note that the integration constants are omitted.

We will now analyze the behavior of the solutions. At large $r \to \infty$, the solutions behave

$$\Phi_1 \sim e^{-g_1 r} \sim e^{-\frac{r}{L_{UV}}}, \qquad A \sim g_1 r \sim \frac{r}{L_{UV}}. \qquad (4.2.21)$$

The solutions are in the form of $\Phi_i \sim e^{-\Delta r/L} + e^{(\Delta-3)r/L}$ which implies that the flow is driven by a relevant operator of dimension $\Delta = 1, 2$ in the UV. In the IR $r \to -\infty$, we find

$$\Phi_1 \sim e^{\frac{g_1 g_2 r}{\sqrt{g_2^2 - g_1^2}}} \sim e^{\frac{r}{L_{IR}}}, \qquad A \sim \frac{g_1 g_2 r}{\sqrt{g_2^2 - g_1^2}} \sim \frac{r}{L_{IR}} \tag{4.2.22}$$

which implies that the operator dual to Φ_1 becomes irrelevant operator with dimension $\Delta = 4$.

We will now consider flows to a large value of $|\Phi_1|$, which correspond to flows from conformal field theory, identified with AdS_4 critical point, to non-conformal gauge theories in IR. For $\Phi_1 \to \infty$, the solution (4.2.20) becomes

$$\begin{aligned} &\Phi_1 \sim -\frac{1}{3} \ln\left[r(g_1 + g_2) + C\right], \qquad A \sim -\Phi_1, \\ &ds^2 = \left[r(g_1 + g_2) + C\right]^{\frac{2}{3}} dx_{1,2}^2 + dr^2 , \end{aligned} \tag{4.2.23}$$

where the constant C can be removed by shifting the coordinate r.

For $\Phi_1 \to -\infty$, the solution becomes

$$\begin{aligned} &\Phi_1 \sim \frac{1}{3} \ln\left[r(g_1 - g_2) + C\right], \qquad A \sim \Phi_1, \\ &ds^2 = \left[r(g_1 - g_2) + C\right]^{\frac{2}{3}} dx_{1,2}^2 + dr^2 . \end{aligned} \tag{4.2.24}$$

This solution has a singularity at $r \sim -\frac{C}{g_1 \pm g_2}$. In this limit, the scalar potential (4.2.8) becomes

$$V(\Phi_1 \to \pm\infty, \Phi_2 = 0) \to -(g_1 \pm g_2)^2 \infty , \tag{4.2.25}$$

which is physically acceptable according to the criterion of [77].

AdS_4 Vacua and RG Flows with $SO(2) \times SO(2)$ Symmetry

We will now consider solutions preserving $SO(2)_{\text{diag}} \subset SO(2) \times SO(2) \subset SO(3) \times SO(3)$ symmetry. There are six singlets corresponding to non-compact generators

$$\begin{aligned} Y_1 &= \hat{Y}_{33}, \qquad Y_2 = \tilde{Y}_{33}, \qquad Y_3 = \hat{Y}_{11} + \hat{Y}_{22}, \\ Y_4 &= \tilde{Y}_{11} + \tilde{Y}_{22}, \qquad Y_5 = \hat{Y}_{21} - \hat{Y}_{12}, \qquad Y_6 = \tilde{Y}_{21} - \tilde{Y}_{12} . \end{aligned} \tag{4.2.26}$$

The coset representative is parametrized by

$$L = e^{\Phi_1 Y_1} e^{\Phi_2 Y_2} e^{\Phi_3 Y_3} e^{\Phi_4 Y_4} e^{\Phi_5 Y_5} e^{\Phi_6 Y_6} \,. \tag{4.2.27}$$

The scalar potential is highly complicated. We will not present the full form of scalar potential but rather show some of its consistent truncations.

We will discuss a truncation with only two $SO(2) \times SO(2)$ singlet scalars corresponding to Φ_1 and Φ_2. The scalar potential, in this case, is given by

$$V = -\frac{1}{2} g_1^2 e^{-2\Phi_1} \left[e^{2\Phi_1} + (1 + e^{4\Phi_1}) \cosh(2\Phi_2) \right] . \tag{4.2.28}$$

This potential has only a trivial critical point, $\Phi_1 = \Phi_2 = 0$, which is $SO(3) \times SO(3)$ critical point.

By following the same procedure, we will set up the relevant BPS equations. In this case, the matrix S_{AB} is given by

$$S_{AB} = \text{diag}(\mathcal{W}_1, \mathcal{W}_1, \mathcal{W}_2) \tag{4.2.29}$$

where

$$\begin{aligned} \mathcal{W}_1 &= -g_1 \cosh \Phi_1 \cosh \Phi_2 \\ \mathcal{W}_2 &= -g_1 (\cosh \Phi_1 \cosh \Phi_2 + i \sinh \Phi_1 \sinh \Phi_2). \end{aligned} \tag{4.2.30}$$

Note that when $\Phi_1 = 0$ or $\Phi_2 = 0$, $\mathcal{W}_1 = \mathcal{W}_2$. For $\Phi_1 \neq 0$ and $\Phi_2 \neq 0$, the $\mathcal{W}_2$ turns out to be the true superpotential since it provides the potential (4.2.28) in the form of

$$V = \frac{1}{2} G^{\alpha\beta} \frac{\partial |\mathcal{W}_2|}{\partial \Phi_\alpha} \frac{\partial |\mathcal{W}_2|}{\partial \Phi_\beta} - \frac{3}{2} |\mathcal{W}_2|^2 \,. \tag{4.2.31}$$

Scalar kinetic metric $G_{\alpha\beta}$, $\Phi_\alpha = (\Phi_1, \Phi_2)$, can be identified from the scalar kinetic terms

$$-\frac{1}{2} P^{Ai}_\mu P^\mu_{iA} = -\frac{1}{2} \left[\cosh^2(2\Phi_2) \Phi_1'^2 + \Phi_2'^2 \right] , \tag{4.2.32}$$

which gives $G_{\alpha\beta} = \text{diag}(-\cosh^2(2\Phi_2), -1)$. Note that $G^{\alpha\beta}$ is the inverse of $G_{\alpha\beta}$.

The equation $\delta\psi_{\mu A} = 0$, $\mu = 0, 1, 2$, gives

$$A' = \mp 2 |\mathcal{W}_2| = \pm \frac{1}{2} g_1 \sqrt{2 + 2\cosh(2\Phi_1)\cosh(2\Phi_2)} \tag{4.2.33}$$

and the phase factor $e^{i\Lambda} = \pm e^{i\omega}$. For Φ_1 and Φ_2 non-vanishing, we have to set $\epsilon_{1,2} = 0$ to satisfy the $\delta\psi_{\mu A} = 0$ equation, only the supersymmetry corresponding to ϵ_3 can be preserved. Therefore, together with the γ^r projection, the flow solution preserves $N = 1$ Poincare supersymmetry in 3 dimensions.

By setting $\epsilon_{1,2} = 0$ the equations $\delta\lambda_{iA}$ are identically satisfied. The equations $\delta\lambda_i$ give

$$\big[e^{i\Lambda}\left[\cosh(2\Phi_2)\Phi_1' + i\Phi_2'\right] + g_1(\sinh\Phi_1\cosh\Phi_2 - i\cosh\Phi_1\sinh\Phi_2)\,\epsilon^3 = 0 \quad (4.2.34)$$

This will give flow equations for Φ_1 and Φ_2. By using the previous result the phase factor $e^{i\Lambda} = \pm e^{i\omega}$, we have verified that the flow equations are in the form of

$$\Phi_\alpha' = \pm G^{\alpha\beta}\frac{\partial|\mathcal{W}_2|}{\partial\Phi_\beta}. \quad (4.2.35)$$

The explicit forms are given by

$$\begin{aligned} \Phi_1' &= \mp\frac{\sinh(2\Phi_1)\mathrm{sech}(2\Phi_2)g_1}{\sqrt{2+\cosh(2\Phi_1)\cosh(2\Phi_2)}}, \\ \Phi_2' &= \mp\frac{\cosh(2\Phi_1)\sinh(2\Phi_2)g_1}{\sqrt{2+\cosh(2\Phi_1)\cosh(2\Phi_2)}}. \end{aligned} \quad (4.2.36)$$

We could not be able to solve these equations completely. However, we find a relation between Φ_1 and Φ_2

$$\coth(2\Phi_2) = \frac{e^{2\Phi_1}}{2-2e^{4\Phi_1}}. \quad (4.2.37)$$

The full flow solution requires some numerical analysis.

We will now discuss asymptotic behaviors of the solution. At large r, the solution becomes

$$\Phi_1 \sim \Phi_2 \sim e^{-g_1 r}, \quad (4.2.38)$$

as expected for a UV fixed point, and this implies dual operators with dimension $\Delta = 1, 2$. In the limit $\Phi_2 \to \pm\infty$, we find

$$\begin{aligned} &\Phi_1 \sim \Phi_0, \qquad \Phi_2 \sim \mp\ln(g_1 r) \\ &ds^2 = r^2 dx_{1,2}^2 + dr^2 \end{aligned} \quad (4.2.39)$$

where Φ_0 is constant. Note that we have put the singularity at $r = 0$ by choosing an integration constant, for simplicity. For limit $\Phi_1 \to \pm\infty$, we find

$$\begin{aligned} &\Phi_1 \sim \mp \ln(g_1 r), \qquad \Phi_2 \sim \Phi_0, \\ &ds^2 = r^2 dx_{1,2}^2 + dr^2. \end{aligned} \tag{4.2.40}$$

All of these flows give $V \to -\infty$ hence they are physical.

As mentioned earlier, for $\Phi_1 = 0$ or $\Phi_2 = 0$, the superpotentials $\mathcal{W}_1$ and $\mathcal{W}_2$ coincide, $\mathcal{W}_1 = \mathcal{W}_2$. This implies the eigenvalues of S_{AB} degenerate. The BPS equations obtained from $\delta\lambda_{iA} = 0$ and $\delta\lambda_i = 0$ are also identical. Therefore, the resulting equations are symmetric for either $\Phi_1 = 0$ or $\Phi_2 = 0$ cases. We will let $\Phi_2 = 0$ in the following analysis. The flow equations (4.2.33) and (4.2.36) reduce to

$$\begin{aligned} \Phi_1' &= -g_1 \sinh \Phi_1, \\ A' &= g_1 \cosh \Phi_1. \end{aligned} \tag{4.2.41}$$

These equations can be solved by

$$\begin{aligned} \Phi_1 &= \pm \ln \left[\frac{e^{g_1 r - C} + 1}{e^{g_1 r - C} - 1} \right], \\ A &= -g_1 r + \ln\big(e^{2g_1 r - 2C} - 1\big). \end{aligned} \tag{4.2.42}$$

At large r, the solutions become

$$\Phi_1 \sim e^{-g_1 r}, \quad A \sim g_1 r, \tag{4.2.43}$$

which is expected for UV AdS_4 fixed point. Near the singularity at $g_1 r \sim C$, the solutions become

$$\begin{aligned} &\Phi_1 \sim \pm \ln(g_1 r - C), \qquad A \sim \ln(g_1 r - C), \\ &ds^2 = (g_1 r - C)^2 dx_{1,2}^2 + dr^2 . \end{aligned} \tag{4.2.44}$$

This solution is also physical and preserves $N = 3$ supersymmetry in three dimensions.

For flow solutions preserving $SO(2)\times SO(2)$ symmetry, there are two classes of deformations. One is with both Φ_1 and Φ_2 non-vanishing, breaking $N=3$ supersymmetry into $N=1$ supersymmetry. Another is with either Φ_1 or Φ_2 non-vanishing, which preserved $N=3$ supersymmetry.

AdS_4 Vacua and RG Flows with $SO(2)$ Symmetry

For $SO(2)_{\text{diag}}$ symmetry, as mentioned before, there are six singlets. The scalar potential and BPS equations are far more complicated than the previous $SO(2)\times SO(2)$ case. We will give the result from $\Phi_2=\Phi_4=\Phi_6=0$ truncation. We have verified that the result is consistent with both the BPS equations and the corresponding field equations.

In this truncation, S_{AB} is diagonal,

$$S_{AB}=\mathcal{W}\delta_{AB}. \tag{4.2.45}$$

The superpotential $\mathcal{W}$ is given by

$$\begin{aligned}\mathcal{W} \;=\; & -\frac{1}{2}g_1\cosh\Phi_1[1+\cosh(2\Phi_3)]\cosh(2\Phi_5)\\ & +g_2[1-\cosh(2\Phi_3)\cosh(2\Phi_5)]\sinh\Phi_1\,. \end{aligned}\tag{4.2.46}$$

Note that in this case, the superpotential is real, $\mathcal{W}=|\mathcal{W}|$. With given scalar kinetic terms,

$$-\frac{1}{2}P^{iA}_{\mu}P^{\mu}_{Ai}=-\frac{1}{2}\Phi_1'^2-\frac{1}{4}e^{-4\Phi_5}(1+e^{4\Phi_5})^2\Phi_3'^2-\Phi_5'^2, \tag{4.2.47}$$

the scalar potential can be written in the form of (4.2.31),

$$\begin{aligned} V \;=\; & -\frac{1}{2}\frac{\partial|\mathcal{W}|}{\partial\Phi_1}\frac{\partial|\mathcal{W}|}{\partial\Phi_1}-\frac{e^{4\Phi_5}}{(1+e^{4\Phi_5})^2}\frac{\partial|\mathcal{W}|}{\partial\Phi_3}\frac{\partial|\mathcal{W}|}{\partial\Phi_3}-\frac{1}{4}\frac{\partial|\mathcal{W}|}{\partial\Phi_5}\frac{\partial|\mathcal{W}|}{\partial\Phi_5}-\frac{3}{2}|\mathcal{W}|^2\\ =\; & \frac{1}{32}\,[-4[1+\cosh(2\Phi_3)\cosh(2\Phi_5)]\,[2\cosh(2\Phi_3)\cosh(2\Phi_5)\\ & +\cosh(2\Phi_1)[1+3\cosh(2\Phi_3)]\cosh(2\Phi_5)]\,g_1^2\\ & -6\left[\cosh(4\Phi_3)+2\cosh^2(2\Phi_3)\cosh(4\Phi_5)-3\right]\sinh(2\Phi_1)g_1g_2,\\ & +2[2\cosh(2\Phi_3)\cosh(2\Phi_5)-2]\,[2\cosh(2\Phi_3)\cosh(2\Phi_5)\\ & +2\cosh(2\Phi_1)[1-3\cosh(2\Phi_3)\cosh(2\Phi_5)]]\,g_2^2\big]\,. \end{aligned}\tag{4.2.48}$$

Similar to $SO(2)\times SO(2)$ case, the equation $\delta\psi_{\mu A}$ yields

$$A' = \pm|\mathcal{W}|. \qquad (4.2.49)$$

Together with the projection $\gamma^r\epsilon_A = \pm\epsilon^A$, the equations $\delta\lambda_{iA}$ and $\delta\lambda_i$ give

$$\begin{aligned}
\Phi_1' &= \pm\frac{\partial|\mathcal{W}|}{\partial\Phi_1}\\
&= \mp\frac{1}{2}\left[g_1[1+\cosh(2\Phi_3)\cosh(2\Phi_5)]\sinh\Phi_1\right.\\
&\quad \left.+g_2\cosh\Phi_1[\cosh(2\Phi_3)\cosh(2\Phi_5)-1]\right], && (4.2.50)\\
\Phi_3' &= \pm\frac{2e^{4\Phi_5}}{(1+e^{4\Phi_5})^2}\frac{\partial|\mathcal{W}|}{\partial\Phi_3}\\
&= \mp\frac{e^{2\Phi_5}}{1+e^{4\Phi_5}}\sinh(2\Phi_3)[g_1\cosh\Phi_1+g_2\sinh\Phi_1], && (4.2.51)\\
\Phi_5' &= \pm\frac{1}{2}\frac{\partial|\mathcal{W}|}{\partial\Phi_5}\\
&= \mp\frac{1}{2}\cosh(2\Phi_3)\sinh(2\Phi_5)[g_1\cosh\Phi_1+g_2\sinh\Phi_1]. && (4.2.52)
\end{aligned}$$

Solutions to these equations preserve $N=3$ supersymmetry in three dimensions.

There are two AdS_4 critical points to the above equations. One is a trivial critical point with all scalars vanishing, all $\Phi_i = 0$. The other one is given by

$$\Phi_1 = \pm\Phi_3 = \frac{1}{2}\ln\left[\frac{g_2+g_1}{g_2-g_1}\right], \quad \Phi_5 = 0, \qquad (4.2.53)$$

which is the $SO(3)_{\text{diag}}$ critical point. Hence, there is no new critical point for this case.

We are not able to solve the equations (4.2.49) and (4.2.52) analytically for general values of g_1 and g_2. However, we found the analytic solution for the truncation $g_1 = g_2$ and $\Phi_5 = 0$,

$$\begin{aligned}
A &= \Phi_1 - \frac{1}{2}\ln\big(e^{4\Phi_1}-1\big),\\
\Phi_3 &= \cosh^{-1}\left[e^{\frac{\Phi_1}{2}}\sqrt{\cosh\Phi_1}\right],\\
g_1 r &= \tan^{-1}e^{\Phi_1} + \frac{1}{2}\ln\left[\frac{e^{\Phi_1}+1}{e^{\Phi_1}-1}\right]. && (4.2.54)
\end{aligned}$$

For $\Phi_1\sim\Phi_3\sim 0$, the solution behaves,

$$\Phi_1 \sim e^{-2g_1 r}, \qquad \Phi_3\sim e^{-g_1 r}, \qquad A\sim g_1 r\,. \qquad (4.2.55)$$

Near the IR singularity $r \sim 0$, the solution becomes

$$\Phi_1 \sim -\ln(g_1 r), \qquad \Phi_3 \sim \Phi_1, \qquad A \sim -\Phi_1 \sim \ln(g_1 r),$$
$$ds^2 = (g_1 r)^2 dx_{1,2}^2 + dr^2 \tag{4.2.56}$$

for $\Phi_1 > 0$ and

$$\Phi_1 \sim \ln(g_1 r), \qquad \Phi_3 \sim \text{constant}, \qquad A \sim \Phi_1 \sim \ln(g_1 r),$$
$$ds^2 = (g_1 r)^2 dx_{1,2}^2 + dr^2 \tag{4.2.57}$$

for $\Phi_1 < 0$. Both of these yield $V \to -\infty$ hence they are physical. These solutions describe RG flows from $N = 3$ SCFT with $SO(3) \times SO(3)$ symmetry to $N = 3$ gauged theory with $SO(2)$ symmetry in three dimensions.

4.2.2 $SO(3,1)$ Gauge Group

We will consider $N = 3$ supergravity coupled to three vector multiplets with $SO(3,1)$ gauge group. The structure constants are given by $f_{\Lambda\Sigma}{}^{\Gamma} = f_{\Lambda\Sigma\Gamma'} J^{\Gamma'\Gamma}$, where

$$f_{\Lambda\Sigma\Gamma} = g(\epsilon_{ABC}, \epsilon_{i+3,j+3,A}). \tag{4.2.58}$$

Note that we use the totally antisymmetric $\epsilon_{i+3,j+3,A}$ with $\epsilon_{345} = \epsilon_{156} = \epsilon_{264} = 1$.

AdS_4 Vacua and RG Flows with $SO(3)$ Symmetry

We are now considering solutions preserving $SO(3) \subset SO(3,1)$ symmetry. The maximal compact subgroup $SO(3)$ is embedding in $SO(3,1)$ as a diagonal subgroup of $SO(3) \times SO(3) \subset SO(3,3)$. Similar to 4.2.5, the 18 scalars transform in the representations

$$\mathbf{3} \times \mathbf{3} + \mathbf{3} \times \mathbf{3} = (\mathbf{1} + \mathbf{3} + \mathbf{5}) + (\mathbf{1} + \mathbf{3} + \mathbf{5}). \tag{4.2.59}$$

In this case, two of $SO(3)$ singlets are given by $SU(3,3)$ non-compact generators,

$$Y_1 = \hat{Y}_{11} - \hat{Y}_{22} + \hat{Y}_{33}, \qquad Y_2 = \tilde{Y}_{11} - \tilde{Y}_{22} + \tilde{Y}_{33}\,. \tag{4.2.60}$$

The coset representative can be parametrized by

$$L = e^{\Phi_1 Y_1} e^{\Phi_2 Y_2}, \tag{4.2.61}$$

which yields the scalar potential,

$$\begin{aligned} V &= -\frac{3}{64} g^2 e^{-6\Phi_1} \left[2e^{6\Phi_1} \left[13\cosh(2\Phi_1) + 3\cosh(6\Phi_1) \right] \cosh(2\Phi_2) \right. \\ & \left. +(e^{4\Phi_1} - 1)^2 \left[(1 + e^{4\Phi_1}) \cosh(6\Phi_2) - 16 e^{2\Phi_1} \cosh^2(2\Phi_2) \right] \right] . \end{aligned} \tag{4.2.62}$$

We found two AdS_4 critical points to the above scalar potential. One is a trivial critical point where $\Phi_1 = \Phi_2 = 0$, which the $SO(3,1)$ gauge symmetry is broken to the maximal compact subgroup $SO(3)$. The cosmological constant and AdS_4 radius are given by

$$V_0 = -\frac{3}{2} g^2, \qquad L^2 = \frac{1}{g^2} . \tag{4.2.63}$$

Scalar masses and their corresponding conformal dimensions are given in Table 4.2. There are three goldstone bosons which indicate one of $SO(3)$ symmetry breaking.

$SO(3)$ representations	m^2L^2	Δ
1	$4,\ -2$	$4,\ (1,2)$
3	$0_{(\times 3)},\ -2_{(\times 3)}$	$3,\ (1,2)$
5	$-2_{(\times 10)}$	$(1,2)$

Table 4.2: Scalar masses and corresponding conformal dimensions at the trivial, $N = 3$ supersymmetric AdS_4 critical point with $SO(3)$ symmetry for $SO(3,1)$ gauge group

Another AdS_4 critical point is at

$$\Phi_1 = \frac{1}{2} \ln \left[\frac{4 \pm \sqrt{7}}{3} \right], \qquad \Phi_2 = 0 \tag{4.2.64}$$

which is a non-supersymmetric critical point. The cosmological constant and the AdS_4 radius are given by

$$V_0 = -\frac{11}{9} g^2, \qquad L^2 = \frac{27}{22 g^2}. \tag{4.2.65}$$

Scalar masses are given in Table 4.3. Note that this critical point is unstable due to some of the scalar masses violate the BF bound.

$SO(3)$ representations	m^2L^2
1	$-\frac{168}{11}$, $-\frac{36}{11}$
3	$0_{(\times 3)}$, $-\frac{36}{11}\Big\vert_{(\times 3)}$
5	$-\frac{24}{11}\Big\vert_{(\times 5)}$, $-\frac{36}{11}\Big\vert_{(\times 5)}$

Table 4.3: Scalar masses at the $N = 3$ non-supersymmetric AdS_4 critical point with $SO(3)$ symmetry for $SO(3,1)$ gauge group

We will set up the BPS equations to consider a supersymmetric RG flows to non-conformal theories since there is no other supersymmetric critical point. The equation $\delta\psi_{\mu A} = 0$ yields

$$\frac{1}{2}A'e^{i\Lambda} + \mathcal{W} = 0\,. \tag{4.2.66}$$

The equations $\delta\lambda_{iA} = 0$ and $\delta\lambda_i = 0$ give

$$\begin{aligned} e^{i\Lambda}\left[\cosh(2\Phi_2)\Phi_1' \pm i\Phi_2'\right] &= g\sinh^3\Phi_1\cosh\Phi_2 + \frac{1}{2}g\cosh\Phi_1\left[\sinh(2\Phi_1)\cosh(3\Phi_2)\right. \\ &\quad \left. -2i\left[1 - 2\sinh^2\Phi_1\cosh(2\Phi_2)\right]\sinh\Phi_2\right]. \end{aligned} \tag{4.2.67}$$

This implies $\Phi_2' = 0$. Consistency with the second-order field equations also requires that $\Phi_2 = 0$. The superpotential, in this case, is real which implies $e^{i\Lambda} = \pm 1$. By choosing $e^{i\Lambda} = 1$, the BPS equations are

$$\Phi_1' = \frac{1}{4}e^{-3\Phi_1}g\left(e^{2\Phi_1} + e^{6\Phi_1} - e^{4\Phi_1} - 1\right), \tag{4.2.68}$$

$$A' = -\frac{1}{4}e^{-3\Phi_1}g\left(1 + e^{6\Phi_1} - 3e^{2\Phi_1} - 3e^{4\Phi_1}\right). \tag{4.2.69}$$

Note that the dual mass dimension of Φ_1 is $\Delta = 4$ which is corresponding to the irrelevant operator. We then expect the AdS_4 critical point to appear in the IR of the RG flow driven by Φ_1.

The solution to (4.2.69) is

$$gr = \ln\left[\frac{e^{\Phi_1} - 1}{e^{\Phi_1} + 1}\right] + \frac{1}{\sqrt{2}}\ln\left[\frac{1 + \sqrt{2}e^{\Phi_1} + e^{2\Phi_1}}{\sqrt{2}e^{\Phi_1} - 1 - e^{2\Phi_1}}\right], \tag{4.2.70}$$

$$A = \Phi_1 + \ln\left(e^{2\Phi_1} - 1\right) - \ln\left(1 + e^{4\Phi_1}\right). \tag{4.2.71}$$

At $\Phi_1 \sim 0$, the solution behaves

$$\Phi_1 \sim e^{\frac{r}{L}}, \qquad A \sim \frac{r}{L} \tag{4.2.72}$$

which is the AdS_4 critical point.

At large $|\Phi_1|$, the behaviors depend on the sign of Φ_1. For $\Phi_1 > 0$, the solution becomes

$$\begin{aligned}
&\Phi_1 \sim -\frac{1}{3}\ln(gr + C), \qquad A \sim -\Phi_1, \\
&ds^2 = (gr + C)^{\frac{2}{3}} dx_{1,2}^2 + dr^2.
\end{aligned} \tag{4.2.73}$$

For $\Phi_1 < 0$, the solution becomes

$$\begin{aligned}
&\Phi_1 \sim \frac{1}{3}\ln(C - gr), \qquad A \sim \Phi_1, \\
&ds^2 = (C - gr)^{\frac{2}{3}} dx_{1,2}^2 + dr^2 .
\end{aligned} \tag{4.2.74}$$

Both of these solutions give $V \to -\infty$ hence they are physical.

AdS_4 Vacua and RG Flows with $SO(2)$ Symmetry

In this case, there are six singlets given by $SU(3,3)$ non-compact generators

$$\begin{aligned}
Y_1 &= \hat{Y}_{33}, \qquad Y_2 = \tilde{Y}_{33}, \qquad Y_3 = \hat{Y}_{11} - \hat{Y}_{22}, \\
Y_4 &= \tilde{Y}_{11} - \tilde{Y}_{22}, \qquad Y_5 = \hat{Y}_{12} + \hat{Y}_{21}, \qquad Y_6 = \tilde{Y}_{12} + \tilde{Y}_{21} .
\end{aligned} \tag{4.2.75}$$

The coset representative is parametrized by

$$L = e^{\Phi_1 Y_1} e^{\Phi_2 Y_2} e^{\Phi_3 Y_3} e^{\Phi_4 Y_4} e^{\Phi_5 Y_5} e^{\Phi_6 Y_6}. \tag{4.2.76}$$

The result of scalar potential is highly complicated. We will again show for a consistent truncation $\Phi_2 = \Phi_4 = \Phi_6 = 0$ with the scalar potential

$$\begin{aligned}
V &= \frac{1}{8}g^2 \left[16\cosh(2\Phi_5)\sinh(2\Phi_1)\sinh(2\Phi_3) - 3\cosh(2\Phi_1)[3 + \cosh(4\Phi_3)]\right. \\
&\quad \left. + 2[2 + (2 - 3\cosh(2\Phi_1))\cosh(4\Phi_5)]\sinh^2(2\Phi_3)\right] .
\end{aligned} \tag{4.2.77}$$

We have not found other supersymmetric critical points aside from the trivial critical point.

We now consider BPS equations. In this truncation, the matrix S_{AB} is diagonal and gives a real superpotential, $\mathcal{W} = W$, with

$$W = -g\cosh\Phi_1 + g\cosh(2\Phi_5)\sinh\Phi_1\sin(2\Phi_3). \tag{4.2.78}$$

Note that the scalar potential can be written in terms of W,

$$V = -\frac{1}{2}\left(\frac{\partial W}{\partial\Phi_1}\right)^2 - e^{4\Phi_5}(1+e^{4\Phi_5})^{-2}\left(\frac{\partial W}{\partial\Phi_3}\right)^2 - \frac{1}{4}\left(\frac{\partial W}{\partial\Phi_5}\right)^2 - \frac{3}{2}W^2\,. \tag{4.2.79}$$

The flow equations are then given by

$$\begin{aligned}
\Phi_1' &= \pm\frac{\partial W}{\partial\Phi_1} = \pm\left[-g\sinh\Phi_1 + g\cosh\Phi_1\cosh(2\Phi_5)\sinh(2\Phi_3)\right], && (4.2.80)\\
\Phi_3' &= \pm 2e^{4\Phi_5}(1+e^{4\Phi_5})^{-2}\frac{\partial W}{\partial\Phi_3} \\
&= \pm\frac{4e^{4\Phi_5}}{(1+e^{4\Phi_5})^2}g\cosh(2\Phi_3)\cosh(2\Phi_5)\sinh\Phi_1, && (4.2.81)\\
\Phi_5' &= \pm\frac{1}{2}\frac{\partial W}{\partial\Phi_5} = \pm g\sinh\Phi_1\sinh(2\Phi_3)\sinh(2\Phi_5), && (4.2.82)\\
A &= \mp 2W\,. && (4.2.83)
\end{aligned}$$

We are not able to solve these equations analytically. We will only discuss the asymptotic behaviors of the solution.

Near the AdS_4 critical point, the solution behaves

$$\Phi_1 \sim \Phi_3 \sim e^{gr} \sim e^{\frac{r}{L}}, \qquad \Phi_5 \sim \text{constant}, \qquad A \sim gr \sim \frac{r}{L}\,. \tag{4.2.84}$$

This implies that Φ_1 and Φ_3 are dual to irrelevant operators with dimension $\Delta = 4$ in three dimensions. The Φ_5 is dual to a marginal operator with dimension $\Delta = 3$ and is also a Goldstone boson.

There is a singularity at a large value of $\Phi_3 \to \pm\infty$. This also gives $\Phi_5' = 0$. We will choose $\Phi_5 = 0$ for simplicity. Near this singularity, the asymptotic behavior of the flow solution is given by

$$\begin{aligned}
&\Phi_1 \sim \pm\Phi_3 \sim \pm\frac{1}{3}\ln\left|C \pm \frac{3}{4}gr\right|, \qquad A \sim \frac{1}{3}\ln\left|C \pm \frac{3}{4}gr\right|, \\
&ds^2 = \left(C \pm \frac{3}{4}gr\right)^{\frac{2}{3}} dx_{1,2}^2 + dr^2\,. && (4.2.85)
\end{aligned}$$

This singularity is also physical.

4.2.3 $SO(2,2)$ Gauge Group

We will consider $N=3$ supergravity coupled to three vector multiplets with gauge group $SO(2,2)\sim SO(2,1)\times SO(1,2)$. The structure constants are given by

$$f_{\Lambda\Sigma}{}^{\Gamma}=(g_1\epsilon_{\bar{A}\bar{B}\bar{D}}\eta^{\bar{D}\bar{C}},g_2\epsilon_{\bar{i}\bar{j}\bar{l}}\eta^{\bar{l}\bar{k}}) \tag{4.2.86}$$

where $\bar{A},\bar{B},\ldots=1,2,6$, $\bar{i},\bar{j},\ldots=3,4,5$. The matrices $\eta^{\bar{A}\bar{B}}$ and $\eta^{\bar{i}\bar{j}}$ are defined by

$$\eta^{\bar{A}\bar{B}}=\mathrm{diag}(1,1,-1)\quad \eta^{\bar{i}\bar{j}}=\mathrm{diag}(1,-1,-1). \tag{4.2.87}$$

AdS_4 Vacua and RG Flows with $SO(2)$ Symmetry

We will consider RG flows with $SO(2)_{\mathrm{diag}}$ symmetry. There are six singlets which are given by

$$\begin{aligned} &Y_1=\hat{Y}_{11}+\hat{Y}_{22},\quad Y_2=\tilde{Y}_{11}+\tilde{Y}_{22},\quad Y_3=\hat{Y}_{33},\\ &Y_4=\tilde{Y}_{33},\quad Y_5=\hat{Y}_{21}-\hat{Y}_{12},\quad Y_6=\tilde{Y}_{21}-\tilde{Y}_{12}. \end{aligned} \tag{4.2.88}$$

The coset representative is parametrized by

$$L=e^{\Phi_1Y_1}e^{\Phi_2Y_2}e^{\Phi_3Y_3}e^{\Phi_4Y_4}e^{\Phi_5Y_5}e^{\Phi_6Y_6}. \tag{4.2.89}$$

The scalar potential is also complicated. We will give only the result from a truncation $\Phi_2=\Phi_4=\Phi_6=0$. The scalar potential in this truncation is

$$\begin{aligned} V\ =\ & \frac{1}{16}\Big[4\cosh(2\Phi_1)\cosh(2\Phi_5)\left[\cosh(2\Phi_1)\cosh(2\Phi_5)(g_1^2-g_2^2)+g_1^2+g_2^2\right]-\\ & 2\cosh(2\Phi_3)\Big[g_1^2+g_2^2+\cosh(2\Phi_1)\cosh(2\Phi_5)\\ & \times\left[3\cosh(2\Phi_1)\cosh(2\Phi_5)\left(g_1^2+g_2^2\right)+4(g_1^2-g_2^2)\right]\Big]\\ & +3g_1g_2\sinh(2\Phi_3)\left[2\cosh(4\Phi_5)\cosh^2(2\Phi_1)+\cosh(4\Phi_1)-3\right]\Big]. \end{aligned} \tag{4.2.90}$$

There is an AdS_4 critical point at $\Phi_i=0$ with

$$V_0=-\frac{1}{2}g_1^2,\quad L^2=\frac{3}{g_1^2}. \tag{4.2.91}$$

However, considering supersymmetry transformations gives

$$\delta\lambda_i = \delta_{i3} g_1 \epsilon^3 \qquad \text{and} \qquad \delta\lambda_{iA} = \delta_{i3} g_1 (\delta_{A2}\epsilon^1 - \delta_{A1}\epsilon^2). \tag{4.2.92}$$

Since the only way to satisfy $\delta\lambda_i = 0$ and $\delta\lambda_{iA} = 0$ is to set $\epsilon^A = 0$, this implies that it is a non-supersymmetric critical point. Scalar masses are given in Table 4.4. This critical point is not stable since some of its masses violate the BF bound.

$SO(2) \times SO(2)$ representations	m^2L^2
$(\mathbf{1}, \mathbf{1})$	-6, -6
$(\mathbf{2}, \mathbf{1})$	$0_{(\times 2)}$, $-\frac{15}{2}\Big\|_{(\times 2)}$
$(\mathbf{1}, \mathbf{2})$	$0_{(\times 2)}$, $-\frac{3g_2^2}{2g_1^2}\Big\|_{(\times 2)}$
$(\mathbf{2}, \mathbf{2})$	$-\frac{3}{2}\frac{g_1^2+g_2^2}{g_1^2}\Big\|_{(\times 8)}$

Table 4.4: Scalar masses at the non-supersymmetric AdS_4 critical point with $SO(2) \times SO(2)$ symmetry for $SO(2,2)$ gauge group

We will now consider a possible half-supersymmetric vacuum in the form of a domain wall. By using the metric ansatz (4.2.2) and the similar procedures, we found that the resulting BPS equations and the scalar potential are intensely complicated. We will consider a simple case with $SO(2) \times SO(2)$ symmetry obtained from a truncation with all $\Phi_i = 0$ except Φ_3 and Φ_4. The scalar potential in this truncation is

$$V = -\frac{1}{2} g_1^2 e^{-2\Phi_3} \left[(1 + e^{4\Phi_3}) \cosh(2\Phi_4) - e^{2\Phi_3} \right]. \tag{4.2.93}$$

The matrix S_{AB} is diagonal,

$$S_{AB} = \mathrm{diag}(\mathcal{W}_1, \mathcal{W}_1, \mathcal{W}_2) \tag{4.2.94}$$

where the superpotentials $\mathcal{W}_1$ and $\mathcal{W}_2$ are given by

$$\mathcal{W}_1 = g_1 \sin\Phi_3 \cosh\Phi_4, \tag{4.2.95}$$

$$\mathcal{W}_2 = g_1 \cosh\Phi_4 \sinh\Phi_3 + i g_1 \cosh\Phi_3 \sinh\Phi_4. \tag{4.2.96}$$

In this case, only supersymmetry corresponding to ϵ_3 is preserved. The BPS equations are given by

$$\begin{aligned}
\Phi_3' &= \pm\cosh^{-2}(2\Phi_4)\frac{\partial W}{\partial\Phi_3} = \pm\frac{g_1\mathrm{sech}(2\Phi_4)\sinh(2\Phi_3)}{\sqrt{2}\sqrt{\cosh(2\Phi_3)\cosh(2\Phi_4)-1}}, && (4.2.97)\\
\Phi_4' &= \mp\frac{\partial W}{\partial\Phi_4} = \mp\frac{g_1\cosh(2\Phi_3)\sinh(2\Phi_4)}{\sqrt{2}\sqrt{\cosh(2\Phi_3)\cosh(2\Phi_4)-1}}, && (4.2.98)\\
A' &= \mp W && (4.2.99)
\end{aligned}$$

where

$$W = |\mathcal{W}_2| = \sqrt{2}g_1\sqrt{\cosh(2\Phi_3)\cosh(2\Phi_4)-1}\,. \qquad (4.2.100)$$

There is no supersymmetric AdS_4 critical point to these equations. A solution for A and Φ_3 can be given as a function of Φ_4,

$$\begin{aligned}
\Phi_3 &= \frac{1}{2}\ln\left[\frac{1}{4}\left[\mathrm{csch}(2\Phi_4)\sqrt{10\cosh(4\Phi_4)-6}-2\coth(2\Phi_4)\right]\right], && (4.2.101)\\
A &= -\frac{1}{2}\ln\sinh(2\Phi_4) - iF(2i\Phi_4, 5) && (4.2.102)
\end{aligned}$$

where F is the elliptic function of the first kind which is given by

$$iF(i\Phi_3, 5) = \int_0^{\Phi_3}\frac{d\chi}{\sqrt{1-25\sinh^3\chi}}\,. \qquad (4.2.103)$$

We are not able to solve for Φ_4.

For further analysis, we will consider $\Phi_4 = 0$. This gives $|\mathcal{W}_1| = |\mathcal{W}_2|$. The BPS equations are reduced to

$$\begin{aligned}
\Phi_3' &= \pm g_1\cosh\Phi_3, && (4.2.104)\\
A' &= \pm g_1\sinh\Phi_3\,. && (4.2.105)
\end{aligned}$$

In this case, the supersymmetry is restored to $N = 3$. An analytic solution to these equations is

$$\begin{aligned}
\Phi_3 &= \ln\tan\left[\frac{g_1 r + C}{2}\right], \qquad A = -\ln\sin(g_1 r + C), && (4.2.106)\\
ds^2 &= \sin^{-2}(g_1 r + C)dx_{1,2}^2 + dr^2\,. && (4.2.107)
\end{aligned}$$

This solution preserves $N = 3$ Poincare supersymmetry in three dimensions.

4.2.4 $SO(2,1)$ Gauge Group

This gauge group can be obtained by coupling $N = 3$ supergravity to one vector multiplet. The structure constants are given by

$$f_{\Lambda\Sigma}{}^{\Gamma} = g\epsilon_{\bar{A}\bar{B}\bar{D}}\eta^{\bar{D}\bar{C}} \tag{4.2.108}$$

where $\bar{A}, \bar{B}, \ldots = 1, 2, 4$ and $\eta_{\bar{A}\bar{B}} = \text{diag}(1, 1, -1)$.

For $SO(2) \subset SO(2,1)$ invariant scalars, the resulting scalar potential and BPS equations are the same as $SO(2,2)$ case with $g_2 = 0$. The scalar potential also admits only a non-supersymmetric critical point where all scalars vanishing. There is also a half-supersymmetric domain wall with $SO(2)$ symmetry in the form of (4.2.107).

4.2.5 $SL(3,R)$ Gauge Group

This gauge group can be obtained from coupling $N = 3$ supergravity to five vector multiplets. The structure constants $f_{\Lambda\Sigma}{}^{\Gamma} = g\tilde{f}_{\Lambda\Sigma}{}^{\Gamma}$ are identified from the $SL(3,\mathbb{R})$ algebra

$$[T_\Lambda, T_\Sigma] = \tilde{f}_{\Lambda\Sigma}{}^{\Gamma} T_\Gamma\,, \tag{4.2.109}$$

where the $SL(3,\mathbb{R})$ generators T_Λ are given in a form of Gell-Mann matrices λ_i,

$$T_\Lambda = (i\lambda_2, i\lambda_7, i\lambda_5, \lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8). \tag{4.2.110}$$

There are 30 scalars transforming as $(\mathbf{3}, \bar{\mathbf{5}}) + (\bar{\mathbf{3}}, \mathbf{5})$ under the $SU(3) \times SU(5)$ local symmetry. The $SO(3)$ maximal compact subgroup can be embedded in the $SL(3,\mathbb{R})$ as

$$\mathbf{3} \to \mathbf{3}, \quad \mathbf{8} \to \mathbf{3} + \mathbf{5}. \tag{4.2.111}$$

The 30 scalars are then transformed under $SO(3)$ as

$$(\mathbf{3} \times \mathbf{5}) + (\mathbf{3} \times \mathbf{5}) = (\mathbf{3} + \mathbf{5} + \mathbf{7}) + (\mathbf{3} + \mathbf{5} + \mathbf{7}), \tag{4.2.112}$$

which gives no singlet under $SO(3)$. We then consider singlets under the $SO(2) \subset SO(3)$ subgroup. By decomposing (4.2.112), each representation gives one singlet, so there are six singlets under $SO(2)$ symmetry. The corresponding non-compact generators of $SU(3,5)$ are given by

$$\begin{aligned} Y_1 &= \hat{Y}_{24}+\hat{Y}_{33}, \qquad Y_2=\hat{Y}_{23}-\hat{Y}_{34}, \qquad Y_3=\hat{Y}_{15}, \\ Y_4 &= \tilde{Y}_{24}+\tilde{Y}_{33}, \qquad Y_5=\tilde{Y}_{23}-\tilde{Y}_{34}, \qquad Y_6=\tilde{Y}_{15}\,. \end{aligned} \tag{4.2.113}$$

The coset representative is parametrized by

$$L = e^{\Phi_1 Y_1}e^{\Phi_2 Y_2}e^{\Phi_3 Y_3}e^{\Phi_4 Y_4}e^{\Phi_5 Y_5}e^{\Phi_6 Y_6}. \tag{4.2.114}$$

The scalar potential is given by

$$\begin{aligned} V &= -\frac{1}{32}e^{-4\Phi_2-4\Phi_3}g^2\Big[16\sqrt{3}e^{2\Phi_2}(e^{4\Phi_2}-1)(e^{4\Phi_3}-1)\cosh(2\Phi_4)\cosh(2\Phi_5)\cosh(2\Phi_6) \\ &\quad +\cosh^2(2\Phi_5)\big[3e^{2\Phi_3}(2e^{4\Phi_2}-3e^{8\Phi_2}-3)-12e^{2\Phi_3}(e^{4\Phi_2})^2\cosh(4\Phi_4) \\ &\quad +(1+e^{4\Phi_3})\big[2(3+e^{4\Phi_2}+3e^{8\Phi_2})+(9-2e^{4\Phi_2}+9e^{8\Phi_2})\cosh(4\Phi_4)\big]\cosh(2\Phi_6)\big] \\ &\quad +(1+e^{4\Phi_3})\cosh(2\Phi_6)\big[3+4e^{4\Phi_2}+3e^{8\Phi_2}+(3-4e^{4\Phi_2}+3e^{8\Phi_2})\sinh^2(2\Phi_5)\big] \\ &\quad -e^{2\Phi_3}\Big[3+14e^{4\Phi_2}+3e^{8\Phi_2}+3(1-6e^{4\Phi_2}+e^{8\Phi_2})\sinh^2(2\Phi_5) \\ &\quad -8\sqrt{3}e^{2\Phi_2}(1+e^{4\Phi_2})\cosh(2\Phi_4)\sinh(4\Phi_5)\sinh(2\Phi_6)\Big]\Big]\,. \end{aligned} \tag{4.2.115}$$

We have not found other AdS_4 critical points besides the trivial critical point. The cosmological constant and the AdS_4 radius at the trivial critical point are given by

$$V_0 = -\frac{3}{2}g^2, \qquad L^2=\frac{1}{g^2}. \tag{4.2.116}$$

The scalar masses are shown in Table 4.5. Note that marginal deformations are corresponding to the scalars in the **7** representation of the broken $SO(3)$ symmetry beside the Goldstone bosons in representation **5**.

The BPS equations, in this case, is also complicated. We will consider a truncation $\Phi_4=\Phi_5=\Phi_6=0$. This truncation gives a real superpotential W,

$$W = -g\left[\cosh\Phi_3+\sqrt{3}\sinh(2\Phi_2)\sinh\Phi_3\right]. \tag{4.2.117}$$

The matrix S_{AB} is diagonal, $S_{AB}=\frac{1}{2}\delta_{AB}W$. Follow the same procedures, the

$SO(3)$ representations	m^2L^2	Δ
3	$10_{(\times 3)},\ -2_{(\times 3)}$	$5,\ (1,2)$
5	$0_{(\times 5)},\ -2_{(\times 5)}$	$3,\ (1,2)$
7	$0_{(\times 7)},\ -2_{(\times 7)}$	$3,\ (1,2)$

Table 4.5: Scalar masses and the corresponding dimensions of the dual operators at the $N=3$ supersymmetric AdS_4 critical point with $SO(3)$ symmetry for $SL(3,\mathbb{R})$ gauge group

BPS equations are given by

$$\Phi_1' = 0, \tag{4.2.118}$$

$$\Phi_2' = \pm\frac{1}{2}\frac{\partial W}{\partial \Phi_2} = \mp\sqrt{3}g\cosh(2\Phi_2)\sinh(\Phi_3), \tag{4.2.119}$$

$$\Phi_3' = \pm\frac{\partial W}{\partial \Phi_3} = \mp g\left[\sqrt{3}\cosh\Phi_3\sinh(2\Phi_2) + \sinh\Phi_3\right], \tag{4.2.120}$$

$$A' = \mp W. \tag{4.2.121}$$

Note that with the scalar kinetic terms

$$-\frac{1}{4}e^{-4\Phi_2}(1+e^{4\Phi_2})^2\Phi_1'^2 - \Phi_2'^2 - \frac{1}{2}\Phi_3'^2, \tag{4.2.122}$$

the scalar potential (4.2.115) can also be written in terms of W,

$$\begin{aligned} V &= -\frac{1}{4}\frac{\partial W}{\partial \Phi_2} - \frac{1}{2}\frac{\partial W}{\partial \Phi_3} - \frac{3}{2}W^2 \\ &= -\frac{1}{4}g^2\left[2+\cosh(2\Phi_3) + \cosh(4\Phi_2 -)[9\cosh(2\Phi_3) - 6]\right. \\ &\quad \left. + 8\sqrt{3}\sinh(2\Phi_2)\sinh(2\Phi_3)\right]. \end{aligned} \tag{4.2.123}$$

Note that we are unable to find an analytic solution for this case.

We now analyze asymptotic behaviors of the solution. Near the trivial AdS_4 critical point, we find

$$\frac{1}{\sqrt{3}}\Phi_2 + \Phi_3 \sim e^{-3g_1 r} \sim e^{-\frac{3r}{L}}, \qquad \Phi_3 - \frac{\sqrt{3}}{2}\Phi_2 \sim e^{2g_1 r} \sim e^{\frac{2r}{L}}, \qquad A \sim g_1 r \sim \frac{r}{L}. \tag{4.2.124}$$

The combination $\frac{1}{\sqrt{3}}\Phi_2 + \Phi_3$ can be interpreted as a marginal operator. The combination $\Phi_3 - \frac{\sqrt{3}}{2}\Phi_2$ is dual to an irrelevant operator of dimension $\Delta = 5$. We expect $\Phi_3 - \frac{\sqrt{3}}{2}\Phi_2$ to drive the flow since the marginal operator does not break conformal symmetry. Note that in this case, the UV SCFT should appear in the IR since the operator driving the flow is irrelevant at the fixed point.

For large $|\Phi_2|$, we find

$$\begin{aligned}
&\Phi_3 \sim \Phi_2 \sim \mp\frac{1}{3}\ln\left[\frac{3\sqrt{3}gr}{4}\right], \qquad A \sim \frac{1}{3}\ln r, \\
&ds^2 = r^{\frac{2}{3}}dx_{1,2}^2 + dr^2\,. \qquad (4.2.125)
\end{aligned}$$

This singularity is physical since it yields $V \to -\infty$. This should describe an RG flow in the dual supersymmetric field theory to a conformal fixed point in the IR.

4.2.6 $SO(2,1) \times SO(2,2)$ Gauge Group

The $SO(2,1)\times SO(2,2) \sim SO(2,1)\times SO(2,1)\times SO(2,1)$ gauge group arises from coupling $N = 3$ supergravity to six vector multiplets. The structure constants for this gauge group are given by

$$f_{\Lambda\Sigma}{}^{\Gamma} = (g_1\epsilon_{\bar{A}\bar{B}\bar{D}}\eta^{\bar{D}\bar{C}}, g_2\epsilon_{\bar{i}\bar{j}\bar{l}}\eta^{\bar{l}\bar{k}}, g_3\epsilon_{\tilde{i}\tilde{j}\tilde{l}}\eta^{\tilde{l}\tilde{k}}) \qquad (4.2.126)$$

where $\bar{A}, \bar{B}, \dots = 1, 4, 5$, $\bar{i}, \bar{j}, \dots = 2, 6, 7$, $\tilde{i}, \tilde{j}, \dots = 3, 8, 9$ and

$$\eta_{\bar{A}\bar{B}} = \text{diag}(1,-1,-1), \qquad \eta_{\bar{i}\bar{j}} = \text{diag}(1,-1,-1), \qquad \eta_{\tilde{i}\tilde{j}} = \text{diag}(1,-1,-1)\,. \qquad (4.2.127)$$

There are 36 scalars in the full $SU(3,6)/SU(3)\times SU(6)\times U(1)$ coset manifold. However, we will consider only 12 singlets under $SO(2)\times SO(2)$ residue symmetry, chosen to be the first two $SO(2)$'s. The corresponding non-compact generators of the $SU(3,6)$ are given by

$$\begin{aligned}
Y_1 &= \hat{Y}_{15}, \qquad Y_2 = \hat{Y}_{16}, \qquad Y_3 = \hat{Y}_{25}, \qquad Y_4 = \hat{Y}_{26}, \\
Y_5 &= \hat{Y}_{35}, \qquad Y_6 = \hat{Y}_{36}, \qquad Y_7 = \tilde{Y}_{15}, \qquad Y_8 = \tilde{Y}_{16}, \\
Y_9 &= \tilde{Y}_{25}, \qquad Y_{10} = \tilde{Y}_{26}, \qquad Y_{11} = \tilde{Y}_{35}, \qquad Y_{12} = \tilde{Y}_{36}\,. \qquad (4.2.128)
\end{aligned}$$

The coset representative is parametrized by

$$L = \prod_{i=1}^{12} e^{\Phi_i Y_i} \,. \tag{4.2.129}$$

The scalar potential is highly complicated. We will not give its explicit form but rather note that the resulting scalar potential admits a Minkowski vacuum, $V = 0$, for all scalars vanishing. It preserves $SO(2) \times SO(2) \times SO(2)$ and $N = 3$ supersymmetry. There are six Goldstone bosons arising from breaking $SO(2,1) \times SO(2,2)$ into $SO(2) \times SO(2) \times SO(2)$.

CHAPTER V

RG Flows from Four-dimensional $N = 4$ Gauged Supergravity

In this chapter, we first review four-dimensional $N = 4$ gauged supergravity. We will then discuss $N = 4$ gauged supergravity obtained from type IIA and IIB superstring compactifications. We also discuss the possible semisimple gaugings. In each case, the scalar potential and its possible supersymmetric AdS_4 vacua will be identified along with their possible holographic RG flows. Note that for type IIB case, we also study examples of supersymmetric Janus solutions.

5.1 $N = 4$ Gauged Supergravity

We first review four-dimensional $N = 4$ gauged supergravity. We follow the general gauging in embedding tensor formalism given in [45]. Specific parametrizations will be given later in each section.

The field contents of $N = 4$ supergravity multiplet are

$$(e^{\hat{\mu}}_\mu, \psi^i_\mu, A^m_\mu, \chi^i, \tau) \tag{5.1.1}$$

given by the graviton $e^{\hat{\mu}}_\mu$, four gravitini ψ^i_μ, six vectors A^m_μ, four spinnor fields χ^i and a complex scalar field τ in the $SL(2,\mathbb{R})/SO(2)$ coset. The complex scalar field τ can be parametrized by two real scalar fields, a dilaton ϕ, and an axion χ. Note that, in this chapter, we use $\mu, \nu = 0, \dots, 3$ for spacetime indices, $\hat{\mu}, \hat{\nu} = 0, \dots, 3$ for tangent indices, $i, j = 1, \dots, 4$ for $SU(4)$ fundamental indices, and $m, n = 1, \dots, 6$

for $SO(6) \sim SU(4)$ vector representation.

Supergravity multiplet can couple to an arbitrary number n of vector multiplets. Each vector multiplet in $N = 4$ supergravity,

$$(A_\mu, \lambda^i, \phi^m), \tag{5.1.2}$$

contains a vector field A_μ, four gravitini λ^i, and six scalar fields ϕ^m. For n vector multiplets, there are $6n$ scalar fields parametrizing the $SO(6,n)/SO(6) \times SO(n)$ coset. Hence there are $2 + 6n$ real scalars parametrizing the $\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(6,n)}{SO(6)\times SO(n)}$ coset. Note that we use $a, b = 1, \ldots, n$ for $SO(n)$ vector indices.

Fermionic fields and supersymmetry parameters in $N = 4$ gauged supergravity transform in the fundamental representation of the $SU(4)_R \sim SO(6)_R$ R-symmetry. The chirality projections of the fundamental fermions are given by

$$\gamma_5 \psi^i_\mu = \psi^i_\mu, \qquad \gamma_5 \chi^i = -\chi^i, \qquad \gamma_5 \lambda^i = \lambda^i\,. \tag{5.1.3}$$

For the anti-fundamental representation of $SU(4)_R$, we have

$$\gamma_5 \psi_{\mu i} = -\psi_{\mu i}, \qquad \gamma_5 \chi_i = \chi_i, \qquad \gamma_5 \lambda_i = -\lambda_i\,. \tag{5.1.4}$$

Gaugings of $N = 4$ supergravity can be described by using the embedding tensor. The embedding tensor encodes all the information for embedding a gauge group G_0 in the global or duality group $G = SL(2,\mathbb{R}) \times SO(6,n)$. Note that gaugings in this formalism are covariant under the symmetry group G. A general gauging can be described by two components of the embedding tensor, $f_{\alpha MNP}$ and $\xi_{\alpha M}$, with $\alpha, \beta = (+,-)$ denoting the fundamental representation of the $SL(2,\mathbb{R})$, and $M, N = (m, a)$ denoting the fundamental representation of the $SO(6,n)$. The embedding tensor component $\Theta_{\alpha MNP}$ can be written as

$$\Theta_{\alpha MNP} = f_{\alpha MNP} + \xi_{\alpha[N}\eta_{P]M}, \tag{5.1.5}$$

where $\eta_{MN} = \mathrm{diag}(-1,-1,-1,-1,-1,-1,1,\ldots,1)$ is the $SO(6,n)$ invariant tensor.

To define a consistent gauging, the gauge generators,

$$X_{\alpha M} = \Theta_{\alpha MNP} t^{NP} - \xi_M^{\beta} t_{\alpha\beta}, \tag{5.1.6}$$

must form a closed algebra. This implies that the embedding tensor has to satisfy the quadratic constraints,

$$\begin{aligned} \xi_\alpha{}^M \xi_{\beta M} &= 0, \\ \xi_{(\alpha}{}^P f_{\beta)PMN} &= 0, \\ 3 f_{\alpha R[MN} f_{\beta PQ]}{}^R + 2\xi_{(\alpha[M} f_{\beta)NPQ]} &= 0, \\ \epsilon^{\alpha\beta}(\xi_\alpha{}^P f_{\beta PMN} + \xi_{\alpha M}\xi_{\beta N}) &= 0, \\ \epsilon^{\alpha\beta}(f_{\alpha MNR} f_{\beta PQ}{}^R - \xi_\alpha{}^R f_{\beta R[M[P}\eta_{Q]N]} - \xi_{\alpha[M} f_{N][PQ]\beta} + \xi_{\alpha[P} f_{Q][MN]\beta}) &= 0. \end{aligned} \tag{5.1.7}$$

The electric vector fields in ungauged Lagrangian appear as $A^{+M} = (A^m_\mu, A^a_\mu)$. Together with the dual magnetic vector fields A^{-M}, they form a doublet under the $SL(2,\mathbb{R})$, denoted by $A^{\alpha M}$. Note that the particular electric-magnetic frame can always be chosen such that A^{+M} and A^{-M} have charges $+1$ and -1, respectively, under the $SO(2)$.

When gauged, the covariant derivative can be written as

$$D_\mu = \nabla_\mu - g A^{\alpha M}_\mu \Theta_{\alpha M}{}^{NP} t_{NP} + g A^{M(\alpha}_\mu \epsilon^{\beta)\gamma} \xi_{\gamma M} t_{\alpha\beta}. \tag{5.1.8}$$

The generators of the $SL(2,\mathbb{R})$ and $SO(6,n)$ are chosen to be

$$(t_{\alpha\beta})_\gamma{}^\delta = 2\delta^\delta_{(\alpha}\epsilon_{\beta)\gamma}, \quad (t_{MN})_P{}^Q = 2\delta^Q_{[M}\eta_{N]P}, \tag{5.1.9}$$

respectively, with $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$. Note that the gauge coupling g can be absorbed in the embedding tensor.

In our work, we will consider solutions with only the metric and scalars non-vanishing. Furthermore, we will consider cases with only $f_{\alpha MNP}$ components of the embedding tensor non-vanishing. The quadratic constraints (5.1.7) are then reduced to

$$f_{\alpha R[MN} f_{\beta PQ]}{}^R = 0, \qquad \epsilon^{\alpha\beta} f_{\alpha MNR} f_{\beta PQ}{}^R = 0\,. \tag{5.1.10}$$

Note that for purely electric gauging, only f_{+MNP} non-vanishing, these constraints reduce to the usual Jacobi identity for $f_{MNP} = f_{+MNP}$ [42] [43].

In general, a subgroup of both $SL(2,\mathbb{R})$ and $SO(6,n)$ can be gauged, with only electric vector fields. However, it has been shown that the purely electric gaugings do not admit AdS_4 vacua [43] [44] [83]. In this case, magnetic components of the embedding tensor, f_{-MNP}, are involved. Since we are interested in gauged supergravity with AdS_4 vacua, we will consider gaugings with both electric and magnetic vector fields.

In $N = 4$ gauged supergravity, there are $2+6n$ scalars parametrizes the scalar coset manifold $SL(2,\mathbb{R})/SO(2) \times SO(6,n)/SO(6) \times SO(n)$. The coset representive of the $SL(2,\mathbb{R})/SO(2)$ sector $\mathcal{V}_\alpha$ can be parametrized by

$$\mathcal{V}_\alpha = \frac{1}{\sqrt{\mathrm{Im}\tau}} \begin{pmatrix} \tau \\ 1 \end{pmatrix}. \tag{5.1.11}$$

This coset representative is equivalent to a symmetric 2×2 matrix,

$$M_{\alpha\beta} = \mathrm{Re}(\mathcal{V}_\alpha \mathcal{V}_\beta^*) = \frac{1}{\mathrm{Im}\tau} \begin{pmatrix} |\tau|^2 & \mathrm{Re}\tau \\ \mathrm{Re}\tau & 1 \end{pmatrix}. \tag{5.1.12}$$

Note that $\mathrm{Im}(\mathcal{V}_\alpha \mathcal{V}_\beta^*) = \epsilon_{\alpha\beta}$. The complex scalar τ can also be written in terms of the dilaton ϕ and the axion χ,

$$\tau = \chi - ie^{-\phi}. \tag{5.1.13}$$

The explicit form is then given by

$$\mathcal{V}_\alpha = e^{\varphi_g/2} \begin{pmatrix} \chi_g - ie^{-\varphi_g} \\ 1 \end{pmatrix}. \tag{5.1.14}$$

The $SO(6,n)/SO(6) \times SO(n)$ factor can be parametrized by the coset representative ${\mathcal{V}_M}^A$, where $A = (m,a)$, which transforming by left and right multiplications under $SO(6,n)$ and $SO(6) \times SO(n)$, respectively. The matrix ${\mathcal{V}_M}^A$ satisfies the relation

$$\eta_{MN} = -{\mathcal{V}_M}^m {\mathcal{V}_N}^m + {\mathcal{V}_M}^a {\mathcal{V}_N}^a \tag{5.1.15}$$

being an element of $SO(6,n)$. The $SO(6,n)/SO(6)\times SO(n)$ can also be parametrized by a symmetric matrix M_{MN}, defined by

$$M_{MN} = \mathcal{V}_M{}^m\mathcal{V}_N{}^m + \mathcal{V}_M{}^a\mathcal{V}_N{}^a\,. \tag{5.1.16}$$

The specific forms of the $SO(6,n)/SO(6)\times SO(n)$ coset representative will be given in each case.

With only the metric and scalars non-vanishing, the bosonic Lagrangian is given by

$$e^{-1}\mathcal{L} = \frac{1}{2}R + \frac{1}{16}\partial_\mu M_{MN}\partial^\mu M^{MN} - \frac{1}{4(\text{Im}\tau)^2}\partial_\mu\tau\partial^\mu\tau^* - V. \tag{5.1.17}$$

The scalar potential can be written in term of the coset representative M_{MN} as

$$\begin{aligned} V \;=\; & \frac{g^2}{16}\left[f_{\alpha MNP}f_{\beta QRS}M^{\alpha\beta}\left[\frac{1}{3}M^{MQ}M^{NR}M^{PS} + \left(\frac{2}{3}\eta^{MQ} - M^{MQ}\right)\eta^{NR}\eta^{PS}\right]\right. \\ & \left. -\frac{4}{9}f_{\alpha MNP}f_{\beta QRS}\epsilon^{\alpha\beta}M^{MNPQRS}\right], \end{aligned} \tag{5.1.18}$$

where M^{MN} is the inverse of M_{MN}, and M^{MNPQRS} is defined by

$$M_{MNPQRS} = \epsilon_{mnpqrs}\mathcal{V}_M{}^m\mathcal{V}_N{}^n\mathcal{V}_P{}^p\mathcal{V}_Q{}^q\mathcal{V}_R{}^r\mathcal{V}_S{}^s, \tag{5.1.19}$$

with indices raised by η^{MN}.

Fermionic supersymmetry transformations of $N=4$ gauged supergravity are given by

$$\delta\psi^i_\mu = 2D_\mu\epsilon^i - \frac{2}{3}gA_1^{ij}\gamma_\mu\epsilon_j, \tag{5.1.20}$$

$$\delta\chi^i = i\epsilon^{\alpha\beta}\mathcal{V}_\alpha D_\mu\mathcal{V}_\beta\gamma^\mu\epsilon^i - \frac{4}{3}igA_2^{ij}\epsilon_j, \tag{5.1.21}$$

$$\delta\lambda^i_a = 2i\mathcal{V}_a{}^M D_\mu\mathcal{V}_M{}^{ij}\gamma^\mu\epsilon_j + 2igA_{2aj}{}^i\epsilon^j\,. \tag{5.1.22}$$

The fermion shift functions are defined by

$$\begin{aligned} A_1^{ij} &= \epsilon^{\alpha\beta}(\mathcal{V}_\alpha)^*\mathcal{V}_{kl}{}^M\mathcal{V}_N{}^{ik}\mathcal{V}_P{}^{jl}f_{\beta M}{}^{NP}, \\ A_2^{ij} &= \epsilon^{\alpha\beta}\mathcal{V}_\alpha\mathcal{V}_{kl}{}^M\mathcal{V}_N{}^{ik}\mathcal{V}_P{}^{jl}f_{\beta M}{}^{NP}, \\ A_{2ai}{}^j &= \epsilon^{\alpha\beta}\mathcal{V}_\alpha\mathcal{V}^M{}_a\mathcal{V}^N{}_{ik}\mathcal{V}_P{}^{jk}f_{\beta MN}{}^P, \end{aligned} \tag{5.1.23}$$

where ${\mathcal{V}_M}^{ij}$ is defined in terms of t' Hooft matrices G^{ij}_m and ${\mathcal{V}_M}^m$,

$$ {\mathcal{V}_M}^{ij} = \frac{1}{2}{\mathcal{V}_M}^m G^{ij}_m. \tag{5.1.24} $$

The inverse ${\mathcal{V}^M}_{ij}$ is defined similarly,

$$ {\mathcal{V}^M}_{ij} = -\frac{1}{2}{\mathcal{V}^M}_m (G^{ij}_m)^*\,. \tag{5.1.25} $$

t' Hooft matrices G^{ij}_m convert an $SO(6)$ vector index m to an anti-symmetric pair of fundamental $SU(4)$ indices $[ij]$. They satisfy the relations

$$ G_{mij} = -(G^{ij}_m)^* = -\frac{1}{2}\epsilon_{ijkl}G^{kl}_m\,. \tag{5.1.26} $$

The explicit form of t' Hooft matrices are given in Appendix A.3. The scalar potential can be written in terms of the fermion shift functions as

$$ V = -\frac{1}{3}A^{ij}_1 A_{1ij} + \frac{1}{9}A^{ij}_2 A_{2ij} + \frac{1}{2}{A_{2ai}}^j {A_{2a}}^i{}_j\,. \tag{5.1.27} $$

Together with the fermionic supersymmetry transformations, it follows that unbroken supersymmetry corresponds to an eigenvalue of A^{ij}_1, α, satisfying the relation $V_0 = -\frac{\alpha^2}{3}$, where V_0 is the value of the scalar potential at the vacuum.

5.2 RG flows from type IIB non-geometric compactification

We will consider $N = 4$ gauged supergravity with six vector multiplets which arise from a non-geometric compactification of type IIB theory on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$. This involves the fluxes of NS and RR three-form fields (H_3, F_3) and non-geometric fluxes (P, Q).

We follow [46] and restrict ourselves to solutions preserving at least $SO(3)$ subgroup of the full gauge group. The residue $SO(3)$ symmetry is embedding in $SO(6, 6)$ as a diagonal subgroup of $SO(3)\times SO(3)\times SO(3)\times SO(3)$. Note that the four factors of $SO(3)$ are subgroups of $SO(6) \times SO(6) \subset SO(6, 6)$. The 36 scalars in the $SO(6,6)/SO(6)\times SO(6)$ sector transform as $(\mathbf{6},\mathbf{6})$ under $SO(6)\times SO(6)$ compact subgroup. The embedding of $SO(3)\times SO(3)$ into $SO(6)$,

$$\mathbf{6}\rightarrow(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3}), \tag{5.2.1}$$

implies that the 36 scalars transform as

$$(\mathbf{6},\mathbf{6})\rightarrow 4\times(\mathbf{1}+\mathbf{3}+\mathbf{5}), \tag{5.2.2}$$

under the unbroken $SO(3)\sim[SO(3)\times SO(3)\times SO(3)\times SO(3)]_{\text{diag}}$. The four singlets are denoted as $(\varphi_1,\varphi_2,\chi_1,\chi_2)$. We will use the explicit parametrization given in [46]. The coset representative is given by

$$\mathcal{V}_M{}^A=\begin{pmatrix} e^T & 0\\ Be^T & e^{-1}\end{pmatrix}\otimes\mathbb{I}_3 \tag{5.2.3}$$

where the two 2×2 matrices e and B are given by

$$e=e^{\frac{1}{2}(\varphi_1+\varphi_2)}\begin{pmatrix}1 & \chi_2\\ 0 & e^{-\varphi_2}\end{pmatrix},\qquad B=\begin{pmatrix}0 & \chi_1\\ -\chi_1 & 0\end{pmatrix}. \tag{5.2.4}$$

The explicit form of the coset representative is then given by

$$\mathcal{V}_M{}^A=\begin{pmatrix} e^{\frac{1}{2}(\varphi_1+\varphi_2)} & 0 & 0 & 0\\ e^{\frac{1}{2}(\varphi_1+\varphi_2)}\chi_2 & e^{\frac{1}{2}(\varphi_1-\varphi_2)} & 0 & 0\\ e^{\frac{1}{2}(\varphi_1+\varphi_2)}\chi_1\chi_2 & e^{\frac{1}{2}(\varphi_1-\varphi_2)}\chi_1 & e^{-\frac{1}{2}(\varphi_1+\varphi_2)} & -e^{\frac{1}{2}(\varphi_2-\varphi_1)}\chi_2\\ -e^{\frac{1}{2}(\varphi_1+\varphi_2)}\chi_1 & 0 & 0 & e^{\frac{1}{2}(\varphi_2-\varphi_1)}\end{pmatrix}\otimes\mathbb{I}_3\,, \tag{5.2.5}$$

in terms of four $SO(3)$ singlet scalars. Note that, in this form, it is clear that φ_1 and φ_2 are singlets under $SO(3)\times SO(3)\subset[SO(3)\times SO(3)]_{\text{diag}}\times[SO(3)\times SO(3)]_{\text{diag}}$.

With these parametrizations and the definition $\phi^i=(\varphi_g,\varphi_1,\varphi_2,\chi_g,\chi_1,\chi_2)$, the scalar kinetic terms can be found to be

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= -\frac{1}{2}K_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\\ &= -\frac{1}{4}\left(\partial_\mu\varphi_g\partial^\mu\varphi_g+3\partial_\mu\varphi_1\partial^\mu\varphi_1+3\partial_\mu\varphi_2\partial^\mu\varphi_2+e^{2\varphi_g}\partial_\mu\chi_g\partial^\mu\chi_g\right.\\ &\quad\left.+3e^{2\varphi_1}\partial_\mu\chi_1\partial^\mu\chi_1+3e^{2\varphi_2}\partial_\mu\chi_2\partial^\mu\chi_2\right),\end{aligned} \tag{5.2.6}$$

where we have defined the scalar kinematic metric K_{ij} for later convenience in setting the BPS equations.

The four $SO(3)$ singlets in $SO(6,6)/SO(6) \times SO(6)$ correspond to non-compact generators of $SO(2,2) \subset SO(6,6)$ that commute with the $SO(3)$ symmetry. We can split the $SO(6,6)$ indices M into $SO(2,2)$ and $SO(3)$ indices, $M = (AI)$, $A = 1,2,3,4$ and $I = 1,2,3$. This implies that the fundamental representation of $SO(6,6)$ can be decomposed as $(\mathbf{4},\mathbf{3})$ under $SO(2,2) \times SO(3)$. The embedding tensor can be written as

$$f_{\alpha MNP} = f_{\alpha AIBJCK} = \Lambda_{\alpha ABC}\epsilon_{IJK}, \tag{5.2.7}$$

where $\Lambda_{\alpha ABC} = \Lambda_{\alpha(ABC)}$. The quadratic constraints (5.1.10) then read

$$\epsilon^{\alpha\beta}\Lambda_{\alpha AB}{}^{C}\Lambda_{\beta DEC} = 0, \qquad \Lambda_{(\alpha A[B}{}^{C}\Lambda_{\beta)D]EC} = 0\,. \tag{5.2.8}$$

The $SO(6,6)$ fundamental indices M, N can also be decomposed into $(m, \bar{m})$, $m, \bar{m} = 1, \ldots, 6$. The index m is used to label the T^6 of the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ internal manifold. It can split into $m = (a, i)$ such that $a = 1,3,5$ and $i = 2,4,6$. The index $\bar{m}$ can be decomposed similarly, $\bar{m} = (\bar{a}, \bar{i})$. Altogether, the indices A, B can be written such that $A = (1,2,3,4) = (a, i, \bar{a}, \bar{i})$. The indices $I, J = 1,2,3$ label the three T^2's inside $T^6 \sim T^2 \times T^2 \times T^2$.

In this case, the $SO(6,6)$ invariant metric and its inverse are chosen to be

$$\eta_{MN} = \eta^{MN} = \begin{pmatrix} 0 & \mathbb{1}_6 \\ \mathbb{1}_6 & 0 \end{pmatrix}. \tag{5.2.9}$$

Note that some extra projections are needed in order to extract the negative and positive eigenvalues of η_{MN}. For example, to compute the scalar potential defined in (5.1.18), we need to project the second index of $\mathcal{V}_M{}^A$ by using the projection matrix

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathbb{1}_6 & \mathbb{1}_6 \\ \mathbb{1}_6 & \mathbb{1}_6 \end{pmatrix}. \tag{5.2.10}$$

Note also that, in this case, we will use the gauge coupling $g = \frac{1}{2}$ as in [46].

The results from [46] show that the effective $N = 4$ gauged supergravity theory is not unique. We will only consider the gauged supergravity admitting the maximally supersymmetric $N = 4$ vacua. In this case, all the gauge and non-geometric fluxes lead to the following components of the embedding tensor

$$\begin{aligned} f_{-\bar{i}\bar{j}\bar{k}} &= \Lambda_{-444} = -\lambda, \qquad f_{+\bar{a}\bar{b}\bar{c}} = \Lambda_{+333} = \lambda, \\ f_{-\bar{i}\bar{j}k} &= \Lambda_{-244} = -\lambda, \qquad f_{+a\bar{b}\bar{c}} = \Lambda_{+133} = \lambda, \end{aligned} \tag{5.2.11}$$

with λ being constant. The first and second lines correspond to (H_3, F_3) and (P, Q) fluxes, respectively. The gauge group that arises from this embedding is $ISO(3) \times ISO(3) \sim [SO(3) \ltimes T^3] \times [SO(3) \ltimes T^3]$. This gauge group is embedded in $SO(6,6)$ via $SO(3,3) \times SO(3,3)$ subgroup.

By using the above gauging, the scalar potential is

$$\begin{aligned} V =& \frac{1}{32} e^{\varphi_1 - 3\varphi_2 - \varphi_g} \lambda^2 \left[e^{2\varphi_1} - 3e^{2\varphi_2} + 6e^{\varphi_1 + 2\varphi_2 + \varphi_g} - 18 e^{3\varphi_2 + \varphi_g} - 3e^{4\varphi_2 + 2\varphi_g} \right. \\ & - 2e^{2\varphi_1 + 3\varphi_2 + \varphi_g}(1 + 3\chi_1^2) + 3e^{2(\varphi_1 + \varphi_2)}(\chi_1 - \chi_2)^2 - 12 e^{5\varphi_2 + \varphi_g}\chi_2^2 \\ & + 3e^{6\varphi_2}\chi_2^4 + e^{2\varphi_1 + 6\varphi_2}\chi_2^4(\chi_2 - 3\chi_1)^2 + 3e^{2\varphi_1 + 4\varphi_2}\chi_2^2(\chi_2 - 2\chi_1)^2 - 3e^{2(\varphi_2 + \varphi_g)}\chi_g^2 \\ & + 6e^{\varphi_1 + 4\varphi_2 + \varphi_g}(1 + \chi_2^2) + e^{2(\varphi_1 + \varphi_g)}\chi_g^2 + 3e^{2(\varphi_1 + \varphi_2 + \varphi_g)}(\chi_1 - \chi_2)^2\chi_g^2 \\ & + 3e^{6\varphi_2 + 2\varphi_g}\chi_2^2(-1 + \chi_2\chi_g)^2 + 3e^{2(\varphi_1 + 2\varphi_2 + \varphi_g)}(\chi_1 - 2\chi_1\chi_2\chi_g + \chi_2^2\chi_g)^2 \\ & \left. + e^{2(\varphi_1 + 3\varphi_2 + \varphi_g)}[1 + \chi_2^3\chi_g - 3\chi_1\chi_2(-1 + \chi_2\chi_g)]^2 \right]. \end{aligned} \tag{5.2.12}$$

This potential admits a trivial critical point at which all scalars are vanishing. The cosmological constant and the AdS_4 radius are given by

$$V_0 = -\frac{3}{8}\lambda^2, \quad L = \frac{2\sqrt{2}}{\lambda}. \tag{5.2.13}$$

The scalar masses and their corresponding dimensions of the dual operators are given in Table 5.1. Note that we have used a different convention for scalar masses from [46]. The scalar masses given in Table 5.1 are obtained by multiplying the masses given in [46] by 3 due to some difference in convention. This critical point preserves $N = 4$ supersymmetry, which could be checked via the A_1^{ij} tensor, and has $SO(3) \times SO(3)$ symmetry, which is the maximal compact subgroup of $ISO(3) \times ISO(3)$ gauge group.

Scalar fields	m^2L^2	Δ
$\varphi_g,\ \chi_g$	-2	$1,2$
$\varphi_1,\ \varphi_2$	4	4
$\chi_1,\ \chi_2$	0	3

Table 5.1: Scalar masses and their corresponding dimensions at the $N=4$ supersymmetric AdS_4 critical point with $SO(3)\times SO(3)$ symmetry

To set up the BPS equations, we will use the metric ansatz

$$ds^2 = e^{2A(r)}dx^2_{1,2} + dr^2. \tag{5.2.14}$$

We will use Majorana representation for gamma matrices, all γ^μ real and γ^5 purely imaginary. This implies that ϵ_i is a complex conjugate of ϵ^i. All scalars are considered to be functions of only the radial coordinate r. The projection condition,

$$\gamma_{\hat{r}}\epsilon^i = e^{i\Lambda}\epsilon^i\,, \tag{5.2.15}$$

is used to solve the $\delta\chi^i = 0$ and $\delta\lambda^i_a = 0$ equations.

From the equation $\delta\psi^i_\mu = 0$, for $\mu = 0,1,2$, we find

$$A' = \pm W, \qquad e^{i\Lambda} = \pm\frac{\mathcal{W}}{W}, \tag{5.2.16}$$

where $W = |\mathcal{W}|$, and $'$ denotes r-derivative. The superpotential $\mathcal{W}$ is defined by

$$\mathcal{W} = \frac{1}{3}\alpha, \tag{5.2.17}$$

where α is the eigenvalue of A_1^{ij} corresponding to the unbroken supersymmetry. The definite signs for A' equation and $e^{i\Lambda}$ will be chosen such that the $N=4$ critical point identified with $N=4$ SCFT in the UV corresponds to $r\to\infty$.

For all four $SO(3)$ singlet scalars non-vanishing, the $N=4$ is broken into $N=1$ corresponding to the Killing spinor ϵ^1. The superpotential is given by

$$\begin{aligned}\mathcal{W} =& \frac{1}{4\sqrt{2}}e^{\frac{1}{2}(\varphi_1-3\varphi_2-\varphi_g)}\left[e^{\varphi_2}[e^{\varphi_2+\varphi_g}(-e^{\varphi_1+\varphi_2}\lambda - 3\lambda(i+e^{\varphi_1}\chi_1)(i+e^{\varphi_2}\chi_2))\right.\\ &\left.-e^{\varphi_1}\lambda(i+e^{\varphi_2}\chi_2)^3(i+e^{\varphi_g}\chi_g) + 3\lambda(i+e^{\varphi_1}\chi_1)(i+e^{\varphi_2}\chi_2)^2(i+e^{\varphi_g}\chi_g)]\right].\end{aligned} \tag{5.2.18}$$

The real superpotential is then

$$W = \frac{1}{8\sqrt{2}}\lambda e^{\frac{1}{2}(\varphi_1-3\varphi_2-\varphi_g)}\Big[[(-3e^{\varphi_2}(-e^{\varphi_1}+2e^{\varphi_2}+e^{2\varphi_2+\varphi_g})\chi_2 - e^{\varphi_1+3\varphi_2}\chi_2^3$$
$$+ e^{\varphi_g}(e^{\varphi_1}-3e^{\varphi_2})\chi_g + 3e^{2\varphi_2+\varphi_g}(-e^{\varphi_1}+e^{\varphi_2})\chi_2^2\chi_g + 3e^{\varphi_1+\varphi_2}\chi_1(-1$$
$$- e^{\varphi_2+\varphi_g} + e^{2\varphi_2}\chi_2^2 + 2e^{\varphi_2+\varphi_g}\chi_2\chi_g))^2 + [e^{\varphi_1}(-1+3e^{2\varphi_2}\chi_2^2)$$
$$- e^{\varphi_1+\varphi_2+\varphi_g}\chi_2(-3+e^{2\varphi_2}\chi_2^2)\chi_g + e^{\varphi_2}(3+3e^{\varphi_2+\varphi_g} - e^{\varphi_1+2\varphi_2+\varphi_g}$$
$$- 3e^{2\varphi_2}\chi_2^2 - 6e^{\varphi_2+\varphi_g}\chi_2\chi_g + 3e^{\varphi_1}\chi_1(-e^{\varphi_2}(2+e^{\varphi_2+\varphi_g})\chi_2$$
$$-e^{\varphi_g}\chi_g + e^{2\varphi_2+\varphi_g}\chi_2^2\chi_g))]^2]\Big]^{\frac{1}{2}}\ . \tag{5.2.19}$$

The scalar potential can be written in terms of W as

$$V = -2K^{ij}\frac{\partial W}{\partial\phi^i}\frac{\partial W}{\partial\phi^j} - 3W^2. \tag{5.2.20}$$

The BPS equations from $\delta\chi^i = 0$ and $\delta\lambda^i_a = 0$ take the form

$$\phi^{i'} = 2K^{ij}\frac{\partial W}{\partial\phi^j}, \tag{5.2.21}$$

where K^{ij} is the inverse of the kinetic metric defined in (5.2.6). Note that, in this form, the BPS equations solve the second-order field equations.

We will first consider subtruncations which preserve some of $SO(3)$'s, and later the full $SO(3)$ singlet sector.

5.2.1 RG flows with $N = 4$ supersymmetry

We now consider RG flows solutions preserving $N = 4$ supersymmetry to $N = 4$ non-conformal theories in the IR. Consistent truncations, in this case, should satisfy the $\delta\psi^i_\mu = 0, \delta\lambda^i_a = 0$, and $\delta\chi^i = 0$ without setting ϵ^i zero. From the analysis, there are two possibilities which preserve $N = 4$ supersymmetry; by setting $\varphi_{1,2} = \chi_{1,2} = 0$, and by setting $\chi_g = \chi_{1,2} = 0$.

$N = 4$ RG flows by relevant deformations

We will consider a truncation with only φ_g and χ_g non-vanishing. The scalars φ_g and χ_g are corresponding to relevant deformations by operators of dimensions 1 or 2. The consistent truncation of the BPS equations gives

$$
\begin{aligned}
\varphi_g' &= -\frac{\lambda e^{-\frac{\varphi_g}{2}}}{2\sqrt{2}}\frac{(e^{2\varphi_g}+e^{2\varphi_g}\chi_g^2-1)}{\sqrt{(1+e^{\varphi_g})^2+e^{2\varphi_g}\chi_g^2}},\\
\chi_g' &= -\frac{\lambda e^{-\frac{\varphi_g}{2}}}{\sqrt{2}}\frac{\chi_g}{\sqrt{(1+e^{\varphi_g})^2+e^{2\varphi_g}\chi_g^2}},\\
A' &= \frac{\lambda e^{-\frac{\varphi_g}{2}}}{4\sqrt{2}}\sqrt{(1+e^{\varphi_g})^2+e^{2\varphi_g}\chi_g^2}\,. \qquad (5.2.22)
\end{aligned}
$$

Since the φ_g and χ_g are scalars in $SL(2,\mathbb{R})/SO(2)$, they are also singlets under $SO(6,6)$. Therefore, they are invariant under the $SO(3)\times SO(3)$ symmetry. The solutions to the above equations preserve the full $SO(3)\times SO(3)$ symmetry.

Solutions to the BPS equations (5.2.22) preserve $N=4$ supersymmetry. It can be checked that with this truncation, the $\delta\lambda_a^i=0$ satisfies identically. The equations $\delta\psi_\mu^i=0$ and $\delta\chi^i=0$ also held for all ϵ^i satisfying the $\gamma_{\hat{r}}$ projector (5.2.15).

By setting $\chi_g=0$, we obtain simpler BPS equations,

$$\varphi_g' = -\frac{\lambda}{2\sqrt{2}}e^{-\frac{\varphi_g}{2}}(e^{\varphi_g}-1), \qquad (5.2.23)$$

$$A' = \frac{\lambda}{4\sqrt{2}}e^{-\frac{\varphi_g}{2}}(1+e^{\varphi_g}). \qquad (5.2.24)$$

The solution to these equations is given by

$$\varphi_g = \ln\left[e^{\frac{r\lambda}{2\sqrt{2}}+C}-1\right]-\ln\left[e^{\frac{r\lambda}{2\sqrt{2}}+C}+1\right], \qquad (5.2.25)$$

$$A = \ln\left[e^{\frac{r\lambda}{2\sqrt{2}}+C}-1\right]-\frac{r\lambda}{2\sqrt{2}}. \qquad (5.2.26)$$

The integration constant C can be removed by shifting the coordinate r. Note that the integration constant for A is neglected since it can be absorbed by scaling the $dx_{1,2}^2$ coordinates.

At large r, we find the solution behaves as

$$\varphi_g \sim e^{-\frac{\lambda r}{2\sqrt{2}}} \sim e^{-\frac{r}{L}}\,, \qquad (5.2.27)$$

which is expected for the dual operators of dimensions $\Delta = 1, 2$. Near the singularity $r \to -\frac{2\sqrt{2}C}{\lambda}$, we find

$$\varphi_g \sim A \sim \ln\left[r + \frac{2\sqrt{2}C}{\lambda}\right]. \tag{5.2.28}$$

This singularity is physical since the scalar potential is bounded above, $V \to -\infty$. This solution describes an RG flow from the dual $N = 4$ SCFT in the UV to an $N = 4$ non-conformal field theory with unbroken $SO(3) \times SO(3)$ symmetry in the IR. The metric in the IR is given by

$$ds^2 = (\lambda r + 2\sqrt{2}C)^2 dx^2_{1,2} + dr^2. \tag{5.2.29}$$

Note that we have absorbed some constants to $dx^2_{1,2}$ coordinates.

To consider RG flows with $\chi_g \neq 0$, we introduce a new variable ρ, defined by

$$\frac{d\rho}{dr} = \frac{\chi_g}{\sqrt{1 - C\chi_g} + \sqrt{1 - \chi_g(2C + \chi_g)}}. \tag{5.2.30}$$

The BPS equations (5.2.22) can be solved by

$$\varphi_g = -\frac{1}{2}\ln[1 - 2C\chi_g - \chi_g^2], \tag{5.2.31}$$

$$\begin{aligned} A &= -\ln\chi_g + \frac{1}{4}\ln[1 - 2C\chi_g - \chi_g^2] \\ &\quad + \frac{1}{2}\ln\left[1 - C\chi_g + \sqrt{1 - 2C\chi_g - \chi_g^2}\right], \end{aligned} \tag{5.2.32}$$

$$\begin{aligned} \rho\lambda[1 - \chi_g(2C + \chi_g)]^{\frac{3}{4}} &= 4(2)^{\frac{1}{4}}(C + \chi_g - \sqrt{1 + C^2})\left[\frac{1 + C^2 + \sqrt{1 + C^2}(C + \chi_g)}{1 + C^2}\right]^{\frac{3}{4}} \times \\ &\quad {}_2F_1\left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{\chi_g + \sqrt{1 + C^2} - C}{2\sqrt{1 + C^2}}\right), \end{aligned} \tag{5.2.33}$$

where ${}_2F_1$ is the hypergeometric function.

The solution above interpolates between $N = 4$ AdS_4 vacuum in the UV as $r \to \infty$, and a singular geometry at a finite value of r in the IR. There are two possibilities for singular in the IR. The first one is given by

$$\begin{aligned} \chi_g &\sim \chi_0, \qquad \varphi_g \sim -2\ln\left[\frac{\sqrt{2}r\lambda(1 + \chi_0^2) - 4\chi_0 C}{8\chi_0}\right], \\ A &\sim \frac{\chi_0}{\sqrt{1 + \chi_0}}\ln[\sqrt{2}r\lambda(1 + \chi_0^2) - 4\chi_0 C], \end{aligned} \tag{5.2.34}$$

where χ_0 is constant. In this case, we have $\phi_g \to \infty$ and $\chi_g \to \chi_0$ such that $\sqrt{2}r\lambda(1+\chi_0^2) \to 4\chi_0 C$. Note that the constant C in these equations is not the same as in the equations (5.2.31), (5.2.32) and (5.2.33).

Another possibility is given by

$$\begin{aligned} \varphi_g &\sim 2\ln\left(\sqrt{2}\lambda r + 4C\right), \qquad \chi_g \sim \frac{\tilde{C}}{4C+\sqrt{2}\lambda r}, \\ A &\sim \ln\left(\sqrt{2}\lambda r + 4C\right). \end{aligned} \tag{5.2.35}$$

There are singularities at $\sqrt{2}r\lambda \to -4C$, which lead to $\varphi_g \to -\infty$ and $\chi_g \to \pm\infty$ depending on the sigh of the constant $\tilde{C}$. These singularities are physical.

$N = 4$ RG flows by relevant and irrelevant deformations

We now consider $N = 4$ supersymmetric RG flows with $\chi_g = \chi_{1,2} = 0$ truncation. Solutions in this truncation should preserve $SO(3) \times SO(3)$ unbroken symmetry since φ_1 and φ_2 are $SO(3)\times SO(3)$ singlets. Note that the truncation with $\chi_{1,2} = 0$ only consistent with $\chi_g = 0$. This implies that $N = 4$ supersymmetry does not allow the operators dual to χ_g and $\chi_{1,2}$ to be turned on simultaneously with ϕ_g and $\phi_{1,2}$. It is interesting to find a description in the dual $N = 4$ SCFT.

In this truncation, the BPS equations are given by

$$\varphi_g' = \frac{\lambda}{4\sqrt{2}}e^{\frac{1}{2}(\varphi_1-3\varphi_2-\varphi_g)}\left(3e^{\varphi_2}-e^{\varphi_1}-3e^{2\varphi_2+\varphi_g}+e^{\varphi_1+3\varphi_2+\varphi_g}\right), \tag{5.2.36}$$

$$\varphi_1' = \frac{\lambda}{4\sqrt{2}}\left(e^{\varphi_1}-e^{\varphi_2}-e^{2\varphi_2+\varphi_g}+e^{\varphi_1+3\varphi_2+\varphi_g}\right), \tag{5.2.37}$$

$$\varphi_2' = \frac{\lambda}{4\sqrt{2}}\left(e^{\varphi_2}-e^{\varphi_1}-e^{2\varphi_2+\varphi_g}+e^{\varphi_1+3\varphi_2+\varphi_g}\right), \tag{5.2.38}$$

$$A' = -\frac{\lambda}{8\sqrt{2}}e^{\frac{1}{2}(\varphi_1-3\varphi_2-\varphi_g)}\left(e^{\varphi_1}-3e^{\varphi_2}-3e^{2\varphi_2+\varphi_g}+e^{\varphi_1+3\varphi_2+\varphi_g}\right). \tag{5.2.39}$$

To solve these equations, we introduce new variables,

$$\tilde{\varphi}_1 = \varphi_1 - \varphi_2, \qquad \tilde{\varphi}_2 = \varphi_1 + \varphi_2. \tag{5.2.40}$$

The above BPS equations then become

$$\begin{aligned}
\tilde{\varphi}_1' &= \frac{\lambda}{2\sqrt{2}}e^{\frac{1}{2}(\tilde{\varphi}_1+\varphi_g)}(e^{\tilde{\varphi}_1}-1), && (5.2.41)\\
\tilde{\varphi}_2' &= \frac{\lambda}{2\sqrt{2}}e^{\frac{1}{2}(\tilde{\varphi}_2-\varphi_g)}(e^{\tilde{\varphi}_2}-1), && (5.2.42)\\
\varphi_g' &= \frac{\lambda}{4\sqrt{2}}e^{-\frac{\varphi_g}{2}}\left(3e^{\frac{\tilde{\varphi}_2}{2}}-e^{\frac{3}{2}\tilde{\varphi}_2}-3e^{\frac{1}{2}\tilde{\varphi}_1+\varphi_g}+e^{\frac{3}{2}\tilde{\varphi}_1+\varphi_g}\right), && (5.2.43)\\
A' &= -\frac{\lambda}{8\sqrt{2}}e^{-\frac{\varphi_g}{2}}\left(e^{\frac{3}{2}\tilde{\varphi}_1+\varphi_g}+e^{\frac{3}{2}\tilde{\varphi}_2}-3e^{\frac{\tilde{\varphi}_2}{2}}-3e^{\frac{1}{2}\tilde{\varphi}_1+\varphi_g}\right). && (5.2.44)
\end{aligned}$$

Combinations of these equations give

$$\begin{aligned}
\frac{dA}{d\tilde{\varphi}_1}-\frac{1}{2}\frac{d\varphi_g}{d\tilde{\varphi}_1} &= \frac{3-e^{\tilde{\varphi}_1}}{2(e^{\tilde{\varphi}_1}-1)}, && (5.2.45)\\
\frac{dA}{d\tilde{\varphi}_2}+\frac{1}{2}\frac{d\varphi_g}{d\tilde{\varphi}_2} &= \frac{3-e^{\tilde{\varphi}_2}}{2(e^{\tilde{\varphi}_2}-1)}. && (5.2.46)
\end{aligned}$$

The above equations can be solved by

$$\begin{aligned}
\varphi_g &= \frac{3}{2}(\tilde{\varphi}_1-\tilde{\varphi}_2)-\ln\big(1-e^{\tilde{\varphi}_1}\big)+\ln\big(1-e^{\tilde{\varphi}_2}\big), && (5.2.47)\\
A &= \frac{\varphi_g}{2}-\frac{3}{2}\tilde{\varphi}_1+\ln\big(1-e^{\tilde{\varphi}_1}\big). && (5.2.48)
\end{aligned}$$

Note that the integration constant for φ_g is chosen to be zero to obtain the AdS_4 critical point with all scalars vanishing. The integration constant for A is irrelevant.

By substitute the φ_g, the combination of (5.2.41) and (5.2.42) gives

$$\frac{d\tilde{\varphi}_1}{d\tilde{\varphi}_2}=e^{2(\tilde{\varphi}_1-\tilde{\varphi}_2)}, \qquad (5.2.49)$$

which can be solved by

$$\tilde{\varphi}_1=-\frac{1}{2}\ln\big(e^{-2\tilde{\varphi}_2}-C_1\big). \qquad (5.2.50)$$

Near the AdS_4 critical point, $\tilde{\varphi}_1\sim\tilde{\varphi}_2\sim 0$, which give $C_1=0$. This implies that $\tilde{\varphi}_1=\tilde{\varphi}_2$, which leads to $\varphi_2=0$ and $\varphi_g=0$. We see that, in this case, the solution is driven by only the irrelevant operator with dimension $\Delta=4$ dual to φ_1. We expect that the $N=4$ SCFT dual to the AdS_4 critical point to appear in the IR. Note that the equation (5.2.43) is consistent for $\varphi_g=0$ only if $\tilde{\varphi}_1=\tilde{\varphi}_2$.

For $\varphi_g = 0$, the equation (5.2.41) becomes

$$\tilde{\varphi}_1' = \frac{\lambda}{2\sqrt{2}} e^{\frac{\tilde{\varphi}_1}{2}} (e^{\tilde{\varphi}_1} - 1). \tag{5.2.51}$$

This equation can be solved for $\tilde{\varphi}_1(r)$,

$$\frac{\lambda r}{2\sqrt{2}} = 2e^{-\frac{\tilde{\varphi}_1}{2}} + \ln\Big(1 - e^{-\frac{\tilde{\varphi}_1}{2}}\Big) - \ln\Big(1 + e^{-\frac{\tilde{\varphi}_1}{2}}\Big) + C\,. \tag{5.2.52}$$

There is a singularity as $r \to \frac{2\sqrt{2}C}{\lambda}$. Asymptotic behavior of the equation near this singularity gives

$$\begin{aligned} \tilde{\varphi}_1 &\sim \tilde{\varphi}_2 \sim -\frac{2}{3}\ln\frac{3}{2}\left[C - \frac{\lambda r}{2\sqrt{2}}\right], \\ A &\sim -\frac{1}{2}\tilde{\varphi}_1 \sim \frac{1}{3}\ln\frac{3}{2}\left[C - \frac{\lambda r}{2\sqrt{2}}\right]. \end{aligned} \tag{5.2.53}$$

This singularity gives $V \to \infty$; hence the solution is unphysical.

We now consider another consistent subtruncation with $\tilde{\varphi}_1 = 0$ or $\tilde{\varphi}_2 = 0$. This is equivalent to setting $\varphi_2 = \pm\varphi_1$. In this case, the solution is found to be

$$\begin{aligned} \varphi_g &= \pm\ln\left[\frac{e^{-\varphi_1} - C_1 e^{3\varphi_1}}{2 - 2e^{2\varphi_1}}\right], \\ A &= -\frac{7}{2}\varphi_1 + \frac{1}{2}\ln\big(1 - e^{2\varphi_1}\big) + \frac{1}{2}\ln\big(1 - C_1 e^{4\varphi_1}\big), \\ \frac{\lambda\rho}{4\sqrt{2}} &= e^{-\varphi_1} + \frac{1}{2}\ln\big(1 - e^{-\varphi_1}\big) - \frac{1}{2}\ln\big(1 + e^{-\varphi_1}\big) + C, \end{aligned} \tag{5.2.54}$$

where we have introduced the new radial coordinate ρ, defined by $d\rho = e^{-\frac{\varphi_g}{2}}dr$. Note that, in this case, the φ_g is not trivial along the flow. The constant $C_1 = 1$ is chosen to make the solution approach AdS_4 critical point. This gives

$$\varphi_g = \pm\ln\cosh\varphi_1\,. \tag{5.2.55}$$

The solution becomes singular as $\rho \to \frac{4\sqrt{2}C}{3\lambda}$ with $\varphi_1 \to \infty$. Near this singularity, we find

$$\varphi_g \sim \pm\varphi_1 \qquad \varphi_1 \sim -\ln\left[C - \frac{3\lambda\rho}{4\sqrt{2}}\right], \qquad A \sim \frac{1}{2}\ln\left[C - \frac{3\lambda\rho}{4\sqrt{2}}\right]. \tag{5.2.56}$$

These singularities are unphysical since they give $V \to \infty$.

5.2.2 RG flows with $N = 1$ supersymmetry

We now consider RG flows with non-trivial χ_1 and χ_2. This RG flow solutions preserve $N = 1$ supersymmetry and break $SO(3) \times SO(3)$ symmetry to its diagonal subgroup. As in the previous $N = 4$ case, we will first consider consistent subtruncations, then move to more general solutions. Note that the truncation with only φ_2 and χ_2 non-vanishing is not consistent. It is interesting to find a description in the dual field theory.

$N = 1$ RG flows by marginal and irrelevant deformations

We begin with the truncation $\varphi_g = \chi_g = \varphi_2 = \chi_2 = 0$. This RG flow solution is driven by irrelevant and marginal operators with dimensions $\Delta = 4$ and $\Delta = 3$, corresponding to φ_1 and χ_1, respectively. We obtain the BPS equations,

$$\varphi_1' = -\frac{\lambda}{2\sqrt{2}} e^{\frac{\varphi_1}{2}} \frac{(3 - 4e^{\varphi_1} + e^{2\varphi_1} + 9\chi_1^2 e^{2\varphi_1})}{\sqrt{(e^{\varphi_1} - 3)^2 + 9\chi_1^2 e^{2\varphi_1}}}, \tag{5.2.57}$$

$$\chi_1' = -\frac{3\lambda}{2\sqrt{2}} \frac{\chi_1 e^{\frac{\varphi_1}{2}}}{\sqrt{(e^{\varphi_1} - 3)^2 + 9\chi_1^2 e^{2\varphi_1}}}, \tag{5.2.58}$$

$$A' = \frac{\lambda}{4\sqrt{2}} e^{\frac{\varphi_1}{2}} \sqrt{(e^{\varphi_1} - 3)^2 + 9\chi_1^2 e^{2\varphi_1}}. \tag{5.2.59}$$

We are not able to analytically solve these equations in full generality. We will give numerical solutions in this case.

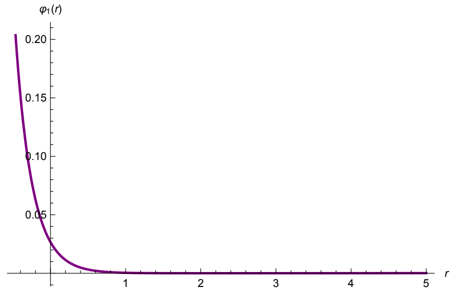
Note that further truncation with $\chi_1 = 0$ yields the BPS equations,

$$\varphi_1' = \frac{\lambda}{2\sqrt{2}} e^{\frac{\varphi_1}{2}} (e^{\varphi_1} - 1) \qquad \text{and} \qquad A' = -\frac{\lambda}{4\sqrt{2}} e^{\frac{\varphi_1}{2}} (e^{\varphi_1} - 3). \tag{5.2.60}$$

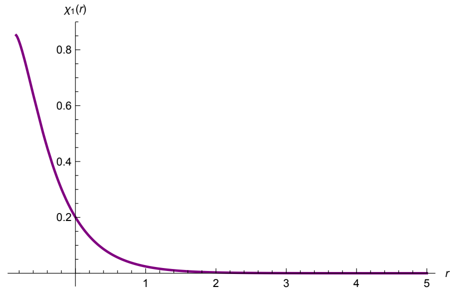
The above equations can be solved by

$$\begin{aligned} A &= -\frac{3}{2}\varphi_1 + \ln(1 - e^{\varphi_1}), \\ \frac{\lambda r}{2\sqrt{2}} &= 2e^{-\frac{\varphi_1}{2}} + \ln\left(1 - e^{-\frac{\varphi_1}{2}}\right) - \ln\left(1 + e^{-\frac{\varphi_1}{2}}\right). \end{aligned} \tag{5.2.61}$$

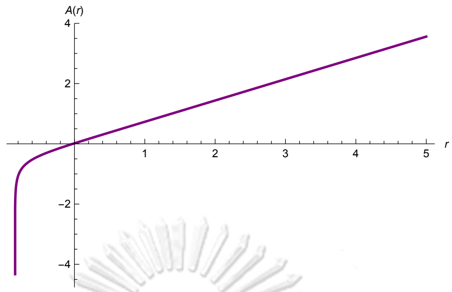
This is the same solution as in the previous section for $\tilde{\varphi}_2 = \tilde{\varphi}_1$. Therefore, we will not further discuss this solution.



(a) Solution for φ_1



(b) Solution for χ_1



(c) Solution for A

Figure 5.1: An $N = 1$ RG flow with irrelevant and marginal deformations from type IIB compactification with $\lambda = 2$

For non-vanishing φ_1 and χ_1, we find an example of these solutions, which is given in Figure 5.1. At large φ_1, the asymptotic behavior of this solution can be determined from the BPS equations,

$$\begin{aligned}
\chi_1 &\sim \chi_0, \qquad \varphi_1 \sim -\frac{2}{3}\ln\left(r\lambda\sqrt{2+18\chi_0^2}-4C_1\right),\\
A &\sim \frac{1}{3}\ln\left(r\lambda\sqrt{2+18\chi_0^2}-4C_1\right),
\end{aligned} \tag{5.2.62}$$

where χ_0 is constant. This singularity gives $V \to \infty$; hence it is unphysical.

$N = 1$ RG flows by relevant, irrelevant and marginal deformations

We now consider RG flow solutions with all six $SO(3)$ scalars non-vanishing. Due to the complication, we will give an explicit form of the BPS equations in Appendix A.4. Note that there could be many possible IR singularities due to the competition between various deformations by operators and vacuum expectation values present in the UV SCFT, similar to the analysis in [84]. We give some examples of the solutions in Figure 5.2, which is given in three different values of λ. The figure shows that the solutions reach the UV SCFT as $r \to \infty$ and approach a singularity at the left end of the flows. This singularity leads to $V \to \infty$; hence it is unphysical.

5.3 Supersymmetric Janus solutions from type IIB compactification

In this section, we discuss another type of solutions with an AdS_3-sliced domain wall ansatz,

$$ds^2 = e^{2A(r)} \left(e^{\frac{2\xi}{\ell}} dx^2_{1,1} + d\xi^2 \right) + dr^2 \,. \tag{5.3.1}$$

This type of solutions, called Janus solutions, describes a conformal interface of co-dimension one within the SCFT dual to the AdS_4 critical point. This solution breaks the three-dimensional conformal symmetry $SO(2,3)$ to $SO(2,2)$ on the $(1,1)$-dimensional interface.

In this case, we need to modify the BPS equations of the previous RG flow cases. In the equation $\delta\psi^i_\mu = 0$, we instead use a $\gamma_{\hat{\xi}}$ projection,

$$\gamma_{\hat{\xi}} \epsilon_i = i\kappa e^{i\Lambda} \epsilon^i. \tag{5.3.2}$$

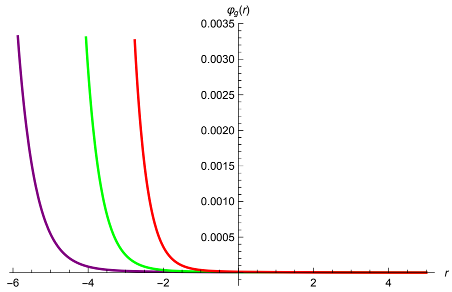
The $\gamma_{\hat{r}}$ projection in the equations $\delta\lambda^i_a = 0$ and $\delta\chi^i = 0$ is still given in (5.2.15), but the phase $e^{i\Lambda}$ modified to

$$e^{i\Lambda} = \frac{\mathcal{W}}{A' + \frac{i\kappa}{\ell} e^{-A}} \,. \tag{5.3.3}$$

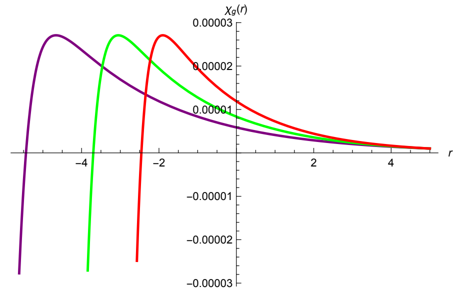
The integrability of $\delta\psi^i_{\hat{0},\hat{1}} = 0$ equations leads to

$$A'^2 + \frac{1}{\ell^2} e^{-2A} = W^2 \,. \tag{5.3.4}$$

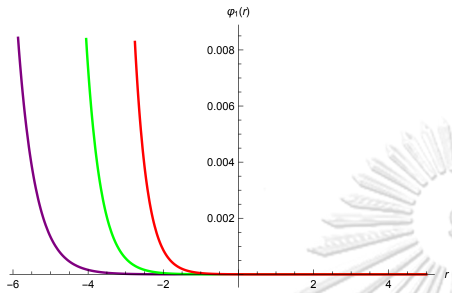
Note that in the limit $\ell \to \infty$, we obtain $A' = \pm W$ and $e^{i\Lambda} = \frac{\mathcal{W}}{A'} = \pm \frac{\mathcal{W}}{W}$ as in the RG flow cases.



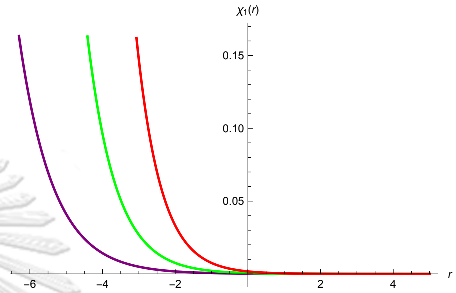
(a) Solution for φ_g



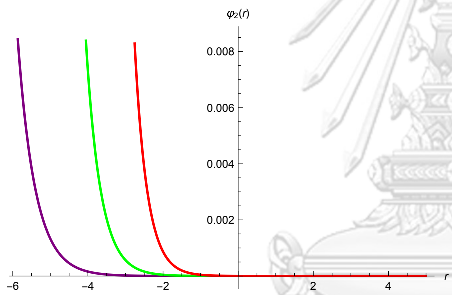
(b) Solution for χ_g



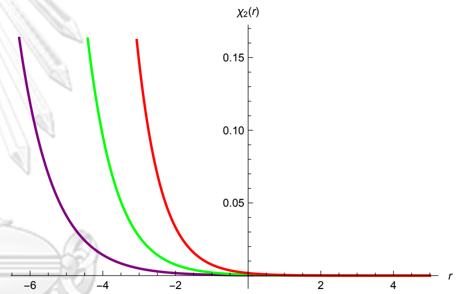
(c) Solution for φ_1



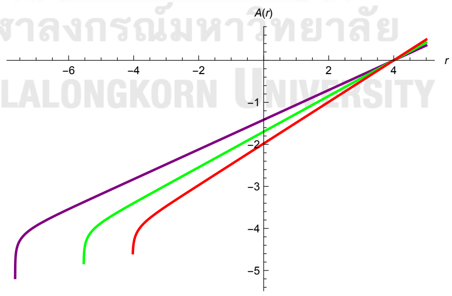
(d) Solution for χ_1



(e) Solution for φ_2



(f) Solution for χ_2



(g) Solution for A

Figure 5.2: An $N = 1$ RG flow from type IIB compactification with all $SO(3)$ singlet scalars and $\lambda = 1$ (purple), $\lambda = 1.2$ (green) and $\lambda = 1.4$ (red)

The constant κ, with $\kappa^2 = 1$, imposes the chirality condition on the Killing spinor corresponding to the unbroken supersymmetry on the $(1,1)$-dimensional interface. Note that the Killing spinors depend on both r and ξ coordinates.

Note that we will not analyze the full BPS equations for supersymmetric Janus solutions since they are usually more complicated than the RG flow cases. We then consider some consistent truncations of the BPS equations. We will consider the cases with (φ_g, χ_g) and (φ_1, χ_1) non-vanishing. As studied in [85] [86] [87], the truncations without the axions or pseudoscalars are not consistent with the Janus BPS equations.

5.3.1 $N = 4$ Janus solution

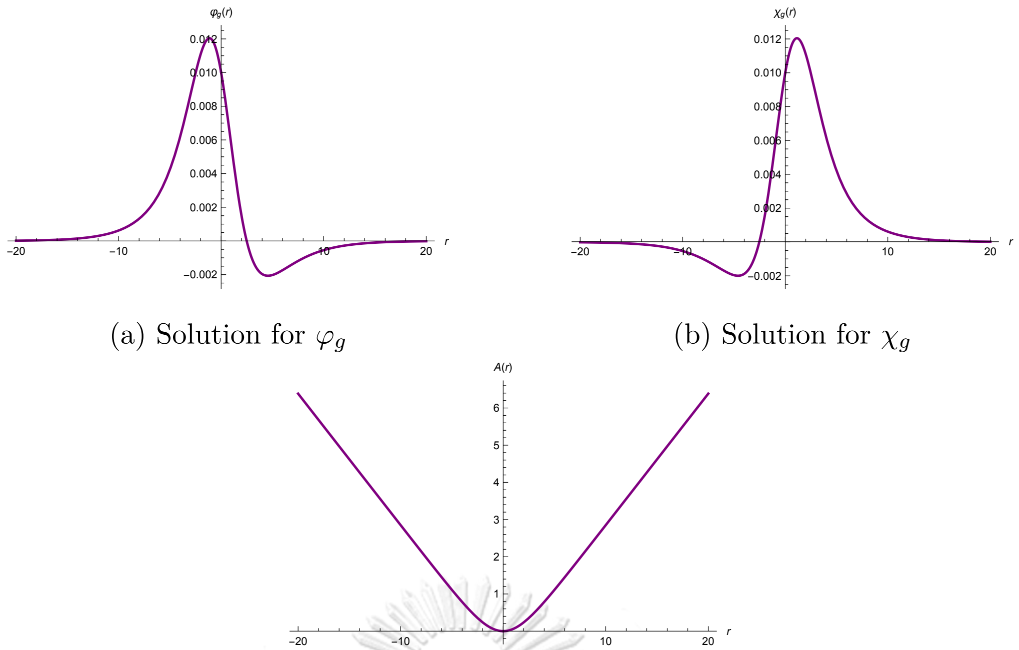
We now consider the case with φ_g and χ_g non-vanishing. By setting $\varphi_1 = \chi_1 = \varphi_2 = \chi_2 = 0$, the equation $\delta\lambda^i_a = 0$ is automatically satisfied. Together with (5.3.4), by solving the real and imaginary parts of the equation $\delta\chi^i = 0$ after applying the phase (5.3.3), we obtain the BPS equations,

$$
\begin{aligned}
\varphi_g' &= -4\frac{A'}{W}\frac{\partial W}{\partial \varphi_g} - 4\kappa e^{-\varphi_g}\frac{e^{-A}}{\ell W}\frac{\partial W}{\partial \chi_g}, \\
&= \frac{-2\ell A'(e^{2\varphi_g} - 1 + 2\chi_g^2 e^{2\varphi_g}) - 4\kappa e^{\varphi_g - A}\chi_g}{\ell\left[(1+e^{\varphi_g})^2 + \chi_g^2 e^{2\varphi_g}\right]}, && (5.3.5) \\
\chi_g' &= -4\frac{A'}{W}e^{-2\varphi_g}\frac{\partial W}{\partial \chi_g} + 4\kappa e^{-\varphi_g}\frac{e^{-A}}{\ell W}\frac{\partial W}{\partial \varphi_g}, \\
&= \frac{2\kappa e^{-A-\varphi_g}(e^{2\varphi_g} - 1 + \chi_g^2 e^{2\varphi_g}) - 4\ell\chi_g A'}{\ell\left[(1+e^{\varphi_g})^2 + \chi_g^2 e^{2\varphi_g}\right]}, && (5.3.6) \\
0 &= A'^2 + \frac{e^{-2A}}{\ell^2} - \frac{\lambda^2}{32}e^{-\varphi_g}\left[(1+e^{\varphi_g})^2 + \chi_g^2 e^{2\varphi_g}\right], && (5.3.7)
\end{aligned}
$$

where the real superpotential is given by

$$
W = \frac{\lambda}{4\sqrt{2}}e^{-\frac{\varphi_g}{2}}\sqrt{(1+e^{\varphi_g})^2 + \chi_g^2 e^{2\varphi_g}}. \qquad (5.3.8)
$$

Note that these equations are similar to the other four-dimensional Janus solutions in [85] [86] [87]. Since the above equations solve the BPS equations for all ϵ^i, $i = 1, \ldots, 4$, any solutions to these equations preserve $N = 4$ supersymmetry. An example of numerical solutions is given in Figure 5.3.



(a) Solution for φ_g

(b) Solution for χ_g

(c) Solution for A

Figure 5.3: An $N = 4$ Janus solution from type IIB compactification within a truncation to φ_g and χ_g with $\lambda = \kappa = 1$ and $\ell = 2\sqrt{2}$

The solution in Figure 5.3 interpolates between $N = 4$ AdS_4 vacua at $r \to \pm\infty$. This can be interpreted as a $(1+1)$-dimensional conformal interface within the $N = 4$ SCFT. The interface preserves $N = (4, 0)$ supersymmetry by choice of the chirality condition $\kappa = 1$. Note that the $SO(3) \times SO(3)$ symmetry remains unbroken along the solution.

5.3.2 $N = 1$ Janus solution

We now consider a truncation with φ_1 and chi_1 non-vanishing. In this case, the $N = 4$ supersymmetry is broken to $N = 1$ supersymmetry and the solution preserves only the $SO(3)$ diagonal subgroup of the full $SO(3) \times SO(3)$ symmetry.

The BPS equations in this truncation are given by

$$\begin{aligned}
\varphi_1' &= -\frac{4}{3}\frac{A'}{W}\frac{\partial W}{\partial \varphi_1} - \frac{4}{3}\kappa e^{-\varphi_1}\frac{e^{-A}}{\ell W}\frac{\partial W}{\partial \chi_1}, \\
&= \frac{2\ell A'(4e^{2\varphi_1} - 3 - 9\chi_1^2 e^{2\varphi_1} - e^{2\varphi_1}) - 12\kappa e^{\varphi_1 - A}\chi_1}{\ell\left[(e^{\varphi_1}-3)^2 + 9\chi_1^2 e^{2\varphi_1}\right]}, \qquad (5.3.9)\\
\chi_1' &= -\frac{4}{3}\frac{A'}{W}e^{-2\varphi_1}\frac{\partial W}{\partial \chi_1} + \frac{4}{3}\kappa e^{-\varphi_1}\frac{e^{-A}}{\ell W}\frac{\partial W}{\partial \varphi_1}, \\
&= \frac{2\kappa e^{-A-\varphi_1}(3 - 4e^{\varphi_1} + e^{2\varphi_1} + 9\chi_1^2 e^{2\varphi_1}) - 12\ell\chi_1 A'}{\ell\left[(e^{\varphi_1}-3)^2 + 9\chi_1^2 e^{2\varphi_1}\right]}, \qquad (5.3.10)\\
0 &= A'^2 + \frac{e^{-2A}}{\ell^2} - \frac{\lambda^2}{32}e^{\varphi_1}\left[(e^{\varphi_1}-3)^2 + 9\chi_1^2 e^{2\varphi_1}\right], \qquad (5.3.11)
\end{aligned}$$

where the real superpotential in this case is

$$W = \frac{\lambda}{4\sqrt{2}}e^{\frac{\varphi_1}{2}}\sqrt{(e^{\varphi_1}-3)^2 + 9\chi_1^2 e^{2\varphi_1}}. \qquad (5.3.12)$$

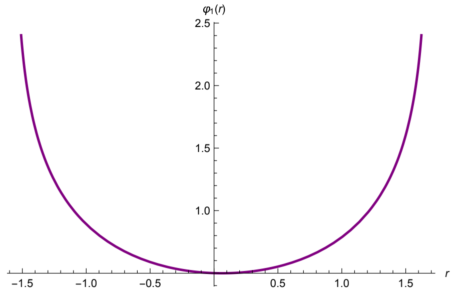
Note that, in this case, after an intensive numerical search, we have not found any solutions interpolating between AdS_4 vacua in the limits $r \to \pm\infty$. All of the solutions found are singular Janus, which connect singular domain walls at two finite values of the radial coordinate. An example of these solutions is shown in Figure 5.4.

The solution in Figure 5.4 should describe a conformal interface between $N = 1$ non-conformal phase of the $N = 4$ SCFT. However, this solution is considered a bad type. An uplift to type IIB is needed to verify if the solution is acceptable in the ten-dimensional context.

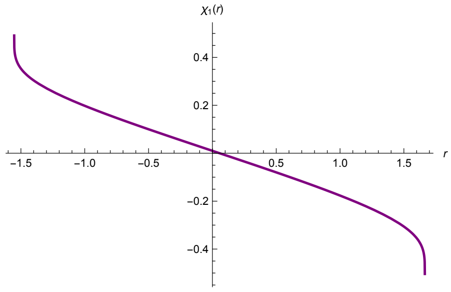
5.4 RG flows from type IIB GKP compactification

There is another class of type IIB compactification, called GKP compactification. It includes the background of (H_3, F_3) and $O3$-plane and/or $D3$-branes in order to cancel a flux-induced tadpole

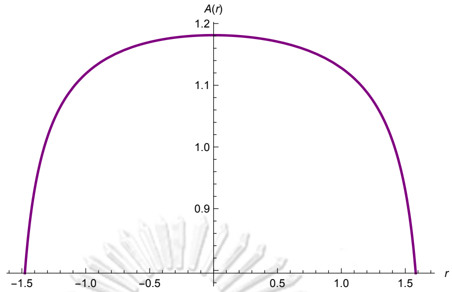
$$\int_{10\mathrm{d}} (H_3 \wedge F_3) \wedge C_4) \Rightarrow N_3 = H_3 \wedge F_3, \qquad (5.4.1)$$



(a) Solution for φ_g



(b) Solution for χ_g



(c) Solution for A

Figure 5.4: An $N = 1$ Janus solution from type IIB compactification within a truncation to φ_1 and χ_1 with $\lambda = \kappa = 1$ and $\ell = 2\sqrt{2}$

for R-R gauge potential C_4. See [88] for more details.

Components of the embedding tensor from this compactification, with an $SO(3)$ truncation of half-maximal supergravity, are given in [46],

$$\begin{aligned} f_{+\bar{i}\bar{j}\bar{k}} = \Lambda_{+333} = -a_0, \qquad f_{-\bar{i}\bar{j}\bar{k}} &= \Lambda_{-333} = -b_0, \\ f_{+\bar{i}\bar{j}\bar{k}} = \Lambda_{+334} = a_1, \qquad f_{-\bar{i}\bar{j}\bar{k}} &= \Lambda_{-334} = b_1, \\ f_{+\bar{i}\bar{j}\bar{k}} = \Lambda_{+344} = -a_2, \qquad f_{-\bar{i}\bar{j}\bar{k}} &= \Lambda_{-344} = -b_2, \\ f_{+\bar{i}\bar{j}\bar{k}} = \Lambda_{+444} = a_3, \qquad f_{-\bar{i}\bar{j}\bar{k}} &= \Lambda_{-444} = b_3. \end{aligned} \tag{5.4.2}$$

By using the same parametrization and procedures as in the previous type IIB non-geometric compactification case, the scalar potential is computed. However, we refrain from giving an explicit form of the scalar potential due to its complexity. At all scalars vanishing, this scalar potential gives a cosmological constant

$$V_0 = \frac{1}{32}((a_0 - b_3)^2 + 3(a_1 + b_2)^2 + 3(a_2 - b_1)^2 + (a_3 + b_0)^2). \tag{5.4.3}$$

As mentioned in [46], due to the stabilization of the imaginary part of the modulus T, the H_3 fluxed background is related to the F_3 via

$$b_3 = a_0, \quad b_2 = -a_1, \quad b_1 = a_2, \quad b_0 = -a_3. \tag{5.4.4}$$

This gives a zero cosmological constant, which corresponds to the Minkowski vacuum. Therefore, we will not give any further analysis for this case.

5.5 RG flows from type IIA geometric compactification

We now consider RG flows from a geometric compactification from type IIA theory. We will repeat the same procedure with the same parametrization from the previous type IIB compactification section. Type IIA compactification involves gauge $(H_3, F_0, F_2, F_4, F_6)$ and the geometric fluxes (ω). The fluxes are more complicated than one in the type IIB case. The components of the embedding tensor given by these fluxes are

$$\begin{aligned}
H_{ijk} &\sim f_{-\bar{a}\bar{b}\bar{c}} = \Lambda_{-333} = \frac{\sqrt{6}}{3}\lambda, \qquad F_{aibjck} \sim f_{+\bar{a}\bar{b}\bar{c}} = \Lambda_{+333} = -\frac{3\sqrt{10}}{2}\lambda, \\
F_{aibj} &\sim f_{+\bar{a}\bar{b}\bar{k}} = \Lambda_{+334} = \frac{\sqrt{6}}{2}\lambda, \qquad F_{ai} \sim f_{+\bar{a}\bar{j}\bar{k}} = \Lambda_{+344} = \frac{\sqrt{10}}{6}\lambda, \\
F_0 &\sim f_{+\bar{i}\bar{j}\bar{k}} = \Lambda_{+444} = \frac{5\sqrt{6}}{6}\lambda, \qquad H_{abk} \sim f_{+\bar{a}\bar{b}k} = \Lambda_{+233} = \frac{\sqrt{6}}{3}\lambda, \\
\omega_{ij}{}^{c} &\sim f_{-\bar{a}\bar{b}\bar{k}} = \Lambda_{-334} = \frac{\sqrt{10}}{3}\lambda, \\
\omega_{ka}{}^{j} = \omega_{bk}{}^{i} &= \omega_{bc}{}^{a} \sim f_{+\bar{a}\bar{j}k} = f_{+\bar{i}\bar{b}k} = f_{+a\bar{b}\bar{c}} = \Lambda_{+234} = \Lambda_{+133} = \sqrt{10}\lambda\,.
\end{aligned} \tag{5.5.1}$$

The $N = 4$ gauged supergravity from type IIA compactification has a non-semisimple gauge group $ISO(3) \ltimes U(1)^6$. It admits the minimal $N = 1$ AdS_4 vacuum at which the gauge group is broken down to $SO(3)$ compact subgroup.

The superpotential for the unbroken $N = 1$ supersymmetry is given by

$$\begin{aligned}\mathcal{W} \;=\;& \frac{\lambda}{24}e^{\frac{1}{2}(\varphi_1-3\varphi_2+\varphi_g)}\Big[2e^{\varphi_1+2\varphi_2-\varphi_g}\Big[3\sqrt{5}i+e^{2\varphi_2}(\sqrt{3}+3\sqrt{5}\chi_2)\Big](i+\chi_g e^{\varphi_g})\\ &-5\sqrt{3}e^{\varphi_1}(i+e^{\varphi_2}\chi_2)^3-3\sqrt{5}e^{\varphi_1+\varphi_2}(i+e^{\varphi_2}\chi_2)^2-9\sqrt{3}ie^{\varphi_1+2\varphi_2}\\ &+18\sqrt{5}e^{2\varphi_2}(i+e^{\varphi_1}\chi_1)(i+e^{\varphi_2}\chi_2)+6\sqrt{3}ie^{3\varphi_2}+9\sqrt{5}e^{\varphi_1+3\varphi_2}\\ &+6\sqrt{3}e^{\varphi_1+3\varphi_2}\chi_1-9\sqrt{3}\chi_2 e^{\varphi_1+3\varphi_2}\Big]. \end{aligned} \tag{5.5.2}$$

The scalar potential can be written in the form

$$V = -\frac{1}{2}K^{ij}\frac{\partial W}{\partial \phi^i}\frac{\partial W}{\partial \phi^j} - \frac{3}{4}W^2\,, \tag{5.5.3}$$

where $W = |\mathcal{W}|$. An explicit form of the scalar potential is given by

$$\begin{aligned}V =& \frac{1}{192}e^{\varphi_1-3\varphi_2-\varphi_g}\lambda^2\big[20e^{2\varphi_1+4\varphi_2}+25e^{2(\varphi_1+\varphi_g)}-240e^{\varphi_1+4\varphi_2+\varphi_g}-180e^{4\varphi_2+2\varphi_g}\\ &+5e^{2(\varphi_1+\varphi_2+\varphi_g)}(1+2\sqrt{15}\chi_2+15\chi_2^2)+12e^{6\varphi_2+2\varphi_g}(1+2\sqrt{15}\chi_2+15\chi_2^2)\\ &+e^{2\varphi_1+6\varphi_2}(4+8\sqrt{15}\chi_2+60\chi_2^2)+e^{2(\varphi_1+2\varphi_2+\varphi_g)}[180\chi_1^2-12\chi_1(3\sqrt{15}\\ &+5\chi_2(2+\sqrt{15}\chi_2)-10\chi_g)+3(9+4\sqrt{15}\chi_2-4\sqrt{15}\chi_g)+5[4\sqrt{15}\chi_2^3\\ &+15\chi_2^4-8\chi_2\chi_g+4\chi_g^2+\chi_2^2(22-4\sqrt{15}\chi_g)]]+e^{2(\varphi_1+3\varphi_2+\varphi_g)}[135\\ &-54\sqrt{15}\chi_2+10\sqrt{15}\chi_2^5+25\chi_2^6+12\sqrt{15}\chi_g-4\chi_2^3(3\sqrt{15}+60\chi_1+20\chi_g)\\ &+8\chi_2(3\chi_1+\chi_g)(18+3\sqrt{15}\chi_1+\sqrt{15}\chi_g)-5\chi_2^4(-21+12\sqrt{15}\chi_1\\ &+4\sqrt{15}\chi_g)+4[9\chi_1(\sqrt{15}+\chi_1)+6\chi_1\chi_g+\chi_g^2]+\chi_2^2\Big[-9-40\sqrt{15}\chi_g\\ &+60[9\chi_1^2-2\chi_1(\sqrt{15}-3\chi_g)+\chi_g^2]\Big]\Big]\Big]. \end{aligned} \tag{5.5.4}$$

At the trivial $N = 1$ AdS_4 critical point, the cosmological constant is

$$V_0 = -\lambda^2. \tag{5.5.5}$$

At this critical point, masses of the six $SO(3)$ singlet scalars are

$$m^2L^2 : \qquad 0, -2, 4 \pm \sqrt{6}, \frac{1}{3}(47 \pm \sqrt{159}). \tag{5.5.6}$$

Note that these masses agree with [46] after the change to our convention with a factor of 3.

In this case, we have verified that the BPS equations can be written as

$$A' = W, \qquad \varphi^{i'} = K^{ij}\frac{\partial W}{\partial \phi^j}\,. \tag{5.5.7}$$

Note that we refrain from giving explicit forms of the BPS equations since they are far more complicated than the previous cases. We are not able to find any consistent subtruncation for this set of equations. An example of RG flows from $N = 1$ SCFT dual to the AdS_4 critical point to non-conformal $N = 1$ field theory in the IR is shown in Figure 5.5.

The numerical analysis near the singularity shown in Figure 5.5 leads to $V \to \infty$. This implies that the singularity is unphysical.

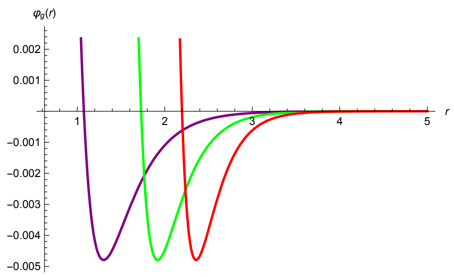
5.6 RG flows from $SO(4) \times SO(4)$ gauged supergravity

We will now consider RG flows from $N = 4$ supergravity coupled to six vector multiplets, with semisimple gauge groups in the form of simple product $G_1 \times G_2$. One of the two factors is embedded as the electric part of $SO(3,3) \subset SO(6,6)$ while another one is embedded as the magnetic part in another $SO(3,3)$ subgroup of the $SO(6,6)$. We will study the cases of $G_1, G_2 = SO(4), SO(3,1), SO(2,2)$. This makes six different gauge groups, $SO(4)\times SO(4)$, $SO(3,1)\times SO(3,1)$, $SO(2,2)\times SO(2,2)$, $SO(4) \times SO(3,1)$, $SO(4) \times SO(2,2)$, and $SO(3,1) \times SO(2,2)$. The embedding tensors for these gauge groups are given in [89].

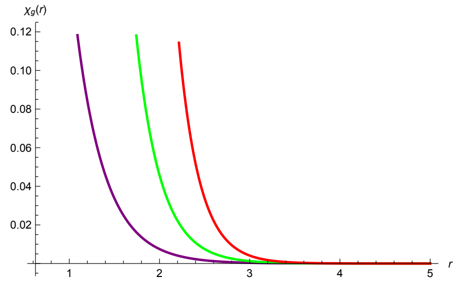
We first consider RG flows from $N = 4$ gauged supergravity with $SO(4) \times SO(4)$ gauge group. Non-zero components of the embedding tensor for this gauge group are given by

$$\begin{aligned} f_{+\hat{m}\hat{n}\hat{p}} &= \sqrt{2}(g_1 - \tilde{g}_1)\epsilon_{\hat{m}\hat{n}\hat{p}}, \qquad f_{+\hat{a}\hat{b}\hat{c}} = \sqrt{2}(g_1 + \tilde{g}_1)\epsilon_{\hat{a}\hat{b}\hat{c}}, \\ f_{-\tilde{m}\tilde{n}\tilde{p}} &= \sqrt{2}(g_2 - \tilde{g}_2)\epsilon_{\tilde{m}\tilde{n}\tilde{p}}, \qquad f_{-\tilde{a}\tilde{b}\tilde{c}} = \sqrt{2}(g_2 + \tilde{g}_2)\epsilon_{\tilde{a}\tilde{b}\tilde{c}}, \end{aligned} \tag{5.6.1}$$

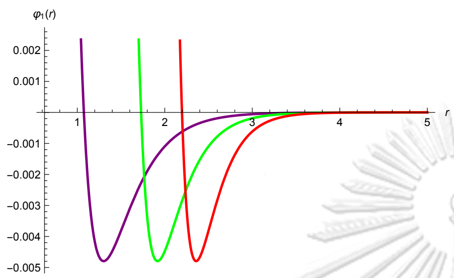
where $M = (m,a) = (\hat{m}, \tilde{m}, \hat{a}, \tilde{a})$, with $\hat{m} = 1,2,3$, $\tilde{m} = 4,5,6$, $\hat{a} = 7,8,9$, and $\tilde{a} = 10,11,12$.



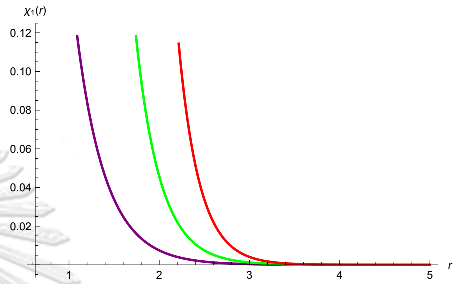
(a) Solution for φ_g



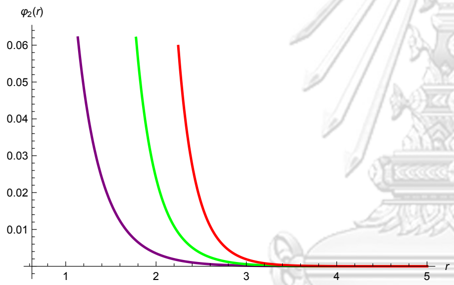
(b) Solution for χ_g



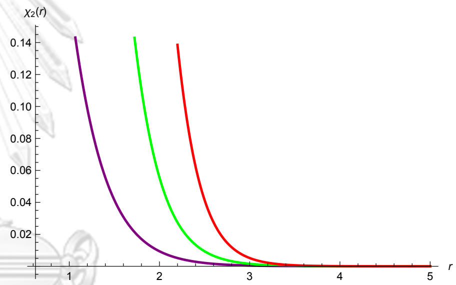
(c) Solution for φ_1



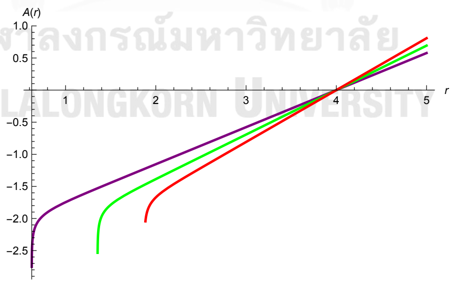
(d) Solution for χ_1



(e) Solution for φ_2



(f) Solution for χ_2



(g) Solution for A

Figure 5.5: An $N = 1$ RG flow from type IIA compactification with $\lambda = 1$ (purple), $\lambda = 1.2$ (green) and $\lambda = 1.4$ (red)

The procedure is the same as in the previous type IIB and type IIA sections. However, we now use a different parametrization involving non-compact generators of $SO(6,6)$,

$$Y_{ma} = e_{m,a+6} + e_{a+6,m}, \tag{5.6.2}$$

where the 12×12 matrices e_{MN} are defined by

$$(e_{MN})_{PQ} = \delta_{MP}\delta_{NQ}\,. \tag{5.6.3}$$

There are 36 scalars in $SO(6,6)/SO(6) \times SO(6)$ transforming as $(\mathbf{6},\mathbf{6})$ under the compact gauge group $SO(6) \times SO(6)$. These scalars transform as

$$(\mathbf{6},\mathbf{6}) \rightarrow (\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{3},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{3}), \tag{5.6.4}$$

under the gauge group $SO(4)_+ \times SO(4)_- \sim SO(3)^1_+ \times SO(3)^2_+ \times SO(3)^1_- \times SO(3)^2_-$. We will consider singlet scalars under the diagonal subgroup $SO(4)_{\text{inv}} \sim [SO(3)^1_+ \times SO(3)^2_+]_D \times [SO(3)^1_- \times SO(3)^2_-]_D$. The scalars transform as

$$2(\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{5}) + (\mathbf{5},\mathbf{1}) + 2(\mathbf{3},\mathbf{3}), \tag{5.6.5}$$

under the $SO(4)_{\text{inv}}$. Solutions with these two singlet scalars should describe RG flows breaking the $SO(4) \times SO(4)$ symmetry to $SO(4)_{\text{inv}} \sim SO(3) \times SO(3)$ symmetry, with each scalar corresponding to breaking each of the $SO(4)$'s. The two singlets correspond to the $SO(6,6)$ non-compact generators

$$\hat{Y}_1 = Y_{11} + Y_{22} + Y_{33}, \qquad \hat{Y}_2 = Y_{44} + Y_{55} + Y_{66}. \tag{5.6.6}$$

The coset representative is given by

$$L = e^{\phi_1 \hat{Y}_1} e^{\phi_2 \hat{Y}_2}\,. \tag{5.6.7}$$

Together with the scalars from the supergravity multiplet, the scalar potential is given by

$$\begin{aligned} V &= \frac{1}{8}e^{-\phi-6\phi_1-6\phi_2}\left[e^{\phi+3\phi_2}\left[e^{\phi+3\phi_2}g_1^2+e^{\phi+12\phi_1+3\phi_2}\tilde{g}_1^2-3e^{\phi+4\phi_1+3\phi_2}(2g_1^2+\tilde{g}_1^2)\right.\right. \\ &\quad -3e^{\phi+8\phi_1+3\phi_2}(g_1^2+2\tilde{g}_1^2)+2e^{3\phi_1}g_1[g_2(1+3e^{4\phi_2})-e^{2\phi_2}\tilde{g}_2(3+e^{4\phi_2})] \\ &\quad +6e^{7\phi_1}g_1[g_2(1+3e^{4\phi_2})-e^{2\phi_2}\tilde{g}_2(3+e^{4\phi_2})]-6e^{5\phi_1}\tilde{g}_1[g_2(1+3e^{4\phi_2}) \\ &\quad \left.-e^{2\phi_2}\tilde{g}_2(3+e^{4\phi_2})-2e^{9\phi_1}\tilde{g}_1[g_2(1+3e^{4\phi_2})-e^{2\phi_2}\tilde{g}_2(3+e^{4\phi_2})]\right] \\ &\quad +e^{6\phi_1}\left[[1-3e^{4\phi_2}(2+e^{4\phi_2})](1+e^{2\phi}\chi^2)g_2^2+16g_2\tilde{g}_2e^{6\phi_2}(1+e^{2\phi}\chi^2)\right. \\ &\quad +e^{4\phi_2}\left[e^{2\phi}(16e^{2\phi_2}g_1\tilde{g}_1-3\tilde{g}_2^2\chi^2-6\tilde{g}_2^2e^{4\phi_2}\chi^2+e^{8\phi_2}\chi^2\tilde{g}_2^2)\right. \\ &\quad \left.\left.\left.+(e^{8\phi_2}-6e^{4\phi_2}-3)\tilde{g}_2^2\right]\right]\right]. \end{aligned} \tag{5.6.8}$$

This scalar potential admits four supersymmetric AdS_4 critical points. One is a maximally supersymmetric AdS_4 critical point with $SO(4)\times SO(4)$ symmetry at

$$\chi=\phi_1=\phi_2=0, \qquad \phi=\ln\left|\frac{g_2-\tilde{g}_2}{g_1-\tilde{g}_1}\right|. \tag{5.6.9}$$

We denote this AdS_4 vacuum by critical point I. Without loss of generality, we can shift the dilaton such that the critical point occurs at $\phi=0$. This implies

$$\tilde{g}_2=g_1+g_2-\tilde{g}_1\,. \tag{5.6.10}$$

The cosmological constant and the AdS_4 radius at this critical point are given by

$$V_0=-6(g_1-\tilde{g}_1)^2 \quad \text{and} \quad L=\frac{1}{\sqrt{2}(\tilde{g}_1-g_1)} \tag{5.6.11}$$

where we have assumed that $\tilde{g}_1>g_1$. All scalars have masses $m^2L^2=-2$, which correspond to relevant operators of dimension $\Delta=1,2$.

The remaining three supersymmetric AdS_4 critical points are listed below:

- II. Critical point with $SO(3)_+\times SO(4)_-$ symmetry

$$\begin{aligned} \phi &= \ln\left[\frac{2\sqrt{g_1\tilde{g}_1}}{g_1+\tilde{g}_1}\right], \qquad \phi_1=\frac{1}{2}\ln\left[\frac{g_1}{\tilde{g}_1}\right], \qquad \phi_2=0, \\ V_0 &= -\frac{3(g_1+\tilde{g}_1)(g_1-\tilde{g}_1)^2}{\sqrt{g_1\tilde{g}_1}}, \qquad L=\frac{(g_1\tilde{g}_1)^{\frac{1}{4}}}{(\tilde{g}_1-g_1)\sqrt{g_1+\tilde{g}_1}}\,. \end{aligned} \tag{5.6.12}$$

- III. Critical point with $SO(4)_+ \times SO(3)_-$ symmetry

$$\begin{aligned}
\phi &= -\ln\left[\frac{2\sqrt{g_2\tilde{g}_2}}{g_2+\tilde{g}_2}\right], \qquad \phi_2 = \frac{1}{2}\ln\left[\frac{g_2}{\tilde{g}_2}\right], \qquad \phi_1 = 0, \\
V_0 &= -\frac{3(g_2+\tilde{g}_2)(g_1-\tilde{g}_1)^2}{\sqrt{g_2\tilde{g}_2}}, \qquad L = \frac{(g_2\tilde{g}_2)^{\frac{1}{4}}}{(\tilde{g}_1-g_1)\sqrt{g_2+\tilde{g}_2}}\,.
\end{aligned} \tag{5.6.13}$$

- IV. Critical point with $SO(4)_{\text{inv}} \sim SO(3)_+ \times SO(3)_-$ symmetry

$$\begin{aligned}
\phi &= \ln\left[\sqrt{\frac{g_1\tilde{g}_1}{g_2\tilde{g}_2}}\frac{g_2+\tilde{g}_2}{g_1+\tilde{g}_1}\right], \qquad \phi_1 = \frac{1}{2}\ln\left[\frac{g_1}{\tilde{g}_1}\right], \qquad \phi_2 = \frac{1}{2}\ln\left[\frac{g_2}{\tilde{g}_2}\right], \\
V_0 &= -\frac{3(g_2+\tilde{g}_2)^2(g_1-\tilde{g}_1)^2}{2\sqrt{g_1\tilde{g}_1g_2\tilde{g}_2}}, \quad L = \frac{\sqrt{2}(g_1\tilde{g}_1g_2\tilde{g}_2)^{\frac{1}{4}}}{(\tilde{g}_1-g_1)\sqrt{(g_1+\tilde{g}_1)(g_2+\tilde{g}_2)}}.
\end{aligned} \tag{5.6.14}$$

Note that $\tilde{g}_2$ in the above equations can be replaced by (5.6.10) to make the critical point I occurs at $\chi = \phi = \phi_1 = \phi_2 = 0$. Scalar masses and the corresponding dimensions at each critical point are shown in Table 5.2, Table 5.3, and Table 5.4, respectively.

Scalar field representations	m^2L^2	Δ
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$-2_{\times 2}$	$1,2$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	4	4
$(\mathbf{3},\mathbf{1},\mathbf{1})$	$0_{\times 3}$	3
$(\mathbf{1},\mathbf{3},\mathbf{3})$	$0_{\times 9}$	3
$(\mathbf{5},\mathbf{1},\mathbf{1})$	$-2_{\times 5}$	$1,2$
$(\mathbf{3},\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{3},\mathbf{1})$	$-2_{\times 18}$	$1,2$

Table 5.2: Scalar masses and the corresponding dimensions of the dual operators at the $N = 4$ supersymmetric AdS_4 critical point with $SO(3)_+ \times SO(4)_-$ symmetry (Critical point II)

As in Table 5.2, there are three massless scalars in the representation $(\mathbf{3},\mathbf{1},\mathbf{1})$ which are Goldstone bosons indicating the symmetry breaking $SO(4)_- \times SO(4)+$ to $SO(3)_+ \times SO(4)_-$. Similarly, Table 5.3 shows Goldstone bosons live in representation $(\mathbf{1},\mathbf{1},\mathbf{3})$ corresponding to the symmetry breaking $SO(4)_- \times SO(4)+$ to

Scalar field representations	m^2L^2	Δ
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$-2_{\times 2}$	$1,2$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	4	4
$(\mathbf{1},\mathbf{1},\mathbf{3})$	$0_{\times 3}$	3
$(\mathbf{3},\mathbf{3},\mathbf{1})$	$0_{\times 9}$	3
$(\mathbf{1},\mathbf{1},\mathbf{5})$	$-2_{\times 5}$	$1,2$
$(\mathbf{1},\mathbf{3},\mathbf{3})+(\mathbf{3},\mathbf{1},\mathbf{3})$	$-2_{\times 18}$	$1,2$

Table 5.3: Scalar masses and the corresponding dimensions of the dual operators at the $N = 4$ supersymmetric AdS_4 critical point with $SO(4)_+ \times SO(3)_-$ symmetry (Critical point III)

Scalar field representations	m^2L^2	Δ
$(\mathbf{1},\mathbf{1})$	$-2_{\times 2}$	$1,2$
$(\mathbf{1},\mathbf{1})$	$4_{\times 2}$	4
$(\mathbf{1},\mathbf{5})+(\mathbf{5},\mathbf{1})$	$-2_{\times 10}$	$1,2$
$(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{1})$	$0_{\times 6}$	3
$(\mathbf{3},\mathbf{3})$	$0_{\times 18}$	3

Table 5.4: Scalar masses and the corresponding dimensions of the dual operators at the $N = 4$ supersymmetric AdS_4 critical point with $SO(4)_{\text{inv}} \sim SO(3) \times SO(3)$ symmetry (Critical point IV)

$SO(4)_+ \times SO(3)_-$. This implies that ϕ_1 and ϕ_2 are corresponding to the deformation with the symmetry breaking $SO(4)_+ \to SO(3)_+$ and $SO(4)_- \to SO(3)_-$, respectively. Table 5.4 also shows six massless scalars indicating both of the $SO(4)$'s breaking simultaneously. Note that the remaining massless scalars in each case are corresponding to marginal deformations in the SCFTs. These deformations should break some amount of supersymmetry since the $N = 4$ AdS_4 vacua have no moduli preserving $N = 4$ supersymmetry [90]. Note that the vacuum structure of this gauged supergravity is similar to two copies of $SO(3) \times SO(3) \sim SO(4)$ gauge group considered in the previous $N = 3$ gauged supergravity Section 4.2.1.

5.6.1 RG flows between $N = 4$ SCFTs

We now consider holographic RG flows interpolating between the previous AdS_4 vacua, I, II, III, and IV. By following the same procedure as in the previous type IIB section, except that for the following the superpotential $\mathcal{W}$ is defined by

$$\mathcal{W} = \frac{2}{3}\alpha, \tag{5.6.15}$$

where α is the eigenvalue of A_1^{ij} corresponding to the unbroken supersymmetry.

For the case with $SO(4)_{\text{inv}}$ singlet scalars, A_1^{ij} is diagonal and takes the form of $\mathcal{W}$,

$$A_1^{ij} = \frac{3}{2}\mathcal{W}\delta^{ij}, \tag{5.6.16}$$

where the superpotential is given by

$$\begin{aligned}\mathcal{W} \;=\; & \frac{1}{4\sqrt{2}}e^{-\frac{\phi}{2}-3\phi_1-3\phi_2}\big[3i\tilde{g}_1 e^{\phi+2\phi_1+3\phi_2} + i\tilde{g}_1 e^{\phi+6\phi_1+3\phi_2} \\ & + e^{3\phi_1}(i+e^{\phi}\chi)[g_2(1+3e^{4\phi}) - \tilde{g}_2 e^{2\phi_2}(3+e^{4\phi_2})] \\ & - ig_1 e^{\phi+3\phi_2} - 3ig_1 e^{\phi+4\phi_1+3\phi_2}\big].\end{aligned} \tag{5.6.17}$$

The equation $\delta\lambda^i_a = 0$ gives the BPS equations

$$\phi_1' = -\frac{i}{2\sqrt{2}}e^{i\Lambda}e^{\frac{\phi}{2}-3\phi_1}(e^{4\phi_1}-1)(e^{2\phi_1}\tilde{g}_1 - g_1), \tag{5.6.18}$$

$$\phi_2' = -\frac{1}{2\sqrt{2}}e^{i\Lambda}e^{-\frac{\phi}{2}-3\phi_2}(e^{4\phi_2}-1)(e^{2\phi_2}\tilde{g}_2 - g_2)(e^{\phi}\chi - i). \tag{5.6.19}$$

The consistency of the equation (5.6.18) implies that the phase $e^{i\Lambda}$ is purely imaginary, $e^{i\Lambda} = \pm i$. With this choice, the equation (5.6.19) requires $\chi = 0$. Note that, with $e^{i\Lambda} = \pm i$, the superpotential given in (5.6.17) is purely imaginary, which is in agreement with the phase defined in (5.2.16).

We will choose the definite sign to identify the critical point I at the limit $r \to \infty$. Together with the equation $\delta\chi^i = 0$, the BPS equations with the above conditions can be written in term of W as

$$\phi' = -4\frac{\partial W}{\partial \phi}, \qquad \chi' = 0, \qquad \phi_1' = -\frac{2}{3}\frac{\partial W}{\partial \phi_1}, \qquad \phi_2' = -\frac{2}{3}\frac{\partial W}{\partial \phi_2}, \qquad A' = W. \tag{5.6.20}$$

Explicit forms of these equations are given by

$$\begin{aligned}
\phi_1' &= -\frac{1}{2\sqrt{2}}e^{\frac{\phi}{2}-3\phi_1}(e^{4\phi_1}-1)(e^{2\phi_1}\tilde{g}_1-g_1), &(5.6.21)\\
\phi_2' &= \frac{1}{2\sqrt{2}}e^{-\frac{\phi}{2}-3\phi_2}(e^{4\phi_2}-1)(e^{2\phi_2}\tilde{g}_2-g_2), &(5.6.22)\\
\phi' &= -\frac{1}{2\sqrt{2}}e^{-\frac{\phi}{2}-3\phi_1-3\phi_2}\big[3\tilde{g}_1e^{\phi+2\phi_1+3\phi_2}-g_1e^{\phi+3\phi_2}-3g_1e^{\phi+4\phi_1+3\phi_2}\\
&\quad+\tilde{g}_1e^{\phi+6\phi_1+3\phi_2}+e^{3\phi_1}[\tilde{g}_2e^{2\phi_2}(3+e^{4\phi_2})-g_2(1+3e^{4\phi_2})]\big], &(5.6.23)\\
A' &= \frac{1}{4\sqrt{2}}e^{-\frac{\phi}{2}-3\phi_1-3\phi_2}\big[3\tilde{g}_1e^{\phi+2\phi_1+3\phi_2}-g_1e^{\phi+3\phi_2}-3g_1e^{\phi+4\phi_1+3\phi_2}\\
&\quad+\tilde{g}_1e^{\phi+6\phi_1+3\phi_2}+e^{3\phi_1}[g_2(1+3e^{4\phi_2})-\tilde{g}_2e^{2\phi_2}(3+e^{4\phi_2})]\big]. &(5.6.24)
\end{aligned}$$

The scalar potential can be written in terms of W as

$$V = 4\left(\frac{\partial W}{\partial\phi}\right)^2+\frac{2}{3}\left(\frac{\partial W}{\partial\phi_1}\right)^2+\frac{2}{3}\left(\frac{\partial W}{\partial\phi_2}\right)^2-3W^2\,. \tag{5.6.25}$$

We will not give the explicit form of the scalar potential due to its complexity.

Solution near the $SO(4)\times SO(4)$ critical point at $r\to\infty$ behaves as

$$\phi,\phi_1,\phi_2\sim e^{-\frac{r}{L_I}}, \tag{5.6.26}$$

which is expected since all of these scalars are dual to operators of dimensions $\Delta = 1,2$. Note that L_I is the AdS_4 radius at the critical point I given in (5.6.11).

RG flow from critical point I to critical point II

The flow between critical points I and II can be solved from the BPS equations (5.6.21) to (5.6.24) with $\phi_2=0$. By considering ϕ and A as functions of ϕ_1, the BPS equations can be written as

$$\begin{aligned}
\frac{d\phi}{d\phi_1} &= -\frac{g_1(1+3e^{4\phi_1})+e^{2\phi_1}[4(g_2-\tilde{g}_2)e^{\phi_1-\phi}-\tilde{g}_1(e^{4\phi_1}+3)]}{(e^{4\phi_1}-1)(\tilde{g}_1e^{2\phi_1}-g_1)} &(5.6.27)\\
\frac{dA}{d\phi_1} &= \frac{g_1(1+3e^{4\phi_1})-e^{2\phi_1}[4(g_2-\tilde{g}_2)e^{\phi_1-\phi}+\tilde{g}_1(3+e^{4\phi_1})]}{2(e^{4\phi_1}-1)(\tilde{g}_1e^{2\phi_1}-g_1)}. &(5.6.28)
\end{aligned}$$

The first equation can be solved by

$$\phi = \ln\left[\frac{g_2-\tilde{g}_2+C_1(e^{4\phi_1}-1)}{\tilde{g}_1e^{3\phi_1}-g_1e^{\phi_1}}\right]. \tag{5.6.29}$$

The integration constant $C_1 = \frac{\tilde{g}_1^2(g_2-\tilde{g}_2)}{\tilde{g}_1^2-g_1^2}$ is chosen to make the solution interpolate between the $SO(4)\times SO(4)$ critical point with $\phi=0$ and the $SO(3)_+\times SO(4)_-$ with $\phi = \ln\left[\frac{\sqrt{g_1\tilde{g}_1}}{g_1+\tilde{g}_1}\right]$. The equation (5.6.29) is then

$$\phi = \ln\left[\frac{(g_2-\tilde{g}_2)(g_1+\tilde{g}_1 e^{2\phi_1})e^{-\phi_1}}{\tilde{g}_1^2-g_1^2}\right]. \tag{5.6.30}$$

With this solution, the second equation in (5.6.28) can be solved by

$$A = \frac{\phi_1}{2} - \ln\big(1-e^{4\phi_1}\big) + \ln\big(g_1-\tilde{g}_1 e^{2\phi_1}\big) + \frac{1}{2}\ln\big(g_1+\tilde{g}_1 e^{2\phi_1}\big). \tag{5.6.31}$$

Note that an irrelevant additive integration constant has been removed.

To find a flow solution for ϕ_1, we introduce a new radial coordinate $\tilde{r}$, defined by $\frac{d\tilde{r}}{dr} = e^{\frac{\phi}{2}}$. The equation (5.6.21) becomes

$$\frac{d\phi_1}{d\tilde{r}} = -\frac{1}{2\sqrt{2}} e^{-3\phi_1}(e^{4\phi_1}-1)(\tilde{g}_1 e^{2\phi_1}-g_1). \tag{5.6.32}$$

The solution to the above equation is given by

$$\begin{aligned}\frac{(g_1^2-\tilde{g}_1^2)\tilde{r}}{\sqrt{2}} &= (g_1-\tilde{g}_1)\tan^{-1} e^{\phi_1} - (g_1+\tilde{g}_1)\tanh^{-1} e^{\phi_1} \\ &\quad +2\sqrt{g_1\tilde{g}_1}\tanh^{-1}\left[\sqrt{\frac{\tilde{g}_1}{g_1}} e^{\phi_1}\right].\end{aligned} \tag{5.6.33}$$

Near critical point II at $r\to-\infty$, the scalars ϕ and ϕ_1 behave as

$$\phi \sim e^{-\frac{r}{L_{\mathrm{II}}}} \qquad \text{and} \qquad \phi_1 \sim e^{\frac{r}{L_{\mathrm{II}}}}. \tag{5.6.34}$$

This implies that the operator dual to ϕ_1 becomes irrelevant with dimension $\Delta=4$. The operator dual to ϕ remains relevant with dimensions $\Delta = 1, 2$.

RG flow from critical point II to critical point IV

We now consider the flow between critical points II and IV. By using similar analysis with $\phi_1 = \frac{1}{2}\ln\frac{g_1}{\tilde{g}_1}$ along the flow, we find solutions of ϕ and A as functions of ϕ_2,

$$\phi = \ln\left[\frac{2\sqrt{g_1\tilde{g}_1}(g_2+\tilde{g}_2)e^{2\phi_2}}{(g_1+\tilde{g}_1)(g_2+\tilde{g}_2 e^{2\phi_2})}\right], \tag{5.6.35}$$

$$A = \frac{\phi_2}{2} - \ln\big(1-e^{4\phi_2}\big) + \ln\big(\tilde{g}_2 e^{2\phi_2} - g_2\big) + \frac{1}{2}\ln\big(g_2+\tilde{g}_2 e^{2\phi_2}\big). \tag{5.6.36}$$

The solution for ϕ_2 is given by

$$\frac{(g_1-\tilde{g}_1)(g_2+\tilde{g}_2)}{\sqrt{2}}\bar{r} = (\tilde{g}_1-g_1)\tan^{-1}e^{\phi_2}-(g_2+\tilde{g}_2)\tanh^{-1}e^{\phi_2} +2\sqrt{g_2\tilde{g}_2}\tanh^{-1}\left[e^{\phi_2}\sqrt{\frac{\tilde{g}_2}{g_2}}\right], \quad (5.6.37)$$

where $\bar{r}$ is defined by $\frac{d\bar{r}}{dr}=e^{-\frac{\phi}{2}}$.

RG flow from critical points I to critical point III

The flow solution between critical points I and III is given by

$$\phi_1 = 0, \quad (5.6.38)$$

$$\phi = \ln\left[\frac{e^{\phi_2}(g_2+\tilde{g}_2)}{g_2+e^{2\phi_2}\tilde{g}_2}\right], \quad (5.6.39)$$

$$A = \frac{\phi_2}{2}-\ln\big(1-e^{4\phi_2}\big)+\ln\big(e^{2\phi_2}\tilde{g}_2-g_2\big)+\frac{1}{2}\ln\big(g_2+\tilde{g}_2e^{2\phi_2}\big), \quad (5.6.40)$$

$$-\frac{(g_2^2-\tilde{g}_2^2)\bar{r}}{\sqrt{2}} = (g_2-\tilde{g}_2)\tan^{-1}e^{\phi_2}-(g_2+\tilde{g}_2)\tanh^{-1}e^{\phi_2} +2\sqrt{g_2\tilde{g}_2}\tanh^{-1}\left[\sqrt{\frac{\tilde{g}_2}{g_2}}e^{\phi_2}\right]. \quad (5.6.41)$$

RG flow from critical points III to critical point IV

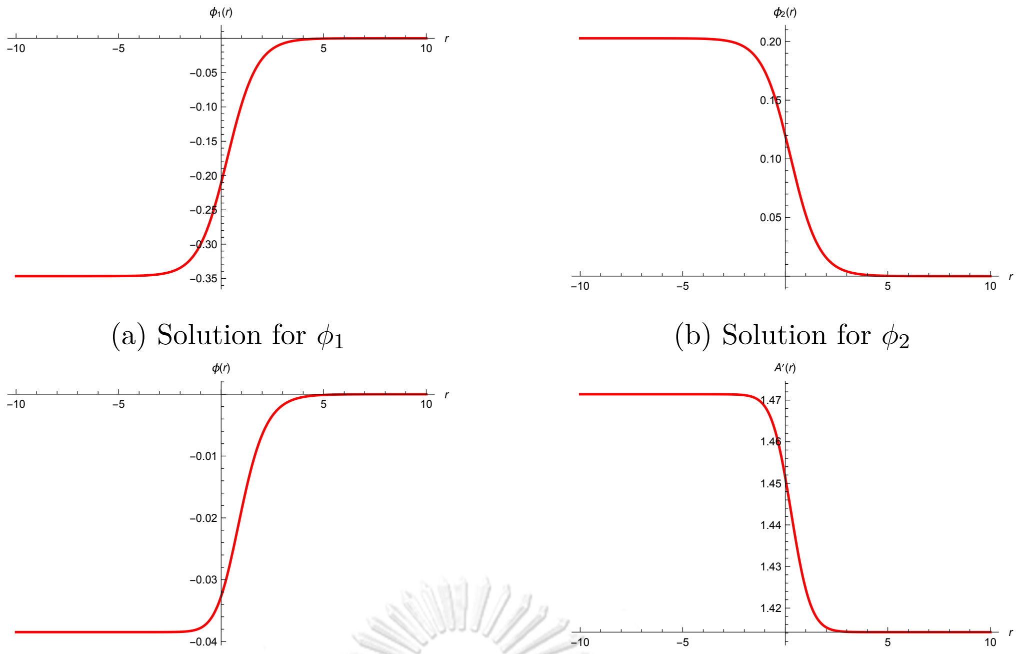
The flow solution between critical points III and IV is given by

$$\phi_2 = \frac{1}{2}\ln\left[\frac{g_2}{\tilde{g}_2}\right], \quad (5.6.42)$$

$$\phi = \ln\left[\frac{e^{-\phi_1}(g_2+\tilde{g}_2)(g_1+\tilde{g}_1e^{2\phi_1})}{2(g_1+\tilde{g}_1)\sqrt{g_2\tilde{g}_2}}\right], \quad (5.6.43)$$

$$A = \frac{\phi_1}{2}-\ln\big(1-e^{4\phi_1}\big)+\ln\big(g_1-e^{2\phi_1}\tilde{g}_1\big)+\frac{1}{2}\ln\big(g_1+\tilde{g}_1e^{2\phi_1}\big), \quad (5.6.44)$$

$$\frac{(g_1^2-\tilde{g}_1^2)\tilde{r}}{\sqrt{2}} = (g_1-\tilde{g}_1)\tan^{-1}e^{\phi_1}-(g_1+\tilde{g}_1)\tanh^{-1}e^{\phi_1} +2\sqrt{g_1\tilde{g}_1}\tanh^{-1}\left[\sqrt{\frac{\tilde{g}_1}{g_1}}e^{\phi_1}\right]. \quad (5.6.45)$$



(a) Solution for ϕ_1 (b) Solution for ϕ_2

(c) Solution for ϕ (d) Solution for A'

Figure 5.6: A numerical RG flow from critical point I to critical point IV with $g_1 = 1$, $\tilde{g}_1 = \tilde{g}_2 = 2$ and $g_2 = 3$

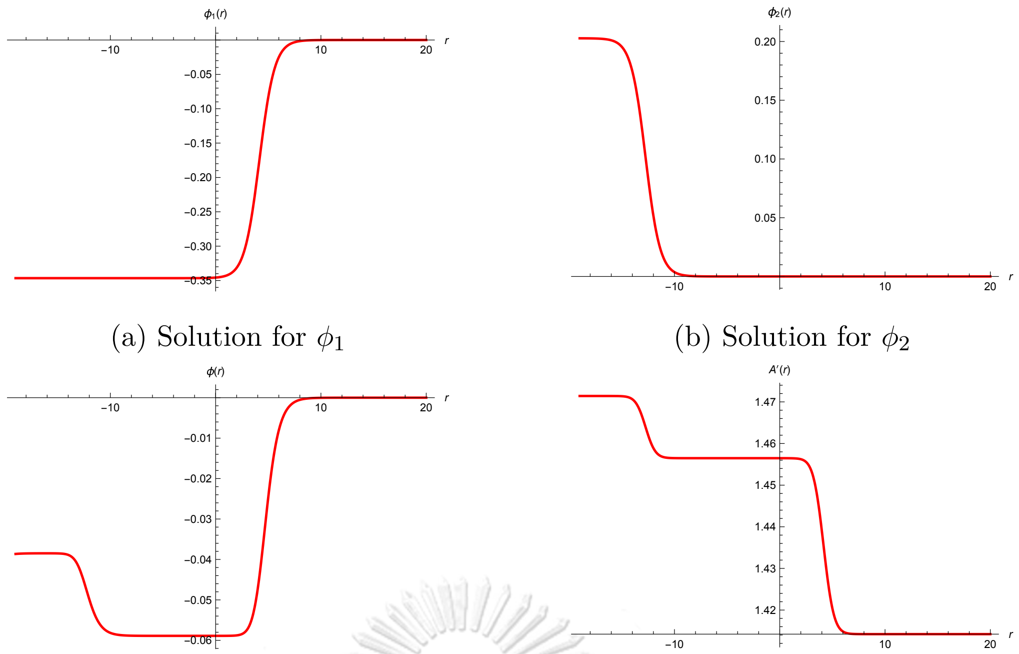
RG flow from critical points I to critical point IV

In this case, we give a numerical solution describing RG flow from critical point I to critical point IV in Figure 5.6.

RG flow from critical points I to critical point II to critical point IV

The numerical solution describing RG flow from critical point I to critical point II, then to critical point IV is given in Figure 5.7.

Note that, by an intensive numerical search, we have not found a solution flow from critical point I to critical point III to critical point IV. It would be interesting to see this description in the dual $N = 4$ SCFT.



(a) Solution for ϕ_1

(b) Solution for ϕ_2

(c) Solution for ϕ

(d) Solution for A'

Figure 5.7: A numerical RG flow from critical point I to critical point II to critical point IV with $g_1 = 1$, $\tilde{g}_1 = \tilde{g}_2 = 2$ and $g_2 = 3$

5.6.2 RG flows to $N = 4$ non-conformal theory

There is another consistent truncation for $N = 4$ supergravity with $SO(4) \times SO(4)$ gauge group, which is obtained by setting $\phi_1 = \phi_2 = 0$. In this case, there are only the scalars from the supergravity multiplet. As mentioned in the previous section, the axion χ cannot be turned on simultaneously with ϕ_1 and ϕ_2.

In this case, the superpotential is given by

$$\mathcal{W} = \frac{1}{\sqrt{2}} e^{-\frac{\phi}{2}} \left[(g_2 - \tilde{g}_2)\chi e^{\phi} - i(\tilde{g}_2 - g_2 + e^{\phi}(g_1 - \tilde{g}_1)) \right]. \tag{5.6.46}$$

The scalar potential is given by

$$\begin{aligned} V &= 4\left(\frac{\partial W}{\partial \phi}\right)^2 + 4e^{-2\phi}\left(\frac{\partial W}{\partial \chi}\right)^2 - 3W^2 \\ &= -(g_1 - \tilde{g}_1)^2 e^{-\phi}[1 + 4e^{\phi} + e^{2\phi}(1 + \chi^2)], \end{aligned} \tag{5.6.47}$$

where we have use $\tilde{g}_2 = g_1 + g_2 - \tilde{g}_1$. This scalar potential admits only one AdS_4 critical point at $\phi = \chi = 0$, which is the same as critical point I in the previous section.

The BPS equations, in this truncation, are given by

$$\phi' = -4\frac{\partial W}{\partial \phi} = -\frac{\sqrt{2}(g_1-\tilde{g}_1)[e^{2\phi}(1+\chi^2)-1]}{\sqrt{(1+e^{\phi})^2+e^{2\phi}\chi^2}}, \tag{5.6.48}$$

$$\chi' = -4e^{-2\phi}\frac{\partial W}{\partial \chi} = -\frac{2\sqrt{2}(g_1-\tilde{g}_1)e^{-\frac{\phi}{2}}\chi}{\sqrt{(1+e^{\phi})^2+e^{2\phi}\chi^2}}, \tag{5.6.49}$$

$$A' = W = \frac{1}{\sqrt{2}}(g_1-\tilde{g}_1)e^{-\frac{\phi}{2}}\sqrt{(1+e^{\phi})^2+e^{2\phi}\chi^2}\,. \tag{5.6.50}$$

Near the AdS_4 critical point, we find

$$\phi \sim \chi \sim e^{-\frac{r}{L_{\text{I}}}}, \tag{5.6.51}$$

which implies that ϕ and χ are corresponding to the relevant operators of dimensions $\Delta = 1, 2$.

By considering ϕ and A as functions of χ, we can combine the BPS equations into

$$\frac{d\phi}{d\chi} = \frac{e^{2\phi}(1+\chi^2)-1}{2\chi}, \tag{5.6.52}$$

$$\frac{dA}{d\chi} = -\frac{1+2e^{\phi}+e^{2\phi}(1+\chi^2)}{4\chi}. \tag{5.6.53}$$

The above equations can be solved by

$$\phi = -\frac{1}{2}\ln\big(1-2C\chi-\chi^2\big), \tag{5.6.54}$$

$$A = -\ln\chi + \frac{1}{2}\ln[1-C\chi+\sqrt{1-2C\chi-\chi^2}] + \frac{1}{4}\ln\big(1-2C\chi-\chi^2\big). \tag{5.6.55}$$

Note that we have neglected an additive integration constant for A. However, we keep the constant $C \neq 0$ to obtain the correct behavior near the AdS_4 critical point as given in (5.6.51).

To solve for χ as a function of r, we need to substitute (5.6.54) into (5.6.49), we find the equation for χ',

$$\chi' = \frac{2(g_1-\tilde{g}_1)(1-2C\chi-\chi^2)^{3/4}\chi}{(1-C\chi+(1-2C\chi-\chi^2)^{1/2})^{1/2}}. \tag{5.6.56}$$

We are not able to solve the equation (5.8) for χ analytically. We will then look for a numerical solution. The equation (5.6.54) gives $\phi \to 0$ in the limit $\chi \to 0$, which is corresponding to the AdS_4 critical point I. The equation (5.6.54) also has singularities at χ_0, which $1 - 2C\chi_0 - \chi_0^2 = 0$. This implies that ϕ flows from the value of 0 at the AdS_4 critical point to the singular value $\phi \to \infty$, while χ flows from the value of 0 to $\chi_0 = -C \pm \sqrt{1+C^2}$. Examples of numerical solutions for χ are given in Figure 5.8. Note that the equation (5.8) also gives $\chi' = 0$ at the value of $\chi = \chi_0$, which is in agreement with the fact that χ flows between the values of 0 to χ_0.

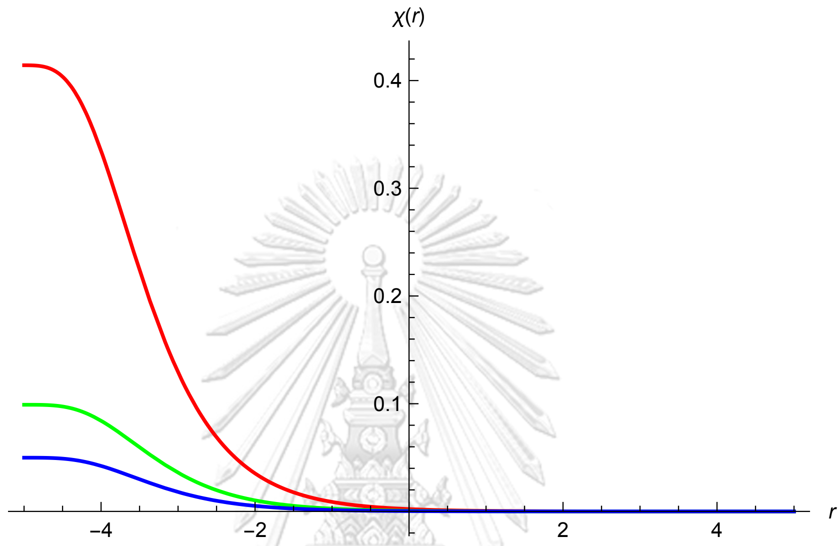


Figure 5.8: Solutions for χ with $g_1 = 1$, $\tilde{g}_1 = 2$ and $\chi_0 = \sqrt{1+C^2} - C$ for $C = 1$ (red), $C = 5$ (green) and $C = 10$ (blue) in $SO(4) \times SO(4)$ gauging

Near the singularity $\phi \to \infty$ and $\chi \to \chi_0$, we find

$$\chi - \chi_0 \sim r^4, \qquad \phi \sim -\ln r^2, \qquad A \sim \ln r\,. \tag{5.6.57}$$

The metric (5.2.14) is then

$$ds^2 = r^2 dx_{1,2}^2 + dr^2\,. \tag{5.6.58}$$

We find that $V \to -\infty$ near this singularity, regardless of the value of χ_0. Hence, the singularity is physical. This solution should describe an RG flow from the $N = 4$ SCFT in the UV to a non-conformal field theory in the IR. Note that this RG flow preserves the $SO(4) \times SO(4)$ symmetry and $N = 4$ Poincare supersymmetry in three dimensions.

5.7 RG flows from $SO(3,1)\times SO(3,1)$ gauged supergravity

We now consider $N=4$ supergravity coupled to six vector multiplets with the gauge group $SO(3,1)\times SO(3,1)$. The non-vanishing components of the embedding tensor are given by

$$\begin{aligned}
f_{+123} &= f_{+189} = f_{+729} = -f_{+783} = \frac{1}{\sqrt{2}}(g_1-\tilde{g}_1),\\
f_{+789} &= f_{+183} = f_{+723} = -f_{+129} = \frac{1}{\sqrt{2}}(g_1+\tilde{g}_1),\\
f_{-456} &= f_{-4,11,12} = f_{-10,5,12} = -f_{-10,11,6} = \frac{1}{\sqrt{2}}(g_2-\tilde{g}_2),\\
f_{-10,11,12} &= f_{-4,11,7} = f_{-10,5,6} = -f_{-45,12} = \frac{1}{\sqrt{2}}(g_2+\tilde{g}_2).
\end{aligned} \tag{5.7.1}$$

We will follow the same procedure as in the previous sections.

To parametrize the coset $SO(6,6)/SO(6)\times SO(6)$, we will use scalars that are invariant under the $SO(3)\times SO(3)\subset SO(3,1)\times SO(3,1)$ subgroup. In this case, there are two $SO(3)\times SO(3)$ singlets which corresponding to $SO(6,6)$ non-compact generators

$$\tilde{Y}_1 = Y_{11}+Y_{22}-Y_{33}, \qquad \tilde{Y}_2 = Y_{44}+Y_{55}-Y_{66}\,. \tag{5.7.2}$$

The coset representative is parametrized by

$$L = e^{\phi_1\tilde{Y}_1}e^{\phi_2\tilde{Y}_2}\,. \tag{5.7.3}$$

The scalar potential is given by

$$\begin{aligned}
V &= \frac{1}{8}e^{-\phi-6\phi_1-6\phi_2}\left[2g_2e^{\phi+3\phi_1+9\phi_2}(e^{6\phi_1}g_1-3g_1e^{2\phi_1}-\tilde{g}_1+3\tilde{g}_1e^{4\phi_1})\right.\\
&\quad -6g_2e^{\phi+3\phi_1+5\phi_2}(g_1e^{6\phi_1}-3g_1e^{2\phi_1}-\tilde{g}_1+3\tilde{g}_1e^{4\phi_1})\\
&\quad +6\tilde{g}_2e^{\phi+3\phi_1+7\phi_2}(3\tilde{g}_1e^{4\phi_1}-3g_1e^{2\phi_1}+g_1e^{6\phi_1}-\tilde{g}_1)\\
&\quad -2\tilde{g}_2e^{\phi+3\phi_1+3\phi_2}(g_1e^{6\phi_1}-3g_1e^{2\phi_1}-\tilde{g}_1+3\tilde{g}_1e^{4\phi_1})\\
&\quad +3e^{6\phi_1+4\phi_2}[e^{4\phi_2}(2g_2^2-\tilde{g}_2^2)(1+\chi^2e^{2\phi})-3(g_2^2-2\tilde{g}_2^2)(1+\chi^2e^{2\phi})]\\
&\quad +g_2^2e^{6\phi_1+6\phi_2}(1+\chi^2e^{2\phi})+\tilde{g}_2^2e^{6\phi_1}(1+\chi^2e^{2\phi})\\
&\quad +e^{6\phi_2}\left[3(2g_1^2-\tilde{g}_1^2)e^{2\phi+8\phi_1}+16e^{6\phi_1}[g_2\tilde{g}_2+e^{2\phi}(g_1\tilde{g}_1+g_2\tilde{g}_2\chi^2)]\right.\\
&\quad \left.\left.+g_1^2e^{2\phi+12\phi_1}\tilde{g}_1^2e^{2\phi}-3(g_1^2-2\tilde{g}_1^2)e^{\phi+4\phi_1}\right]\right]. \qquad (5.7.4)
\end{aligned}$$

This scalar potential admits an AdS_4 critical point at

$$\begin{aligned}
\phi &= \frac{1}{2}\ln\left[\frac{g_1\tilde{g}_1(g_2^2+\tilde{g}_2^2)^2(g_1^2+\tilde{g}_1^2)^2}{g_2\tilde{g}_2}\right], \qquad \chi=0,\\
\phi_1 &= \frac{1}{2}\ln\left[-\frac{\tilde{g}_1}{g_1}\right], \qquad \phi_2=\frac{1}{2}\ln\left[-\frac{\tilde{g}_2}{g_2}\right]. \qquad (5.7.5)
\end{aligned}$$

Note that this critical point preserves $N=4$ supersymmetry and $SO(3)\times SO(3)$ maximal subgroup of the $SO(3,1)\times SO(3,1)$ gauge group. We can also shift the scalars to make the critical point occurs at $\phi=\chi=\phi_1=\phi_2=0$, which can be done by setting

$$\tilde{g}_1=-g_1, \qquad \tilde{g}_2=-g_2, \qquad g_2=-g_1\,. \qquad (5.7.6)$$

The cosmological constant and the AdS_4 radius with these values are given by

$$V_0=-6g_1^2 \qquad \text{and} \qquad L^2=\frac{1}{2g_1^2}\,. \qquad (5.7.7)$$

Note that there is another choice with

$$\tilde{g}_1=-g_1, \quad \tilde{g}_2=-g_2, \qquad (5.7.8)$$

at which the critical point occurs at $\phi=\chi=\phi_1=\phi_2=0$ but this is a dS_4 with $V_0=2g_1^2$.

Scalar masses and their corresponding dimensions of the dual operators are given in Table 5.5. Note that these scalars are in the representation of the $SO(3)\times$

$SO(3)$. We see that there are two singlets scalars in the representation $(\mathbf{1},\mathbf{1})$, corresponding to ϕ_1 and ϕ_2, dual to the irrelevant operators of dimensions $\Delta = 4$, and six Goldstone bosons in the representation $(\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1})$. This indicates that the $SO(3,1) \times SO(3,1)$ breaks down into its maximal compact subgroup $SO(3) \times SO(3)$.

Scalar field representations	m^2L^2	Δ
$(\mathbf{1},\mathbf{1})$	$-2_{\times 2}$	$1, 2$
$(\mathbf{1},\mathbf{1})$	$4_{\times 2}$	4
$(\mathbf{1},\mathbf{5}) + (\mathbf{5},\mathbf{1})$	$-2_{\times 10}$	$1, 2$
$(\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1})$	$0_{\times 6}$	3
$(\mathbf{3},\mathbf{3})$	$0_{\times 18}$	3

Table 5.5: Scalar masses and their corresponding dimensions at the $N = 4$ supersymmetric AdS_4 critical point with $SO(3) \times SO(3)$ symmetry for $SO(3,1) \times SO(3,1)$ gauge group

5.7.1 RG flows without vector multiplet scalars

Since there is only one supersymmetric critical point, there is no supersymmetric flow between the dual SCFTs. We will instead consider RG flows from the SCFT dual to $N = 4$ AdS_4 critical point. For a truncation $\phi_1 = \phi_2 = 0$, with $SO(3) \times SO(3)$ symmetry, the superpotential in this truncation is given by

$$\mathcal{W} = \frac{3i}{2\sqrt{2}} g_1 e^{-\frac{\phi}{2}} [1 + e^{\phi}(1 - i\chi)]. \tag{5.7.9}$$

The scalar potential can be written in the form of $W = |\mathcal{W}|$,

$$\begin{aligned} V &= \frac{16}{9}\left(\frac{\partial W}{\partial \phi}\right)^2 + \frac{16}{9} e^{-2\phi}\left(\frac{\partial W}{\partial \chi}\right)^2 - \frac{4}{3}W^2 \\ &= -g_1^2 e^{-\phi}[1 + 4e^{\phi} + e^{2\phi}(1 + \chi^2)]. \end{aligned} \tag{5.7.10}$$

The BPS equations in this truncation are given by

$$\phi' = -\frac{8}{3}\frac{\partial W}{\partial \phi} = -\frac{\sqrt{2}g_1 e^{-\frac{\phi}{2}}[e^{2\phi}(1+\chi^2)-1]}{\sqrt{(1+e^{\phi})^2+e^{2\phi}\chi^2}}, \tag{5.7.11}$$

$$\chi' = -\frac{8}{3}e^{-2\phi}\frac{\partial W}{\partial \chi} = -\frac{2\sqrt{2}g_1 e^{-\frac{\phi}{2}}\chi}{\sqrt{(1+e^{\phi})^2+e^{2\phi}\chi^2}}, \tag{5.7.12}$$

$$A' = W = \frac{3}{2\sqrt{2}}g_1 e^{-\frac{\phi}{2}}\sqrt{(1+e^{\phi})^2+e^{2\phi}\chi^2}\,. \tag{5.7.13}$$

The solution to the above equations near AdS_4 critical point behaves

$$\phi \sim \chi \sim e^{-\sqrt{2}g_1 r} \sim e^{-\frac{r}{L}}, \tag{5.7.14}$$

which is expected for the dual operators of dimensions $\Delta = 1, 2$.

Similar to the $SO(4) \times SO(4)$ case in Section 5.6.2, the solution to (5.7.11) to (5.7.13) is given by

$$\phi = -\frac{1}{2}\ln\big(1-\chi^2-2C\chi\big), \tag{5.7.15}$$

$$A = -\frac{3}{2}\ln\chi + \frac{3}{8}\ln\big(1-2C\chi-\chi^2\big) + \frac{3}{4}\ln\Big(1-C\chi+\sqrt{1-2C\chi-\chi^2}\Big). \tag{5.7.16}$$

We give an example of numerical solutions for χ in Figure 5.9. As shown in the figure, the solution for χ flows from $\chi = 0$ to $\chi = \chi_0 = -C \pm \sqrt{1+C^2}$. With these values of χ, the equation (5.7.15) implies that ϕ flows from $\phi \to 0$ to $\phi \to \infty$. This singularity gives $V_0 \to -\infty$, hence it is physical. This solution should describe an RG flow from $N = 4$ SCFT in the UV with $SO(3) \times SO(3)$ symmetry to a non-conformal field theory in the IR.

5.7.2 RG flows with vector multiplet scalars

We now consider RG flows with non-vanishing ϕ_1 and ϕ_2. Note that in this case, we need to set $\chi = 0$ to make the solutions of the BPS equations consistent with

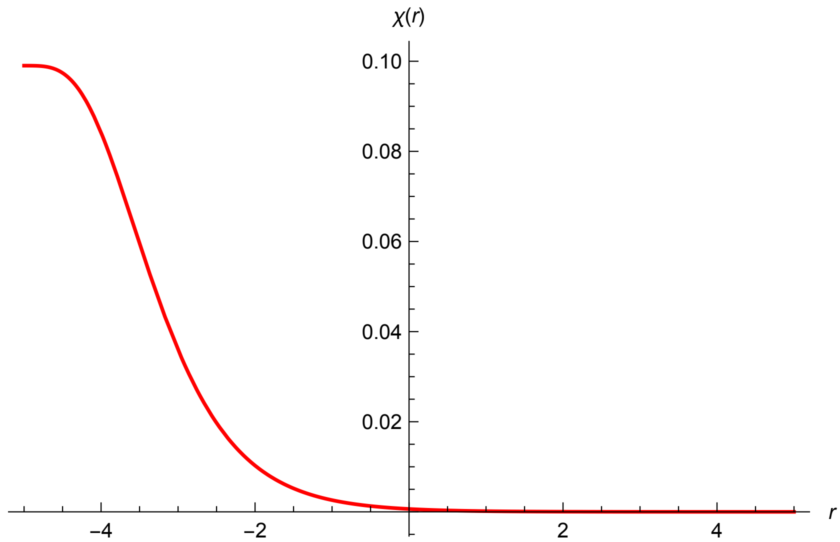


Figure 5.9: Solutions for χ with $g_1 = 1$, $\tilde{g}_1 = 2$ and $\chi_0 = \sqrt{1+C^2} - C$ for $C = 5$ in $SO(3,1) \times SO(3,1)$ gauging

the second-order field equations. The BPS equations, in this case, are given by

$$\phi_1' = \sqrt{2} g_1 e^{\frac{\phi}{2}} \cosh(2\phi_1) \sinh \phi_1, \quad (5.7.17)$$

$$\phi_2' = \sqrt{2} g_1 e^{-\frac{\phi}{2}} \cosh(2\phi_2) \sinh \phi_2, \quad (5.7.18)$$

$$\phi' = \sqrt{2} g_1 e^{-\frac{\phi}{2}} \left[e^{\phi} \cosh \phi_1 (\cosh(2\phi_1) - 2) - \cosh \phi_2 (\cosh(2\phi_2) - 2) \right], \quad (5.7.19)$$

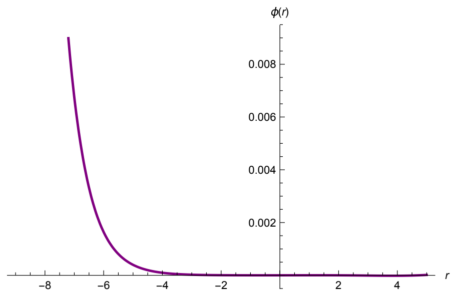
$$A' = \frac{1}{\sqrt{2}} g_1 e^{-\frac{\phi}{2}} \left[e^{\phi} \cosh \phi_1 (\cosh(2\phi_1) - 2) + \cosh \phi_2 (\cosh(2\phi_2) - 2) \right]. \quad (5.7.20)$$

These equations can be solved numerically with suitable boundary conditions. An example of numerical solutions is given in Figure 5.10. Note that the singularity shown in Figure 5.10 leads to $V \to \infty$, hence the singularity is unphysical. We will look at some consistent truncations in which the BPS equations can be solved analytically.

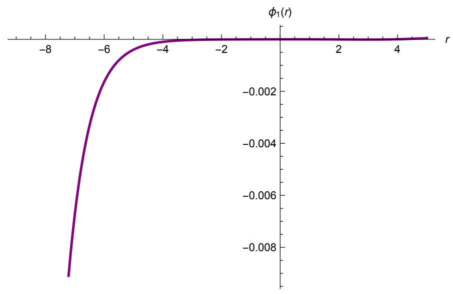
We first consider a truncation with $\phi = 0$ and $\phi_1 = \phi_2$. The BPS equations are then

$$\phi_1' = \sqrt{2} g_1 \cosh(2\phi_1) \sinh \phi_1, \quad (5.7.21)$$

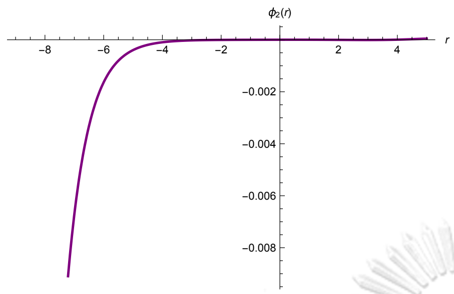
$$A' = \sqrt{2} g_1 \cosh \phi_1 [\cosh(2\phi_1) - 2]. \quad (5.7.22)$$



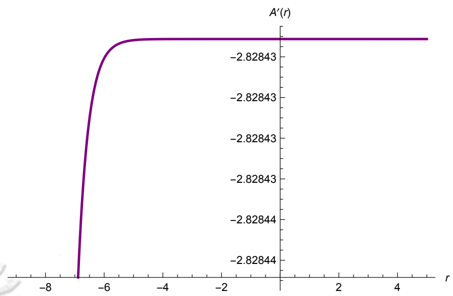
(a) Solution for ϕ



(b) Solution for ϕ_1



(c) Solution for ϕ_2



(d) Solution for A'

Figure 5.10: A numerical flow from AdS_4 critical point with $g_1 = 1$ in $SO(3,1) \times SO(3,1)$ gauging

A solution to the above equations is given by

$$2g_1 r = \ln\left[\frac{1-\sqrt{2}\cosh\phi_1}{1+\sqrt{2}\cosh\phi_1}\right] - 2\sqrt{2}\tanh^{-1} e^{\phi_1}, \tag{5.7.23}$$

$$A = \ln\left(1+e^{4\phi_1}\right) - \ln\left(1-e^{2\phi_1}\right) - \phi_1\,. \tag{5.7.24}$$

The solution for ϕ_1 has a singularity at a finite value of r. Near this singularity, we find

$$\phi_1 \sim \pm\frac{1}{3}\ln\left[C - \frac{3g_1 r}{2\sqrt{2}}\right], \quad A \sim -\frac{1}{3}\ln\left[C - \frac{3g_1 r}{2\sqrt{2}}\right], \tag{5.7.25}$$

where C is a constant. Note that ϕ_1 and ϕ_2 are dual to irrelevant operators of dimensions $\Delta = 4$. In this case, the $N = 4$ SCFT should appear in the IR. However, the singularity leads to $V \to \infty$; therefore, this singularity is unphysical.

Another truncation is obtained by setting $\phi_2 = 0$. The BPS equations in this truncation are

$$\begin{aligned}
\phi_1' &= \sqrt{2}g_1 e^{\frac{\phi}{2}}\cosh(2\phi_1)\sinh\phi_1, && (5.7.26)\\
\phi' &= \sqrt{2}g_1 e^{-\frac{\phi}{2}}(1+e^{\phi}\cosh\phi_1)[\cosh(2\phi_1)-2], && (5.7.27)\\
A' &= \frac{1}{\sqrt{2}}g_1 e^{-\frac{\phi}{2}}[e^{\phi}\cosh\phi_1(\cosh(2\phi_1)-2)-1]. && (5.7.28)
\end{aligned}$$

A solution to the above equations is given by

$$\begin{aligned}
\phi &= \ln\left[\cosh\phi_1 - \frac{1}{2}C\cosh(2\phi_1)\mathrm{csch}\phi_1\right], && (5.7.29)\\
\sqrt{2}g_1\tilde{r} &= \ln[C-\tanh(2\phi_1)], && (5.7.30)\\
A &= \ln[\cosh(2\phi_1)] - \frac{1}{2}\ln(\sinh\phi_1)\\
&\quad -\frac{1}{2}\ln[C\cosh(2\phi_1)-\sinh(2\phi_1)] && (5.7.31)
\end{aligned}$$

where the coordinate $\tilde{r}$ is defined by $\frac{d\tilde{r}}{dr} = e^{-\frac{\phi}{2}}$. The constant $C \neq 0$ is needed to obtain the correct behavior of ϕ and ϕ_1 near the AdS_4 critical point. Note that this solution is singular at a finite value of $\tilde{r}$. Near the singularity, we find

$$\phi_1 \sim \pm\frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r}-\tilde{C}\right|, \qquad (5.7.32)$$

where $\tilde{C}$ is constant.

The behaviors of ϕ and A depend on the value of the constant C. For the case $\phi \to \infty$, we find

$$\begin{aligned}
\phi &\sim -\phi_1 \sim \frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r}-\tilde{C}\right|,\\
A &\sim \phi_1 \sim -\frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r}-\tilde{C}\right|, && (5.7.33)
\end{aligned}$$

for $C = 2$, and

$$\begin{aligned}
\phi &\sim \phi_1 \sim -\frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r}-\tilde{C}\right|,\\
A &\sim \phi_1 \sim -\frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r}-\tilde{C}\right|, && (5.7.34)
\end{aligned}$$

for $C \neq 2$. Both of these singularities lead to $V \to \infty$. Therefore, they are unphysical.

For the case $\phi \to -\infty$, we find

$$\begin{aligned}\phi &\sim \phi_1 \sim \frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r} - \tilde{C}\right|, \\ A &\sim -\phi_1 \sim -\frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r} - \tilde{C}\right|, \end{aligned} \tag{5.7.35}$$

for $C = -2$ and

$$\begin{aligned}\phi &\sim -\phi_1 \sim -\frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r} - \tilde{C}\right|, \\ A &\sim -\phi_1 \sim -\frac{1}{4}\ln\left|\sqrt{2}g_1\tilde{r} - \tilde{C}\right| \end{aligned} \tag{5.7.36}$$

for $C \neq -2$. These singularities lead to $V \to \infty$, hence they are unphysical. The solutions in this particular truncation do not describe RG flows in $N = 4$ SCFT.

A similar truncation with $\phi_1 = 0$ also leads to unphysical singularities. It would be interesting to uplift these solutions to ten or eleven dimensions and determine if these singularities are resolved.

5.8 RG flows from $SO(2,2) \times SO(2,2)$ gauged supergravity

We now consider $N = 4$ supergravity coupled to six vector multiplets with $SO(2,2)\times SO(2,2)$ gauge group. The components of the embedding tensor in this gauging are given by

$$\begin{aligned} &f_{+189} = \frac{1}{\sqrt{2}}(g_1 - \tilde{g}_1), \qquad f_{+723} = \frac{1}{\sqrt{2}}(g_1 + \tilde{g}_1), \\ &f_{-10,5,6} = \frac{1}{\sqrt{2}}(g_2 + \tilde{g}_2), \qquad f_{-4,11,12} = \frac{1}{\sqrt{2}}(g_2 - \tilde{g}_2). \end{aligned} \tag{5.8.1}$$

We will consider four $SO(2) \times SO(2) \times SO(2) \times SO(2)$ singlet scalars corresponding to the $SO(6,6)$ non-compact generators,

$$Y_1 = Y_{1,7}, \quad Y_2 = Y_{1,10}, \quad Y_3 = Y_{4,10}, \quad Y_4 = Y_{4,7}. \tag{5.8.2}$$

The coset representative is parametrized by

$$L = e^{\phi_1 Y_1} e^{\phi_2 Y_2} e^{\phi_3 Y_3} e^{\phi_4 Y_4}. \tag{5.8.3}$$

The scalar potential in this gauging is given by

$$
\begin{aligned}
V \;=\;& \frac{1}{4}\left(\sinh\phi\left(\left(\chi^2-1\right)\left(\tilde{g}_2+g_2\right)^2+\left(\tilde{g}_1+g_1\right)^2\right)+\cosh\phi\left(\left(\chi^2+1\right)\left(\tilde{g}_2+g_2\right)^2\right.\right.\\
&\left.+\left(\tilde{g}_1+g_1\right)^2\right)+4\left(\tilde{g}_1+g_1\right)\left(\tilde{g}_2+g_2\right)\left(\sinh\phi_1\sinh\phi_3\cosh\phi_4\right.\\
&\left.\left.-\sinh\phi_2\sinh\phi_4\cosh\phi_1\right)\right). \qquad (5.8.4)
\end{aligned}
$$

However, in order to obtain consistent BPS equations, we need to impose the relation

$$\tilde{g}_1=-g_1, \quad \tilde{g}_2=-g_2. \qquad (5.8.5)$$

This relation gives $V=0$ for any values of the scalars; hence this gauging gives a Minkowski vacuum. We will not perform any further analysis since it does not give a holographic description.

5.9 RG flows from $SO(4)\times SO(3,1)$ gauged supergravity

We now consider $N=4$ supergravity coupled to six vector multiplets with $SO(4)\times SO(3,1)$ gauge group. The $SO(4)$ and $SO(3,1)$ are electrically and magnetically embedded in the $SO(3,3)\times SO(3,3)$, respectively. The components of the embedding tensor in this gauging are given by

$$
\begin{aligned}
&f_{+123}=\sqrt{2}(g_1-\tilde{g}_1), \qquad f_{+789}=\sqrt{2}(g_1+\tilde{g}_1),\\
&f_{-456}=f_{-4,11,12}=f_{-10,5,12}=-f_{-10,11,6}=\frac{1}{\sqrt{2}}(g_2-\tilde{g}_2),\\
&f_{-10,11,12}=f_{-4,11,7}=f_{-10,5,6}=-f_{-45,12}=\frac{1}{\sqrt{2}}(g_2+\tilde{g}_2). \qquad (5.9.1)
\end{aligned}
$$

We will consider scalars fields that invariant under $SO(4)_{\text{inv}}\sim SO(3)\times SO(3)\subset SO(4)\times SO(3)\subset SO(4)\times SO(3,1)$. These scalars correspond to $SO(6,6)$ non-compact generators

$$\hat{Y}_1=Y_{11}+Y_{22}+Y_{33}, \qquad \tilde{Y}_2=Y_{44}+Y_{55}-Y_{66}. \qquad (5.9.2)$$

The coset representative is parametrized by

$$L = e^{\phi_1 \hat{Y}} e^{\phi_2 \tilde{Y}_2}. \tag{5.9.3}$$

The scalar potential, in this case, is given by

$$\begin{aligned} V = & \frac{1}{8} e^{-\phi-6\phi_1-6\phi_2} \left[(g_1+g_2)^2 e^{2\phi+12\phi_1+6\phi_2} - 3(3g_1^2+2g_1g_2+g_2^2) e^{2\phi+4\phi_1+6\phi_2} \right. \\ & + e^{6\phi_1} \left[g_2^2 (1+e^{2\phi}\chi^2)(1+e^{4\phi_2})^3 + 16 e^{6\phi_2} [e^{2\phi}(g_1^2+g_1g_2-g_2^2\chi^2) - g_2^2] \right] \\ & + 8 g_2 e^{\phi+3\phi_1+6\phi_2} [g_1(e^{2\phi_1}-1)^3 + g_2 e^{2\phi_1}(3+e^{4\phi_1})] \cosh\phi_2 \times \\ & \left. [\cosh(2\phi_2)-2] - 3(3g_1^2+4g_1g_2+2g_2^2) e^{2\phi+8\phi_1+6\phi_2} + g_1^2 e^{2\phi+6\phi_2} \right]. \end{aligned} \tag{5.9.4}$$

Note that we have imposed the relations

$$\tilde{g}_1 = g_1 + g_2 \quad \text{and} \quad \tilde{g}_2 = -g_2 \tag{5.9.5}$$

in order to obtain an $N = 4$ supersymmetric AdS_4 critical point with $SO(4) \times SO(3)$ symmetry for all scalars vanishing.

This scalar potential admits two supersymmetric AdS_4 critical points with $N = 4$ supersymmetry. The first is a trivial critical point, at which all scalars are vanishing, with $SO(4) \times SO(3)$ symmetry. The cosmological constant and the AdS_4 radius at this critical point are given by

$$V_0 = -6g_2^2, \quad L = \frac{\sqrt{3}}{g_2}. \tag{5.9.6}$$

Scalar masses and the corresponding dimensions of the dual operators are given in Table 5.6.

Another one is a non-trivial critical point, which is given by

$$\begin{aligned} \phi_2 &= \chi = 0, \qquad \phi_1 = \frac{1}{2} \ln \left[\frac{g_1}{g_1+g_2} \right], \\ \phi &= \frac{1}{2} \ln \left[\frac{4g_1(g_1+g_2)}{(2g_1+g_2)^2} \right]. \end{aligned} \tag{5.9.7}$$

At this critical point, the cosmological constant and the AdS_4 radius are given by

$$V_0 = -\frac{3g_2^2(2g_1+g_2)}{\sqrt{g_1(g_1+g_2)}}, \quad L = \frac{\sqrt{3}(g_1(g_1+g_2))^{\frac{1}{4}}}{g_2(2g_1+g_2)^{\frac{1}{2}}}. \tag{5.9.8}$$

This critical point preserves an $SO(3) \times SO(3)$ symmetry. Scalar masses and the corresponding dimensions of the dual operators are given in Table 5.7

Scalar field representations	m^2L^2	Δ
$(\mathbf{1},\mathbf{1},\mathbf{1})$	$-2_{\times 2}$	$1,2$
$(\mathbf{1},\mathbf{1},\mathbf{1})$	4	4
$(\mathbf{1},\mathbf{1},\mathbf{3})$	$0_{\times 3}$	3
$(\mathbf{3},\mathbf{3},\mathbf{1})$	$0_{\times 9}$	3
$(\mathbf{1},\mathbf{1},\mathbf{5})$	$-2_{\times 5}$	$1,2$
$(\mathbf{1},\mathbf{3},\mathbf{3})+(\mathbf{3},\mathbf{1},\mathbf{3})$	$-2_{\times 18}$	$1,2$

Table 5.6: Scalar masses and the corresponding dimensions of the dual operators at the $N=4$ supersymmetric AdS_4 critical point with $SO(4)\times SO(3)$ symmetry for $SO(4)\times SO(3,1)$ gauge group

Scalar field representations	m^2L^2	Δ
$(\mathbf{1},\mathbf{1})$	$-2_{\times 2}$	$1,2$
$(\mathbf{1},\mathbf{1})$	$4_{\times 2}$	4
$(\mathbf{1},\mathbf{5})+(\mathbf{5},\mathbf{1})$	$-2_{\times 10}$	$1,2$
$(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{1})$	$0_{\times 6}$	3
$(\mathbf{3},\mathbf{3})$	$0_{\times 18}$	3

Table 5.7: Scalar masses and the corresponding dimensions of the dual operators at the $N=4$ supersymmetric AdS_4 critical point with $SO(3)\times SO(3)$ symmetry for $SO(4)\times SO(3,1)$ gauge group

5.9.1 RG flow between $SO(4)\times SO(3)$ and $SO(3)\times SO(3)$ critical points

We now consider a supersymmetric RG flow between the AdS_4 critical points with $SO(4)\times SO(3)$ and $SO(3)\times SO(3)$ symmetries. As in the previous cases, we need to set $\chi=0$ in the presence of vector multiplet scalars.

With $\chi=0$, the superpotential is given by

$$\begin{aligned}\mathcal{W} &= \frac{i}{4\sqrt{2}}e^{-\frac{\phi}{2}-3\phi_1-3\phi_2}\left[g_1e^{\phi+3\phi_2}+3g_1e^{\phi+4\phi_1+3\phi_2}-3(g_1+g_2)e^{\phi+2\phi_1+3\phi_2}\right.\\ &\quad \left.+g_2e^{3\phi_1}(1+e^{2\phi_2})(1-4e^{2\phi_2}+e^{4\phi_2})-(g_1+g_2)e^{\phi+6\phi_1+3\phi_2}\right]. \qquad (5.9.9)\end{aligned}$$

The scalar potential can be written as

$$V = 4\left(\frac{\partial W}{\partial \phi}\right)^2 + \frac{2}{3}\left(\frac{\partial W}{\partial \phi_1}\right)^2 + \frac{2}{3}\left(\frac{\partial W}{\partial \phi_2}\right)^2 - 3W^2\,. \tag{5.9.10}$$

The BPS equations, in this case, are given by

$$\begin{aligned}
\phi_1' &= -\frac{2}{3}\frac{\partial W}{\partial \phi_1} = -\frac{1}{2\sqrt{2}}e^{\frac{\phi}{2}-3\phi_1}(e^{4\phi_1}-1)(e^{2\phi_1}(g_1+g_2)-g_1), && (5.9.11)\\
\phi_2' &= -\frac{2}{3}\frac{\partial W}{\partial \phi_2} = \frac{1}{2\sqrt{2}}g_2 e^{-\frac{\phi}{2}-3\phi_1}(e^{2\phi_2}-1)(e^{4\phi_1}+1), && (5.9.12)\\
\phi' &= -4\frac{\partial W}{\partial \phi} = -\frac{1}{2\sqrt{2}}e^{-\frac{\phi}{2}-3\phi_1}\left[4g_2 e^{3\phi_1}\cosh\phi_2[\cosh(2\phi_2)-2]\right. \\
&\quad \left. +e^{\phi}\left[[(e^{2\phi_1}-1)^3 g_1 + e^{2\phi_1}(3+e^{4\phi_1})g_2]\right]\right], && (5.9.13)\\
A' &= \frac{1}{4}\sqrt{2}e^{-\frac{\phi}{2}-3\phi_1}\left[e^{\phi}[(e^{2\phi_1}-1)^3 g_1 + e^{2\phi_1}(3+e^{4\phi_1})g_2]\right. \\
&\quad \left. -4g_2 e^{3\phi_1}\cosh\phi_2[\cosh(2\phi_2)-2]\right]. && (5.9.14)
\end{aligned}$$

Since both of the critical points have $\phi_2 = 0$, we can consistently truncate ϕ_2 out. Note that ϕ_2 is dual to an irrelevant operator with dimension $\Delta = 4$, which can also be seen from the linearized BPS equations,

$$\phi \sim \phi_1 \sim e^{-\frac{r}{L}}, \qquad \phi_2 \sim e^{\frac{r}{L}}\,. \tag{5.9.15}$$

The solution with $\phi_2 = 0$ is given by

$$\begin{aligned}
g_2(2g_1+g_2)\tilde{r} &= \sqrt{2}g_2\tan^{-1}e^{\phi_1} + \sqrt{2}(2g_1+g_2)\tanh^{-1}e^{\phi_1} \\
&\quad -2\sqrt{2g_1(g_1+g_2)}\tanh^{-1}\left[e^{\phi_1}\sqrt{\frac{g_1+g_2}{g_1}}\right], && (5.9.16)\\
\phi &= \ln\left[\frac{e^{-\phi_1}g_1 + e^{\phi_1}(g_1+g_2)}{2g_1+g_2}\right], && (5.9.17)\\
A &= \frac{1}{2}\phi_1 - \ln\left(1-e^{4\phi_1}\right) + \ln[(e^{2\phi_1}-1)g_1 + e^{2\phi_1}g_2] \\
&\quad +\frac{1}{2}\ln[g_1 + (g_1+g_2)e^{2\phi_1}] && (5.9.18)
\end{aligned}$$

where $\tilde{r}$ is defined by $\frac{d\tilde{r}}{dr} = e^{\frac{\phi}{2}}$. Note that this solution preserves $N = 4$ supersymmetry in three dimensions. This should describe $N = 4$ RG flow from $N = 4$ SCFT in the UV with $SO(4)\times SO(3)$ symmetry to another $N = 4$ SCFT in the IR with $SO(3)\times SO(3)$ symmetry. Note that the flavor symmetry $SO(3)$ in the UV is broken by the relevant operator dual to ϕ_1.

We can also truncate out the vector multiplet scalars. However, it leads to a similar structure as in the previous case. Hence we will not consider this truncation.

5.10 RG flows from $SO(4)\times SO(2,2)$ gauged supergravity

We now consider $N=4$ supergravity coupled to six vector multiplets with $SO(4)\times SO(2,2)$ gauge group. The $SO(4)$ and $SO(2,2)$ are electrically and magnetically embedded in the $SO(3,3)\times SO(3,3)$, respectively. The components of the embedding tensor in this gauging are given by

$$\begin{aligned} f_{+123} &= \sqrt{2}(g_1-\tilde{g}_1), \qquad f_{+789}=\sqrt{2}(g_1+\tilde{g}_1), \\ f_{-10,5,6} &= \frac{1}{\sqrt{2}}(g_2+\tilde{g}_2), \qquad f_{-4,11,12}=\frac{1}{\sqrt{2}}(g_2-\tilde{g}_2). \end{aligned} \tag{5.10.1}$$

We will consider four singlet scalars corresponding to the $SO(6,6)$ non-compact generators,

$$Y_1=Y_{1,7}+Y_{2,8}+Y_{3,9}, \quad Y_2=Y_{4,10}, Y_3=Y_{5,11}+Y_{6,12}, \quad Y_4=Y_{5,12}-Y_{6,11}.$$

The coset representative is parametrized by

$$L=e^{\phi_1Y_1}e^{\phi_2Y_2}e^{\phi_3Y_3}e^{\phi_4Y_4}. \tag{5.10.2}$$

The scalar potential in this gauging is given by

$$\begin{aligned}
V \;=\; & \frac{1}{8}e^{-\phi-6(\phi_1+\phi_2)}\left(\chi^2 e^{2(\phi+3\phi_1+6\phi_2)}\tilde{g}_2^2 - 3\chi^2 e^{2\phi+6\phi_1+4\phi_2}\left(\tilde{g}_2^2+2g_2^2\right)\right.\\
& -3\chi^2 e^{2\phi+6\phi_1+8\phi_2}\left(2\tilde{g}_2^2+g_2^2\right)+16e^{2(\phi+3(\phi_1+\phi_2))}\left(g_2\chi^2\tilde{g}_2+g_1\tilde{g}_1\right)\\
& +e^{2(\phi+6\phi_1+3\phi_2)}\tilde{g}_1^2+e^{6(\phi_1+2\phi_2)}\tilde{g}_2^2-6g_2e^{\phi+5\phi_1+3\phi_2}\tilde{g}_1\\
& -2g_2e^{\phi+9\phi_1+3\phi_2}\tilde{g}_1-18g_2e^{\phi+5\phi_1+7\phi_2}\tilde{g}_1-6g_2e^{\phi+9\phi_1+7\phi_2}\tilde{g}_1\\
& -3e^{2\phi+4\phi_1+6\phi_2}\left(\tilde{g}_1^2+2g_1^2\right)-3e^{2\phi+8\phi_1+6\phi_2}\left(2\tilde{g}_1^2+g_1^2\right)\\
& -6g_1e^{\phi+3\phi_1+5\phi_2}\tilde{g}_2-18g_1e^{\phi+7\phi_1+5\phi_2}\tilde{g}_2-2g_1e^{\phi+3\phi_1+9\phi_2}\tilde{g}_2\\
& -6g_1e^{\phi+7\phi_1+9\phi_2}\tilde{g}_2+16g_2e^{6(\phi_1+\phi_2)}\tilde{g}_2+6e^{\phi+9\phi_1+5\phi_2}\tilde{g}_1\tilde{g}_2\\
& +6e^{\phi+5\phi_1+9\phi_2}\tilde{g}_1\tilde{g}_2+18e^{\phi+5(\phi_1+\phi_2)}\tilde{g}_1\tilde{g}_2+2e^{\phi+9(\phi_1+\phi_2)}\tilde{g}_1\tilde{g}_2\\
& -3e^{6\phi_1+4\phi_2}\left(\tilde{g}_2^2+2g_2^2\right)-3e^{6\phi_1+8\phi_2}\left(2\tilde{g}_2^2+g_2^2\right)+g_2^2\chi^2e^{2\phi+6\phi_1}\\
& +g_1^2e^{2\phi+6\phi_2}+g_2^2e^{6\phi_1}+6g_1g_2e^{\phi+7\phi_1+3\phi_2}+6g_1g_2e^{\phi+3\phi_1+7\phi_2}\\
& \left.+2g_1g_2e^{\phi+3(\phi_1+\phi_2)}+18g_1g_2e^{\phi+7(\phi_1+\phi_2)}\right). \qquad (5.10.3)
\end{aligned}$$

Note that, in order to find consistent BPS equations, we have to impose the relation

$$\tilde{g}_1 = g_1, \quad \tilde{g}_2 = g_2. \qquad (5.10.4)$$

This leads to the Minkowski vacuum, $V_0 = 0$, for all scalars vanishing. We will not perform any further analysis since it does not give a holographic description.

5.11 RG flows from $SO(3,1)\times SO(2,2)$ gauged supergravity

We now consider $N = 4$ supergravity coupled to six vector multiplets with $SO(3,1)\times SO(2,2)$ gauge group. The $SO(3,1)$ and $SO(2,2)$ are electrically and magnetically embedded in the $SO(3,3)\times SO(3,3)$, respectively. The components of the embedding tensor in this gauging are given by

$$f_{+123} = f_{+189} = f_{+729} = -f_{+783} = \frac{1}{\sqrt{2}}(g_1 - \tilde{g}_1),$$
$$f_{+789} = f_{+183} = f_{+723} = -f_{+129} = \frac{1}{\sqrt{2}}(g_1 + \tilde{g}_1),$$
$$f_{-10,5,6} = \frac{1}{\sqrt{2}}(g_2 + \tilde{g}_2), \qquad f_{-4,11,12} = \frac{1}{\sqrt{2}}(g_2 - \tilde{g}_2). \qquad (5.11.1)$$

We will consider eight singlet scalars corresponding to the $SO(6,6)$ non-compact generators,

$$Y_1 = Y_{1,7} + Y_{2,8}, \quad Y_2 = -Y_{1,8} + Y_{2,7}, \quad Y_3 = Y_{3,9}, \quad Y_4 = Y_{3,10}$$
$$Y_5 = Y_{4,9}, \quad Y_6 = Y_{4,10}, \quad Y_7 = Y_{5,11} + Y_{6,12}, \quad Y_8 = Y_{5,12} - Y_{6,11} \qquad (5.11.2)$$

The coset representative is parametrized by

$$L = e^{\phi_1 Y_1} e^{\phi_2 Y_2} e^{\phi_3 Y_3} e^{\phi_4 Y_4} e^{\phi_5 Y_5} e^{\phi_6 Y_6} e^{\phi_7 Y_7} e^{\phi_8 Y_8}. \qquad (5.11.3)$$

The scalar potential computed with these scalars is highly complicated. We refrain from giving its explicit form here. However, in order to obtained $N = 4$ supersymmetric critical point with all scalars vanishing, we need to impose the relation,

$$g_1 = 0, \quad \tilde{g}_1 = 0, \quad \tilde{g}_2 = -g_2. \qquad (5.11.4)$$

This leads to the Minkowski vacuum, $V_0 = 0$. We will not perform any further analysis since it does not give a holographic description.

CHAPTER VI

Summary

In this dissertation, we have mainly focused on the study of holographic RG flows from $N = 3$ and $N = 4$ gauged supergravities in four dimensions. We have reviewed the AdS/CFT correspondence, which includes $N = 4$ super Yang-Mills theory, type IIB string theory on $AdS_5 \times S^5$. We have adopted the principle to study holographic RG flows in the context of the AdS_4/CFT_3 correspondence. We have reviewed some general features of $N > 2$ gauged supergravities in four dimensions, mainly focused on $2 < N \leq 4$.

We have studied holographic RG flows from $N = 3$ gauged supergravity in four dimensions. We have studied various types of semisimple gauge groups and identified their vacua. For $SO(3) \times SO(3)$ gauge group, we have found two supersymmetric AdS_4 critical points. One is a supersymmetric AdS_4 critical point, at which all scalars vanish, and a non-trivial AdS_4 critical point with $SO(3)_{\text{diag}}$ symmetry and unbroken $N = 3$ supersymmetry. We have given a holographic RG flow interpolating between $SO(3) \times SO(3)$ and $SO(3)_{\text{diag}}$ critical points. We have also given a number of RG flows to non-conformal theories, preserving $N = 3$ or $N = 1$ supersymmetry. For $SO(3, 1)$ gauge group, we have found a supersymmetric AdS_4 critical point and a non-supersymmetric AdS_4 critical point with $SO(3)$ symmetry. A supersymmetric RG flow to a non-conformal theory is given. For $SO(2, 2)$ gauge group, we have found a non-supersymmetric AdS_4 critical point when all scalars vanish. We have also given a half-supersymmetric domain wall solution preserving $N = 3$ supersymmetry. For $SO(2, 1)$ gauge group, the scalar potential admits only non-supersymmetric critical point. There is also a half-supersymmetric domain wall with $SO(2)$ symmetry in the same form as in $SO(2,2)$ case. For $SL(3,R)$ gauge group, we have found an $N=3$ supersymmetric AdS_4 critical point with $SO(3)$ symmetry. For $SO(2,1)\times SO(2,2)$ gauge group, the scalar potential admits a Minkowski vacuum with all scalars vanishing.

We have also studied holographic RG flows from $N=4$ gauged supergravity in four dimensions, obtained from type IIA and IIB string theories compactified on $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ with non-semisimple gaugings. For the gauged supergravity from type IIB compactification, the scalar potential admits a trivial AdS_4 critical point which preserves $N=4$ supersymmetry. In this case, we have given various RG flows together with examples of supersymmetric Janus solutions. The scalar potential obtained from type IIB GKP compactification admits only a Minkowski vacuum when all scalars vanish. The scalar potential from the gauged supergravity obtained from type IIA compactification admits an $N=1$ AdS_4 critical point at all scalar vanishing. We have also given some examples of numerical flows.

We have considered the $N=4$ gauged supergravity with various semisimple gaugings. For $SO(4)\times SO(4)$ gauge group, we have found four supersymmetric AdS_4 critical points, with $SO(4)\times SO(4)$, $SO(3)\times SO(4)$, $SO(4)\times SO(3)$, and $SO(4)_{\text{diag}}$ symmetries. We have given a number of RG flows between these critical points together with RG flows to $N=4$ non-conformal theories. For $SO(3,1)\times SO(3,1)$ gauge group, we have found a supersymmetric AdS_4 critical point and discussed RG flows to non-conformal theories. The scalar potential for $SO(2,2)\times SO(2,2)$ gives a class of Minkowski vacuum. For $SO(4)\times SO(3,1)$, we have found two supersymmetric AdS_4 critical points with $N=4$ supersymmetry. The RG flow between critical points with $SO(4)\times SO(3)$ and $SO(3)\times SO(3)$ is given. For $SO(4)\times SO(2,2)$ and $SO(3,1)\times SO(2,2)$ gauge groups, we have found only Minkowski vacua in both cases.

We would like to note that the gaugings considered in chapter 4 are electric gaugings, in which only electric gauge fields are involving. It would be interesting to apply the embedding tensor formalism to the $N=3$ gauged supergravity, similar to $N=4$ gauged supergravity in chapter 5, and look for more general gaugings, such as magnetic or dyonic gaugings in which magnetic gauge fields participate.

There are many possibilities for future investigations. It would be interesting to identify the SCFTs or non-conformal gauge theories dual to the gravity solutions obtained here. This should allow us to identify the dual operators driving the RG flows obtained in this dissertation. It could be interesting to look for more general solutions by restoring the truncated scalars or turn on more scalars. Uplifting the solutions found in this dissertation to higher dimensions could be interesting, as it will give new AdS_4 backgrounds in the context of string/M-theory. The unphysical singularities found in this dissertation could be also checked by identifying the g_{00} component of the ten-dimensional metric according to the criterion proposed by Maldacena and Nunez. Finding other types of solutions, such as Janus solutions or flows across dimensions to $AdS_2 \times \Sigma_2$, with Σ_2 being a Riemann surface, could be useful in the holographic study of defect SCFTs and black hole physics.

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APPENDIX

APPENDIX A

Appendix

A.1 General relativity formula

A derivative of frame field $de^a = \frac{1}{2}(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu)dx^\mu \wedge dx^\nu$ does not transform as vector under local Lorentz transformation. We can add a term with spin connection $\omega^a{}_b$ to restore the property,

$$de^a + \omega^a{}_b \wedge e^b \equiv T^a, \tag{A.1.1}$$

which implies spin connection local Lorentz transformation $\omega'_\mu{}^a{}_b = \Lambda^{-1a}{}_c d\Lambda^c{}_b + \Lambda^{-1a}{}_c \omega^c{}_d \Lambda^d{}_c$. The torsion 2-form T^a is now transformed as a vector under local Lorentz transformation, $T'^a = \Lambda^{-1a}{}_b T^b$.

The covariant derivative of an arbitrary tensor is defined by a partial derivative plus correction terms. For example the standard covariant derivative acts on rank $(1,1)$,

$$\nabla_\mu X^\rho{}_\sigma = \partial_\mu X^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\lambda X^\lambda{}_\sigma - \Gamma_\mu{}^\lambda{}_\sigma X^\rho{}_\lambda \,. \tag{A.1.2}$$

We can also define a local Lorentz covariant derivative using the spin connection in the same way the standard covariant derivative does, i.e.

$$D_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - X^a{}_c \omega_\mu{}^c{}_b \,. \tag{A.1.3}$$

Note that for type (p,q) tensors, p connection terms contract on the left while the remain q connection terms contract on the right. One can show that it relates to Christoffel connection by using $\nabla_\mu V^\nu = e^a{}_\mu D_a V^\nu$

$$\omega_\mu{}^a{}_b = e^a{}_\nu e^\lambda{}_b \Gamma_\mu{}^\nu{}_\lambda - e^\lambda{}_b \partial_\mu e^a{}_\lambda, \tag{A.1.4}$$

which is also known as ”tetrad postulate” [91] by simple manipulation of A.1.4.

In tetrad language, the Riemann curvature tensor can be written as $(1,1)$-tensor-valued two-form $R^a{}_{b\mu\nu}$,

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\,. \tag{A.1.5}$$

It is equivalent to the standard Riemann tensor,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_\nu{}^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\alpha \Gamma_\nu{}^\alpha{}_\sigma - \partial_\nu \Gamma_\mu{}^\rho{}_\sigma - \Gamma_\nu{}^\rho{}_\alpha \Gamma_\mu{}^\alpha{}_\sigma, \tag{A.1.6}$$

which can be written in 2-form as

$$R^\rho{}_\sigma = R^\rho{}_{\sigma\mu\nu} dx^\mu \wedge dx^\nu\,. \tag{A.1.7}$$

The Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ can be defined as a vector-value-1-form, $R^a{}_\mu dx^\mu$, in vierbein language. The curvature scalar is then $R = g^{\mu\nu} R_{\mu\nu} = e^\mu{}_a R^a{}_\mu(\omega)$.

A.1.1 Non-linear σ-model

One of an important application of differential geometry is non-linear σ-model which describe dynamics of scalar fields in spacetime. In general, it is a field theory in which fields are restricted to a manifold as a map from spacetime to target space or internal space. For a map from coordinates x^μ, $\mu = 0, \ldots, D-1$ of a flat spacetime M_D to local coordinates $\phi^i(x^\mu)$, $i = 1, \ldots, n$ of n-dimensional Riemannian manifold M_n. Hence it is a field theory with n scalar fields.

The action describing the dynamics of scalar fields (maps) is given by

$$S[\phi] = -\frac{1}{2} \int d^D x g_{ij}(\phi) \eta^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \tag{A.1.8}$$

where $g_{ij}(\phi)$ is the metric tensor of local coordinates and $\eta_{\mu\nu}$ is the Minkowski metric. The equations of motion are in the form

$$\Box \phi^i + \Gamma^i_{jk}(\phi) \partial^\mu \phi^j \partial_\mu \phi^k = 0\,. \tag{A.1.9}$$

A.2 Anti-de Sitter Space in Various Dimensions

Anti-de Sitter space (AdS) is an example of maximally symmetric space. A maximally symmetric space in D dimensions of arbitrary signature (i.e. Euclidean or Minkowski) is a space with $\frac{1}{2}D(D+1)$ Killing vectors. In our definition, its metric obeys

$$R_{MNPQ} = k(g_{MP}g_{NQ} - g_{MQ}g_{NP}) \tag{A.2.1}$$

where k is a constant known as curvature constant. Recall that

$$\Gamma^P_{MN} = \frac{1}{2}g^{PQ}(\partial_M g_{QN} + \partial_N g_{QM} - \partial_Q g_{MN}) \tag{A.2.2}$$

$$R_{MNP}{}^{Q} = \partial_M \Gamma^Q_{NP} - \partial_N \Gamma^Q_{MP} + \Gamma^Q_{MR}\Gamma^R_{NP} - \Gamma^Q_{NR}\Gamma^R_{MP}. \tag{A.2.3}$$

Any two spaces with the same k and same signature are isomorphic.

The value of curvature constant k determines the maximal space geometry. In Euclidean signature, $k > 0$ is a sphere S^D, $k = 0$ is a Euclidean space $\mathbb{R}^D$, and $k < 0$ is a hyperbolic space H^D. In Minkowski signature, $k > 0$ is the de Sitter space dS_D, $k = 0$ is the Minkowski space $\mathbb{R}^{D,1}$ and $k < 0$ is the anti-de Sitter space AdS_D.

A maximally symmetric space obeys Einstein equation with a cosmological constant Λ,

$$R_{MN} - \frac{1}{2}g - MNR + \frac{\Lambda}{2}g_{MN} = 0 \tag{A.2.4}$$

which is given by a variation $\delta S = 0$ of the action

$$S = \frac{1}{2\kappa^2}\int d^D x\sqrt{g}(R - \Lambda). \tag{A.2.5}$$

By the definition A.2.1, one can obtain Ricci tensor and scalar,

$$R_{MN} = g^{PQ}R_{MPNQ} = k(D-1)g_{MN} \tag{A.2.6}$$

$$R = g^{MN}R_{MN} = kD(D-1). \tag{A.2.7}$$

Therefore by A.2.4, we can get the curvature constant as a function of cosmological constant $k = \frac{\Lambda}{(D-2)(D-1)}$. This relation is for $D > 2$ only. Thus we see that $\Lambda > 0$ for de Sitter space and $\Lambda < 0$ for anti-de Sitter space.

Maximally symmetric space can be embedded into a flat space one dimension higher as hyperboloids $\mathbb{R}^{p,q}$ with invariant metric

$$\eta = diag(\underbrace{-1, \dots, -1}_{p}, \underbrace{+1, \dots, 1}_{q}). \tag{A.2.8}$$

A maximally symmetric space of signature $(p, q-1)$ can be embedded as

$$\eta_{AB} Y^A Y^B = -(\vec{Y}_p)^2 + (\vec{Y}_q)^2 = L^2 \tag{A.2.9}$$

where $A, B = 1, \dots, D+1$ and $p + q = D + 1$. Its metric also be embedded as

$$ds^2 = -\sum_{i=1}^{p} (dY_i)^2 + \sum_{j=1}^{q} (dY_j)^2 \tag{A.2.10}$$

for mostly negative signature. The space is manifest isometry $SO(p, q)$ since it is invariant under coordinate transformation $Y'^A = \Lambda^A{}_B Y^B$. List of embedded spaces are shown in Table A.1. For example, the space S^5 has isometry $SO(6)$ and AdS_5 has isometry $SO(4, 2)$. Note that the cosmological constant Λ can be written as

$$\Lambda = -\frac{(D-1)(D-2)}{L^2}. \tag{A.2.11}$$

It can be found that the equation of motion for the action (A.2.5) is

$$R_{\mu\nu} = -\frac{D-1}{L^2} g_{\mu\nu}. \tag{A.2.12}$$

A.2.1 AdS_5 metric

Let's discuss on AdS_5 metric. From the embedded space A.2.9, we can write a coordinate relation for AdS_5,

$$-(Y_1)^2 - \cdots - (Y_4)^2 + (Y_5)^2 + (Y_6)^2 = L^2. \tag{A.2.13}$$

Relation	Space	Isometry
$+(Y_1)^2+(Y_2)^2+\cdots+(Y_{D+1})^2=L^2$	Sphere S^D	$SO(D+1)$
$-(Y_1)^2+(Y_2)^2+\cdots+(Y_{D+1})^2=L^2$	de Sitter dS_D	$SO(1,D)$
$\vdots$	$\vdots$	$\vdots$
$-(Y_1)^2-\cdots-(Y_{D-1})^2+(Y_D)^2+(Y_{D+1})^2=L^2$	anti de Sitter AdS_D	$SO(D-1,2)$
$-(Y_1)^2-\cdots-(Y_D)^2+(Y_{D+1})^2=L^2$	Sphere S^D	$SO(D,1)$
$-(Y_1)^2+\cdots-(Y_{D+1})^2=L^2$	no solution	

Table A.1: Relations of coordinates of D dimensions embedded in D+1 dimensions and their isometry

We can redefine the coordinate Y_A into 5-dimensional spacetime coordinates (t, x^i) by using

$$\begin{aligned} &Y_i = rx^i, \quad \text{where} \quad \sum_{i=1}^{4}(x^i)^2 = 1 \\ &Y_5 = \sqrt{L^2+r^2}sin(t/L), \quad Y_6 = \sqrt{L^2+r^2}cos(t/L). \end{aligned} \tag{A.2.14}$$

The metric is then

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right)dt^2 + \left(1+\frac{r^2}{L^2}\right)^{-1}dr^2 + r^2 d\Omega_3 \tag{A.2.15}$$

A patch of AdS_5 metric can be written as

$$ds^2 = \frac{r^2}{L^2}(dx^\mu)^2 + \frac{L^2}{r^2}dr^2. \tag{A.2.16}$$

It is usually more practical to change variable r to $z = L^2/r$ for which

$$ds^2 = \frac{L^2}{z^2}\left((dx^\mu)^2 + dz^2\right) \tag{A.2.17}$$

which makes it clear that the boundary $z = 0$ has a Minkowski signature. This metric does not cover the whole hyperboloid

$$\begin{aligned} Y_1 = \frac{Lx^1}{z}, \quad Y_2 = \frac{Lx^2}{z}, \quad Y_3 = \frac{Lx^3}{z}, \\ Y_4 = \frac{z}{2}\left(1+\frac{L^2-x^{\mu 2}}{z^2}\right), \quad Y_5 = \frac{Lx^0}{z}, \quad Y_6 = \frac{z}{2}\left(1+\frac{L^2+x^{\mu 2}}{z^2}\right). \end{aligned} \tag{A.2.18}$$

It is also possible to find a global metric covering the whole hyperboloid,

$$\begin{aligned} Y_1 = L\sinh\rho\hat{n}_1, \quad Y_2 = L\sinh\rho\hat{n}_2, \quad Y_3 = L\sinh\rho\hat{n}_3, \\ Y_4 = L\sinh\rho\hat{n}_4, \quad Y_5 L\cosh\rho\cos\tau, \quad Y_6 = L\cosh\rho\sin\tau \end{aligned} \tag{A.2.19}$$

with $\hat{n}^2 = 1$ parameterizing a 3-sphere. The metric is then

$$ds^2 = L^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3\right). \tag{A.2.20}$$

Notice that to have a mostly plus metric one must define ds^2 as the pull-back of $-\eta_{AB}dY^A dY^B$ for AdS.

A.2.2 AdS_4 metric

For AdS_4 we repeat the same procedure. Starting from relation

$$-(Y_1)^2 - \cdots - (Y_3)^2 + (Y_4)^2 + (Y_5)^2 = L^2. \tag{A.2.21}$$

Redefine Y_i as 4-dimensional spacetime coordinate

$$Y_i = rx^i, \quad \text{where} \quad \sum_{i=1}^{3}(x^i)^2 = 1 \tag{A.2.22}$$

$$Y_4 = \sqrt{L^2+r^2}sin(t/L), \quad Y_5 = \sqrt{L^2+r^2}cos(t/L). \tag{A.2.23}$$

The AdS_4 metric is then

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right)dt^2 + \left(1+\frac{r^2}{L^2}\right)^{-1}dr^2 + r^2 d\Omega_2 \tag{A.2.24}$$

A.3 ’t Hooft Matrices

To describe the $SO(6)$ spinor representation in $\mathcal{N} = 4$ supergravity theory, the $SO(6)$ indices are converted to a fundamental $SU(4)$ indices due to $(\mathbf{4}\otimes\mathbf{4})_{\text{antisymmetric}} \cong \mathbf{6}$. An $SO(6)$ vector index m can be converted to a pair of anti-symmetric $SU(4)$ indices $[ij]$ in the following ways

$$\phi^{ij} = \frac{1}{2}\sum_{m=1}^{6}[G_m]^{ij}\phi_m, \quad \phi_{ij} = -\frac{1}{2}\sum_{m=1}^{6}[G_m]_{ij}\phi_m \tag{A.3.1}$$

where ϕ_m is a generic $SO(6)$ vector, and G's are the 't Hooft matrices with the following explicit form

$$[G_1]^{ij} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad [G_2]^{ij} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$[G_3]^{ij} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad [G_4]^{ij} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \tag{A.3.2}$$

$$[G_5]^{ij} = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \qquad [G_6]^{ij} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}.$$

To be consistent with $(\phi_{ij})^* = \phi^{ij}$, these matrices satisfy the relation

$$[G_m]_{ij} = -\frac{1}{2}\epsilon_{ijkl}[G_m]^{kl} = -([G_m]^{ij})^* . \tag{A.3.3}$$

$$\mathcal{V}_M{}^{ij} = \frac{1}{2}\sum_{m=1}^{6}[G_m]^{ij}\mathcal{V}_M{}^m \tag{A.3.4}$$

$$\mathcal{V}^M{}_{ij} = -\frac{1}{2}\sum_{m=1}^{6}[G_m]_{ij}\mathcal{V}^M{}_m \tag{A.3.5}$$

A.4 BPS equations for type IIB compactification

In this section, we give the full BPS equations form the non-geometric compactification of type IIB theory. The BPS equations are given by

$$
\begin{aligned}
\varphi_g' =& -\frac{1}{32W}e^{\varphi_1-3\varphi_2-\varphi_g}\lambda^2[-e^{2\varphi_1}-9e^{2\varphi_2}+6e^{\varphi_1+\varphi_2}+e^{2(\varphi_1+3\varphi_2+\varphi_g)}\\
&+9e^{4\varphi_2+2\varphi_g}-6e^{\varphi_1+5\varphi_2+2\varphi_g}+6e^{3\varphi_2+2\varphi_g}(2e^{\varphi_1}-3e^{\varphi_2}-e^{2\varphi_1+\varphi_2}\\
&+2e^{\varphi_1+2\varphi_2})\chi_2\chi_g+2e^{5\varphi_2+2\varphi_g}(6e^{\varphi_1}-9e^{\varphi_2}+e^{2\varphi_1+\varphi_2})\chi_2^3\chi_g+e^{2(\varphi_1+\varphi_g)}\chi_g^2\\
&+9e^{2(\varphi_2+\varphi_g)}\chi_g^2-6e^{\varphi_1+\varphi_2+2\varphi_g}\chi_g^2+3e^{4\varphi_2}(e^{2\varphi_1}+3e^{2\varphi_2}-2e^{\varphi_1+\varphi_2})\chi_2^4(-1\\
&+e^{2\varphi_g}\chi_g^2)+e^{2\varphi_1+6\varphi_2}\chi_2^6(-1+e^{2\varphi_g}\chi_g^2)+3e^{2\varphi_2}\chi_2^2(-e^{2\varphi_1}-6e^{2\varphi_2}+4e^{\varphi_1+\varphi_2}\\
&+3e^{4\varphi_2+2\varphi_g}+e^{2\varphi_g}(e^{2\varphi_1}+6e^{2\varphi_2}-4e^{\varphi_1+\varphi_2})\chi_g^2)+9e^{2(\varphi_1+\varphi_2)}\chi_1^2(1\\
&+e^{2\varphi_2}\chi_2^2)(-1+e^{2(\varphi_2+\varphi_g)}-2e^{2(\varphi_2+\varphi_g)}\chi_2\chi_g+e^{2\varphi_g}\chi_g^2+e^{2\varphi_2}\chi_2^2(-1+e^{2\varphi_g}\chi_g^2))\\
&-6e^{2(\varphi_1+\varphi_2)}\chi_1(-e^{2\varphi_g}(-1+e^{2\varphi_2})\chi_g+e^{4\varphi_2+2\varphi_g}\chi_2^2\chi_g-e^{4\varphi_2+2\varphi_g}\chi_2^4\chi_g\\
&-\chi_2(1+e^{4\varphi_2+2\varphi_g}-e^{2\varphi_g}\chi_g^2)+2e^{2\varphi_2}\chi_2^3(-1+e^{2\varphi_g}\chi_g^2)+e^{4\varphi_2}\chi_2^5(-1+e^{2\varphi_g}\chi_g^2))]
\end{aligned}
\tag{A.4.1}
$$

$$
\begin{aligned}
\chi_g' =& -\frac{1}{16W}e^{\varphi_1-3\varphi_2-\varphi_g}\lambda^2[3e^{3\varphi_2}(2e^{\varphi_1}-3e^{\varphi_2}-e^{2\varphi_1+\varphi_2}+2e^{\varphi_1+2\varphi_2})\chi_2\\
&+e^{5\varphi_2}(6e^{\varphi_1}-9e^{\varphi_2}+e^{2\varphi_1+\varphi_2})\chi_2^3+(e^{\varphi_1}-3e^{\varphi_2})^2\chi_g+3e^{2\varphi_2}(e^{2\varphi_1}+6e^{2\varphi_2}\\
&-4e^{\varphi_1+\varphi_2})\chi_2^2\chi_g+3e^{4\varphi_2}(e^{2\varphi_1}+3e^{2\varphi_2}-2e^{\varphi_1+\varphi_2})\chi_2^4\chi_g+e^{2\varphi_1+6\varphi_2}\chi_2^6\chi_g\\
&+9e^{2(\varphi_1+\varphi_2)}\chi_1^2(1+e^{2\varphi_2}\chi_2^2)(-e^{2\varphi_2}\chi_2+\chi_g+e^{2\varphi_2}\chi_2^2\chi_g)-3e^{2(\varphi_1+\varphi_2)}\chi_1(1\\
&-e^{2\varphi_2}+e^{4\varphi_2}\chi_2^2-e^{4\varphi_2}\chi_2^4+2\chi_2\chi_g+4e^{2\varphi_2}\chi_2^3\chi_g+2e^{4\varphi_2}\chi_2^5\chi_g)]
\end{aligned}
\tag{A.4.2}
$$

$$
\begin{aligned}
\varphi_1' = &-\frac{1}{32W}e^{\varphi_1-3\varphi_2-\varphi_g}\lambda^2[e^{2\varphi_1}+3e^{2\varphi_2}-4e^{\varphi_1+\varphi_2}-4e^{\varphi_1+2\varphi_2+\varphi_g}+6e^{3\varphi_2+\varphi_g}\\
&+e^{2(\varphi_1+3\varphi_2+\varphi_g)}+2e^{2\varphi_1+3\varphi_2+\varphi_g}-4e^{\varphi_1+4\varphi_2+\varphi_g}+3e^{4\varphi_2+2\varphi_g}-4e^{\varphi_1+5\varphi_2+2\varphi_g}\\
&+2e^{3\varphi_2+2\varphi_g}(4e^{\varphi_1}-3e^{\varphi_2}-3e^{2\varphi_1+\varphi_2}+4e^{\varphi_1+2\varphi_2})\chi_2\chi_g+2e^{5\varphi_2+2\varphi_g}(4e^{\varphi_1}\\
&-3e^{\varphi_2}+e^{2\varphi_1+\varphi_2})\chi_2^3\chi_g+e^{2(\varphi_1+\varphi_g)}\chi_g^2+3e^{2(\varphi_2+\varphi_g)}\chi_g^2-4e^{\varphi_1+\varphi_2+2\varphi_g}\chi_g^2\\
&+e^{2\varphi_1+6\varphi_2}\chi_2^6(1+e^{2\varphi_g}\chi_g^2)+e^{2\varphi_2}\chi_2^2(3e^{2\varphi_1}+6e^{2\varphi_2}-8e^{\varphi_1+\varphi_2}+6e^{3\varphi_2+\varphi_g}\\
&-6e^{2\varphi_1+3\varphi_2+\varphi_g}+4e^{\varphi_1+4\varphi_2+\varphi_g}+3e^{4\varphi_2+2\varphi_g}+e^{2\varphi_g}(3e^{2\varphi_1}+6e^{2\varphi_2}\\
&-8e^{\varphi_1+\varphi_2})\chi_g^2)+e^{4\varphi_2}\chi_2^4(3e^{2\varphi_1}+3e^{2\varphi_2}-4e^{\varphi_1+\varphi_2}+4e^{\varphi_1+2\varphi_2+\varphi_g}\\
&+e^{2\varphi_g}(3e^{2\varphi_1}+3e^{2\varphi_2}-4e^{\varphi_1+\varphi_2})\chi_g^2)+9e^{2(\varphi_1+\varphi_2)}\chi_1^2(1+e^{2\varphi_2}\chi_2^2)((1+e^{\varphi_2+\varphi_g})^2\\
&-2e^{2(\varphi_2+\varphi_g)}\chi_2\chi_g+e^{2\varphi_g}\chi_g^2+e^{2\varphi_2}\chi_2^2(1+e^{2\varphi_g}\chi_g^2))-6e^{2(\varphi_1+\varphi_2)}\chi_1(-e^{2\varphi_g}(-1\\
&+e^{2\varphi_2})\chi_g+e^{4\varphi_2+2\varphi_g}\chi_2^2\chi_g-e^{4\varphi_2+2\varphi_g}\chi_2^4\chi_g+e^{4\varphi_2}\chi_2^5(1+e^{2\varphi_g}\chi_g^2)\\
&+2e^{2\varphi_2}\chi_2^3(1+e^{\varphi_2+\varphi_g}+e^{2\varphi_g}\chi_g^2)+\chi_2(1+2e^{\varphi_2+\varphi_g}-2e^{3\varphi_2+\varphi_g}\\
&-e^{4\varphi_2+2\varphi_g}+e^{2\varphi_g}\chi_g^2))]
\end{aligned}
\tag{A.4.3}
$$

$$
\begin{aligned}
\varphi_2' = &-\frac{1}{32W}e^{\varphi_1-3\varphi_2-\varphi_g}\lambda^2[-e^{2\varphi_1}-3e^{2\varphi_2}+4e^{\varphi_1+\varphi_2}+2e^{\varphi_1+2\varphi_2+\varphi_g}\\
&+e^{2(\varphi_1+3\varphi_2+\varphi_g)}-2e^{\varphi_1+4\varphi_2+\varphi_g}+3e^{4\varphi_2+2\varphi_g}-4e^{\varphi_1+5\varphi_2+2\varphi_g}+2e^{4\varphi_2+2\varphi_g}(-3\\
&-e^{2\varphi_1}+4e^{\varphi_1+\varphi_2})\chi_2\chi_g+2e^{5\varphi_2+2\varphi_g}(4e^{\varphi_1}-9e^{\varphi_2}+e^{2\varphi_1+\varphi_2})\chi_2^3\chi_g\\
&-e^{2(\varphi_1+\varphi_g)}\chi_g^2-3e^{2(\varphi_2+\varphi_g)}\chi_g^2+4e^{\varphi_1+\varphi_2+2\varphi_g}\chi_g^2+e^{2\varphi_1+6\varphi_2}\chi_2^6(1+e^{2\varphi_g}\chi_g^2)\\
&+e^{2\varphi_2}\chi_2^2(-e^{2\varphi_1}+6e^{2\varphi_2}+12e^{3\varphi_2+\varphi_g}-4e^{2\varphi_1+3\varphi_2+\varphi_g}+6e^{\varphi_1+4\varphi_2+\varphi_g}\\
&+9e^{4\varphi_2+2\varphi_g}-e^{2\varphi_g}(e^{2\varphi_1}-6e^{2\varphi_2})\chi_g^2)+e^{4\varphi_2}\chi_2^4(e^{2\varphi_1}+9e^{2\varphi_2}-4e^{\varphi_1+\varphi_2}\\
&+6e^{\varphi_1+2\varphi_2+\varphi_g}+e^{2\varphi_g}(e^{2\varphi_1}+9e^{2\varphi_2}-4e^{\varphi_1+\varphi_2})\chi_g^2)-2e^{2(\varphi_1+\varphi_2)}\chi_1(-e^{2\varphi_g}(1\\
&+e^{2\varphi_2})\chi_g+3e^{4\varphi_2+2\varphi_g}\chi_2^2\chi_g-3e^{4\varphi_2+2\varphi_g}\chi_2^4\chi_g+3e^{4\varphi_2}\chi_2^5(1+e^{2\varphi_g}\chi_g^2)\\
&+2e^{2\varphi_2}\chi_2^3(1+2e^{\varphi_2+\varphi_g}+e^{2\varphi_g}\chi_g^2)-\chi_2(1+4e^{3\varphi_2+\varphi_g}+3e^{4\varphi_2+2\varphi_g}+e^{2\varphi_g}\chi_g^2))\\
&+3e^{2(\varphi_1+\varphi_2)}\chi_1^2(-1+e^{2(\varphi_2+\varphi_g)}-2e^{2(\varphi_2+\varphi_g)}\chi_2\chi_g-6e^{4\varphi_2+2\varphi_g}\chi_2^3\chi_g-e^{2\varphi_g}\chi_g^2\\
&+3e^{4\varphi_2}\chi_2^4(1+e^{2\varphi_g}\chi_g^2)+e^{2\varphi_2}\chi_2^2(2+4e^{\varphi_2+\varphi_g}+3e^{2(\varphi_2+\varphi_g)}+2e^{2\varphi_g}\chi_g^2))]
\end{aligned}
\tag{A.4.4}
$$

$$\chi_1' = \frac{1}{16W} e^{\varphi_1-\varphi_2-\varphi_g}\lambda^2[-e^{2\varphi_g}(-1+e^{2\varphi_2})\chi_g + e^{4\varphi_2+2\varphi_g}\chi_2^2\chi_g - e^{4\varphi_2+2\varphi_g}\chi_2^4\chi_g + e^{4\varphi_2}\chi_2^5(1+e^{2\varphi_g}\chi_g^2) + 2e^{2\varphi_2}\chi_2^3(1+e^{\varphi_2+\varphi_g}+e^{2\varphi_g}\chi_g^2) + \chi_2(1+2e^{\varphi_2+\varphi_g} - 2e^{3\varphi_2+\varphi_g} - e^{4\varphi_2+2\varphi_g} + e^{2\varphi_g}\chi_g^2) - 3\chi_1(1+e^{2\varphi_2}\chi_2^2)((1+e^{\varphi_2+\varphi_g})^2 - 2e^{2(\varphi_2+\varphi_g)}\chi_2\chi_g + e^{2\varphi_g}\chi_g^2 + e^{2\varphi_2}\chi_2^2(1+e^{2\varphi_g}\chi_g^2))] \tag{A.4.5}$$

$$\chi_2' = -\frac{1}{16W} e^{\varphi_1-3\varphi_2-\varphi_g}\lambda^2[e^{\varphi_2+2\varphi_g}(2e^{\varphi_1} - 3e^{\varphi_2} - e^{2\varphi_1+\varphi_2} + 2e^{\varphi_1+2\varphi_2})\chi_g + e^{3\varphi_2+2\varphi_g}(6e^{\varphi_1} - 9e^{\varphi_2} + e^{2\varphi_1+\varphi_2})\chi_2^2\chi_g + e^{2\varphi_1+4\varphi_2}\chi_2^5(1+e^{2\varphi_g}\chi_g^2) + \chi_2(e^{2\varphi_1} + 6e^{2\varphi_2} - 4e^{\varphi_1+\varphi_2} + 6e^{3\varphi_2+\varphi_g} - 2e^{2\varphi_1+3\varphi_2+\varphi_g} + 2e^{\varphi_1+4\varphi_2+\varphi_g} + 3e^{4\varphi_2+2\varphi_g} + e^{2\varphi_g}(e^{2\varphi_1} + 6e^{2\varphi_2} - 4e^{\varphi_1+\varphi_2})\chi_g^2) + 2e^{2\varphi_2}\chi_2^3(e^{2\varphi_1} + 3e^{2\varphi_2} - 2e^{\varphi_1+\varphi_2} + 2e^{\varphi_1+2\varphi_2+\varphi_g} + e^{2\varphi_g}(e^{2\varphi_1} + 3e^{2\varphi_2} - 2e^{\varphi_1+\varphi_2})\chi_g^2) - e^{2\varphi_1}\chi_1(1 + 2e^{\varphi_2+\varphi_g} - 2e^{3\varphi_2+\varphi_g} - e^{4\varphi_2+2\varphi_g} + 2e^{4\varphi_2+2\varphi_g}\chi_2\chi_g - 4e^{4\varphi_2+2\varphi_g}\chi_2^3\chi_g + e^{2\varphi_g}\chi_g^2 + 5e^{4\varphi_2}\chi_2^4(1+e^{2\varphi_g}\chi_g^2) + 6e^{2\varphi_2}\chi_2^2(1+e^{\varphi_2+\varphi_g}+e^{2\varphi_g}\chi_g^2)) + 3e^{2(\varphi_1+\varphi_2)}\chi_1^2(-e^{2\varphi_g}\chi_g - 3e^{2(\varphi_2+\varphi_g)}\chi_2^2\chi_g + 2e^{2\varphi_2}\chi_2^3(1+e^{2\varphi_g}\chi_g^2) + \chi_2(2+2e^{\varphi_2+\varphi_g} + e^{2(\varphi_2+\varphi_g)} + 2e^{2\varphi_g}\chi_g^2))] \tag{A.4.6}$$

where W is given in (5.2.19).

Vitae

Mr. Khem Upathambhakul was born in Bangkok on 16 March 1988. He received his Bachelor's degree in physics from Chulalongkorn University in 2010 and Master's degree in physics from Chulalongkorn University in 2014. He has studies quantum field theory, general relativity, supersymmetry, supergravity, and string theory for his PhD. His research interests are in high energy physics, particularly in the area of gauged supergravity and AdS/CFT correspondence.

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