ลักษณะเฉพาะเชิงการนับที่เกี่ยวข้องกับวงศ์ของฟังก์ชันและการเรียงสับเปลี่ยน

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CARDINAL CHARACTERISTICS ASSOCIATED WITH FAMILIES OF FUNCTIONS AND PERMUTATIONS

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สมมติฐานความต่อเนื่องกล่าวว่า ขนาดของเซตของจำนวนจริงทั้งหมด $\mathfrak{c}$ เป็นจำนวนเชิงการนับนับไม่ได้ที่เล็กที่สุด นั่นคือ $\mathfrak{c} = \aleph_1$ เมื่อปราศจากสมมติฐานความต่อเนื่อง เป็นไปได้ที่จะมีจำนวนเชิงการนับซึ่งมีค่าอยู่ระหว่าง $\aleph_1$ กับ $\mathfrak{c}$ มีจำนวนเชิงการนับเหล่านี้มากมายที่เป็นจำนวนเชิงการนับของวงศ์อนันต์ซึ่งเกี่ยวข้องกับแนวคิดบางประการในคณิตศาสตร์เชิงการจัดอนันต์ เรียกจำนวนเชิงการนับเหล่านี้ว่า ลักษณะเฉพาะเชิงการนับ ซึ่งส่วนใหญ่จะนิยามบนวงศ์ของเซตของจำนวนธรรมชาติที่เป็นเซตอนันต์ เราศึกษาวงศ์ของฟังก์ชันและการเรียงสับเปลี่ยนบนเซตของจำนวนธรรมชาติทั้งหมดที่มีสมบัติเชิงการนับบางประการ และลักษณะเฉพาะเชิงการนับที่เกี่ยวข้อง เราแสดงความสัมพันธ์ระหว่างจำนวนเชิงการนับเหล่านี้กับอันที่เป็นที่รู้จักกันดีอื่น ๆ พร้อมทั้งผลด้านความไม่แย้งกันที่เกี่ยวข้อง

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NATTAPON SONPANOW : CARDINAL CHARACTERISTICS ASSOCIATED WITH FAMILIES OF FUNCTIONS AND PERMUTATIONS. ADVISOR : ASSOC. PROF. PIMPEN VEJJAJIVA, Ph.D., 31 pp.

The Continuum Hypothesis (CH) states that the size of the set of real numbers $\mathfrak{c}$ is the least uncountable cardinal, i.e. $\mathfrak{c} = \aleph_1$. In the absence of CH, it is possible that there are cardinals that lie between $\aleph_1$ and $\mathfrak{c}$. Many of them are cardinals of infinite families related to some concepts in infinite combinatorics, called cardinal characteristics. Most of these cardinals are defined on families of infinite sets of natural numbers. We study families of functions and permutations on the set of natural numbers with some combinatorial properties and associated cardinal characteristics. We give relations among these cardinals and other well-known ones as well as related consistency results.

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Contents

CHAPTER I

INTRODUCTION

The Continuum Hypothesis (CH) states that the size of the set of real numbers $\mathfrak{c}$ is the least uncountable cardinal, i.e. $\mathfrak{c} = \aleph_1$. It is well-known that CH is relatively independent from the Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC).

There are many cardinals of infinite families related to some concepts in infinite combinatorics that lie between $\aleph_1$ and $\mathfrak{c}$, called cardinal characteristics. Without assuming CH, the exact values of these cardinals are impossible to be determined. Relations among them as well as related consistency results were widely studied. Most of these cardinals are defined on families of infinite sets of natural numbers.

In Zhang's work ([12] and [13], for example), almost disjoint families of functions and permutations on the set of natural numbers ω were studied. This inspires us to study families of functions and permutations on ω with other combinatorial properties.

We first provide some basic background in Chapter II. Our new results are in Chapter III and are divided into several sections: Sections 1 to 3 introduce new cardinal characteristics and show, in ZFC, relations among our new cardinals and other well-known ones, and Section 4 shows some related consistency results. Chapter IV summarizes our results and gives some open problems.

CHAPTER II

PRELIMINARIES

In this thesis, we use $a, b, c, ..., A, B, C, ...$ for sets. $\mathcal{P}(A), \text{Sym}(A), (A, B), {}^{B}A$, and $F \restriction A$ denote the power set of A, the set of permutations (bijections) on A, the ordered-pair of A and B, the set of all functions from B into A, and the restriction of a functions F to A. ZFC denotes the Zermelo-Fraenkel set theory with the Axiom of Choice (AC). Throughout the thesis, we shall work in ZFC. Proofs of all theorems in this chapter will be omitted. They can be found in [8] or [9].

2.1 Ordinal Numbers

Natural numbers are constructed as follows:

$$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, ..., n + 1 = \{0, 1, ..., n\}, ...,$$

and ω denotes the set of all natural numbers.

A *(strict) partial ordering* on a set A is a binary relation on A which is irreflexive and transitive. A *linear ordering* on A is a partial ordering on A whose every two members are comparable. A *well-ordering* R on A is a linear ordering on A such that every nonempty subset of A has an R-least element. A set A is *well-ordered* if there is a well-ordering on A.

Definition. A set A is *transitive* if each element of A is a subset of A.

Definition. A set is an *ordinal (number)* if it is transitive and well-ordered by $\in$.

Note that every natural number and ω are ordinals.

Theorem 2.1.1. *Every well-ordered set is isomorphic to a unique ordinal.*

Definition. For any ordinals α and β, we say that

1. α is *less than* β, written $\alpha < \beta$, if $\alpha \in \beta$.
2. α is *less than or equal to* β, written $\alpha \leq \beta$, if $\alpha < \beta$ or $\alpha = \beta$.

Theorem 2.1.2. *Let α, β, and γ be ordinals.*

1. Every member of α is an ordinal.

2. $\alpha \nless \alpha$.

3. If $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$.

4. Exactly one of the following holds: $\alpha < \beta$, $\alpha = \beta$, $\alpha > \beta$.

Theorem 2.1.3. *Every nonempty set of ordinals has a least element.*

Definition. For any ordinal α, the *successor* of α, denoted by $\alpha + 1$, is defined by

$$\alpha + 1 = \alpha \cup \{\alpha\}.$$

Definition. An ordinal α is a *successor ordinal* if $\alpha = \beta + 1$ for some ordinal β. An ordinal $\alpha \neq 0$ which is not a successor is called a *limit ordinal*.

Note that ω is the least limit ordinal.

2.2 Cardinal Numbers

Definition. For any sets A and B, we say that A is *equinumerous* to B, denoted by $A \approx B$, if there is a bijection from A onto B.

Intuitively, the *cardinality* of a set is the number of all elements of the set. One form of AC states that every set can be well-ordered. So, by Theorems 2.1.1 and 2.1.3, the following definition is well-defined.

Definition. For any set A, the *cardinality* of A, denoted by $|A|$, is the least ordinal κ such that $A \approx \kappa$. We say that κ is a *cardinal* (number) if $\kappa = |A|$ for some set A.

Note that every natural number and ω are cardinals.

Theorem 2.2.1. *For any sets A and B, $|A| = |B|$ if and only if $A \approx B$.*

Definition. A set A is said to be *finite* if $|A| = n$ for some $n \in \omega$. A set which is not finite is said to be *infinite*. Natural numbers are said to be *finite cardinals*. Cardinals which are not finite are said to be *infinite cardinals*. A set is said to be *denumerable* if its cardinality is ω.

Theorem 2.2.2. *For any cardinal κ, $|\kappa| = \kappa$ and if κ is infinite, then κ is a limit ordinal.*

Theorem 2.2.3. *For any ordinal α, there is a cardinal κ such that $|\alpha| < \kappa$.*

Notation. $\mathfrak{c}$ is the cardinality of the set of real numbers $\mathbb{R}$, $\aleph_0$ is the cardinality of ω, and $\aleph_1$ is the least cardinal which is greater than $\aleph_0$.

Definition. Let $\kappa = |A|$ and $\lambda = |B|$. We define

1. $\kappa + \lambda = |A \cup B|$ where $A \cap B = \emptyset$,
2. $\kappa \cdot \lambda = |A \times B|$,
3. $\kappa^\lambda = |{}^B A|$.

Theorem 2.2.4. *(Absorption Law) For any cardinals κ and λ such that κ or λ is infinite,*

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

Theorem 2.2.5. *For any set A, $|\mathcal{P}(A)| = 2^{|A|}$.*

Theorem 2.2.6. *For any infinite set A, $|\mathrm{Sym}(A)| = 2^{|A|}$.*

Definition. For any limit ordinal α, the *cofinality* of α, denoted by $\text{cf}(\alpha)$, is the least ordinal β such that there is a function $f : \beta \to \alpha$ so that $\text{ran}(f)$ is unbounded in α, i.e.

$$\forall\gamma < \alpha\ \exists\delta < \beta\ (f(\delta) > \gamma).$$

Theorem 2.2.7**.** *For any limit ordinal* α*,* $\text{cf}(\alpha)$ *is a cardinal and* $\text{cf}(\text{cf}(\alpha)) = \text{cf}(\alpha)$*.*

Definition. An infinite cardinal κ is *regular* if $\text{cf}(\kappa) = \kappa$; otherwise, it is *singular*.

Theorem 2.2.8**.** *For any regular cardinal* κ *and any set* $\mathcal{A}$*, if* $|\mathcal{A}| < \kappa$ *and* $|A| < \kappa$ *for all* $A \in \mathcal{A}$*, then* $|\bigcup\mathcal{A}| < \kappa$*.*

2.3 The Continuum Hypothesis

Theorem 2.3.1**.** *(Cantor)* $\kappa < 2^\kappa$ *for any cardinal* κ*.*

The Continuum Hypothesis (CH) states that there is no cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$, i.e. $\aleph_1 = 2^{\aleph_0}$. The Generalized Continuum Hypothesis (GCH) states that, for any infinite cardinal λ, there is no cardinal κ such that $\lambda < \kappa < 2^\lambda$.

Notation. Throughout this thesis, we use $\alpha, \beta, \gamma, ...$ for ordinal numbers, $\kappa, \lambda, \mu, ...$ for cardinal numbers, and $k, l, m, ...$ for natural numbers, unless otherwise stated.

2.4 Cardinal Characteristics

Some concepts in infinite combinatorics lead to cardinal characteristics which lie inclusively between $\aleph_1$ and $\mathfrak{c}$. So, without CH, it is interesting to know properties of these cardinals. Some of these combinatorial concepts and associated cardinal characteristics are as follows. For more information, see Chapter 9 of [8].

Notation. For any set A and any cardinal κ,

$$[A]^\kappa = \{X \in \mathcal{P}(A) : |X| = \kappa\},$$
$$[A]^{<\kappa} = \{X \in \mathcal{P}(A) : |X| < \kappa\}, \text{ and}$$
$${}^{<\kappa}A = \bigcup\{{}^{\alpha}A : \alpha < \kappa\}.$$

Definition. For any two functions $f, g \in {}^\omega\omega$, we say that g *dominates* f if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A family $\mathcal{D} \subseteq {}^\omega\omega$ is a *dominating family* if each function in ${}^\omega\omega$ is dominated by some member of $\mathcal{D}$, and a family $\mathcal{B} \subseteq {}^\omega\omega$ is an *unbounded family* if there is no function in ${}^\omega\omega$ which dominates every member of $\mathcal{B}$. The *dominating number* $\mathfrak{d}$ and the *bounding number* $\mathfrak{b}$ are defined as follows:

$$\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \subseteq {}^\omega\omega \text{ is a dominating family}\},$$
$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq {}^\omega\omega \text{ is an unbounded family}\}.$$

Definition. For any two sets $X, Y \in [\omega]^\omega$, we say that Y *splits* X if $X \cap Y$ and $X \setminus Y$ are infinite. A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* if each member of $[\omega]^\omega$ is split by some member of $\mathcal{S}$, and a family $\mathcal{R} \subseteq [\omega]^\omega$ is a *reaping family* if there is no set in $[\omega]^\omega$ which splits every member of $\mathcal{R}$. The *splitting number* $\mathfrak{s}$ and the *reaping number* $\mathfrak{r}$ are defined as follows:

$$\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \text{ is a splitting family}\},$$
$$\mathfrak{r} = \min\{|\mathcal{R}| : \mathcal{R} \subseteq [\omega]^\omega \text{ is a reaping family}\}.$$

Definition. Two sets $X, Y \in [\omega]^\omega$ are *almost disjoint* if $X \cap Y$ is finite. An infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is an *almost disjoint family* if its members are pairwise almost disjoint. Such a family $\mathcal{A}$ is a *maximal almost disjoint family* if it is maximal with respect to the inclusion. The *almost disjoint number* $\mathfrak{a}$ is defined as follows:

$$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ is a maximal almost disjoint family}\}.$$

Definition. An infinite family $\mathcal{I} \subseteq [\omega]^\omega$ is an *independent family* if, for any two finite disjoint sets $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{I}$, $\bigcap \mathcal{X} \setminus \bigcup \mathcal{Y}$ is infinite (here $\bigcap \emptyset = \omega$). Such a family $\mathcal{I}$ is a *maximal independent family* if it is maximal with respect to the inclusion. The *independent number* $\mathfrak{i}$ is defined as follows:

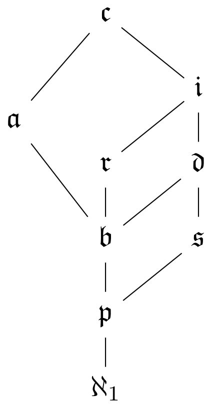
$$\mathfrak{i} = \min\{|\mathcal{I}| : \mathcal{I} \subseteq [\omega]^\omega \text{ is a maximal independent family}\}.$$

Definition. A family $\mathcal{E} \subseteq [\omega]^\omega$ has the *strong finite intersection property (sfip)* if, for any finite set $\mathcal{F} \subseteq \mathcal{E}$, $\bigcap \mathcal{F}$ is infinite (here $\bigcap \emptyset = \omega$). A set $Z \in [\omega]^\omega$ is a *pseudo-intersection* of such a family $\mathcal{E}$ if $Z \setminus X$ is finite for all $X \in \mathcal{E}$. The *pseudo-intersection number* $\mathfrak{p}$ is defined as follows:

$$\mathfrak{p} = \min\{|\mathcal{E}| : \mathcal{E} \subseteq [\omega]^\omega \text{ has the sfip but has no pseudo-intersection}\}.$$

Theorem 2.4.1**.** *Relations between these cardinals, provable in ZFC, are in the following diagram. A line connecting two cardinals indicates that the lower cardinal is less than or equal to the upper cardinal. Rigorously,*

$$\aleph_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{i} \leq \mathfrak{c},\ \ \mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{i},\ \ \mathfrak{p} \leq \mathfrak{s} \leq \mathfrak{d}, \text{ and } \mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}.$$



2.5 Some Background in Logic

This section gives some informal concepts in first-order logic. For a precise explanation, see [6]. We write $\Gamma \vdash \varphi$ if a formula φ can be proved from a set of formulas Γ.

Definition. A set of formulas Γ is *consistent* if there is no formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$. We denote the statement "Γ is consistent" by $Con(\Gamma)$.

Definition. For a set M and a set of formulas Γ, we say that M *is a model of* Γ, or M *satisfies* Γ, if every formula $\varphi \in \Gamma$ holds in M.

Notation. We write $M \vDash \varphi$ meaning that φ holds in M.

Theorem 2.5.1. *A set of formulas Γ is consistent if and only if there exists a model M satisfying Γ.*

Theorem 2.5.2. *Let φ be a formula and Γ be a set of formulas. Then $\Gamma \cup \{\neg\varphi\}$ is consistent if and only if $\Gamma \nvdash \varphi$.*

Thus, to show that a formula φ cannot be proved from a set of formulas Γ, we instead show that $\Gamma \cup \{\neg\varphi\}$ is consistent. The details for consistency proofs are very deep in logic and set theory. One of the widely used method is forcing.

2.6 Forcing

We give a brief information about forcing, which will be used in Section 3.4. See [9] or [8] for the details and proofs.

From now on, we let M be a transitive model of ZFC (this means a finite fragment of ZFC). In this section, a *partial order* is a pair $(\mathbb{P}, \leq)$ such that $\mathbb{P} \neq \emptyset$ and $\leq$ is a relation on $\mathbb{P}$ which is transitive and reflexive.

Definition. A *forcing poset* is composed of a set $\mathbb{P}$ with a partial order $\leq$ and a largest element $\mathbb{1}$. Elements in $\mathbb{P}$ are called *forcing conditions*. A subset $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ if

$$\forall p \in \mathbb{P}\ \exists q \in D\ (q \leq p).$$

In the following, $\mathbb{P}$ is a set with a partial order $\leq$ and a largest element $\mathbb{1}$.

Definition. A nonempty $F \subseteq \mathbb{P}$ is a *filter* on $\mathbb{P}$ if

1. for any $p, q \in F$, there is an $r \in F$ such that $r \leq p$ and $r \leq q$, and
2. for any $p, q \in \mathbb{P}$, $p \leq q$ and $p \in F$ implies $q \in F$.

Definition. A filter $G \subseteq \mathbb{P}$ is $\mathbb{P}$-*generic* over M if for any $D \in M$ which is dense in $\mathbb{P}$, $G \cap D \neq \emptyset$.

Theorem 2.6.1. *If M is a countable transitive model of ZFC and $\mathbb{P} \in M$ is a forcing poset, then there exists a $\mathbb{P}$-generic filter G over M.*

In the following, $M[G]$ will be constructed from a $\mathbb{P}$-generic filter G over M by applying set-theoretic processes definable in M. Each element of $M[G]$ will have a *name* in M. The following two definitions are defined recursively. In order to keep things simple, we omit the details.

Definition. τ is a $\mathbb{P}$-*name* if τ is a relation and

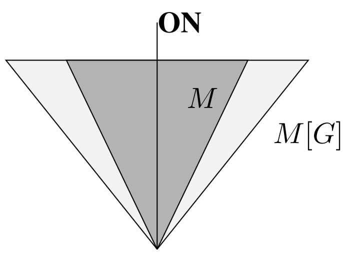
$$\forall(\sigma, p) \in \tau\ (\sigma \text{ is a } \mathbb{P}\text{-name} \wedge p \in \mathbb{P}).$$

Definition. Suppose G is a $\mathbb{P}$-generic filter over M and $\mathbb{P} \in M$.

- For any $\mathbb{P}$-name τ, $\tau_G = \{\sigma_G : (\sigma, p) \in \tau \text{ for some } p \in G\}$.
- $M[G] = \{\tau_G : \tau \in M \text{ is a } \mathbb{P}\text{-name}\}$.

We sometimes use $\mathring{f}$ for a $\mathbb{P}$-name where $\mathring{f}_G = f \in M[G]$.

The model M is regarded as the *ground model*, and the model $M[G]$ is called a *generic model* (or a *generic extension of* M). $M[G]$ will be the least extension of M to a transitive model of ZFC containing G, where the set of ordinals in M and in $M[G]$ are the same, but some cardinals might be different. For example, it could happen that $2^{\aleph_0} = \aleph_1$ in M but $2^{\aleph_0} = \aleph_2$ in $M[G]$.



Definition. For any formula $\varphi(x_1, ..., x_n)$ and $\mathbb{P}$-names $\tau_1, ..., \tau_n \in M$, we say that p *forces* $\varphi(\tau_1, ..., \tau_n)$, denoted by $p \Vdash \varphi(\tau_1, ..., \tau_n)$, if

for any $\mathbb{P}$-generic filter G over M with $p \in G$, $\varphi(\tau_{1G}, ..., \tau_{nG})$ holds in $M[G]$.

Theorem 2.6.2. *For any formula $\varphi(x_1, ..., x_n)$ and $\mathbb{P}$-names $\tau_1, ..., \tau_n \in M$, if G is $\mathbb{P}$-generic over M, then*

$\varphi(\tau_{1G}, ..., \tau_{nG})$ holds in $M[G]$ if and only if there is a $p \in G$ such that $p \Vdash \varphi(\tau_1, ..., \tau_n)$.

By the definition, $\mathbb{1} \Vdash \varphi(\tau_1, ..., \tau_n)$ tells us that $\varphi(\tau_{1G}, ..., \tau_{nG})$ holds in $M[G]$ for any $\mathbb{P}$-generic filter G over M since $\mathbb{1} \in G$ for any filter G. In general, $p \Vdash \varphi(\tau_1, ..., \tau_n)$ tells us that the possibility that $\varphi(\tau_{1G}, ..., \tau_{nG})$ holds in $M[G]$ is related to the possibility that $p \in G$. For example, in many situations, we may consider a set $D = \{p \in \mathbb{P} : p \Vdash \varphi(\tau_1, ..., \tau_n)\}$ and try to prove that D is dense in $\mathbb{P}$. If this has been done and G is a $\mathbb{P}$-generic filter over M then there exists a $p \in G \cap D$, and hence $\varphi(\tau_{1G}, ..., \tau_{nG})$ holds in $M[G]$.

In the following theorem, $ZFC + \psi$ denotes the union of the set of ZFC axioms and $\{\psi\}$ where ψ is a sentence.

Theorem 2.6.3. *Suppose M is a countable transitive model of $ZFC + \psi$, $\mathbb{P}$ is a forcing poset and G is a $\mathbb{P}$-generic filter over M and $\mathbb{P} \in M$. Then $M[G]$ is a countable transitive model of ZFC, $M \subseteq M[G]$ and $G \in M[G]$. Moreover, if a sentence φ holds in the model $M[G]$, then we conclude that*

$$Con(ZFC + \psi) \rightarrow Con(ZFC + \varphi).$$

The statement $Con(ZFC) \rightarrow Con(ZFC + \varphi)$ can be read as "φ is relatively consistent with ZFC". By Theorem 2.5.2, the statement means that if ZFC is consistent, then $ZFC \nvdash \neg\varphi$.

Definition. Let $\mathbb{P}$ be a forcing poset.

- A set $A \subseteq \mathbb{P}$ is an *antichain* in $\mathbb{P}$ if for any distinct $p, q \in A$ there is no $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$.
- The poset $\mathbb{P}$ satisfies the *countable chain condition* (or $\mathbb{P}$ is *ccc*) if every antichain in $\mathbb{P}$ is countable.

Definition. For any $\mathbb{P}$-name τ, a *nice name for a subset of* τ is a $\mathbb{P}$-name of the form

$$\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \text{dom}(\tau)\}$$

where each A_σ is an antichain in $\mathbb{P}$.

Theorem 2.6.4. *If $\mathbb{P} \in M$ and $\tau, \mu \in M$ are $\mathbb{P}$-names, then there is a nice $\mathbb{P}$-name $\vartheta \in M$ for a subset of τ such that $\mathbb{1} \Vdash (\mu \subseteq \tau \rightarrow \mu = \vartheta)$.*

Example. (Cohen Forcing)
In a ground model M, consider the poset $\mathbb{P} = {}^{<\omega}\omega$ with the order $\supseteq$. Clearly the largest element $\mathbb{1} = \emptyset$. Note that $\mathbb{P}$ is ccc. Suppose G is a $\mathbb{P}$-generic filter over M and $g = \bigcup G$. Then it can be shown that $g \in M[G]$ is a surjective function on ω.

Definition. $\text{Fn}(I, J) = \{p \subseteq I \times J : p \text{ is a finite function}\}$.

Example. (Another Cohen Forcing)
In a ground model M, consider a cardinal $\kappa \neq 0$ and the poset $\mathbb{P} = \text{Fn}(\kappa \times \omega, 2)$ with the order $\supseteq$. Clearly the largest element $\mathbb{1} = \emptyset$. Note that $\mathbb{P}$ is ccc. Suppose G is a $\mathbb{P}$-generic filter over M and $g = \bigcup G$. Then $g \in M[G]$ is a function from $\kappa \times \omega$ to 2. In addition, if M satisfies GCH and κ is regular, then $2^{\aleph_0} = \kappa$ holds in $M[G]$.

Roughly speaking, two posets are *forcing equivalent* if they produce the same generic extension and the same interpretation of names. For example, $({}^{<\omega}\omega, \supseteq)$ and $(\text{Fn}(\omega, \omega), \supseteq)$ are forcing equivalent.

2.7 Finite-Support Iterated Forcing

In this section, M is a countable transitive model of ZFC. We first want to obtain a two-step iterated forcing. Intuitively, we start with a poset $\mathbb{P} \in M$ and a $\mathbb{P}$-generic filter G over M, which give us a generic extension $M[G]$. Then we want to get a poset $\mathbb{Q} \in M[G]$ in order to obtain a $\mathbb{Q}$-generic filter H over $M[G]$ and further generic extension $M[G][H]$. However, since $\mathbb{Q} \in M[G]$, there must be a $\mathbb{P}$-name corresponding to $\mathbb{Q}$. This idea leads to the following definitions.

Definition. If $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ is a forcing poset, then a *$\mathbb{P}$-name for a forcing poset* is a triple of $\mathbb{P}$-names $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$ such that $\mathring{\mathbb{1}}_{\mathbb{Q}} \in \text{dom}(\mathring{\mathbb{Q}})$ and

$$\mathbb{1}_{\mathbb{P}} \Vdash [\mathring{\mathbb{1}}_{\mathbb{Q}} \in \mathring{\mathbb{Q}} \wedge \mathring{\leq}_{\mathbb{Q}} \text{ is a partial order of } \mathring{\mathbb{Q}} \text{ with largest element } \mathring{\mathbb{1}}_{\mathbb{Q}}].$$

From now on, let $\mathbb{P}$ be a forcing poset and $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$ be a $\mathbb{P}$-name for a forcing poset. Sometimes we write $\mathring{\mathbb{Q}}$ for $(\mathring{\mathbb{Q}}, \mathring{\leq}_{\mathbb{Q}}, \mathring{\mathbb{1}}_{\mathbb{Q}})$. Note that $\mathring{\mathbb{Q}}_G$ is a forcing poset in $M[G]$.

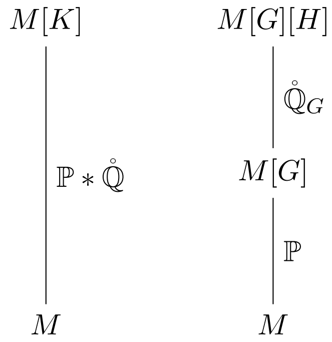
Definition. The product $\mathbb{P} * \mathring{\mathbb{Q}}$ is the triple $(\mathbb{R}, \leq, \mathbb{1})$ where

$$\mathbb{R} = \{(p, \mathring{q}) \in \mathbb{P} \times \text{dom}(\mathring{\mathbb{Q}}) : p \Vdash \mathring{q} \in \mathring{\mathbb{Q}}\}, \ \mathbb{1} = (\mathbb{1}_{\mathbb{P}}, \mathring{\mathbb{1}}_{\mathbb{Q}}), \text{ and}$$
$$(p_1, \mathring{q}_1) \leq (p_2, \mathring{q}_2) \text{ if and only if } p_1 \leq_{\mathbb{P}} p_2 \text{ and } p_1 \Vdash [\mathring{q}_1 \mathring{\leq}_{\mathbb{Q}} \mathring{q}_2].$$

Note that $\mathbb{P} * \mathring{\mathbb{Q}}$ is a forcing poset.

Theorem 2.7.1. *Let K be $\mathbb{P} * \mathring{\mathbb{Q}}$-generic over M. Let $G = \{p \in \mathbb{P} : (p, \mathring{\mathbb{1}}_{\mathbb{Q}}) \in \mathbb{P} * \mathring{\mathbb{Q}}\}$ and let $H = \{\mathring{q}_G : \mathring{q} \in \text{dom}(\mathring{\mathbb{Q}}) \wedge \exists p (p, \mathring{q}) \in K\}$. Then G is $\mathbb{P}$-generic over M, H is $\mathring{\mathbb{Q}}_G$-generic over $M[G]$, and $M[K] = M[G][H]$.*

The following figure illustrates this theorem. Two-step iterated forcing, by $\mathbb{P} \in M$ and then by $\mathbb{Q} \in M[G]$, is the same as one-step forcing by a product $\mathbb{P} * \mathring{\mathbb{Q}} \in M$.



In the following definition, if p is a sequence of length η, then we write $(p)_\mu$ to denote the μ-th component of p. (It is $p(\mu)$ if we regard p as a function with $\mathrm{dom}(p) = \eta$.)

Definition. For any ordinal α, a *finite-support iteration of length* α is a pair of sequences of the form

$$\left(\left\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \alpha\right\rangle, \left\langle(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_{\mathring{\mathbb{Q}}_\xi}, \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi}) : \xi < \alpha\right\rangle\right)$$

satisfying the following conditions.

1. Each $(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi)$ is a forcing poset.

2. Each $(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_{\mathring{\mathbb{Q}}_\xi}, \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\xi})$ is a $\mathbb{P}_\xi$-name for a forcing poset.

3. For all ordinals $\xi < \alpha$, $\mathbb{P}_{\xi+1}$ is the set of all sequences $p^\frown \mathring{q}$ such that $p \in \mathbb{P}_\xi$, $\mathring{q} \in \mathrm{dom}(\mathring{\mathbb{Q}}_\xi)$ and $p \Vdash_{\mathbb{P}_\xi} \mathring{q} \in \mathring{\mathbb{Q}}_\xi$. Here $p^\frown \mathring{q}$ is the concatenation of the sequence p and the length-one sequence $\langle \mathring{q} \rangle$.

4. For all limit ordinals $\eta \leq \alpha$, $\mathbb{P}_\eta$ is the set of all sequences $p = \langle \mathring{q}_\xi : \xi < \eta \rangle$ of length η such that, for some $\xi < \eta$, $p{\restriction}\xi \in \mathbb{P}_\xi$ and $(p)_\mu = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$ whenever $\xi \leq \mu < \eta$.

5. If $p, p' \in \mathbb{P}_\xi$, then $p \leq_\xi p'$ if and only if $p{\restriction}\mu \Vdash_{\mathbb{P}_\mu} [(p)_\mu \leq (p')_\mu]$ for all $\mu < \xi$.

6. $\mathbb{1}_\xi$ is the sequence $\left\langle \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu} : \mu < \xi \right\rangle$.

From 3, $\mathbb{P}_{\xi+1}$ and $\mathbb{P}_\xi * \mathring{\mathbb{Q}}_\xi$ are forcing equivalent. From 4, for all limit ordinals $\eta \leq \alpha$ and all $p = \langle \mathring{q}_\mu : \mu < \eta \rangle \in \mathbb{P}_\eta$, the set $\left\{\mu < \eta : \mathring{q}_\mu \neq \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}\right\}$ is finite. This indicates a property of finite-support iteration.

Note that we can consider $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ as an $\subseteq$-increasing sequence of forcing posets: If $\xi < \eta$ and $p \in \mathbb{P}_\xi$, then we can regard p as an element $\hat{p} \in \mathbb{P}_\eta$ so that $\hat{p} \restriction \xi = p$ and $(\hat{p})_\mu = \mathring{\mathbb{1}}_{\mathring{\mathbb{Q}}_\mu}$ whenever $\xi \leq \mu < \eta$.

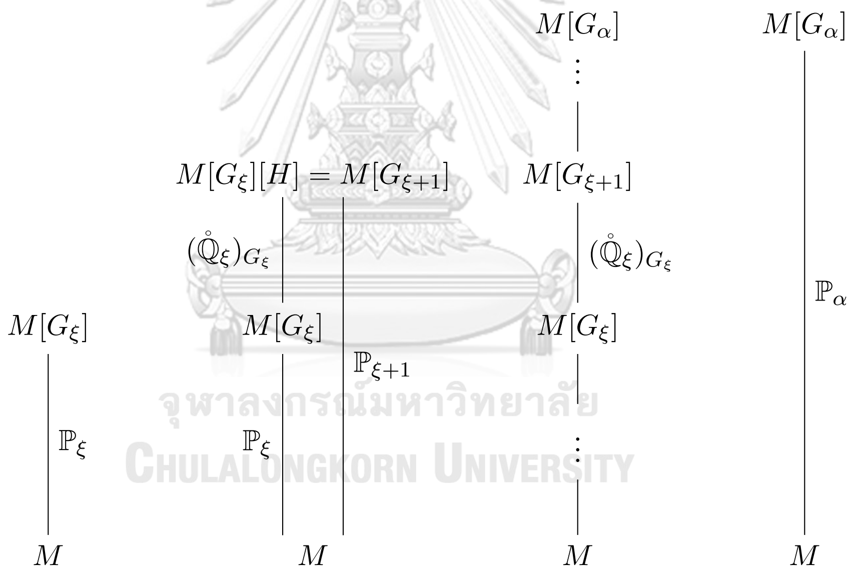
Theorem 2.7.2**.** *In a finite-support iteration of length* α*, if* $\mathbb{1}_\xi \Vdash \left[\mathring{\mathbb{Q}}_\xi \text{ is ccc}\right]$ *for all* $\xi < \alpha$*, then* $\mathbb{P}_\alpha$ *is ccc.*

Theorem 2.7.3. *Let* $\left(\left\langle \mathbb{P}_\xi : \xi \leq \alpha \right\rangle, \left\langle \mathring{\mathbb{Q}}_\xi : \xi < \alpha \right\rangle\right)$ *be a finite-support iteration of length* α*, and* G *be a* $\mathbb{P}_\alpha$*-generic filter over* M*. For each* $\xi \leq \alpha$*, let* $G_\xi = \{p{\restriction}\xi : p \in \mathbb{P}_\alpha\}$ *be the restriction of* G *to* $\mathbb{P}_\xi$*.*

1. $\langle M[G_\xi] : \xi \leq \alpha \rangle$ *is an increasing* $\subseteq$*-chain of generic extensions of* M*.*

2. *For each* $\xi < \alpha$*, there is a filter* H *which is* $(\mathring{\mathbb{Q}}_\xi)_{G_\xi}$*-generic over* $M[G_\xi]$ *and* $M[G_\xi][H] = M[G_{\xi+1}]$.

The following figure illustrates a finite-support iteration of length α. In many situations, the length of the iteration is a cardinal (or regular cardinal) and the iterands $\mathbb{Q} = (\mathring{\mathbb{Q}}_\xi)_{G_\xi}$ are the same (while the name $\mathring{\mathbb{Q}}_\xi$ might be different according to different previous posets and models). In such cases, although $M[G_\alpha]$ is not the union of the previous all $M[G_\xi]$'s, some important sets in $M[G_\alpha]$ can be shown that they are actually in some $M[G_\xi]$ where $\xi < \alpha$. This feature yields a good result if the single-step iteration $\mathbb{Q}$ is good enough.



CHAPTER III

CARDINAL CHARACTERISTICS ASSOCIATED WITH FAMILIES OF FUNCTIONS AND PERMUTATIONS

In this chapter, we introduce eight new cardinal characteristics associated with some families of functions and permutations. In the forthcoming sections, we show our results on these new cardinals. First, recall some definitions from Chapter II.

For any sets A and B, we say that A *splits* B if $B \cap A$ and $B \setminus A$ are infinite, and A and B are *almost disjoint* if $A \cap B$ is finite. For any functions $f, g \in {}^{\omega}\omega$, we say that f *dominates* g, denoted by $g \leq^* f$, if $g(n) \leq f(n)$ for all but finitely many $n < \omega$.

Let $\mathcal{X}$ be a set such that $\bigcup \mathcal{X}$ is a denumerable set. We generalize some combinatorial concepts given in Chapter II to subfamilies of $\mathcal{X}$ as follows:

- A family $\mathcal{A} \subseteq \mathcal{X}$ is an *almost disjoint family* if its members are pairwise almost disjoint.
- A family $\mathcal{I} \subseteq \mathcal{X}$ is an *independent family* if, for any disjoint finite sets $A, B \subseteq \mathcal{I}$, $\bigcap A \setminus \bigcup B$ is infinite. We interpret $\bigcap \emptyset = \bigcup \mathcal{X}$.
- A family $\mathcal{S} \subseteq \mathcal{X}$ is a *splitting family* (in $\mathcal{X}$) if each member of $\mathcal{X}$ is split by some member of $\mathcal{S}$, and a family $\mathcal{R} \subseteq \mathcal{X}$ is a *reaping family* (in $\mathcal{X}$) if there is no set in $\mathcal{X}$ which splits every member of $\mathcal{R}$.
- For the case $\mathcal{X} \subseteq {}^{\omega}\omega$, a family $\mathcal{D} \subseteq \mathcal{X}$ is a *dominating family* if each function in $\mathcal{X}$ is dominated by some member of $\mathcal{D}$, and a family $\mathcal{B} \subseteq \mathcal{X}$ is an *unbounded family* if there is no function in $\mathcal{X}$ which dominates every function in $\mathcal{B}$.

Definition. We define

$$\begin{aligned}
\mathfrak{a}(\mathcal{X}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{X} \text{ is a maximal almost disjoint family}\},\\
\mathfrak{i}(\mathcal{X}) &= \min\{|\mathcal{I}| : \mathcal{I} \subseteq \mathcal{X} \text{ is a maximal independent family}\},\\
\mathfrak{s}(\mathcal{X}) &= \min\{|\mathcal{S}| : \mathcal{S} \subseteq \mathcal{X} \text{ is a splitting family}\},\\
\mathfrak{r}(\mathcal{X}) &= \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathcal{X} \text{ is a reaping family}\},\\
\mathfrak{d}(\mathcal{X}) &= \min\{|\mathcal{D}| : \mathcal{D} \subseteq \mathcal{X} \text{ is a dominating family}\},\\
\mathfrak{b}(\mathcal{X}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{X} \text{ is an unbounded family}\},
\end{aligned}$$

where the maximality is considered under the inclusion.

Some well-known cardinals introduced in Section 2.4 can be written in these terminologies as follows:

$$\mathfrak{a} = \mathfrak{a}([\omega]^\omega),\ \mathfrak{i} = \mathfrak{i}([\omega]^\omega),\ \mathfrak{s} = \mathfrak{s}([\omega]^\omega),\ \mathfrak{r} = \mathfrak{r}([\omega]^\omega),\ \mathfrak{d} = \mathfrak{d}({}^\omega\omega), \text{ and } \mathfrak{b} = \mathfrak{b}({}^\omega\omega);$$

see [8] for more details. The cardinals $\mathfrak{a}_e = \mathfrak{a}({}^\omega\omega)$ and $\mathfrak{a}_p = \mathfrak{a}(\mathrm{Sym}(\omega))$ were introduced by Zhang in [12] and were also studied in [5]. Our main work is to study the following eight cardinals.

$\mathfrak{i}_f = \mathfrak{i}({}^\omega\omega)$, $\mathfrak{i}_p = \mathfrak{i}(\mathrm{Sym}(\omega))$, $\mathfrak{s}_f = \mathfrak{s}({}^\omega\omega)$, $\mathfrak{s}_p = \mathfrak{s}(\mathrm{Sym}(\omega))$, $\mathfrak{r}_f = \mathfrak{r}({}^\omega\omega)$, $\mathfrak{r}_p = \mathfrak{r}(\mathrm{Sym}(\omega))$, $\mathfrak{d}_p = \mathfrak{d}(\mathrm{Sym}(\omega))$, and $\mathfrak{b}_p = \mathfrak{b}(\mathrm{Sym}(\omega))$.

3.1 Splitting and Reaping Families

First note that $\mathfrak{s}_f$, $\mathfrak{s}_p$, $\mathfrak{r}_f$, and $\mathfrak{r}_p$ are well-defined since ${}^\omega\omega$ and $\mathrm{Sym}(\omega)$ are splitting and reaping families of functions and permutations respectively.

We first show our results of $\mathfrak{s}_f$ and $\mathfrak{r}_f$. Recall that the *covering number* of the meagre ideal $\mathcal{M}$, $\mathrm{cov}(\mathcal{M})$, is the smallest size of a family of meager subsets of $\mathbb{R}$ whose union is $\mathbb{R}$, and the *uniformity* of $\mathcal{M}$, $\mathrm{non}(\mathcal{M})$, is the smallest size of a nonmeager subset of $\mathbb{R}$; see [3] or Chapter III of [9] for more details.

The following is Theorem 5.9 of [3]. The first statement is also from Corollary 1.8 (page 233) of [1] and Related Result 117 (Chapter 22) of [8].

Theorem 3.1.1.

$$\mathrm{cov}(\mathcal{M}) = \min\{|\mathcal{C}| : \mathcal{C} \subseteq {}^\omega\omega \wedge \neg\exists f \in {}^\omega\omega\ \forall g \in \mathcal{C}\ [f \cap g \textit{ is infinite}]\}, \textit{ and}$$
$$\mathrm{non}(\mathcal{M}) = \min\{|\mathcal{C}| : \mathcal{C} \subseteq {}^\omega\omega \wedge \forall f \in {}^\omega\omega\ \exists g \in \mathcal{C}\ [f \cap g \textit{ is infinite}]\}.$$

Theorem 3.1.2. *$\mathfrak{s}_f = \mathrm{non}(\mathcal{M})$ and $\mathfrak{r}_f = \mathrm{cov}(\mathcal{M})$.*

Proof. Notice that if $\mathcal{C} \subseteq {}^\omega\omega$ is a splitting family, then for any $f \in {}^\omega\omega$, there is a $g \in \mathcal{C}$ such that $f \cap g$ is infinite. By the previous theorem, $\mathrm{non}(\mathcal{M}) \leq \mathfrak{s}_f$. By the same theorem, $\mathfrak{r}_f \leq \mathrm{cov}(\mathcal{M})$ since $\mathcal{R} \subseteq {}^\omega\omega$ is a reaping family of functions if there is no $f \in {}^\omega\omega$ such that $f \cap g$ is infinite for all $g \in \mathcal{R}$.

To show that $\mathfrak{s}_f \leq \mathrm{non}(\mathcal{M})$, let $\mathcal{C} \subseteq {}^\omega\omega$ be an infinite family such that

$$\forall f \in {}^\omega\omega\ \exists g \in \mathcal{C}\ [f \cap g \text{ is infinite}].$$

For each $g \in \mathcal{C}$, define $\tilde{g} \in {}^\omega\omega$ by

$$\tilde{g}(n) = \begin{cases} g(n) & \text{if } n \text{ is even,} \\ g(n)+1 & \text{if } n \text{ is odd.} \end{cases}$$

Let $\mathcal{D} = \mathcal{C} \cup \{\tilde{g} : g \in \mathcal{C}\}$. It remains to show that $\mathcal{D}$ is a splitting family of functions.

Let $f \in {}^\omega\omega$. By the property of $\mathcal{C}$, there is a $g \in \mathcal{C}$ such that $f \cap g$ is infinite. If $f \setminus g$ is finite, then there is an $n_0 < \omega$ such that $f(n) = g(n)$ for all $n \geq n_0$, and hence $\tilde{g}$ splits f. Otherwise, g splits f. Thus $\mathfrak{s}_f \leq |\mathcal{D}| = |\mathcal{C}|$. Since $\mathcal{C}$ is arbitrary, $\mathfrak{s}_f \leq \text{non}(\mathcal{M})$.

To show that $\text{cov}(\mathcal{M}) \leq \mathfrak{r}_f$, let $\mathcal{C} \subseteq {}^\omega\omega$ be an infinite family such that $|\mathcal{C}| < \text{cov}(\mathcal{M})$. We shall show that $\mathcal{C}$ is not a reaping family.

For each $g \in \mathcal{C}$, let $g \oplus 1 \in {}^\omega\omega$ be defined by $(g \oplus 1)(n) = g(n) + 1$. Let
$$\mathcal{D} = \mathcal{C} \cup \{g \oplus 1 : g \in \mathcal{C}\}.$$
Then $\mathcal{D} \subseteq {}^\omega\omega$ and $|\mathcal{D}| = |\mathcal{C}| < \text{cov}(\mathcal{M})$. By the above theorem, there is an $f \in {}^\omega\omega$ such that $f \cap h$ is infinite for any $h \in \mathcal{D}$.

Consider a $g \in \mathcal{C}$. Since $f \cap (g \oplus 1)$ is infinite, there are infinitely many $k \in \omega$ such that $f(k) = g(k) + 1$, so $f(k) \neq g(k)$. Hence $g \setminus f$ is infinite. Since $f \cap g$ is infinite, f splits g. Therefore, $\mathcal{C}$ is not a reaping family. □

From the facts that $\mathfrak{b} \leq \text{non}(\mathcal{M}) \leq \mathfrak{a}_e, \mathfrak{a}_p$ (see Theorem 2.2 and Proposition 4.6 in [5]) and $\mathfrak{p} \leq \text{cov}(\mathcal{M}) \leq \mathfrak{d}$ (see Proposition 5.5, Theorem 7.12 and 7.13 of [3]), by the above theorem, we obtain the following corollary.

Corollary 3.1.3. *$\mathfrak{b} \leq \mathfrak{s}_f \leq \mathfrak{a}_e, \mathfrak{a}_p$ and $\mathfrak{p} \leq \mathfrak{r}_f \leq \mathfrak{d}$.*

Next, we shall show that $\text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$ and give a lower bound of $\mathfrak{s}_p$. The proofs make use of Martin's Axiom. We start with some relevant definitions and known facts. The following is Definition III.3.11 of [9].

Definition. $MA_{\mathbb{P}}(\kappa)$ is the statement that whenever $\mathcal{D}$ is a family of dense subsets of a poset $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

By the Generic Filter Existence Lemma (Lemma III.3.14 in [9]), we obtain the following theorem.

Theorem 3.1.4. *$MA_{\mathbb{P}}(\kappa)$ holds for any poset $\mathbb{P}$ and $\kappa \leq \aleph_0$.*

Definition. A subset C of a poset $\mathbb{P}$ is *centered* if, for any $n \in \omega$ and any $p_1, p_2, ..., p_n \in C$ there is a $q \in \mathbb{P}$ such that $q \leq p_i$ for all i. $\mathbb{P}$ is *σ-centered* if $\mathbb{P}$ is a countable union of centered subsets of $\mathbb{P}$.

Definition. $\mathfrak{m}_\sigma$ is the least κ such that there is a σ-centered poset $\mathbb{P}$ for which $MA_{\mathbb{P}}(\kappa)$ fails, and $\mathfrak{m}_{\text{ctbl}}$ is the least κ such that there is a countable poset $\mathbb{P}$ for which $MA_{\mathbb{P}}(\kappa)$ fails.

It is easy to see that every countable poset is σ-centered, and the following two posets are countable.

Notation. Let $\mathrm{Fn}(\omega, \omega) = \{s \subseteq \omega \times \omega : s \text{ is a finite function}\}$ and
$\mathrm{Fn}_{1-1}(\omega, \omega) = \{s \in \mathrm{Fn}(\omega, \omega) : s \text{ is injective}\}$.

The following theorem is from Bell ([2]), and is also Theorem III.3.61 in [9].

Theorem 3.1.5. $\mathfrak{m}_\sigma = \mathfrak{p}$.

It is well-known that $\mathfrak{p} \leq \mathfrak{s}$ (see Chapter 9 of [8]). Now, we shall use the above fact to show that $\mathfrak{p}$ is also a lower bound of $\mathfrak{s}_p$.

Theorem 3.1.6. $\mathfrak{p} \leq \mathfrak{s}_p$.

Proof. It suffices to show that $\mathfrak{m}_\sigma \leq \mathfrak{s}_p$. To show this, let $\mathcal{C} \subseteq \mathrm{Sym}(\omega)$ be such that $\aleph_0 \leq |\mathcal{C}| < \mathfrak{m}_\sigma$. Define the poset $\mathbb{P} = \mathrm{Fn}_{1-1}(\omega, \omega) \times [\mathcal{C}]^{<\omega}$, where $(s, E) \leq (t, F)$ iff

$$s \supseteq t,\ E \supseteq F \text{ and } \forall n \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)\ \forall f \in F\ [s(n) \neq f(n)].$$

Clearly this poset is σ-centered, as the set $\{(s, E) \in \mathbb{P} : E \in [\mathcal{C}]^{<\omega}\}$ is centered for any fixed s and $\mathrm{Fn}_{1-1}(\omega, \omega)$ is countable.

For each $n \in \omega$ and $f \in \mathcal{C}$, let

$$\begin{aligned} A_n &= \{(s, E) \in \mathbb{P} : n \in \mathrm{dom}(s) \cap \mathrm{ran}(s)\}, \\ B_f &= \{(s, E) \in \mathbb{P} : f \in E\}. \end{aligned}$$

Since for all $(s, E) \in \mathbb{P}$, $(s, E \cup \{f\}) \leq (s, E)$ for all $f \in \mathcal{C}$, B_f is dense in $\mathbb{P}$ for all $f \in \mathcal{C}$.

To show that A_n is dense in $\mathbb{P}$ for any $n \in \omega$, let $n \in \omega$ and $(s, E) \in \mathbb{P}$. Since s is a finite function and E is a finite set of injections, we can pick $k \in \omega \setminus \mathrm{dom}(s)$ and $\ell \in \omega \setminus \mathrm{ran}(s)$ so that $(k, n), (n, \ell) \notin \bigcup E$. We choose

$$t = \begin{cases} s & \text{if } n \in \mathrm{dom}(s) \cap \mathrm{ran}(s), \\ s \cup \{(k, n)\} & \text{if } n \in \mathrm{dom}(s) \setminus \mathrm{ran}(s), \\ s \cup \{(n, \ell)\} & \text{if } n \in \mathrm{ran}(s) \setminus \mathrm{dom}(s), \\ s \cup \{(k, n), (n, \ell)\} & \text{if } n \notin \mathrm{dom}(s) \cup \mathrm{ran}(s). \end{cases}$$

Then $(t, E) \leq (s, E)$ and $(t, E) \in A_n$. So A_n is dense in $\mathbb{P}$.

Since $\mathcal{D} = \{A_n : n \in \omega\} \cup \{B_f : f \in \mathcal{C}\}$ is of size $|\mathcal{C}| < \mathfrak{m}_\sigma$, there is a filter G on $\mathbb{P}$ such that $G \cap A_n \neq \emptyset \neq G \cap B_f$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let $g = \bigcup \mathrm{dom}(G)$.

To show that g is a function, suppose that $(x, y_1), (x, y_2) \in g$. Then there are $(s_1, E_1), (s_2, E_2) \in G$ such that $(x, y_1) \in s_1$ and $(x, y_2) \in s_2$. Since G is a filter, there is a $(s, E) \in G$ such that $s_1, s_2 \subseteq s$. So $(x, y_1), (x, y_2) \in s$. Since s is a function, $y_1 = y_2$. Therefore, g is a function. Since s is injective for any $s \in \text{dom}(\mathbb{P})$, we can show similarly that g is injective.

To show that $\text{dom}(g) = \text{ran}(g) = \omega$, let $n \in \omega$. Since $G \cap A_n \neq \emptyset$, there is a $(s, E) \in G \cap A_n$. So $s \in \text{dom}(G)$ and $n \in \text{dom}(s) \cap \text{ran}(s)$. Hence $s \subseteq g$, and the desired result follows. Thus $g \in \text{Sym}(\omega)$.

Next, we shall show that $g \cap f$ is finite for any $f \in \mathcal{C}$.

Let $f \in \mathcal{C}$. Since $G \cap B_f \neq \emptyset$, there is a $(s, E) \in G$ such that $f \in E$. Let $m \in \text{dom}(g) \setminus \text{dom}(s)$. We shall show that $g(m) \neq f(m)$. Since $(m, g(m)) \in g = \bigcup \text{dom}(G)$, there is a $(t, F) \in G$ such that $(m, g(m)) \in t$. Since G is a filter, there is a $(s', E') \in G$ such that $(s', E') \leq (s, E)$ and $(s', E') \leq (t, F)$. Then $m \in \text{dom}(s') \setminus \text{dom}(s)$ and hence, by the definition of the order $\leq$ of $\mathbb{P}$, $g(m) = t(m) = s'(m) \neq f(m)$. Therefore, $g(m) \neq f(m)$ for any $m \in \text{dom}(g) \setminus \text{dom}(s)$. So $\{m : g(m) = f(m)\} \subseteq \text{dom}(s)$, which implies that $g \cap f$ is finite. Therefore, $\mathcal{C}$ is not a splitting family. □

We have shown, in Theorem 3.1.2, that $\mathfrak{r}_f = \text{cov}(\mathcal{M})$. Now, we shall show that $\text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$ by using the following theorem which is Proposition (d) of [7].

Theorem 3.1.7. $\mathfrak{m}_{\text{ctbl}} = \text{cov}(\mathcal{M})$.

Theorem 3.1.8. $\text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$.

Proof. It suffices to show that $\mathfrak{m}_{\text{ctbl}} \leq \mathfrak{r}_p$. To show this, let $\mathcal{C} \subseteq \text{Sym}(\omega)$ be such that $\aleph_0 \leq |\mathcal{C}| < \mathfrak{m}_{\text{ctbl}}$. Consider the countable poset $\mathbb{P} = \text{Fn}_{1-1}(\omega, \omega)$.

For each $n \in \omega$ and $f \in \mathcal{C}$, let

$$A_n = \{p \in \mathbb{P} : n \in \text{dom}(p) \cap \text{ran}(p)\},$$
$$B_{n,f} = \{p \in \mathbb{P} : \exists k \geq n \, \exists \ell \geq n \, [p(k) = f(k) \wedge p(\ell) \neq f(\ell)]\}.$$

Then A_n and $B_{n,f}$ are dense in $\mathbb{P}$ for any $n \in \omega$ and $f \in \mathcal{C}$.

Since $\mathcal{D} = \{A_n : n \in \omega\} \cup \{B_{n,f} : n \in \omega, f \in \mathcal{C}\}$ is of size $< \mathfrak{m}_{\text{ctbl}}$, there is a filter G on $\mathbb{P}$ such that $G \cap A_n \neq \emptyset \neq G \cap B_{n,f}$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let $g = \bigcup G$. Since $G \cap A_n \neq \emptyset$ for all $n \in \omega$, g is a bijection on ω, i.e. $g \in \text{Sym}(\omega)$. Moreover, for any $n \in \omega$ and $f \in \mathcal{C}$, we have that $g(k) = f(k)$ and $g(\ell) \neq f(\ell)$ for some $k, \ell \geq n$. Hence $f \cap g$ and $f \setminus g$ are infinite for any $f \in \mathcal{C}$, and thus $\mathcal{C}$ is not a reaping family of permutations. □

3.2 Dominating and Unbounded Families

We investigate two lemmas before our main results of $\mathfrak{b}_p$ and $\mathfrak{d}_p$. In this section, for any $f, g \in {}^\omega\omega$, we say that $f =^* g$ if $f(n) = g(n)$ for all but finitely many $n < \omega$.

Lemma 3.2.1. *For any $f \in {}^\omega\omega$, $\{g \in {}^\omega\omega : f =^* g\}$ is countable.*

Proof. Let $f \in {}^\omega\omega$. Define $A_n = \{g \in {}^\omega\omega : f(k) = g(k)$ for all $k \geq n\}$ for each $n < \omega$. Then, for each $n < \omega$, a map from A_n to ${}^n\omega$ defined by $g \mapsto g \restriction n$ is bijective, so A_n's are countable. Hence $\{g \in {}^\omega\omega : f =^* g\} = \bigcup_{n<\omega} A_n$ is also countable. □

Lemma 3.2.2. *For any $f, g \in {}^\omega\omega$, if f is injective, g is bijective and $f \leq^* g$, then $f =^* g$.*

Proof. Let $f, g \in {}^\omega\omega$ be such that f is injective and g is bijective. Suppose to the contrary that $f \leq^* g$ but $\{k < \omega : f(k) \neq g(k)\}$ is infinite. Since $f \leq^* g$,

$\{k < \omega : f(k) > g(k)\}$ is finite and so $\{k < \omega : f(k) < g(k)\}$ is infinite.

Define $A = \{n < \omega : f \circ g^{-1}(n) > n\}$ and $B = \{n < \omega : f \circ g^{-1}(n) < n\}$. Consider the map $\varphi : \{n < \omega : f \circ g^{-1}(n) \neq n\} \to \{k < \omega : f(k) \neq g(k)\}$ defined by $\varphi(n) = g^{-1}(n)$. Then φ is bijective (since g^{-1} is bijective), $\varphi[A] = \{k < \omega : f(k) > g(k)\}$ and $\varphi[B] = \{k < \omega : f(k) < g(k)\}$. So A is finite and B is infinite. Pick an $\ell \in B$ such that $\ell > \max f \circ g^{-1}[A]$. Notice that, for any $i \leq \ell$,

- if $i \in B$, then $f \circ g^{-1}(i) < i \leq \ell$;
- if $i \in A$, then $f \circ g^{-1}(i) \in f \circ g^{-1}[A]$, so $f \circ g^{-1}(i) < \ell$ (since $\ell > \max f \circ g^{-1}[A]$);
- if $i \notin A \cup B$, then $f \circ g^{-1}(i) = i < \ell$ (since $\ell \in B$).

So $f \circ g^{-1} \restriction (\ell + 1) : (\ell + 1) \to \ell$ and is injective, which is impossible. □

While $\mathfrak{p} \leq \mathfrak{b}$, $\mathfrak{b}_p$ turns out to be so small as shown in the following theorem.

Theorem 3.2.3. $\mathfrak{b}_p = 2$.

Proof. Notice that, for any $f \in \mathrm{Sym}(\omega)$, $\{f\}$ is not an unbounded family of permutations since $f \leq^* f$. So $\mathfrak{b}_p \geq 2$. To show that $\mathfrak{b}_p \leq 2$, define

$$f_0 = \{(2k, 2k+1) : k < \omega\} \cup \{(2k+1, 2k) : k < \omega\}$$

and consider the family $\{\mathrm{id}_\omega, f_0\}$. If there is an $f \in \mathrm{Sym}(\omega)$ which dominates both id_ω and f_0, then $\mathrm{id}_\omega =^* f =^* f_0$ by the previous lemma, but $\mathrm{id}_\omega =^* f_0$ is impossible. □

Unlike the result of unbounded families of permutations, the cardinal associated with dominating families of permutations is as big as $\mathfrak{c}$. Recall that $\mathfrak{d} \leq \mathfrak{i}$ (see [11] or Theorem 9.1 of [8]).

Theorem 3.2.4. $\mathfrak{d}_p = \mathfrak{c}$.

Proof. Clearly $\mathrm{Sym}(\omega)$ is a dominating family of permutations. To show that $\mathfrak{d}_p = \mathfrak{c}$, notice that the above two lemmas imply that $\{g \in \mathrm{Sym}(\omega) : g \leq^* f\}$ is countable for any $f \in \mathrm{Sym}(\omega)$. So, for any family $\mathcal{D} \subseteq \mathrm{Sym}(\omega)$ of infinite size κ, the set

$$\{g \in \mathrm{Sym}(\omega) : \exists f \in \mathcal{D}\ (g \leq^* f)\} = \bigcup_{f \in \mathcal{D}} \{g \in \mathrm{Sym}(\omega) : g \leq^* f\}$$

is of size at most κ. Since $|\mathrm{Sym}(\omega)| = \mathfrak{c}$, any family of permutations of size $< \mathfrak{c}$ is not a dominating family of permutations. □

3.3 Independent Families

Let us first give the following theorem which confirms that $\mathfrak{i}_f$ and $\mathfrak{i}_p$ are well-defined.

Theorem 3.3.1. *There is an independent family of permutations of size* $\mathfrak{c}$. *Consequently, there is an independent family of functions of size* $\mathfrak{c}$.

Proof. Recall that there is an independent family $\mathcal{I} \subseteq [\omega]^\omega$ of size $\mathfrak{c}$ (see Proposition 9.9 in [8]). For each $X \in \mathcal{I}$, define $f_X \in \mathrm{Sym}(\omega)$ by

$$f_X(2k) = \begin{cases} 2k+1 & \text{if } k \in X, \\ 2k & \text{if } k \notin X, \end{cases} \quad \text{and} \quad f_X(2k+1) = \begin{cases} 2k & \text{if } k \in X, \\ 2k+1 & \text{if } k \notin X, \end{cases}$$

Since for any $A, B \in \mathrm{fin}(\mathcal{I})$ such that $A \cap B = \emptyset$,

$$\bigcap\{f_X : X \in A\} \setminus \bigcup\{f_X : X \in B\} \supseteq \{(2k, 2k+1) : k \in \bigcap A \setminus \bigcup B\},$$

where $\bigcap A \setminus \bigcup B$ is infinite, the family $\{f_X : X \in \mathcal{I}\}$ is an independent family of permutations of the same size as $\mathcal{I}$, which is $\mathfrak{c}$. □

For the case of almost disjoint families of functions and permutations, relations between $\mathfrak{a}_e$, $\mathfrak{a}_p$ and other well-known cardinal characteristics provable in ZFC which have been shown so far are that $\mathrm{non}(\mathcal{M})$ is a lower bound of both $\mathfrak{a}_e$ and $\mathfrak{a}_p$ (see Theorem 2.2 and Proposition 4.6 in [5]). Zhang showed in [12] that each of $\mathfrak{a} < \mathfrak{a}_e$ and $\mathfrak{a} < \mathfrak{a}_p$ is relatively consistent with ZFC. As a result, each of $\mathfrak{a}_e \leq \mathfrak{a}$ and $\mathfrak{a}_p \leq \mathfrak{a}$ is not provable from ZFC. Surprisingly, for the case of independent families of functions and permutations, it turns out that $\mathfrak{i}$ is an upper bound of both $\mathfrak{i}_f$ and $\mathfrak{i}_p$. The following lemma is needed for the proofs.

Lemma 3.3.2. *There is an almost disjoint family* $\mathcal{A} \subseteq {}^\omega\omega$ *of cardinality* $\mathfrak{c}$ *such that, for any* $f \in \mathcal{A}$ *and* $n < \omega$,

$$1 \leq f(n) \leq 2^{n+1}.$$

Proof. For each $g \in {}^\omega 2$, define $f_g \in {}^\omega\omega$ by

$$f_g(n) = 1 + \sum_{i=0}^{n} g(i) \cdot 2^i.$$

It is easy to see that, for any $g, h \in {}^\omega 2$, if $g(N) \neq h(N)$ for some $N < \omega$, then $f_g(n) \neq f_h(n)$

for all $n \geq N$. Hence $\mathcal{A} = \{f_g : g \in {}^\omega 2\}$ is an almost disjoint family. Moreover, for any $g \in {}^\omega 2$ and $n < \omega$,

$$1 \leq f_g(n) \leq 1 + \sum_{i=0}^{n} 2^i = 2^{n+1}.$$

So $\mathcal{A}$ is the desired family. □

Definition. For any two sets A and B, we say that A is *almost contained* in B, written $A \subseteq^* B$, if $A \setminus B$ is finite.

Theorem 3.3.3. $\mathfrak{i}_f \leq \mathfrak{i}$.

Proof. Let $\aleph_0 \leq \kappa < \mathfrak{i}_f$ and $\mathcal{C} \subseteq [\omega]^\omega$ be an independent family such that $|\mathcal{C}| = \kappa$. Say $\mathcal{C} = \{X_\xi : \xi < \kappa\}$. We shall show that $\mathcal{C}$ is not maximal.

Let $\mathcal{A} = \{f_\xi : \xi < \mathfrak{c}\}$ be an almost disjoint family of functions as in the above lemma. Then

(i) $(\omega \times \{0\}) \cap f_\xi = \emptyset$ for all $\xi < \mathfrak{c}$,

(ii) $f_\alpha \cap f_\beta$ is finite for any distinct $\alpha, \beta < \mathfrak{c}$.

For each $\xi < \kappa$, define $g_\xi \in {}^\omega\omega$ by

$$g_\xi = (X_\xi \times \{0\}) \cup f_\xi \restriction (\omega \setminus X_\xi).$$

Let $A, B \in \mathrm{fin}(\kappa)$ be disjoint and let

$$g_{A,B} = \bigcap\{g_\alpha : \alpha \in A\} \setminus \bigcup\{g_\beta : \beta \in B\} \text{ and}$$
$$X_{A,B} = \bigcap\{X_\alpha : \alpha \in A\} \setminus \bigcup\{X_\beta : \beta \in B\}.$$

By (i), $X_{A,B} \times \{0\} \subseteq g_{A,B}$. Since $\mathcal{C}$ is an independent family, $X_{A,B}$ is infinite, and so is $g_{A,B}$. Hence $\mathcal{D} = \{g_\xi : \xi < \kappa\}$ is an independent family of functions. Since $|\mathcal{D}| = \kappa < \mathfrak{i}_f$, $\mathcal{D}$ is not maximal. Then $\mathcal{D} \cup \{h\}$ is an independent family of functions for some $h \notin \mathcal{D}$. Let $H = h^{-1}[\{0\}]$.

We next show that $X_{A,B} \cap H$ and $X_{A,B} \setminus H$ are infinite. Since $\mathcal{C}$ is infinite and $X_{A,B} \supseteq X_{A',B}$ for any $A' \in \mathrm{fin}(\kappa)$ such that $A \subseteq A'$, we may assume that $|A| \geq 2$.

By (ii), we have $g_{A,B} \subseteq^* X_{A,B} \times \{0\}$. Thus

$$g_{A,B} \cap h \subseteq^* \big(X_{A,B} \times \{0\}\big) \cap h = \big(X_{A,B} \cap H\big) \times \{0\},$$
$$g_{A,B} \setminus h \subseteq^* \big(X_{A,B} \times \{0\}\big) \setminus h = \big(X_{A,B} \setminus H\big) \times \{0\}.$$

Since $\mathcal{D} \cup \{h\}$ is an independent family, $g_{A,B} \cap h$ and $g_{A,B} \setminus h$ are infinite, and so are $X_{A,B} \cap H$ and $X_{A,B} \setminus H$. Hence $\mathcal{C} \cup \{H\}$ is an independent family. Moreover, since A is arbitrary and $X_{A,B} \setminus H$ is infinite, $H \notin \mathcal{C}$. So $\mathcal{C}$ is not a maximal independent family of functions. □

In the following proof, we write σ^n for the composition of n copies of a permutation σ and $(x_1; x_2; ...; x_n)$ for the cyclic permutation $x_1 \mapsto x_2 \mapsto ... \mapsto x_n \mapsto x_1$.

Theorem 3.3.4. $\mathfrak{i}_p \leq \mathfrak{i}$.

Proof. Let $\aleph_0 \leq \kappa < \mathfrak{i}_p$ and $\mathcal{C} \subseteq [\omega]^\omega$ be an independent family which is of cardinality κ, say $\mathcal{C} = \{X_\xi : \xi < \kappa\}$. We shall show that $\mathcal{C}$ is not maximal. In order to construct an independent family of permutations of size κ, we partition ω as follows.

For each $n < \omega$, let $k_n = 2^{n+2} + 3n - 4$ and $P_n = \{x < \omega : k_n \leq x < k_{n+1}\}$. Then $\{P_n : n < \omega\}$ is a partition of ω. For convenience, for each $n < \omega$, we write $P_n = \{a_{n,i} : i \in |P_n|\}$ where $\langle a_{n,i}\rangle_{i<|P_n|}$ is strictly increasing. For each $n < \omega$, define $\varphi_n, \psi_n \in \text{Sym}(P_n)$ by

$$\varphi_n = (a_{n,1}; a_{n,2}; ...; a_{n,|P_n|-1}) \text{ and}$$
$$\psi_n = (a_{n,0}; a_{n,1}; a_{n,2}; ...; a_{n,|P_n|-1}).$$

For each $n < \omega$, let $\ell_n = 2^{n+1}$. Then $|P_n| = k_{n+1} - k_n = 2\ell_n + 3$, $\varphi_n^i \setminus \{(a_{n,0}, a_{n,0})\} \subseteq \psi_n^i \cup \psi_n^{i+1}$ for all $n < \omega$ and all $1 \leq i \leq \ell_n$, and for any $1 \leq i, j \leq \ell_n$,

(i) $\varphi_n^i \cap \varphi_n^j = \{(a_{n,0}, a_{n,0})\}$ whenever $i \neq j$,

(ii) $\psi_n^{\ell_n+2+i} \cap \psi_n^{\ell_n+2+j} = \emptyset$ whenever $i \neq j$,

(iii) $\varphi_n^i \cap \psi_n^{\ell_n+2+j} = \emptyset$.

Let $\mathcal{A} = \{f_\xi : \xi < \mathfrak{c}\}$ be an almost disjoint family of functions as in Lemma 3.3.2. For each $\xi < \kappa$, define $g_\xi \in \text{Sym}(\omega)$ so that

$$g_\xi \restriction P_n = \begin{cases} \varphi_n^{f_\xi(n)} & \text{if } n \in X_\xi, \\ \psi_n^{\ell_n+2+f_\xi(n)} & \text{if } n \notin X_\xi. \end{cases}$$

For each $\xi < \kappa$ and $n < \omega$, since $1 \leq f_\xi(n) \leq 2^{n+1} = \ell_n$ and $\psi_n^m \cap \text{id}_{P_n} = \emptyset$ for all $0 < m < |P_n| = 2\ell_n + 3$,

$$n \in X_\xi \text{ if and only if } (a_{n,0}, a_{n,0}) \in g_\xi. \tag{$*$}$$

Let $A, B \in \text{fin}(\kappa)$ be disjoint. Define $g_{A,B}$ and $X_{A,B}$ as in the proof of the previous theorem. For any $Y \subseteq \omega$, let $I(Y) = \{(a_{n,0}, a_{n,0}) : n \in Y\}$. So $I(X_{A,B}) \subseteq g_{A,B}$. Since $\mathcal{C}$ is an independent family, $X_{A,B}$ is infinite, and so is $g_{A,B}$. Hence $\mathcal{D} = \{g_\xi : \xi < \kappa\}$ is not maximal. Then $\mathcal{D} \cup \{h\}$ is an independent family of permutations for some $h \notin \mathcal{D}$. Let $H = \{n < \omega : (a_{n,0}, a_{n,0}) \in h\}$.

For distinct $\xi, \eta < \kappa$, since $f_\xi \cap f_\eta$ is finite, there is an $N < \omega$ such that $f_\xi(n) \neq f_\eta(n)$ for all $n \geq N$. Recall that $1 \leq f_\xi(n) \leq \ell_n$ for all $\xi < \kappa$ and $n < \omega$. Thus, for $\xi \neq \eta$, we have

$$
\begin{aligned}
g_\xi \cap g_\eta &= \bigcup_{n<\omega} (g_\xi \restriction P_n \cap g_\eta \restriction P_n) \\
&= \bigcup_{n \in X_\xi \cap X_\eta} (g_\xi \restriction P_n \cap g_\eta \restriction P_n) \cup \bigcup_{n \notin X_\xi \cap X_\eta} (g_\xi \restriction P_n \cap g_\eta \restriction P_n) \\
&\subseteq^* \bigcup_{n \in (X_\xi \cap X_\eta) \setminus N} (g_\xi \restriction P_n \cap g_\eta \restriction P_n) \cup \bigcup_{n \notin (X_\xi \cap X_\eta) \cup N} (g_\xi \restriction P_n \cap g_\eta \restriction P_n) \\
&= \bigcup_{n \in (X_\xi \cap X_\eta) \setminus N} (\varphi_n^{f_\xi(n)} \cap \varphi_n^{f_\eta(n)}) && \text{(by (ii) and (iii))} \\
&= \{(a_{n,0}, a_{n,0}) : n \in (X_\xi \cap X_\eta) \setminus N\} \subseteq I(X_\xi \cap X_\eta). && \text{(by (i))}
\end{aligned}
$$

Hence, by $(*)$, if $|A| \geq 2$, then $g_{A,B} \subseteq^* I(X_{A,B})$.

We next show that $X_{A,B} \cap H$ and $X_{A,B} \cap (\omega \setminus H)$ are infinite. As in the proof of Theorem 3.3.3, we may assume that $|A| \geq 2$. Then

$$
\begin{aligned}
g_{A,B} \cap h &\subseteq^* I(X_{A,B}) \cap h = I(X_{A,B} \cap H) \\
g_{A,B} \setminus h &\subseteq^* I(X_{A,B}) \setminus h = I(X_{A,B} \setminus H)
\end{aligned}
$$

Since $\mathcal{D} \cup \{h\}$ is an independent family, $g_{A,B} \cap h$ and $g_{A,B} \setminus h$ are infinite, and so are $X_{A,B} \cap H$ and $X_{A,B} \setminus H$. Thus $\mathcal{C}$ is not a maximal independent family of permutations. □

We have shown that $\mathfrak{i}$ is an upper bound of $\mathfrak{i}_f$ and $\mathfrak{i}_p$. Next we shall show that $\mathfrak{p}$ is a lower bound of both of them.

Definition. Let X be a denumerable set. For any family $\mathcal{E} \subseteq [X]^\omega$, we say that an infinite set $K \subseteq X$ is a *pseudo-intersection* of $\mathcal{E}$ if $K \subseteq^* E$ for all $E \in \mathcal{E}$, and we say that $\mathcal{E}$ has the *strong finite intersection property (sfip)* if $\bigcap \mathcal{F}$ is infinite for any $\mathcal{F} \in [\mathcal{E}]^{<\omega}$ (we interpret $\bigcap \emptyset = X$).

First, we state a generalization of Lemma III.1.23 in [9] which will be used for the theorem below.

Lemma 3.3.5. *Let X be a denumerable set. Fix $\mathcal{E} \subseteq [X]^\omega$ with $|\mathcal{E}| < \mathfrak{p}$. Also, fix a nonempty set $\mathcal{H} \subseteq [X]^\omega$ such that $|\mathcal{H}| < \mathfrak{p}$ and assume that for all $H \in \mathcal{H}$, $\{Z \cap H : Z \in \mathcal{E}\}$ has the strong finite intersection property. Then $\mathcal{E}$ has a pseudo-intersection K such that $K \cap H$ is infinite for all $H \in \mathcal{H}$.*

Notation. For an infinite family $\mathcal{C} \subseteq {}^\omega\omega$, let

$$bc(\mathcal{C}) = \{\bigcap A \setminus \bigcup B : A, B \in \text{fin}(\mathcal{C}), A \cap B = \emptyset \text{ and } A \neq \emptyset\}.$$

Then each member of $bc(\mathcal{C})$ is a function and is an injection if $\mathcal{C}$ is a family of permutations. Notice that $\mathcal{C}$ is an independent family if and only if every member of $bc(\mathcal{C})$ is infinite.

In the following proof, for $a, b < \omega$, let $[a, b)$ denote $\{i < \omega : a \leq i < b\}$.

Theorem 3.3.6. *$\mathfrak{p} \leq \mathfrak{i}_p$.*

Proof. Let $\aleph_0 \leq \kappa < \mathfrak{p}$ and $\mathcal{C} \subseteq \mathrm{Sym}(\omega)$ be an independent family of permutations such that $|\mathcal{C}| = \kappa$. Then each member of $bc(\mathcal{C})$ is an infinite injection and $|bc(\mathcal{C})| = \kappa$. We shall show that $\mathcal{C}$ is not maximal.

For each $x \in bc(\mathcal{C})$ and $n < \omega$, let

$$Z_{x,n} = \big\{s \in \mathrm{Fn}_{1-1}(\omega, \omega) : \exists k, \ell \geq n\big(k, \ell \in \mathrm{dom}(x) \cap \mathrm{dom}(s) \wedge s(k) = x(k) \wedge s(\ell) \neq x(\ell)\big)\big\},$$

$$H_n = \{s \in \mathrm{Fn}_{1-1}(\omega, \omega) : \mathrm{dom}(s) = \mathrm{ran}(s) = [n, k) \text{ for some } k > n\}.$$

Let $\mathcal{E} = \{Z_{x,n} : x \in bc(\mathcal{C}), n < \omega\}$ and $\mathcal{H} = \{H_m : m < \omega\}$. We shall show that $\mathcal{E}$ has a pseudo-intersection K such that $K \cap H_m$ is infinite for all $m < \omega$ by using Lemma 3.3.5 with $X = \mathrm{Fn}_{1-1}(\omega, \omega)$.

Claim. For each $m < \omega$, $\{Z \cap H_m : Z \in \mathcal{E}\}$ has the sfip.

Let $m < \omega$ and consider $(Z_{x_1,n_1} \cap Z_{x_2,n_2} \cap ... \cap Z_{x_N,n_N}) \cap H_m$. Let $K = \max\{n_1, ..., n_N, m\}$.

- Pick $k_i \in \mathrm{dom}(x_i)$ for all $1 \leq i \leq N$ such that $K \leq k_1 < k_2 < ... < k_N$, and $m \leq x_1(k_1), x_2(k_2), ..., x_N(k_N)$ are distinct. (This is possible since x_i's are infinite and are injective functions.)

- Pick $\ell_i \in \mathrm{dom}(x_i)$ for all $1 \leq i \leq N$ which are distinct from k_i's such that $k \leq \ell_1 < \ell_2 < ... < \ell_N$.

- Pick distinct $p_1, p_2, ..., p_N$ which are distinct from $x_i(k_i)$'s and $p_i \neq x_i(\ell_i)$ for all $1 \leq i \leq N$.

Let $M = \max\{k_N, \ell_N, x_1(k_1), ..., x_N(k_N), p_1, ..., p_N\}$. Then we can pick $s \in \mathrm{Fn}_{1-1}(\omega, \omega)$ such that

$$\mathrm{dom}(s) = \mathrm{ran}(s) = [m, M+1),\ s(k_i) = x_i(k_i) \text{ and } s(\ell_i) = p_i \text{ for all } 1 \leq i \leq N.$$

Then $s \in \bigcap_{1 \leq i \leq N} Z_{x_i,n_i} \cap H_m$. Moreover, if $t \in H_m$ and $t \supseteq s$, then t also belongs to this set. As there are infinitely many such t, $\bigcap_{1 \leq i \leq N} Z_{x_i,n_i} \cap H_m$ is infinite. $\dashv_{Claim}$

By Lemma 3.3.5, $\mathcal{E}$ has a pseudo-intersection K such that $K \cap H_m$ is infinite for all $m < \omega$. Let $k_0 = 0$. We recursively pick $s_i \in K \cap H_{k_i}$ and $k_{i+1} > k_i$ such that $\mathrm{dom}(s_i) = \mathrm{ran}(s_i) = [k_i, k_{i+1})$. Define $f = \bigcup_{i<\omega} s_i$. Then $f \in \mathrm{Sym}(\omega)$.

To show that for all $x \in bc(\mathcal{C})$, $x \cap f$ and $x \setminus f$ are infinite, let $x \in bc(\mathcal{C})$ and $n < \omega$. Since $\{s_i : i < \omega\} \subseteq K \subseteq^* Z_{x,n}$, there is an $i_0 < \omega$ such that $s_{i_0} \in Z_{x,n}$. Since $s_{i_0} \subseteq f$, $f \in Z_{x,n}$. This implies that

$$\exists k, \ell \geq n\big((k, x(k)) \in x \cap f \wedge (\ell, x(\ell)) \in x \setminus f\big).$$

Since n is arbitrary, $x \cap f$ and $x \setminus f$ are infinite, and so $f \notin bc(\mathcal{C})$. Thus we conclude that $f \notin \mathcal{C}$ and $\mathcal{C} \cup \{f\}$ is an independent family of permutations. □

By replacing $\mathrm{Fn}_{1-1}(\omega, \omega)$ in the above proof by $\mathrm{Fn}(\omega, \omega)$ and simplifying the proof, we obtain the following theorem.

Theorem 3.3.7. $\mathfrak{p} \leq \mathfrak{i}_f$.

The above proof shows directly that $\mathfrak{p}$ is a lower bound of $\mathfrak{i}_p$. However, lower bounds of both of $\mathfrak{i}_p$ and $\mathfrak{i}_f$ can be improved as shown in the theorem below since $\mathfrak{p} \leq \mathrm{cov}(\mathcal{M})$ (see Theorem 22.5 in [8]).

Theorem 3.3.8. $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{i}_p$.

Proof. Recall from Theorem 3.1.7 that $\mathfrak{m}_{\mathrm{ctbl}} = \mathrm{cov}(\mathcal{M})$. So it suffices to show that $\mathfrak{m}_{\mathrm{ctbl}} \leq \mathfrak{i}_p$. Let $\mathcal{C} \subseteq \mathrm{Sym}(\omega)$ be an independent family of permutations such that $\aleph_0 \leq |\mathcal{C}| < \mathfrak{m}_{\mathrm{ctbl}}$. We shall show that $\mathcal{C}$ is not maximal. Consider the countable poset $\mathbb{P} = \mathrm{Fn}_{1-1}(\omega, \omega)$ with the ordering $\leq$ defined by $p \leq q$ if and only if $p \supseteq q$. For each $n < \omega$ and $x \in bc(\mathcal{C})$, let

$$D_{x,n} = \big\{p \in \mathbb{P} : \exists k, \ell \geq n\big(k, \ell \in \mathrm{dom}(x) \cap \mathrm{dom}(p) \wedge p(k) = x(k) \wedge p(\ell) \neq x(\ell)\big)\big\},$$
$$A_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p) \cap \mathrm{ran}(p)\}.$$

For each $x \in bc(\mathcal{C})$ and $n < \omega$, since for any $p \in \mathbb{P}$, we can pick distinct $k, \ell \geq n$ such that $k, \ell \in \mathrm{dom}(x) \setminus \mathrm{dom}(p)$ where $x(k) \neq x(\ell) + 1$, and define $q = p \cup \{(k, x(k)), (\ell, x(\ell) + 1)\} \in D_{x,n}$, $D_{x,n}$ is dense in $\mathbb{P}$. Similar to the proof in Theorem 3.1.6, A_n is dense in $\mathbb{P}$ for all $n < \omega$.

Let $\mathcal{D} = \{A_n : n < \omega\} \cup \{D_{x,n} : n < \omega, x \in bc(\mathcal{C})\}$. Since $|\mathcal{D}| = |\mathcal{C}| < \mathfrak{m}_{\mathrm{ctbl}}$, there exists a filter G on $\mathbb{P}$ such that $A_n \cap G \neq \emptyset \neq D_{x,n} \cap G$ for any $n < \omega$ and $x \in bc(\mathcal{C})$. Let $g = \bigcup G$. Since G is a filter and $A_n \cap G \neq \emptyset$ for any $n < \omega$, as shown in the proof of Theorem 3.1.6, we have that $g \in \mathrm{Sym}(\omega)$. Since $D_{x,n} \cap G \neq \emptyset$ for any $n < \omega$ and $x \in bc(\mathcal{C})$, $x \cap g$ and $x \setminus g$ are infinite for any $x \in bc(\mathcal{C})$, so $g \notin \mathcal{C}$. Thus $\mathcal{C} \cup \{g\}$ is still an independent family of permutations, and hence $\mathcal{C}$ is not maximal. □

By replacing $\mathrm{Fn}_{1-1}(\omega, \omega)$ in the above proof by $\mathrm{Fn}(\omega, \omega)$ and simplifying the proof, we obtain the following theorem.

Theorem 3.3.9. $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{i}_f$.

The results in this section are also in [10].

3.4 **Consistency Results**

In this section, we shall give models of ZFC in which our new cardinal characteristics are greater than $\aleph_1$ or less than $\mathfrak{c}$. In fact, the consequences of these results can also be obtained by the results from Sections 3.1–3.3 together with known consistency results concerning relations among well-known cardinal characteristics. However, to see the models in which they are separated from $\aleph_1$ or $\mathfrak{c}$ directly makes us see the behavior of each corresponding family in those models.

Lemma 3.4.1**.** *Let M be a ground model satisfying ZFC and $\mathcal{C} \in M$ be a subset of $[\omega \times \omega]^\omega$ whose members are infinite injections. Let $\mathbb{P}$ be the Cohen poset ${}^{<\omega}\omega$ and G be $\mathbb{P}$-generic over M. Then, in $M[G]$, there is an $h \in \mathrm{Sym}(\omega)$ which splits all members of $\mathcal{C}$ and $h \notin \mathcal{C}$.*

Proof. Define $g = \bigcup G$. Then $g \in {}^\omega\omega \cap M[G]$ and g is surjective. Define $h \in \mathrm{Sym}(\omega)$ recursively by

$$h(i) = g\left(\min\{j < \omega : g(j) \notin \mathrm{ran}(h{\restriction}i)\}\right).$$

That is, h is the one-to-one sequence obtained from g by removing all repetitions of each occurrence of $g(i)$ except its first one. Since g is in $M[G]$ and surjective, so is h. Thus $h \in \mathrm{Sym}(\omega) \cap M[G]$. For each $x \in \mathcal{C}$ and $n < \omega$, let

$$D_{x,n} = \left\{p \in \mathbb{P} : \exists k, \ell \geq n\big(k, \ell \in \mathrm{dom}(x) \wedge p \Vdash \mathring{h}(k) = x(k) \wedge \mathring{h}(\ell) \neq x(\ell)\big)\right\}.$$

To show that each $D_{x,n}$ is dense in $\mathbb{P}$, let $x \in \mathcal{C}, n < \omega$, and $p \in \mathbb{P}$. Pick distinct $k, \ell \geq \max\{n, \mathrm{dom}(p)\}$ such that $k, \ell \in \mathrm{dom}(x)$ and $k < \ell$ where $x(k)$ and $x(\ell)$ are not in $\mathrm{ran}(p)$. Choose a $q \in \mathbb{P}$ such that $q \supseteq p$ and the k-th and the ℓ-th unrepeated elements are equal to $x(k)$ and not equal to $x(\ell)$, respectively. Rigorously, let $s = \mathrm{dom}(p)$, $t = |\mathrm{ran}(p)|$, pick distinct $a_0, a_1, ..., a_{k-t-1}, b_0, b_1, ..., b_{\ell-k-1} \in \omega \setminus (\mathrm{ran}(p) \cup \{x(k), x(\ell)\})$, and define

$$q = p \cup \{(s+i, a_i) : i < k-t\} \cup \{(s-t+k, x(k))\} \cup \{(s-t+k+1+j, b_j) : j < \ell-k\}.$$

Thus $q \Vdash \mathring{h}(k) = x(k) \wedge \mathring{h}(\ell) \neq x(\ell)$, so $q \in D_{x,n}$.

Since G is $\mathbb{P}$-generic, we can pick a $p_{x,n} \in G \cap D_{x,n}$. By the definition of $D_{x,n}$, there are $k, \ell \geq n$ such that $p_{x,n} \Vdash \mathring{h}(k) = x(k) \wedge \mathring{h}(\ell) \neq x(\ell)$. Since $p_{x,n} \in G$, $h(k) = x(k)$ and $h(\ell) \neq x(\ell)$ in $M[G]$. Thus $x \cap h$ and $x \setminus h$ are infinite for all $x \in \mathcal{C}$. This also implies that $h \notin \mathcal{C}$. □

Corollary 3.4.2**.** *Let M be a ground model satisfying ZFC and $\mathcal{C} \in M$ be a subset of* $\mathrm{Sym}(\omega)$*. Let $\mathbb{P}$ be the Cohen poset ${}^{<\omega}\omega$ and G be $\mathbb{P}$-generic over M.*

1. *$\mathcal{C}$ is not a reaping family of permutations in $M[G]$.*
2. *If $\mathcal{C}$ is an independent family of permutations in M, then $\mathcal{C}$ is not maximal in $M[G]$.*

Proof. The first statement follows directly from the previous lemma. The second one is obtained by applying the lemma to $bc(\mathcal{C})$. (By the definition of independency, if $\mathcal{C}$ is an independent family of permutations and $h \notin \mathcal{C}$ splits all members of $bc(\mathcal{C})$, then $\{h\} \cup \mathcal{C}$ is also an independent family of permutations.) □

Theorem 3.4.3**.** *Let M be a ground model satisfying $ZFC + GCH$. In M, let $\kappa > \aleph_1$ be a regular cardinal and $\mathbb{P}$ be a finite-support iteration of length κ of Cohen posets. If G is $\mathbb{P}$-generic over M, then*

$$\aleph_1 < \kappa = \mathfrak{i}_f = \mathfrak{i}_p = \mathfrak{r}_f = \mathfrak{r}_p = \mathfrak{c}$$

holds in $M[G]$.

Proof. Since $\mathbb{P}$ is a finite-support iteration of length κ of Cohen posets, $\mathbb{P}$ is forcing equivalent to $\mathrm{Fn}(\kappa \times \omega, 2)$ and since M is a model of GCH, $M[G] \vDash \aleph_1 < \kappa = \mathfrak{c}$ (see Theorem IV.3.13 in [9]). It remains to show that $M[G] \vDash \kappa \leq \mathfrak{i}_f, \mathfrak{i}_p, \mathfrak{r}_f, \mathfrak{r}_p$.

For each $\alpha \leq \kappa$, let $\mathbb{P}_\alpha$ be a finite-support iteration of length α of Cohen posets (so $\mathbb{P} = \mathbb{P}_\kappa$) and let G_α be the restriction of G to $\mathbb{P}_\alpha$. Let $\mathcal{C} \in M[G]$ be an independent family of permutations such that $|\mathcal{C}| < \kappa$.

To show that $\mathcal{C} \in M[G_\alpha]$ for some $\alpha < \kappa$, let τ be a name for $\omega \times \omega$ such that all forcing conditions in τ is $\mathbb{1}$ and $\mathrm{dom}(\tau)$ is countable (the detail is omitted). Consider each $f \in \mathcal{C}$. There is a $\mathbb{P}$-name $\mathring{f}$ so that $\mathring{f}_G = f$. As $f \subseteq \omega \times \omega$, by Theorem 2.6.4, we may choose the $\mathring{f}$ so that it is a nice name for a subset of τ. So

$$\mathring{f} = \bigcup \{\{\sigma\} \times A_\sigma : \sigma \in \mathrm{dom}(\tau)\},$$

where each A_σ is an antichain. Since $\mathbb{P}$ is ccc (Theorem 2.7.2), each A_σ is countable. By the fact that $\mathrm{dom}(\tau)$ is countable, we conclude that there are countably many forcing conditions (elements of $\mathbb{P}$) occurring in $\mathring{f}$. Let S be the set of all forcing conditions occurring in $\{\mathring{f} : f \in \mathcal{C}\}$. Since there are $< \kappa$ many of these $\mathring{f}$'s (as $|\mathcal{C}| < \kappa$) and κ is uncountable, by Absorption Law, $|S| < \kappa$. Recall that we can consider $\langle \mathbb{P}_\xi : \xi \leq \kappa \rangle$ as an $\subseteq$-increasing sequence of forcing posets. For each forcing condition $p \in S$, let $\eta(p)$ be the least ordinal such that $p \in \mathbb{P}_{\eta(p)}$. Since κ is regular and $\{\eta(p) : p \in S\} \subseteq \kappa$ is of size $< \kappa$, there is an ordinal α such that $\{\eta(p) : p \in S\} \subseteq \alpha$. This means that all forcing conditions occurring in S can be regarded as they are all in $\mathbb{P}_\alpha$. So $f \in M[G_\alpha]$, and hence $\mathcal{C} \in M[G_\alpha]$ as we claimed.

If H is Cohen generic over $M[G_\alpha]$, by Corollary 3.4.2, $\mathcal{C}$ is not maximal in $M[G_\alpha][H] = M[G_{\alpha+1}]$. So $\mathcal{C}$ is not maximal in $M[G]$. Thus $M[G] \vDash \kappa \leq \mathfrak{i}_p$. By the same method and the same corollary, $M[G] \vDash \kappa \leq \mathfrak{r}_p$. Moreover, we can get analogous lemma and corollary to obtain $M[G] \vDash \kappa \leq \mathfrak{i}_f, \mathfrak{r}_f$. □

Let us move to another poset $\mathbb{Q} = \mathrm{Fn}_{1-1}(\omega, \omega) \times [\mathrm{Sym}(\omega)]^{<\omega}$, where $(s, E) \leq (t, F)$ iff

$$s \supseteq t,\ E \supseteq F \text{ and } \forall n \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)\ \forall f \in F\ [s(n) \neq f(n)].$$

This is the same poset as in the proof of Theorem 3.1.6 with $\mathcal{C} = \mathrm{Sym}(\omega)$. This poset is σ-centered, and hence is ccc.

Lemma 3.4.4. *Let M be a ground model satisfying ZFC and the poset $\mathbb{Q} \in M$ be defined as above. If G is $\mathbb{Q}$-generic over M then, in $M[G]$, there is a $g \in \mathrm{Sym}(\omega)$ which is not split by any $f \in \mathrm{Sym}(\omega) \cap M$.*

Proof. The arguments which are omitted in this proof can be found in the proof of Theorem 3.1.6. For each $n \in \omega$ and $f \in \mathrm{Sym}(\omega) \cap M$, let

$$A_n = \{(s, E) \in \mathbb{Q} : n \in \mathrm{dom}(s) \cap \mathrm{ran}(s)\},$$
$$B_f = \{(s, E) \in \mathbb{Q} : f \in E\}.$$

Clearly $A_n, B_f \in M$ since $\mathbb{Q}, n, f \in M$. It can be shown that A_n and B_f are dense in $\mathbb{Q}$ for all $n \in \omega$ and $f \in \mathrm{Sym}(\omega) \cap M$. Since G is $\mathbb{Q}$-generic over M, $G \cap A_n \neq \emptyset \neq G \cap B_f$ for all $n \in \omega$ and $f \in \mathrm{Sym}(\omega) \cap M$.

Let $g = \bigcup \mathrm{dom}(G)$. Using the fact that $G \cap A_n \neq \emptyset$ for all $n \in \omega$ and G is a filter, it can be shown that $g \in \mathrm{Sym}(\omega)$. Using the fact that $G \cap B_f \neq \emptyset$ for all $f \in \mathrm{Sym}(\omega) \cap M$, it can be shown that $g \cap f$ is finite, hence g is not split by any $f \in \mathrm{Sym}(\omega) \cap M$. □

Theorem 3.4.5. *Let M be a ground model satisfying ZFC. In M, let $\kappa > \aleph_1$ be a regular cardinal and $\mathbb{P}$ be a finite-support iteration of length κ of $\mathbb{Q}$. If G is $\mathbb{P}$-generic over M, then*

$$\aleph_1 < \kappa \leq \mathfrak{s}_p$$

holds in $M[G]$.

Proof. For each $\alpha \leq \kappa$, let $\mathbb{P}_\alpha$ be a finite-support iteration of length α of $\mathbb{Q}$ (so $\mathbb{P} = \mathbb{P}_\kappa$) and let G_α be the restriction of G to $\mathbb{P}_\alpha$.

Let $\mathcal{C} \in M[G]$ be a family of permutations such that $|\mathcal{C}| < \kappa$. By the same argument as in the proof of Theorem 3.4.3, $\mathcal{C} \in M[G_\alpha]$ for some $\alpha < \kappa$. If H is $\mathbb{Q}$-generic over $M[G_\alpha]$, by Lemma 3.4.4, there is a $g \in \mathrm{Sym}(\omega) \cap M[G_\alpha][H] = \mathrm{Sym}(\omega) \cap M[G_{\alpha+1}]$ which is not split by any element in $\mathrm{Sym}(\omega) \cap M[G_\alpha]$. In particular, g is not split by any element in $\mathcal{C}$. So $\mathcal{C}$ is not a splitting family in $M[G]$. Thus $M[G] \vDash \kappa \leq \mathfrak{s}_p$ as desired. □

Theorem 3.4.6. *Let M be a ground model satisfying $ZFC + \aleph_1 < \mathfrak{c}$. In M, let $\mathbb{P}$ be a finite-support iteration of length $\aleph_1$ of $\mathbb{Q}$. If G is $\mathbb{P}$-generic over M, then*

$$\aleph_1 = \mathfrak{r}_p < \mathfrak{c}$$

holds in $M[G]$.

Proof. Since $\aleph_1 \leq \mathfrak{r}_p$, it suffices to show that $\mathfrak{r}_p \leq \aleph_1$. As always, for each $\alpha \leq \aleph_1$, let $\mathbb{P}_\alpha$ be a finite-support iteration of length α of $\mathbb{Q}$ (so $\mathbb{P} = \mathbb{P}_{\aleph_1}$) and let G_α be the restriction of G to $\mathbb{P}_\alpha$. By Lemma 3.4.4, for each $\alpha < \aleph_1$ there is a $g_\alpha \in \mathrm{Sym}(\omega) \cap M[G_{\alpha+1}]$ such that g_α is not split by any element in $\mathrm{Sym}(\omega) \cap M[G_\alpha]$. Let $\mathcal{C} = \{g_\alpha : \alpha < \aleph_1\}$. Clearly $\mathcal{C} \in M[G]$ and

$|\mathcal{C}| \leq \aleph_1$. Since $\aleph_1$ is regular, by the same argument as in the proof of Theorem 3.4.3, for any $f \in \text{Sym}(\omega) \cap M[G]$, $f \in M[G_\alpha]$ for some $\alpha < \aleph_1$, so f does not split g_α. Therefore, $\mathcal{C}$ is a reaping family of permutations in $M[G]$. □

Remark. The poset $\mathbb{Q}$ actually depends on the ground model since $\text{Sym}(\omega)$ might be different in various models (while the Cohen poset ${}^{<\omega}\omega$ is the same in any model). We may write $\mathbb{Q}_\alpha$ to denote each $\mathbb{Q} \in M[G_\alpha]$ in each step α. Those $\mathbb{Q}_\alpha$'s are different sets but they are defined by the same definition, so Lemma 3.4.7 can be applied at any step α. Moreover, the $\mathbb{Q}$-genericity is strong enough to give us, at each step α, a new $g_\alpha \in \text{Sym}(\omega)$ which is not split by any $f \in \text{Sym}(\omega) \cap M[G_\alpha]$.

Similarly, consider a poset $\mathbb{Q}' = \text{Fn}(\omega, \omega) \times [{}^\omega\omega]^{<\omega}$, where $(s, E) \leq (t, F)$ iff

$$s \supseteq t,\, E \supseteq F \text{ and } \forall n \in \text{dom}(s) \setminus \text{dom}(t)\ \forall f \in F\ [s(n) \neq f(n)].$$

Similar to $\mathbb{Q}$, this poset is σ-centered, and hence is ccc.

Lemma 3.4.7. *Let M be a ground model satisfying ZFC and the poset $\mathbb{Q}' \in M$ be defined as above. If G is $\mathbb{Q}'$-generic over M then, in $M[G]$, there is a $g \in {}^\omega\omega$ which is not split by any $f \in {}^\omega\omega \cap M$.*

Proof. This is the same as in Lemma 3.4.4, replacing $\text{Fn}_{1-1}(\omega, \omega)$ by $\text{Fn}(\omega, \omega)$ and $\text{Sym}(\omega)$ by ${}^\omega\omega$. (We might also relax the set A_n to be $\{(s, E) \in \mathbb{Q}' : n \in \text{dom}(s)\}$ since the surjectivity is not required here.) □

Theorem 3.4.8. *Let M be a ground model satisfying ZFC. In M, let $\kappa > \aleph_1$ be a regular cardinal and $\mathbb{P}$ be a finite-support iteration of length κ of $\mathbb{Q}'$. If G is $\mathbb{P}$-generic over M, then*

$$\aleph_1 < \kappa \leq \mathfrak{s}_f$$

holds in $M[G]$.

Proof. This is the same as in Theorem 3.4.5, replacing $\mathbb{Q}$ by $\mathbb{Q}'$ and $\text{Sym}(\omega)$ by ${}^\omega\omega$. □

Theorem 3.4.9. *Let M be a ground model satisfying $ZFC + \aleph_1 < \mathfrak{c}$. In M, let $\mathbb{P}$ be a finite-support iteration of length $\aleph_1$ of $\mathbb{Q}'$. If G is $\mathbb{P}$-generic over M, then*

$$\aleph_1 = \mathfrak{r}_f < \mathfrak{c}$$

holds in $M[G]$.

Proof. This is the same as in Theorem 3.4.6, replacing $\mathbb{Q}$ by $\mathbb{Q}'$ and $\text{Sym}(\omega)$ by ${}^\omega\omega$. □

Corollary 3.4.10. *Each of the following statements is relatively consistent with ZFC.*

1. $\aleph_1 < \mathfrak{i}_f = \mathfrak{i}_p = \mathfrak{r}_f = \mathfrak{r}_p = \mathfrak{c}$.

2. $\aleph_1 < \mathfrak{s}_p$.

3. $\aleph_1 = \mathfrak{r}_p < \mathfrak{c}$.

4. $\aleph_1 < \mathfrak{s}_f$.

5. $\aleph_1 = \mathfrak{r}_f < \mathfrak{c}$.

Proof. Follows by Theorem 2.6.3 together with

1. Theorem 3.4.3 and the fact that $Con(ZFC) \to Con(ZFC + GCH)$ (see Theorem II.6.24 in [9]).

2. Theorem 3.4.5.

3. Theorem 3.4.6 and the fact that $Con(ZFC) \to Con(ZFC + \aleph_1 < \mathfrak{c})$ (by Cohen forcing, see Corollary IV.3.14 in [9]).

4. Theorem 3.4.8.

5. Theorem 3.4.9 and the fact that $Con(ZFC) \to Con(ZFC + \aleph_1 < \mathfrak{c})$. □

It is known that there is a forcing poset which produces a model of $\aleph_1 = \mathfrak{i} < \mathfrak{c}$ (the poset is rather complicated and is not used here, so we refer the reader to Proposition 18.11 in [8]). Since $\mathfrak{i}_f, \mathfrak{i}_p \leq \mathfrak{i}$,

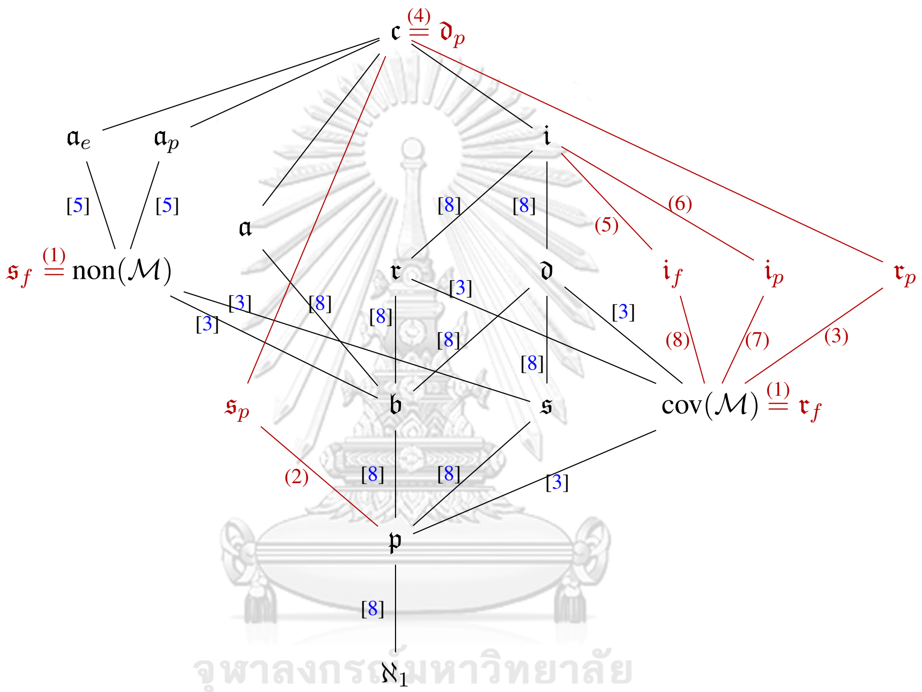
$$\aleph_1 = \mathfrak{i}_f = \mathfrak{i}_p < \mathfrak{c}$$

holds in the model as well. The direct proof of this fact uses the same idea as that of $\mathfrak{i} < \mathfrak{c}$ in the model.

CHAPTER IV

CONCLUSIONS AND FURTHER RESEARCH

The following diagram summarizes our results, together with the results of $\mathfrak{a}_e$ and $\mathfrak{a}_p$ from [12] and [5]. A line connecting two cardinals indicates that the lower cardinal is less than or equal to the upper cardinal. Our new cardinals and results are in red. Since $\mathfrak{b}_p = 2$ (Theorem 3.2.3), it does not occur in the diagram.



From the diagram,

(1) Theorem 3.1.2.

(2) Theorem 3.1.6.

(3) Theorem 3.1.8.

(4) Theorem 3.2.4.

(5) Theorem 3.3.3.

(6) Theorem 3.3.4.

(7) Theorem 3.3.8.

(8) Theorem 3.3.9.

We also give models of ZFC in which each of the following statements holds:

- $\aleph_1 < \mathfrak{i}_f = \mathfrak{i}_p = \mathfrak{r}_f = \mathfrak{r}_p = \mathfrak{c}$,
- $\aleph_1 = \mathfrak{i}_f = \mathfrak{i}_p < \mathfrak{c}$,
- $\aleph_1 = \mathfrak{r}_f < \mathfrak{c}$,
- $\aleph_1 = \mathfrak{r}_p < \mathfrak{c}$,
- $\aleph_1 < \mathfrak{s}_f$,
- $\aleph_1 < \mathfrak{s}_p$.

Together with some known-facts in forcing, we can conclude consistency results as follows.

- By Cohen forcing,
  $$\aleph_1 = \mathfrak{a} = \mathfrak{s} = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \mathfrak{r} = \mathfrak{c}$$
  is relatively consistent with ZFC (see pages 472–473, Section 11.3, in [3]). Therefore, the following statement is relatively consistent with ZFC:
  $$\aleph_1 = \mathfrak{p} = \mathfrak{b} = \mathfrak{s}_f = \mathfrak{s} = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \mathfrak{r} = \mathfrak{r}_f = \mathfrak{r}_p = \mathfrak{i}_f = \mathfrak{i}_p = \mathfrak{d} = \mathfrak{i} = \mathfrak{c}.$$
- By Random forcing,
  $$\aleph_1 = \mathfrak{s} = \text{cov}(\mathcal{M}) < \text{non}(\mathcal{M}) = \mathfrak{r} = \mathfrak{c}$$
  is relatively consistent with ZFC (see pages 473–474, Section 11.4, in [3]). Therefore, the following statement is relatively consistent with ZFC:
  $$\aleph_1 = \mathfrak{p} = \mathfrak{r}_f = \mathfrak{s} = \text{cov}(\mathcal{M}) < \text{non}(\mathcal{M}) = \mathfrak{r} = \mathfrak{s}_f = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{i} = \mathfrak{c}.$$

Finally, some open problems are listed below.

1. Is it provable in ZFC that $\mathfrak{r}_p = \text{cov}(\mathcal{M})$?
2. Is it provable in ZFC that $\mathfrak{s}_p = \text{non}(\mathcal{M})$ (or at least $\mathfrak{s}_p \leq \text{non}(\mathcal{M})$)?
3. Is there any lower bound of $\mathfrak{i}_f$ or $\mathfrak{i}_p$ other than $\text{cov}(\mathcal{M})$?
4. Is there any model of ZFC in which $\mathfrak{i}_f$ or $\mathfrak{i}_p$ is separated from $\text{cov}(\mathcal{M})$?
5. Is each of $\mathfrak{i}_f < \mathfrak{i}$ and $\mathfrak{i}_p < \mathfrak{i}$ relatively consistent with ZFC?
6. Are any strict inequalities between $\mathfrak{i}_f$ and $\mathfrak{i}_p$ relatively consistent with ZFC?
7. Does analogous result in [13] hold for independent families?
   (Zhang showed in [13] that it is consistent with ZFC$+\neg$CH that there is a maximal almost disjoint family of permutations which can be extended to an almost disjoint (eventually distinct) family of functions of greater cardinality.)

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