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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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ปีการศึกษา 2562

COVERS FOR LEO MOSER'S WORM PROBLEM

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ในปี ค.ศ. 1966 ลีโอ โมเชอร์ ได้ตั้งปัญหาทางเรขาคณิตซึ่งมีใจความว่า บริเวณที่มีพื้นที่น้อยที่สุดที่สามารถปิดทับเส้นโค้งหนึ่งหน่วยใด ๆ ได้คืออะไร ซึ่งเป็นปัญหาที่สามารถทำความเข้าใจได้ง่าย แต่เป็นไปได้ยากที่จะหาบริเวณที่เป็นแผ่นปิดทับที่มีพื้นที่น้อยสุด ซึ่งในปี ค.ศ. 2018 ณัฐพล พลอยมะกล่ำและวัชรินทร์ วิชิรมาลา ได้นำเสนอแผ่นปิดทับที่มีพื้นที่น้อยที่สุด ณ ปัจจุบัน โดยแผ่นปิดทับนี้ได้มีการปรับปรุงมาจากแผ่นปิดทับของนอร์วูดและพูล ในงานวิจัยนี้เราได้มีการปรับปรุงแผ่นปิดทับของนอร์วูลและพูล และแผ่นปิดทับของพลอยมะกล่ำและวิชิรมาลา โดยมีการปรับเปลี่ยนขอบบนของแผ่นปิดทับ เพื่อให้ได้บริเวณที่คาดว่าจะเป็นแผ่นปิดทับใหม่ แต่บริเวณดังกล่าวยังคงสอดคล้องกับสมบัติบางประการที่มีในงานของนอร์วูดและพูล นอกจากนี้เราได้นำเสนอการพิสูจน์การเป็นแผ่นปิดทับของบริเวณที่มีขอบบนเป็นเส้นโค้งโคไซน์ ซึ่งมีพื้นที่ประมาณ 0.26009 ยิ่งไปกว่านั้นเราได้ทำการทดลองสร้างบริเวณที่คาดว่าเป็นแผ่นปิดทับโดยเริ่มจากการกำหนดขอบล่าง อย่างไรก็ตามแนวทางการสร้างบริเวณนี้ยังไม่ประสบความสำเร็จเท่าที่ควร เนื่องจากการคำนวณมีความซับซ้อนมาก

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาหลัก ....

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THITIPONG KANCHAI : COVERS FOR LEO MOSER'S WORM PROBLEM
ADVISOR : ASSOCIATE PROFESSOR WACHARIN WICHIRAMALA, Ph.D.

Leo Moser's worm problem was posted in 1966 stating that "What is the region of the smallest area that can cover every unit arc?". In 2018, N. Ploymaklam and W. Wichiramala illustrated a new cover, which is currently smallest. Their cover was adapted from the cover of R. Norwood and G. Poole. In this work, we modify the region from the work of R. Norwood and G. Poole and the work of N. Ploymaklam and W. Wichiramala. Our regions are modified by changing their upper boundary. However, the regions remain satisfying properties in the work of R. Norwood and G. Poole. In addition, we construct the region using cosine function as the upper boundary. This region can cover every unit arc which has area approximately 0.26009. Moreover, we try to construct the region starting from the lower boundary. However, we confront with the complex numerical computation. Thus, we cannot construct the region using this idea.

Department: Mathematics and Computer Science	Student's Signature: ....
Field of Study: ....Mathematics......	Advisor's Signature: ....
Academic Year: ....2019.........	

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CONTENTS

CHAPTER I
INTRODUCTION

In 1966, Leo Moser published a number of problems, one of which is the following: "What is the region of the smallest area which will accommodate every unit arc?". This problem is also known as "Leo Moser's worm problem ". Although there are a lot of mathematicians who have investigated this problem, the problem remains unsolved to this day. It is difficult to solve Moser's worm problem without any conditions because we don't know the proper structure and shape of a region. Furthermore, the methods to prove can be difficult. Therefore, many researchers focus on conditions in which divide the problem into 3 criteria as follows:

i. Patterns of the cover.

   Using this criterion, we consider the region according to our interest. For instance,

   – The region which contains unit arcs is a triangle.

   – The region which contains unit arcs is convex.

   – The region which contains unit arcs is without condition.

   Note that a set B can be **covered** by set A if there exists an isometry f on $\mathbb{R}^2$ such that $B \subseteq f(A)$.

ii. The ways to cover.

   Under this criterion, we are interested in the ways to cover. For instance,

   – Allow only translation.

   – Allow translation and rotation but not flipping.

   – Allow translation, rotation and flipping.

iii. Patterns of arc to be covered.

We consider arcs according to our interest. For instance,

- Closed arcs.
- Convex arcs.
- Arbitrary arcs.

One of the easiest approaches to find the smaller covers is to use the most recent regions of other works that are well-supported by lemmas as references to improve further and find the smaller region.

Next, we will demonstrate the regions which can cover every unit arcs.

In 1973, J. Gerriets and G. Poole [1] created the convex region which is a rhombus (see Figure 1.1). This region has area approximately 0.28610.

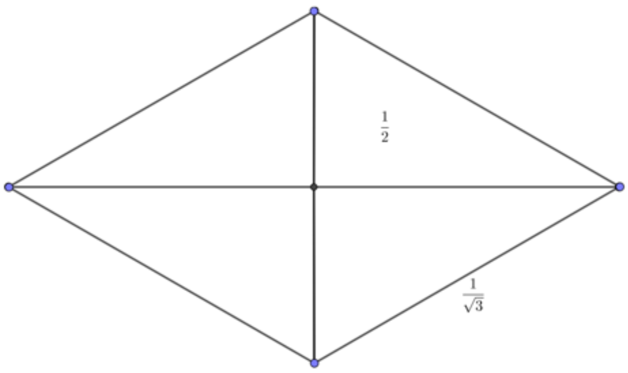

Figure 1.1: The convex region of J. Gerriets and G. Poole.

In 1992, R. Norwood, G. Poole, and M. Laidacker [2] created the convex region which was modified from the rhombus of J. Gerriets and G. Poole (see Figure 1.2). This region has area approximately 0.27524.

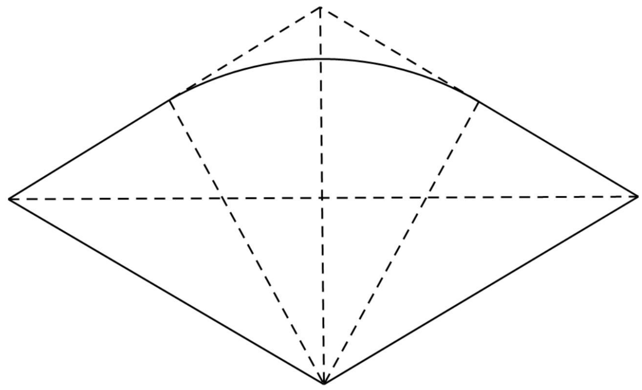

Figure 1.2: The convex region of R. Norwood, G. Poole, and M. Laidacker.

In 2003, R. Norwood and G. Poole [3] created the convex region and the non-convex region. They constructed the convex region from the non-convex region. This non-convex region can cover every unit arcs, whose area is approximately 0.26044 (see Figure 1.3). For the convex region, this region has area approximately 0.27381 (see Figure 1.4).

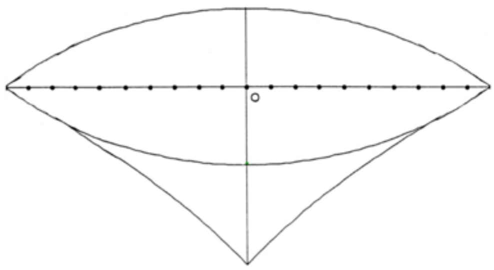

Figure 1.3: The non-convex region of R. Norwood and G. Poole.

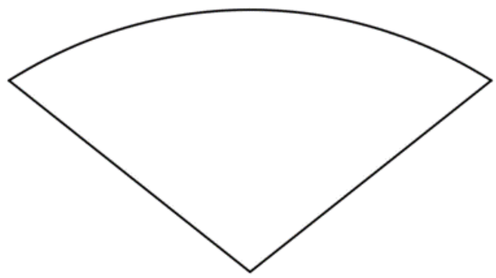

Figure 1.4: The convex region of R. Norwood and G. Poole.

In 2006, W. Wang [7] created the convex region (see Figure 1.5). This region has area approximately 0.27091.

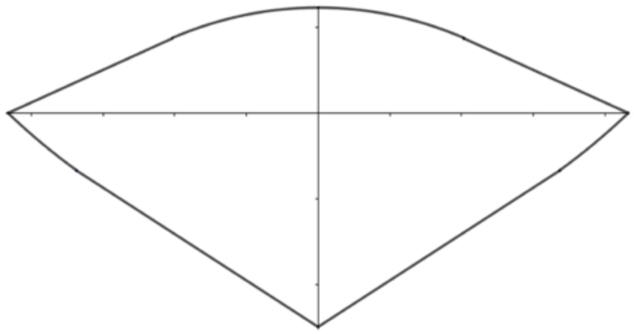

Figure 1.5: The convex region of W. Wang.

In 2018, N. Ploymaklam and W. Wichiramala [6] created the non-convex region, modified from the cover by Norwood and Poole (see Figure 1.6). This region is currently smallest. The upper boundary of this region is an elliptic arc instead of a circular arc. This region has area approximately 0.26007.

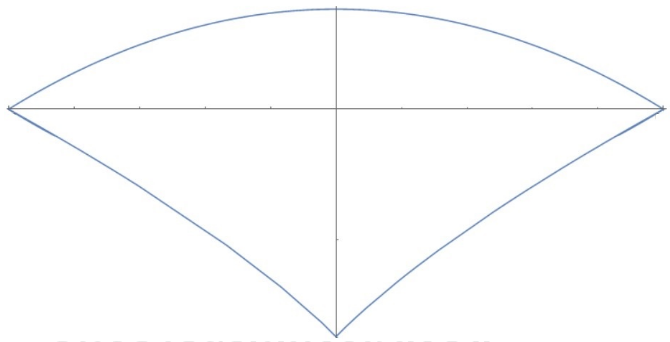

Figure 1.6: The non-convex region of N. Ploymaklam and W. Wichiramala.

In 2019, C. Panraksa and W. Wichiramala created the convex region (see Figure 1.7). This region is a 30° circular sector of unit radius. This region has area approximately 0.26180.

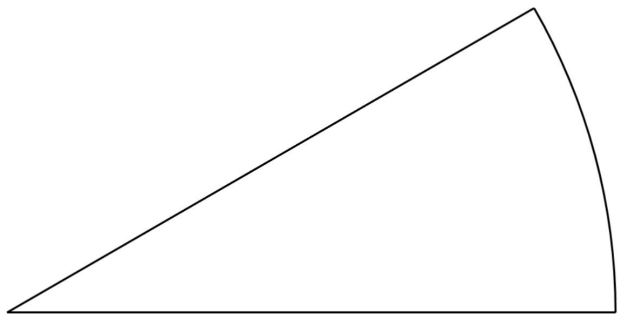

Figure 1.7: The convex region of C. Panraksa and W. Wichiramala.

In this work, we are interested in non-convex regions on $\mathbb{R}^2$. For the patterns of arcs and the ways to cover, they can be anything. In the case of convex covers, the Blaschke selection theorem guarantees the existence of such a solution. We modify the covers in [3] and [6] by replacing the upper boundary with the graphs of conic sections and cosine. The details of these graphs will be described in Chapter III. However, our work focuses on constructing new regions with new upper boundaries rather than finding the new smallest area of the region compared to the current by the smallest one. Thus, our proposed regions do not guarantee that they are the smallest area.

In Chapter II, we recall some basic definitions and ideas of the construction in the literature.

In Chapter III, we demonstrate the construction concept of the new regions starting from the upper boundary and construct a new covering set that contains a congruent copy of every unit arc on the plane.

In Chapter IV, we show the concept of construction of the new regions starting from the lower boundary and the problems of constructing the new region from the lower boundary.

In the last chapter, we provide conclusions and further research about this work.

CHAPTER II
PRELIMINARIES

In this chapter, we recall some definitions used in this work. First, we introduce definitions about convex sets and arcs related to this work.

Definition 2.1. A set A is said to be **convex** if for every points x and y in A, the segment $\overline{xy} \subseteq A$.

The next definitions are the definition about arc.

Definition 2.2. Define the continuous function $\lambda : [-1, 1] \to \mathbb{R}^2$ to be an arc and $l(\lambda)$ to be the length of an arc. If $l(\lambda) = 1$, λ is called a **unit arc**. Note that an arc λ refers to the set $[0, 1]$.

Definition 2.3. If both end points of an arc λ are the same point, then λ is called a **closed arc**.

Definition 2.4. An arc γ is called **simple** if γ is not self-intersecting. Whereas a simple arc is called **simple closed** if its starting point and end point are alike.

Definition 2.5. Let A, A' and B be sets in $\mathbb{R}^2$. A' is said to be a congruent copy of A if there exists an isometry f on $\mathbb{R}^2$ such that $f(A) = A'$. A set B can be **covered** by set A if there exists a congruent copy B which is a subset of A.

Some ideas of the construction in the literature

We will show the region of Norwood, Poole, and Laidacker. The region of Norwood, Poole, and Laidacker was adapted from the region of Gerriets and Poole. In the work of Gerriets and Poole, the region is the rhombus which was composed of two adjacent equilateral triangles with sides of length $\frac{\sqrt{3}}{3}$. This rhombus had two end points $\left(-\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$. Moreover, it passed through points C' and V which lie on y-axis, as shown in Figure 2.1.

Norwood, Poole, and Laidacker modified the region of Gerriets and Poole by snipping off one corner of the rhombus. Thus, the region of Norwood, Poole, and Laidacker contains a 60°-sector with radius 0.5 ($\overset{\frown}{DVB}$). Figure 2.2 demonstrates the region of Norwood, Poole, and Laidacker.

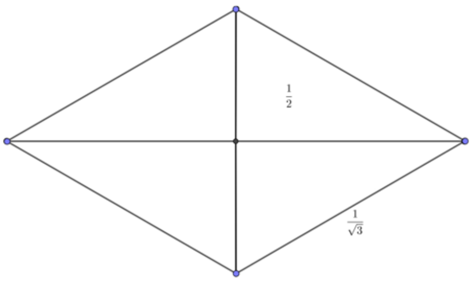

Figure 2.1: The convex region of J. Gerriets and G. Poole.

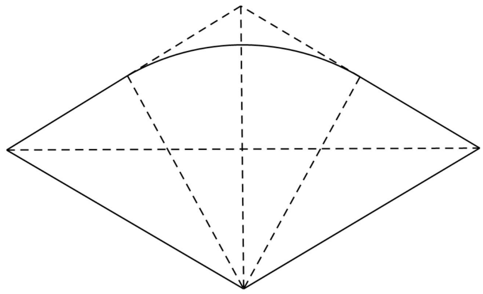

Figure 2.2: The region of Norwood, Poole, and Laidacker.

We will show the region of Norwood and Poole. The region of Norwood and Poole contained two parts which are the upper and the lower boundaries. The upper boundary is the circular arc of the equation $x^2 + (y-c)^2 = r^2$ where $-\frac{1}{2} \leq x \leq \frac{1}{2}$, passing through the points $\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0\right)$ and $(0, t)$. Then it can conclude that $c = t^2 - \frac{1}{4}$ and $r = t - \frac{t^2 - \frac{1}{4}}{2t}$. For the lower boundary, it consisted of

1. the union of two parabolic arcs which are
   - the right parabolic arc is the graph $y = \sqrt{2wx + w^2} + c$ where $0 \leq x \leq 0.37046$ and $w = r - \frac{1}{2}$
   - the left parabolic arc is the reflection of the the right parabolic arc across the y-axis
2. the circular arc came from the reflection of the circular upper boundary over the x-axis where $x > 0.37046$.

Figure 2.3 illustrates the region that can contain every unit arc proposed by Norwood and Poole. It is obvious that the upper and lower boundary of the region shown in Figure 2.3 depends on variable t. The approximated value of t is 0.1527985 such that the region satisfies some properties. Moreover, this value of t results in the smallest area of the region. The idea of this construction will be described more in Chapter 3.

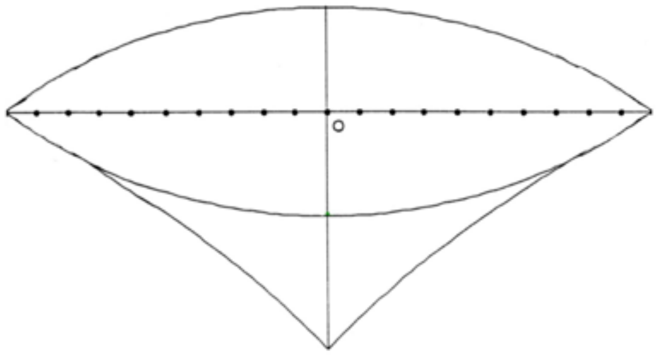

Figure 2.3: The region of Norwood and Poole.

CHAPTER III

MAIN RESULTS

In the works of [3] and [6], the authors constructed non-convex regions to cover every unit arc. They started by creating the upper boundary and then form the lower boundary which provides the important properties for the proof. The upper boundaries in these two works are different. In [3], they proposed the top boundary using the arc of a circle with the equation $x^2+(y-c)^2=r^2$ where $-\frac{1}{2}\leq x\leq\frac{1}{2}$ and passing through the point $\left(-\frac{1}{2},0\right)$ and $\left(\frac{1}{2},0\right)$, as the right half of the bottom boundary consists of a parabolic arc and a circular arc. While [6] presented the top boundary using the elliptic arc with the equation $\left(\frac{x}{a}\right)^2+\left(\frac{y-y_c}{b}\right)^2=1$ where $-\frac{1}{2}\leq x\leq\frac{1}{2}$ and passing through the point $\left(-\frac{1}{2},0\right)$ and $\left(\frac{1}{2},0\right)$. Therefore, we would like to find the other shapes of the upper boundary which differ from [3] and [6]. Next section we show the steps of construction for our proposed regions based on the ideas from [3] and [6].

3.1 Construction of C^*

We construct the new region named C^* based on the idea from [3]. To construct C^*, first we consider the function of the upper boundary, says f where f contains the points $\left(-\frac{1}{2},0\right)$ and $\left(\frac{1}{2},0\right)$.

For the construction of the lower boundary, we consider the right half of the lower boundary. The other half is the reflection of the right half of the lower boundary about the y-axis. We construct the right half of the lower boundary g in order to have the specific property. That is, no half of a unit arc starting from y-axis can touch the bottom of the region and then escape the top of the region. We define the locus of points $(\bar{x},\bar{y})$ by L, which the sum of the minimum distance from the y-axis to L and the minimum distance from $(\bar{x}, \bar{y})$ to the arc of the function f is $\frac{1}{2}$. Therefore, the straight line passing through point (x, y) and $(\bar{x}, \bar{y})$ must be pendendicular to the upper boundary which has a slope equals $f'(x)$. Form this property, we have the equation as follows:

$$\bar{x} + \sqrt{(x-\bar{x})^2 + (y-\bar{y})^2} = \frac{1}{2} \text{ and } \frac{y-\bar{y}}{x-\bar{x}} = \frac{-1}{f'(x)} \text{ where } (x, y) \in f.$$

Moreover, the distance of any arc from the origin to the top of the region and then goes back to meet the bottom of the region is greater than $\frac{1}{2}$ (see Figure 3.1).

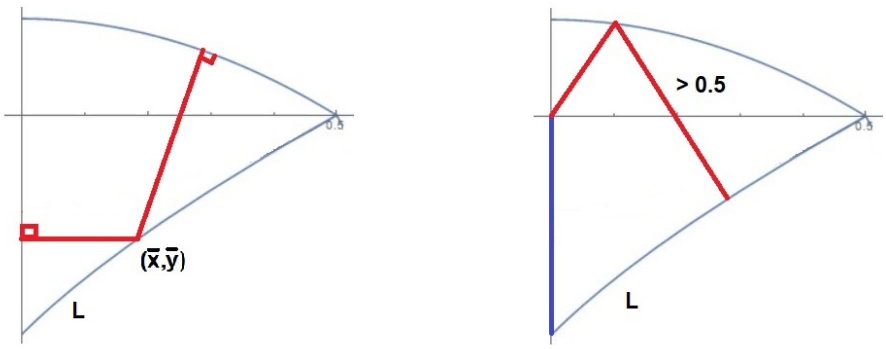

Figure 3.1: Construction of C^*.

Let $l(x)$ be the function of graph L where $0 \le x \le \frac{1}{2}$. The lower boundary of our proposed region is defined by $g(x) = \min\left(l(x), -f(x)\right)$ (see Figure 3.2).

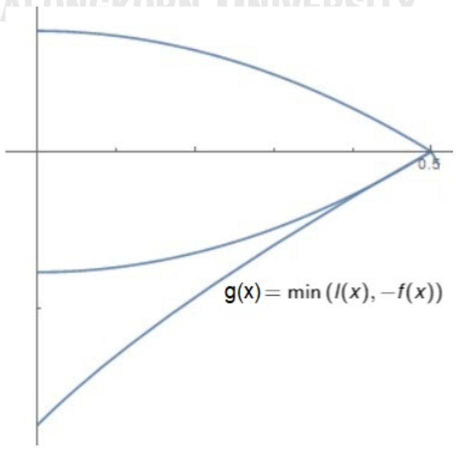

Figure 3.2: The lower boundary.

3.2 Properties of the cover C^*

Let γ be a unit arc with midpoint m on the y-axis. From the construction with specific constants, we get the relations between the upper boundary and the lower boundary of C^* as follows:

- The shortest distance from the y-axis to the lower boundary and then meets the upper boundary is at least $\frac{1}{2}$. Thus, we have the following property.

  Property A: No half of γ can touch the bottom of C^* and then escapes the top of C^* (see Figure 3.3).

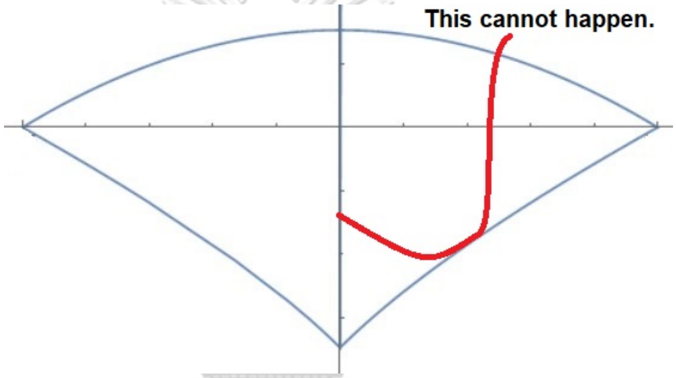

Figure 3.3: Property A.

- The shortest distance from the origin to the upper boundary and then meets the lower boundary is at least $\frac{1}{2}$. Thus, we have the following property.

  Property B: No half of γ whose midpoint is below the origin can touch the top of C^* and then escapes the bottom of C^* (see Figure 3.4).

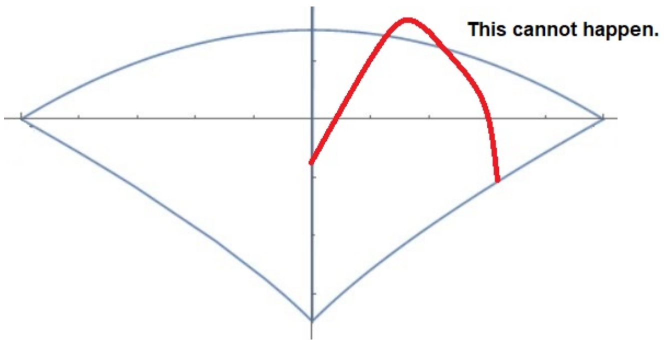

Figure 3.4: Property B.

We construct the region that satisfies Property A and Property B. Hence, we can apply the idea of the proof of Norwood and Poole in [3] and Ploymaklam and Wichiramala in [6] to show that the region can cover every unit arc. If the region does not satisfy Property A and Property B, we cannot guarantee that the region can cover every unit arc. Therefore, our proposed regions need to satisfy Property A and Property B.

3.3 The regions satisfy Property A

Next, we present the regions that we investigated in this research. Note that all regions satisfy Property A and we show only the right half of the regions. The left half of the regions comes from the reflection of the right half about the y-axis.

1. **The upper boundary is a circular arc.**
   Equation: $(x-x_0)^2+(y-y_0)^2=r^2$ where $0 \leq x \leq \frac{1}{2}$ and pass through the point $\left(\frac{1}{2},0\right)$. The lower boundary is described in Appendix 2.1.
   Figure 3.5 shows the region with the circular upper boundary where $x_0 = 0, r = 0.89447, y_0 = -0.5\sqrt{-1+4r^2+4x_0-4x_0^2}$.

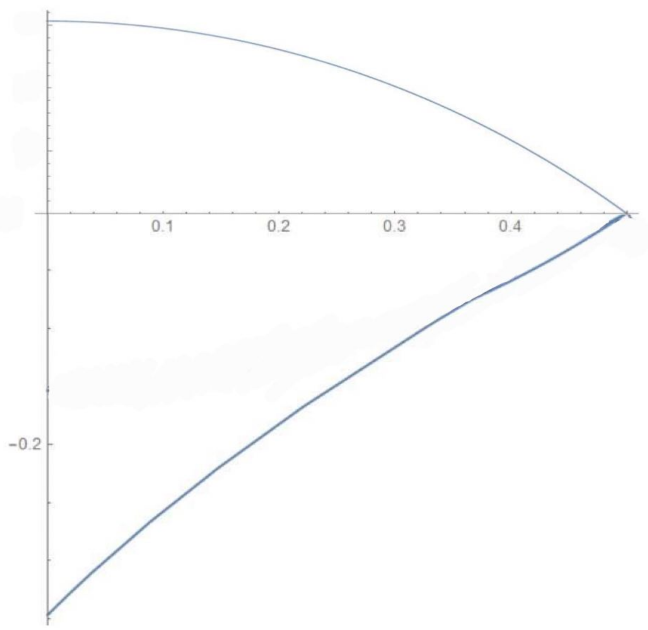

Figure 3.5: the region with the circular upper boundary.

In this case, these constants are caused by the region satisfying Property B. This region satisfies Property A and Property B which has area approximately 0.260437.

2. **The upper boundary is an elliptic arc.**
   Equation: $\left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{b}\right)^2=1$ where $0\leq x\leq\frac{1}{2}$ and pass through the point $\left(\frac{1}{2},0\right)$. The lower boundary is described in Appendix 2.2.
   Figure 3.6 shows the region with the elliptic upper boundary where $a=1.95272, b=4.58588, x_0=0, y_0=-\frac{b\sqrt{a^2-(0.5-x_0)^2}}{a}$.

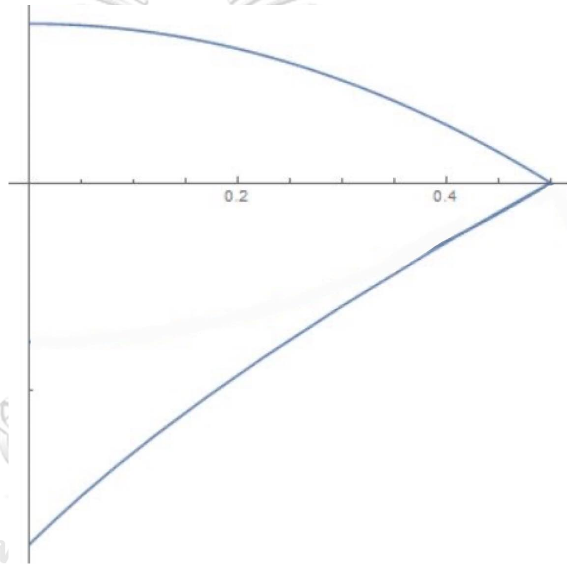

Figure 3.6: The region with an elliptic upper boundary.

In this case, these constants are caused by the region satisfying Property B. This region satisfies Property A and Property B which has area approximately 0.26007.

3. **The upper boundary is a parabolic arc.**
   Equation 1: $(x-x_0)^2=4c(y-y_0)$ where $0\leq x\leq\frac{1}{2}$ and pass through the point $\left(\frac{1}{2},0\right)$. The lower boundary is described in Appendix 2.3.
   Figure 3.7 shows the region with the parabolic upper boundary where $x_0=-0.4, c=-0.77, y_0=-\frac{(0.5-x_0)^2}{4c}$.

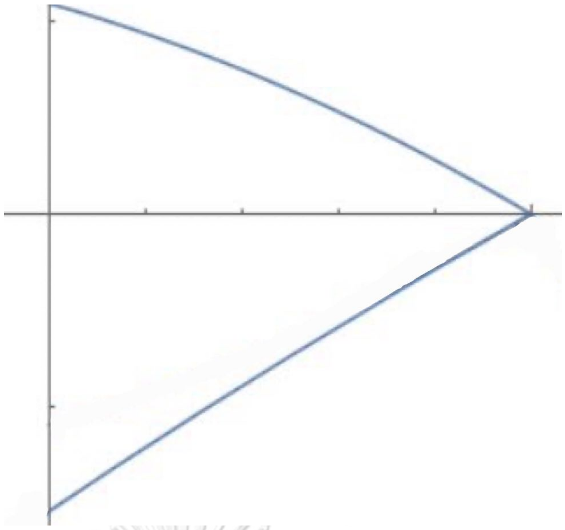

Figure 3.7: The region with the parabolic of the Equation 1 upper boundary.

In this case, these constants are caused by the region satisfying Property B. This region satisfies Property A and Property B which has area approximately 0.270511. In addition, we can get a smaller region compared to Figure 3.7 by snipping off one of the corners of the region. Details of snipping off will be presented in the next section.

Equation 2: $(y - y_0)^2 = 4c(x - x_0)$ where $0 \leq x \leq \dfrac{1}{2}$ and pass through the point $\left(\dfrac{1}{2}, 0\right)$. The lower boundary is described in Appendix 2.4.

Figure 3.8 shows the region with the parabolic upper boundary where $x_0 = 1, c = -0.17, y_0 = -\sqrt{2c - 4cx_0}$.

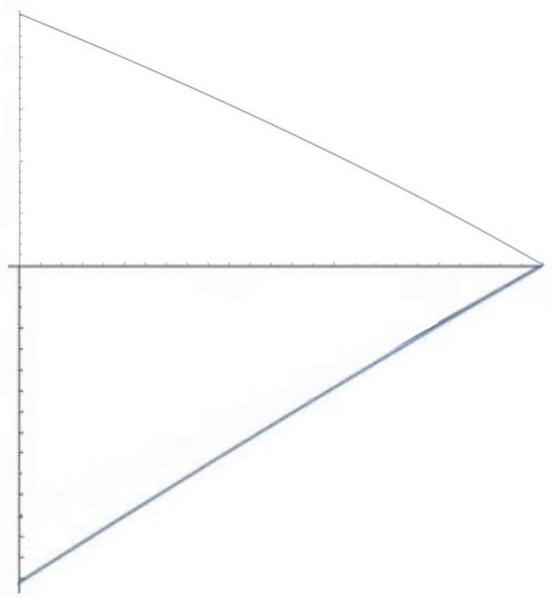

Figure 3.8: The region with the parabolic of the Equation 2 upper boundary.

In this case, these constants are caused by the region satisfying Property B. This region satisfies Property A and Property B which has area approximately 0.276963.

4. **The upper boundary is a hyperbolic arc**

   Equation 1: $\left(\frac{y-y_0}{b}\right)^2 - \left(\frac{x-x_0}{a}\right)^2 = 1$ and pass through the point $\left(\frac{1}{2}, 0\right)$.
   The lower boundary is described in Appendix 2.5.
   Figure 3.9 shows the region with the parabolic upper boundary where $x_0 = 0, a = 5.35, b = 35.4, y_0 = \frac{b\sqrt{1+4a^2-4x_0+4x_0^2}}{2a}$.

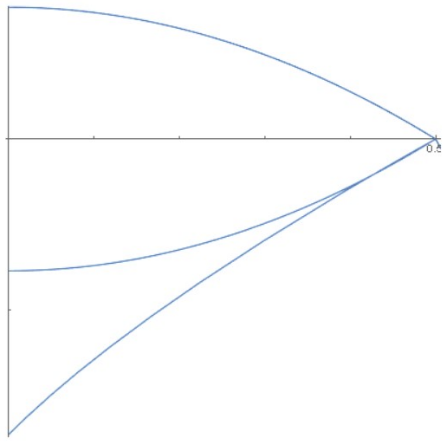

Figure 3.9: The region with the hyperbolic of the Equation 1 upper boundary.

In this case, these constants are caused by the region satisfying Property B. This region satisfies Property A and Property B which has area approximately 0.260299.

Equation 2: $\left(\frac{x-x_0}{a}\right)^2 - \left(\frac{y-y_0}{b}\right)^2 = 1.$

In this case, we can define the equation of this upper boundary as $\left(\frac{x-x_0}{a}\right)^2 - \left(\frac{y-y_0}{b}\right)^2 = 1$ where $0 \leq x \leq \frac{1}{2}$ and pass through the point $\left(\frac{1}{2}, 0\right)$. For the lower boundary, we use Mathematica to find the lower boundary that satisfies Property A. Although we can find the general form of the lower boundary, we have a problem to find the constants that make the region satisfies Property A and Property B.

5. **The upper boundary is a graph of cosine.**

Equation: $y = a\cos bx + k$ where $0 \leq x \leq \frac{1}{2}$ and pass through the point $\left(\frac{1}{2}, 0\right)$. The lower boundary is described in Appendix 2.5.

Figure 3.10 shows the region with a graph of cosine upper boundary where $a = 25, b = 0.22144, k = -a\cos\frac{b}{2}$.

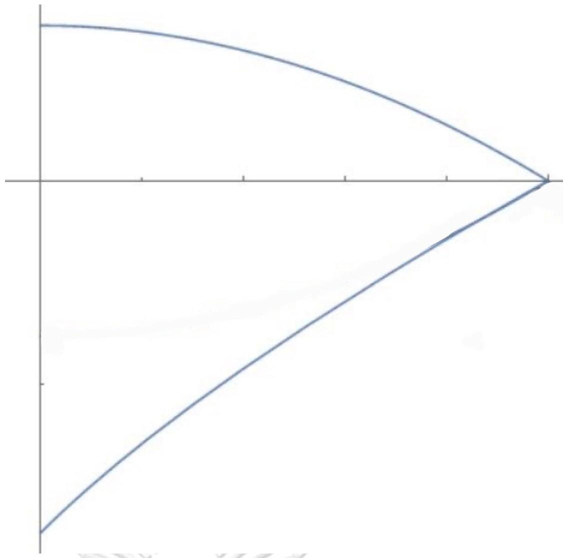

Figure 3.10: The region with a graph of cosine upper boundary.

In this case, these constants are caused by the region satisfying Property B. This region satisfies Property A and Property B which has area approximately 0.26009.

Each region is constructed based on the above idea. We experiment on several functions for the upper boundary. Each of these functions provides a region with the lower boundary equation in which satisfies some certain properties. Therefore, we would like to find the values of the above functions in order to give the smallest area of the region.

3.4 The snipping off of the upper boundary

We show the adaptation of our proposed regions in order to get the smaller regions. The idea of adaptation is by snipping off the peak of the regions. We apply this idea from Norwood, Poole and Laidacker [2]. In addition, the adapted regions still satisfy Property A and Property B after modification. To snip the peak off, we use a circular arc with the center at the vertex of the lower boundary and radius $\frac{1}{2}$ as the work of Norwood, Poole and Laidacker. The reason that we use this arc is that we want the modified regions still satisfying Property A. However, after the modification, we have to calculate the shortest distance from the origin to the modified upper boundary and then go back to the lower boundary. The distance must be greater than $\frac{1}{2}$ in order to make sure that the modified regions satisfy Property B.

Next, we illustrate an example of the modified regions where it still satisfies Property A and Property B.

The parabolic upper boundary is snipped off its peak. The equation of parabolic upper boundary is $(x - x_0)^2 = 4c(y - y_0)$ where $0 \leq x \leq \frac{1}{2}$ and pass through the point $\left(\frac{1}{2}, 0\right)$. The form of lower boundary can be seen in Appendix 2.6. Figure 3.11 shows this region where $x_0 = -0.4, c = -0.77, y_0 = -\frac{(0.5 - x_0)^2}{4c}$.

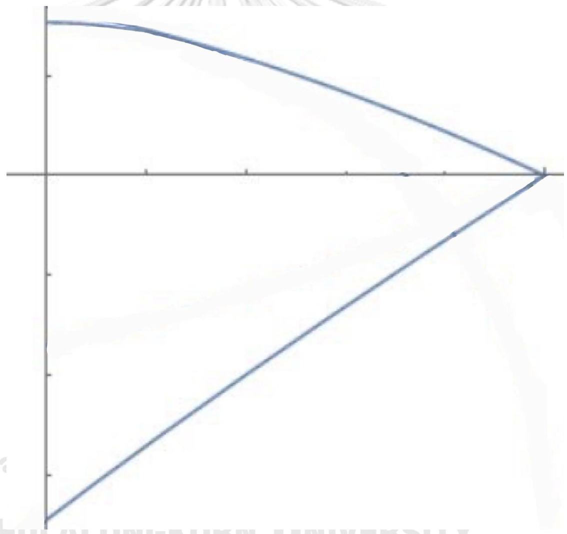

Figure 3.11: The snipping off of the parabolic upper boundary.

In this case, these constants are caused by the region satisfying Property A and Property B. This region satisfies Property A and Property B which has area approximately 0.267663.

3.5 The upper boundary represented by the graph of cosine

It can be clearly seen that most of previous works constructed the upper boundary using circular arc and elliptical arc. Thus, we want to find the upper boundary which is not from conic section. Finally, we come up with the graph of cosine. In this work, we present the upper boundary represented by the graph of cosine $f(x) = a\cos bx + k$.

One of our regions has the upper boundary defined by the equation $f(x) = a\cos bx + k$. According to this upper boundary, we construct the lower boundary such that the region satisfies Property A and Property B. The lower boundary can be written as $g(x) = \min(l(x), -f(x))$ where $l(x)$ is the function of the locus of point $(\bar{x}, \bar{y})$. More the details and calculations of $(\bar{x}, \bar{y})$ can be found in Appendix. Obviously, this upper and lower boundaries depend on the variables a, b and k. If we adjust the value of a, b and k, this process will affect the characteristic of the upper and lower boundaries. Thus, the area of region will be affected as well. Since the upper boundary $f(x) = a\cos bx + k$ passes through the point $\left(\frac{1}{2}, 0\right)$, the constant k can be written in the form of a and b. That is $k = -a\cos\frac{b}{2}$. Therefore, there are two independent variables affecting to the area which are a and b. To find the smallest area, one of the independent variables need to be fixed. In this case, we fix the value of a and adjust the value of b. Also, the value of variable a is adjusted until it leads us to the smallest area of the region.

We use numerical computation to find the values of a, b and k. Thus, we obtain $a = 25$, $b = 0.22144$ and $k = -24.8469$. The right half of the lower boundary of C^* is defined by $l(x)$ where $0 \leq x \leq 0.4324$ and by $-f(x)$ where $0.4324 \leq x \leq 0.5$. This gives C^* of area 0.26009 (see Figure 3.12).

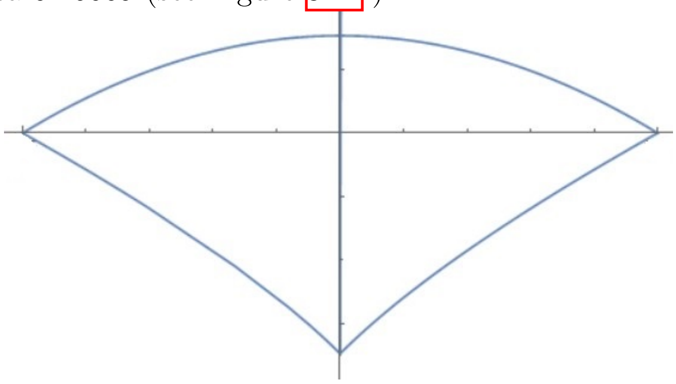

Figure 3.12: The new region C^*.

We follow the proof of Norwood and Poole in [3] and Ploymaklam and Wichiramala in [6] and get the improved result.

Theorem 3.1. *The region C^* which has area approximately 0.26009 can cover every unit arc.*

Proof. We assume that γ is simple, i.e., γ is non-self intersecting. To show that C^* can cover γ, we first assume that γ cannot fit in C^* and then find a contradiction.

Let α and β be the two halves of γ. Assume that γ cannot fit in C^*. We consider the situation where the midpoint m always lies on y-axis. We may rotate and move γ until it touches the bottom of C^* and γ is above the bottom of C^*.

The proof is split into two cases according to how γ touches the bottom of C^*.

Case 1. Only one half of γ can touch the bottom of C^*. We consider the situation when m is as low as possible, we call this the minimal positioning scheme 1 (MPS 1). Without loss of generality, we call that half β. Suppose that γ is not covered. Thus, there are some parts of γ which escape C^*.

Case 1A. Assume that a part of β is not in C^*. Note that we use the idea from Norwood and Poole in [3] to prove this case. By Property A, no half of γ can touch the bottom of C^* and then escape the top of C^*. Thus, the possible position of β is that it must escape the top of C^* and then back to touch the bottom of C^*. By Property B, m must be above the origin and β gets through the top of C^* before it touches the bottom of C^* (see Figure 3.13).

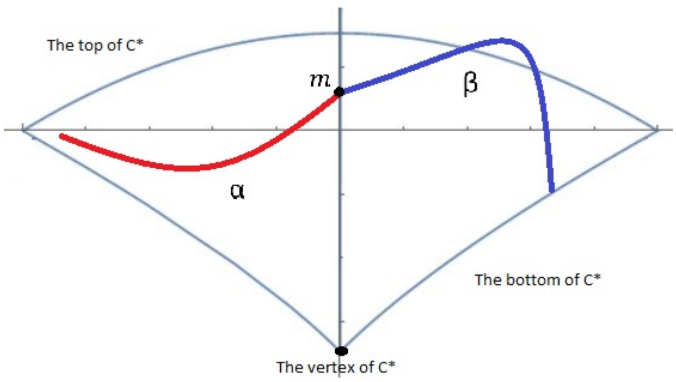

Figure 3.13: β is not in C^* and γ satisfies MPS 1.

We now let C^{**} be the reflection of C^{*} about the x-axis. We define the notations used in this work as below:

- B^{*} is the bottom of C^{*} and B^{**} is the reflection of B^{*}.
- T^{*} is the top of C^{*} and T^{**} is the reflection of T^{*}.
- For points s and t in arc δ, a part of δ between s and t is called δ_{st}.
- Let x_p be the distance from y-axis to the point p.
- Let $l(\gamma_{st})$ be the length of a part of γ from the point s to the point t.

Consider when γ does not cross B^{**}, we may move γ upward to touch B^{**}. From Case 1, we know that β must touch first. Then α cannot touch B^{**} first. It is clear that m is between the origin and B^{**}. Since β touches B^{**}, it contradicts to MPS 1. This implies that β must cross both T^{*} and B^{**} (see Figure 3.14).

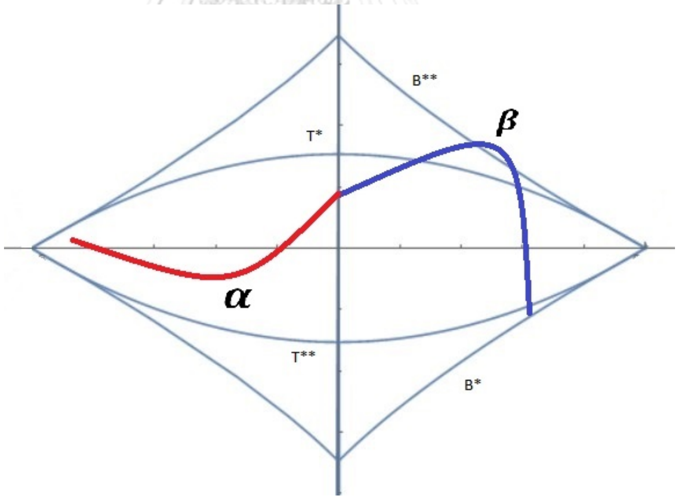

Figure 3.14: β must cross both T^{*} and B^{**}.

Next, we move γ downward until it touches B^{**}, not crossing. Since β touches B^{**} and escapes T^{**} (see Figure 3.15), it contradicts to property A. Therefore, β is covered by C^{*}.

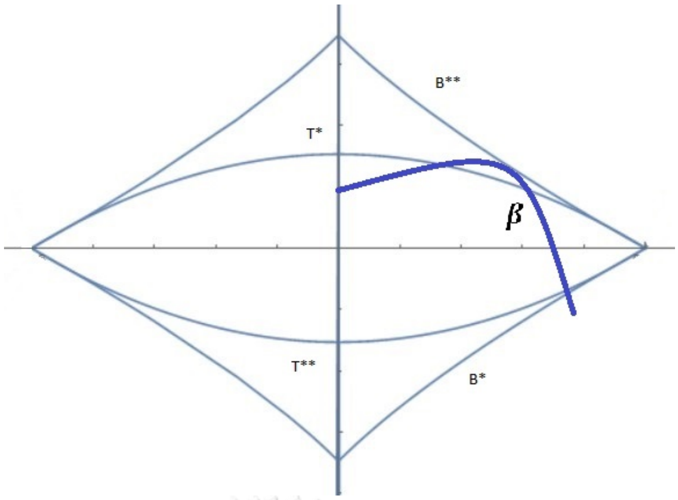

Figure 3.15: β touches B^{**} and escapes T^{**}.

Case 1B. Assume that a part of α is not in C^*. We know that α does not touch B^{**}. Then α must escape T^*. Let a be the point such that α crosses T^* and let L_1 be a tangent line of T^* at point a (see Figure 3.16).

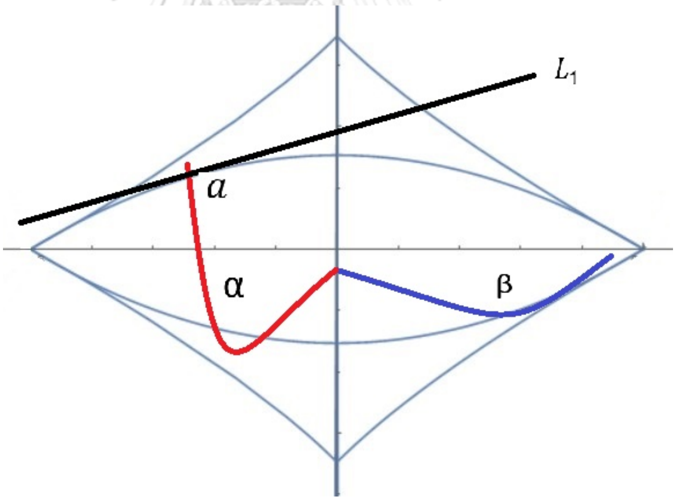

Figure 3.16: α is not in C^* and a is the point such that α crosses T^*.

We may translate β until it touches B^{**}. When β touches B^{**}, there are two incidents that can occur. We apply the concept of Norwood and Poole in [3] and Ploymaklam and Wichiramala in [6] to validate the first and the second subcases as follows, respectively.

Subcase 1B(i). β is in C^{**}. If α is in C^{**}, then γ is covered by C^{**}. Thus, we consider when α escapes C^{**}. Since β must touch C^{**} first, α must get through T^{**}. Let b be a point such that α crosses T^{**} (see Figure 3.17).

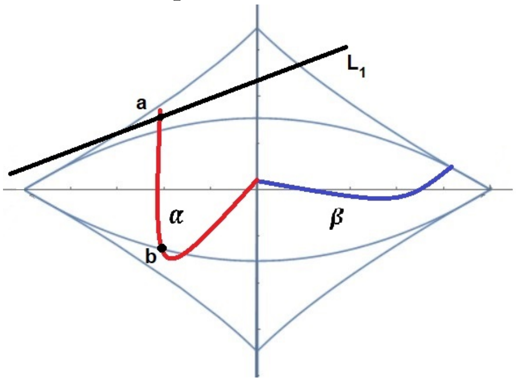

Figure 3.17: β is in C^{**} and α get through C^{**}.

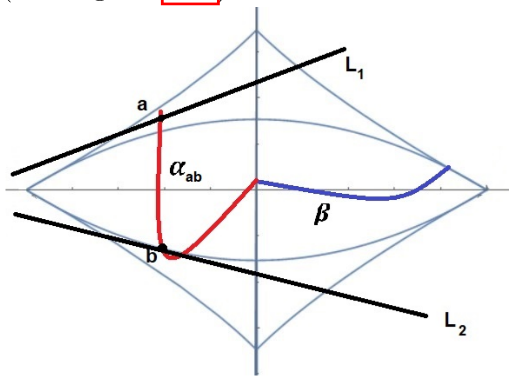

Figure 3.18: α_{ab} divides the region between L_1 and L_2.

We see that if we move γ up, then L_1 is also moved upward. Since the original position of β is covered by C^*, L_1 does not cross β. By the assumption, β does not cross α_{ab}. Let L_2 be a tangent line of T^{**} at point b. Since β is covered by C^{**}, L_2 does not cross β. We use α_{ab} to divide the region between L_1 and L_2 into two parts. Thus, β must be in either of the two sides of separated region because γ is simple (see Figure 3.18).

We rotate γ while m still lies on y-axis until the straight line passing through a and b is horizontal and β is above α_{ab}. Next, we move γ down until it touches B^*. We see that α must touch B^* first, which is a contradiction. Thus, β is not covered by C^{**} (see Figure 3.19).

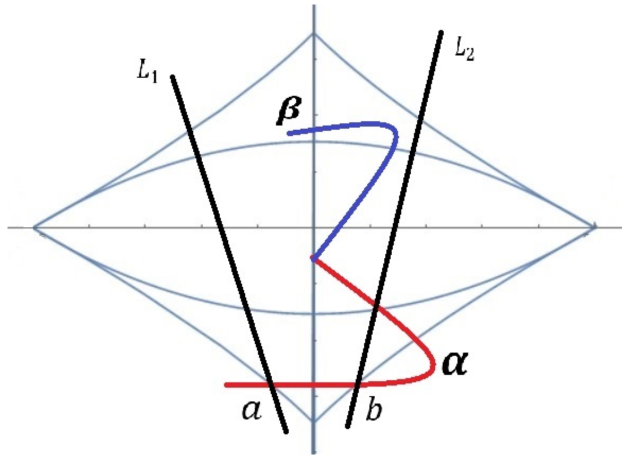

Figure 3.19: a and b are horizontal.

Subcase 1B(ii). β is not in C^{**}. Let c be a point where β touches B^{**}. By Property A and Property B, m is below the origin and β escapes T^{**} at a point d and then touches B^{**} (see Figure 3.20).

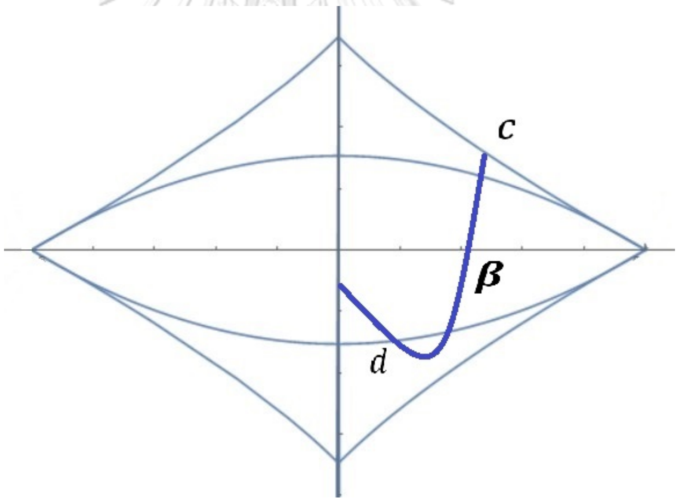

Figure 3.20: β is not in C^{**}.

Without loss of generality, suppose that α is on the left of β_{cd}. We see that α cannot be above c because if we translate γ upward until it touches B^{**}, β must touch B^{**} first. Similarly, we get α that cannot be below d. Next, we translate C^{**} and γ downward until β touches B^{*}. Thus, α must be between B^{*} and B^{**}. Since α escapes T^{*}, α can only escape to the shaded region (see Figure 3.21).

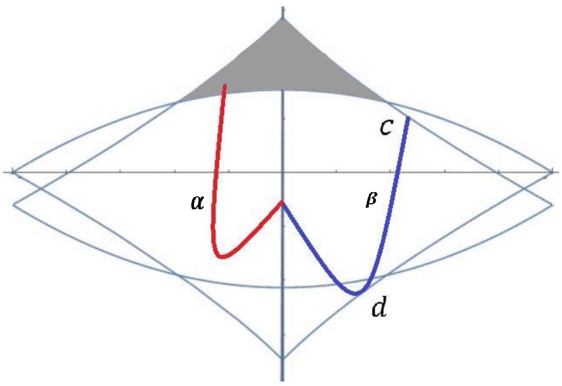

Figure 3.21: α is on the left of γ_{cd}.

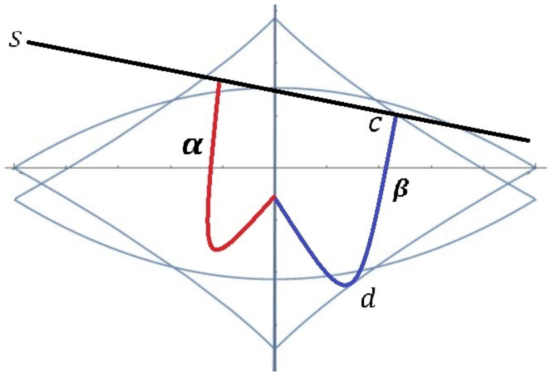

Figure 3.22: S is a tangent line touching α and β.

Let S be the tangent line touching α and β from above (see Figure 3.22). We rotate and move γ and S until the slope of S equals to the slope of the right of B^{**} at x equals to 0. This implies that S is parallel to the tangent line on the right side of B^{**} (see Figure 3.23).

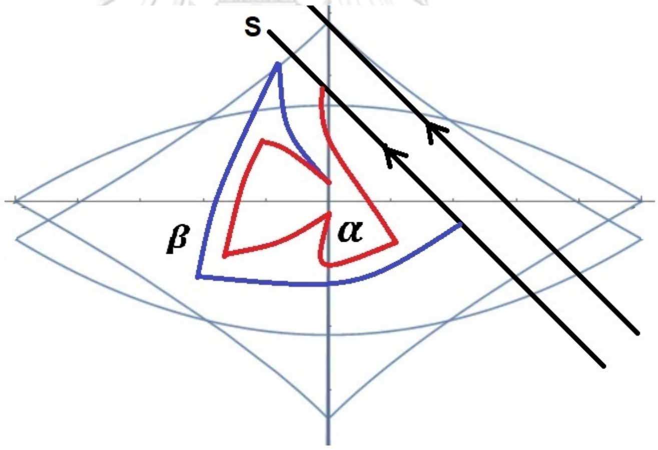

Figure 3.23: S is parallel to the tangent line of the right of B^{**}.

When we move γ up, β must touch the left half of B^{**} first. Next, we rotate γ counterclockwise while keeping β touching B^{**} until β also touches the left and the right halves of B^{**} at point e and f, respectively. From this situation, α is above β_{ef}. By assumption, β is not in C^{**} and β touches both the right and the left of B^{**}. This contradicts to Property A (see Figure 3.24).

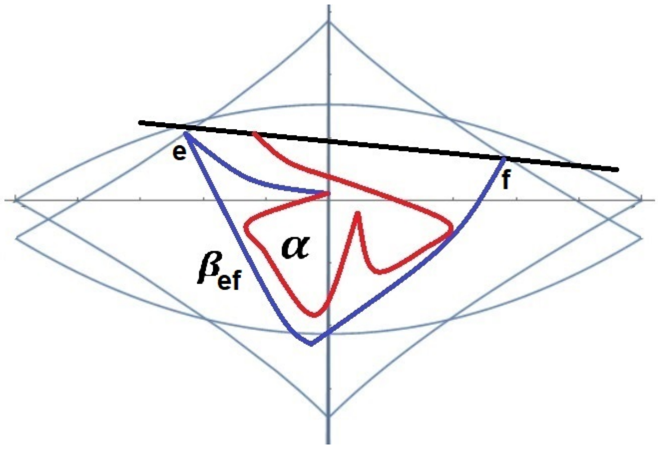

Figure 3.24: α is above β_{ef} and β is not in C^{**}.

Case 2. Both α and β may touch C^*. We follow the proof of Ploymaklam and Wichiramala in [6]. By the continuity, we have at least one orientation which both α and β touch B^*.

Assume that both α and β touch B^* and γ is not in C^*. Let a be a boint of β which touches B^*. We consider the situation that m is as low as possible and both α and β touch B^*. We call this the minimal positioning scheme 2 (MPS 2). By Property A and Property B, m is above the origin and there are at least one half of γ which escapes T^* and then they touch B^*. Without loss of generality, assume that β is not in C^* and the point which β touches B^* is on the right of the point which α touches B^*. Let b be a point such that β crosses T^* and p is the endpoint of β. In this position, γ must be below B^{**} because if γ touches or cross B^{**}, this contradicts to Property A. In this situation, we are interested in m which is above the origin (see Figure 3.25).

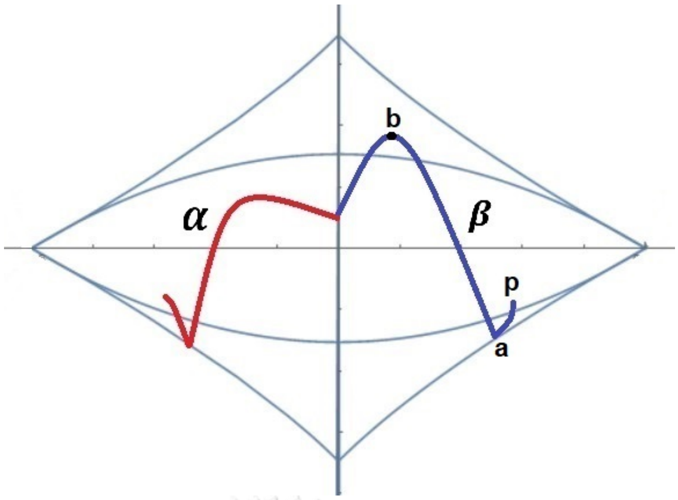

Figure 3.25: Both α and β touch B^*.

Next, we translate γ upward until γ touches B^{**}, if needed. If α and β touch B^{**}, this contradicts to MPS 2. So either α or β touches B^{**}.

Case 2A. β touches B^{**} at point b'. Then β is in C^{**}. Meanwhile, α is not in C^{**} but is above B^*. Let c' be a point of α which is below T^{**} (see Figure 3.26).

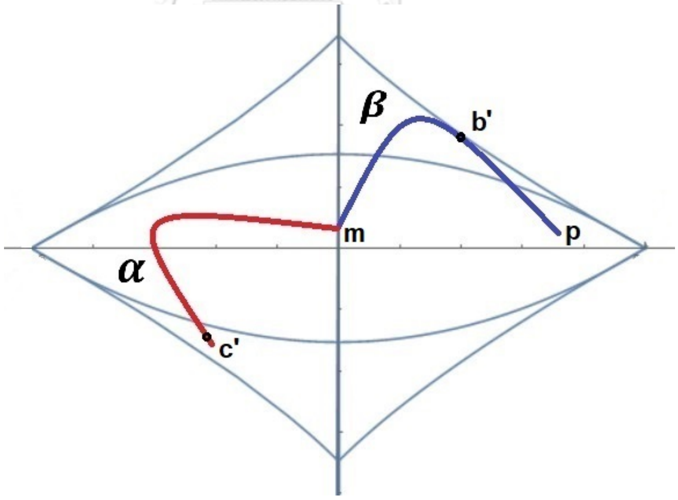

Figure 3.26: β touches B^{**} at point b' and c' be a point of α which is below T^{**}.

We rotate γ clockwise while β still touches the right half of B^{**}. While subarc from c' to b' is on the left of the remaining of β, α must go up to touch B^{**}. Otherwise, β must touch the left half of B^{**}. From this situation, α is above a part of β. Similar to the end of case 1B(ii), this contradicts to Property A.

Next, we consider the position of m while β and α touch B^{**}. If m still lies above the origin when β and α touch B^{**}, it contradicts to MPS 2. Then we consider the position of m while we rotate γ. Assume that while we rotate γ, m is on the origin where β touches B^{**} at point b''.

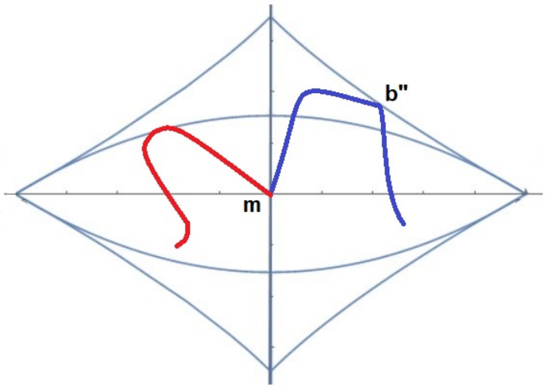

Figure 3.27: m is on the origin where β touches B^{**} at point b''.

Since the shortest distance from the origin to B^{**} is greater than 0.266639, $l(\gamma_{mb''}) > 0.266639$ and then $l(\gamma_{b''p}) < 0.233361$. Since the length of $\gamma_{b''p}$ greater than the original position of γ from the point b to a, $l(\gamma_{ba}) < 0.233361$. By the numerical computation, $x_b > 0.239093$. From the reasoning in [3], $x_{b''} > 0.157124$. Thus, m move up, not down. Hence, we have the position that m is closer to B^{**} than B^* at the original position. This contradicts to MPS 2.

Case 2B. α touches B^{**}. The way to prove is similar to case 2A.

This completes the proof.

□

CHAPTER IV
THE PROBLEM OF THE ALTERNATIVE WAY TO CONSTRUCT THE REGIONS

In Chapter 3, we apply the idea of Norwood and Poole in order to construct the regions which contain every unit arc. To construct the regions, we start by identifying the upper boundary. Then we define the lower boundary such that our regions satisfy Property A and Property B. In this chapter, we adapt the concept from Norwood and Poole by starting from lower boundaries. We found that it is too difficult to find the region satisfying Property A and Property B.

4.1 Concepts of construction the new regions starting from the lower boundaries

We wish to construct regions satisfying Property A and Property B. The constructed region still satisfies Property A and Property B like the region in Chapter 3 i.e., **Property A**: No half of γ can touch the bottom of the region and then escapes the top of C^*. **Property B**: No half of γ whose midpoint is below the origin can touch the top of the region and escapes the bottom of C^*.

This construction leads to the relation between the lower boundary g passing through $(\frac{1}{2}, 0)$ and the upper boundary f passing through $(\frac{1}{2}, 0)$. Moreover, the upper boundary $(\widehat{x}, \widehat{y}) \in f$ is defined by

$$x + \sqrt{(x - \widehat{x})^2 + (y - \widehat{y})^2} = \frac{1}{2} \text{ and } \frac{y - \widehat{y}}{x - \widehat{x}} = \frac{-1}{f'(\widehat{x})}, \text{ where } (x, y) \in g.$$

We design the equations which help to find the solutions.

- **Plan A.** We consider

$$x+\sqrt{(x-\widehat{x})^2+(y-\widehat{y})^2}=\frac{1}{2} \text{ and } \frac{y-\widehat{y}}{x-\widehat{x}}=\frac{-1}{f'(\widehat{x})}, \text{ where } (\widehat{x},\widehat{y})\in f.$$

  Then

$$x+(\widehat{x}-x)\sqrt{1+\left(\frac{y-\widehat{y}}{x-\widehat{x}}\right)^2}=\frac{1}{2} \text{ and } \frac{d\,\widehat{y}}{d\,\widehat{x}}=-\frac{x-\widehat{x}}{y-\widehat{y}}, \text{ where } (\widehat{x},\widehat{y})\in f.$$

  Let $\widehat{y}'=\dfrac{d\,\widehat{y}}{dx}$ and $\widehat{x}'=\dfrac{d\,\widehat{x}}{dx}$. Then $\widehat{y}'=-\dfrac{x-\widehat{x}}{y-\widehat{y}}\,\widehat{x}'$.

  This implies that $\sqrt{1+\left(\dfrac{\widehat{x}'}{\widehat{y}'}\right)^2}=\dfrac{\frac{1}{2}-x}{\widehat{x}-x}$ and then $\widehat{x}'^2+\widehat{y}'^2=\left(\dfrac{\frac{1}{2}-x}{\widehat{x}-x}\right)^2\widehat{y}'^2$.

- **Plan B.** We consider

$$x+\sqrt{(x-\widehat{x})^2+(y-\widehat{y})^2}=\frac{1}{2} \text{ and } \frac{y-\widehat{y}}{x-\widehat{x}}=\frac{-1}{f'(\widehat{x})}, \text{ where } (\widehat{x},\widehat{y})\in f.$$

  Let $\widehat{y}=f(\widehat{x})$, $y=g(x)$ and $\widehat{x}=h(x)$.

  We will arrange the equation in terms of $\widehat{x}$. Then

$$H(\widehat{x})+\sqrt{(H(\widehat{x})-\widehat{x})^2+(g(H(\widehat{x}))-f(\widehat{x}))^2}=\frac{1}{2} \text{ and } f'(\widehat{x})=-\frac{H(\widehat{x})-\widehat{x}}{g(H(\widehat{x}))-f(\widehat{x})}$$

  where H is the inverse of the function h.

We observe that the equations of Plan A and Plan B have complication on solving differential equations. Hence, we experiment with the simple lower boundary such as the linear lower boundary passing through the points $(0,t)$ and $(\frac{1}{2},0)$ where $t<0$ and the parabolic lower boundary proposed by Norwood and Poole. However, we are not able to find the upper boundaries which satisfy these lower boundaries. This procedure resulted in the upper boundaries which are in the form of derivative and contain a lot of variables. Thus, we applied substitution and simplifying. However, we confronted many errors occurred in Mathematica.

CHAPTER V

CONCLUSIONS AND FURTHER RESEARCH

This paper explores Leo Moser's worm problem, which is to find the region with the smallest area that can cover every unit arc. We modify the work of Ploymaklam and Wichiramala [6] together with the work of Norwood and Poole [3]. The region is modified by changing the upper boundary. In addition, the adapted regions still satisfy Property A and Property B after modification.

One of our region contains the upper boundary according to function $f(x) = a\cos bx + k$ and this region satisfies Property A and Property B. By numerical computation, we have $a = 25$, $b = 0.22144$, $k = -24.8469$ and the cover of area 0.26009.

Our work does not ensure that the proposed regions have the smallest area. Instead, we present the region with the new upper boundary. Hence, we advise some ideas to extend this research as follows:

1. In our way to prove, we may use other upper boundaries to construct the new region in order to find the region with the smallest area. Also, the region remains satisfied Property A and Property B.

2. In case we would like to construct the region starting from the lower boundary, we may use the appropriate lower boundary that is easy to find the whole region.

3. Other properties should be examined since it may lead to the constuction of new region .

4. In this work, we propose the non-convex region. We could modify the region to be a convex region by using a straingh line as a part of the lower boundary.

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CHAPTER VI
APPENDIX

In this thesis, we use Wolframe Mathematica to calculate the constants for each upper boundary proposed in Chapter 3. Therefore, we describe the commands in Mathematica that we use and the codes that we generate.

- Solve[expr,vars]: To identify the variables vars by solving the system expressions of the equations or inequalities.
- Reduce[expr,vars]: To identify vars and eliminate quantifiers by reducing the statement expressions by solving equations or inequalities.
- Manipulate[expr,$\{u, u_{min}, u_{max}\}$]: To allow interactive manipulation of the value of u. by generating a version of expr with control added.
- ParametricPlot[$\{f_x, f_y\}, \{u, u_{min}, u_{max}\}$]: To create a parametric plot of a curve with x and y coordinators f_x and f_y as a function of u.
- FullSimplify[expr]: To find the simplest form by trying a wide range of transformations on expr involving elementary and special function.

1. The steps to construct the regions and find the area.

We described the codes constructing to find the regions and the areas. In this case, we would like to construct the region starting from a graph of cosine upper boundary.

1.1 Define the upper boundary function f.

Let f (x) = a Cos [b x] + k

1.2 Simplify the above function f in terms of x.

```
Solve[a Cos[b x] + k == y, y]
```

```
{{y → k + a Cos[b x]}}
```

1.2 Find the derivative of y with respect to x.

```
D[k + a Cos[b x], x]
```

```
-a b Sin[b x]
```

1.3 Find the locus of a part of the lower boundary satisfying Property A.

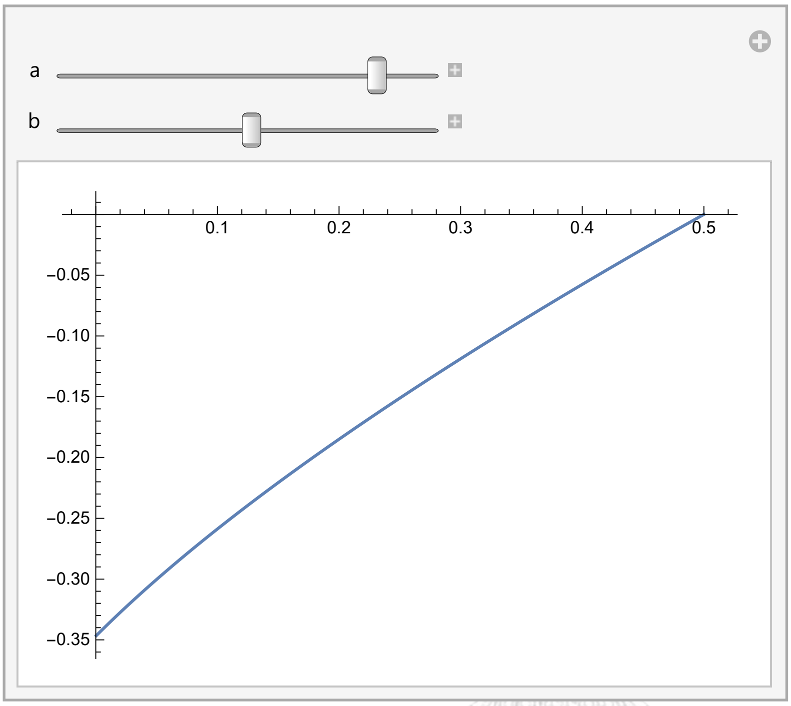
```
d[{x1_, y1_}, {x2_, y2_}] := Sqrt[(x1 - x2)^2 + (y1 - y2)^2];
p = {x, y};
p̅ = {x̅, y̅};
m[{x1_, y1_}, {x2_, y2_}] := (y1 - y2)/(x1 - x2);
d2[{x1_, y1_}, {x2_, y2_}] := (x1 - x2)^2 + (y1 - y2)^2;
FullSimplify[Solve[d2[p̅, p /. y → k + a Cos[b x]] == (1/2 - x̅)^2 &&
    m[p̅, p /. y → k + a Cos[b x]] == -1/(-a b Sin[b x]), {x̅, y̅}]]
```

```
{{x̅ → 1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2),
  y̅ → k + a Cos[b x] - 1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]},
 {x̅ → 1/2 (2 x + a b (a b (-1 + 2 x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2),
  y̅ → k + 1/2 (a b (-1 + 2 x) + 2 a Cot[b x] + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]}}
```

1.4 Figure shows the locus of point $(\overline{x}, \overline{y})$ which is a part of the lower boundary.

```
Remove["Global`*"]
```

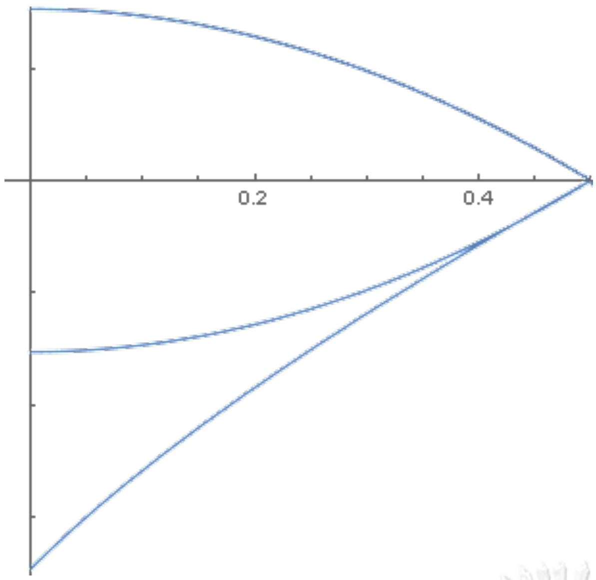
```
Manipulate[k = - a Cos[b 1/2];
 ParametricPlot[{1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2),
   k + a Cos[b x] - 1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]},
  {x, 0.001, 1/2}], {{a, -1}, -10, 30}, {{b, 1}, -10, 10}]
```




```
Remove["Global`*"]
```


1.5 Figure shows the right half of the region with a graph of cosine upper boundary where a = 25, b = 0.22144 and k = - a Cos[b 1/2].

```
a = 25; b = 0.22144; k = - a Cos[b 1/2];
Show[ParametricPlot[
  {1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2), k + a Cos[b x] -
    1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]}, {x, 0.001, 2.1}],
 Plot[{k + a Cos[b x]}, {x, 0, 1/2}], Plot[{-k - a Cos[b x]}, {x, 0, 1/2}]]
```




```
Remove["Global`*"]
```

1.6 Find the lower boundary satisfying Property B. The value of variables are adjusted until the region satisfy Property B.

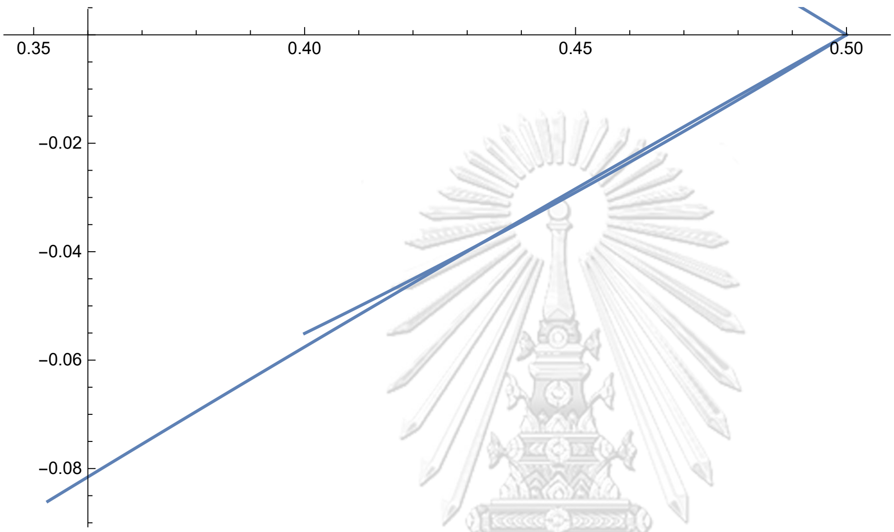
```
x = i/100;
a = 25;
b = 0.22144;
k = - a Cos[b 1/2];
Table[NMinimize[
  Sqrt[((v - 1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2))^2 +
      (k + a Cos[b v] - (k + a Cos[b x] - 1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)])
            Sin[b x]))^2)] + Sqrt[(v)^2 + (k + a Cos[b v])^2], {v, 0, 0.5}], {i, 50}]
```

```
{{0.649195, {v → 0.00129581}}, {0.645281, {v → 0.00263923}},
 {0.641334, {v → 0.00403378}}, {0.637352, {v → 0.00548307}},
 {0.633334, {v → 0.00699098}}, {0.629277, {v → 0.00856178}},
 {0.625181, {v → 0.0101999}}, {0.621044, {v → 0.0119101}},
 {0.616866, {v → 0.0136977}}, {0.612647, {v → 0.0155681}},
 {0.608387, {v → 0.0175273}}, {0.604086, {v → 0.0195819}},
 {0.599748, {v → 0.0217387}}, {0.595373, {v → 0.0240054}},
 {0.590965, {v → 0.0263903}}, {0.586527, {v → 0.0289021}},
 {0.582064, {v → 0.0315509}}, {0.577582, {v → 0.0343472}},
 {0.573087, {v → 0.0373029}}, {0.568587, {v → 0.0404312}},
 {0.56409, {v → 0.0437463}}, {0.559607, {v → 0.0472646}},
 {0.555149, {v → 0.0510038}}, {0.550728, {v → 0.0549845}},
 {0.54636, {v → 0.0592292}}, {0.542061, {v → 0.0637642}},
 {0.537846, {v → 0.068619}}, {0.533737, {v → 0.0738278}},
 {0.529753, {v → 0.07943}}, {0.525916, {v → 0.0854717}},
 {0.522251, {v → 0.0920071}}, {0.518783, {v → 0.0990999}},
 {0.515536, {v → 0.106826}}, {0.512537, {v → 0.115278}},
 {0.509813, {v → 0.124566}}, {0.507389, {v → 0.134825}},
 {0.505289, {v → 0.146221}}, {0.503531, {v → 0.158959}},
 {0.50213, {v → 0.17329}}, {0.501093, {v → 0.189518}}, {0.500412, {v → 0.208008}},
 {0.500066, {v → 0.229165}}, {0.500012, {v → 0.2534}}, {0.500182, {v → 0.281032}},
 {0.500484, {v → 0.312141}}, {0.500804, {v → 0.346432}}, {0.50102, {v → 0.383221}},
 {0.501018, {v → 0.42161}}, {0.500702, {v → 0.460748}}, {0.5, {v → 0.5}}}
```

```
Remove["Global`*"]
```

1.7 Find the intersection point between the locus of point $(\overline{x}, \overline{y})$ and the reflection of the upper boundary over the x-axis.

```
a = 25; b = 0.22144; k = - a Cos[b 1/2];
Show[
 ParametricPlot[{1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2),
   k + a Cos[b x] - 1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]},
  {x, 0.42, 1/2}], Plot[{k + a Cos[b x]}, {x, 0.42, 1/2}],
 Plot[{-k - a Cos[b x]}, {x, 0.4, 1/2}]]
```



```
x = 0.42 + i/1000; a = 25; b = 0.22144; k = - a Cos[b 1/2];
Table[{v = 1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2),
  k + a Cos[b x] - 1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x],
  -k - a Cos[b v]}, {i, 50}]
```

```
{{0.354217, -0.0850671, -0.0762132}, {0.355835, -0.0840816, -0.07551},
 {0.357457, -0.0830938, -0.0748012}, {0.359086, -0.0821038, -0.0740867},
 {0.36072, -0.0811115, -0.0733666}, {0.362359, -0.0801169, -0.0726408},
 {0.364004, -0.07912, -0.0719092}, {0.365655, -0.0781209, -0.0711718},
 {0.367311, -0.0771195, -0.0704286}, {0.368972, -0.0761158, -0.0696796},
 {0.37064, -0.0751098, -0.0689246}, {0.372312, -0.0741015, -0.0681637},
 {0.373991, -0.0730909, -0.0673967}, {0.375675, -0.072078, -0.0666238},
 {0.377365, -0.0710629, -0.0658448}, {0.37906, -0.0700454, -0.0650596},
 {0.380761, -0.0690256, -0.0642683}, {0.382468, -0.0680035, -0.0634709},
 {0.38418, -0.0669791, -0.0626671}, {0.385898, -0.0659524, -0.0618571},
 {0.387622, -0.0649234, -0.0610408}, {0.389352, -0.063892, -0.0602181},
 {0.391087, -0.0628583, -0.059389}, {0.392828, -0.0618223, -0.0585534},
 {0.394575, -0.060784, -0.0577113}, {0.396328, -0.0597433, -0.0568627},
 {0.398087, -0.0587003, -0.0560075}, {0.399851, -0.0576549, -0.0551457},
 {0.401621, -0.0566072, -0.0542772}, {0.403397, -0.0555572, -0.0534019},
 {0.405179, -0.0545048, -0.05252}, {0.406967, -0.0534501, -0.0516311},
 {0.408761, -0.052393, -0.0507355}, {0.410561, -0.0513335, -0.0498329},
 {0.412366, -0.0502717, -0.0489234}, {0.414178, -0.0492075, -0.0480069},
 {0.415995, -0.048141, -0.0470833}, {0.417819, -0.0470721, -0.0461527},
 {0.419649, -0.0460008, -0.0452149}, {0.421484, -0.0449271, -0.0442699},
 {0.423326, -0.0438511, -0.0433177}, {0.425173, -0.0427727, -0.0423582},
 {0.427027, -0.0416918, -0.0413914}, {0.428886, -0.0406087, -0.0404172},
 {0.430752, -0.0395231, -0.0394356}, {0.432624, -0.0384351, -0.0384465},
 {0.434502, -0.0373447, -0.0374499}, {0.436386, -0.036252, -0.0364457},
 {0.438276, -0.0351568, -0.0354339}, {0.440173, -0.0340592, -0.0344144}}
```

```
Remove["Global`*"]
```

1.8 Find the area of the region.

```
a = 25; b = 0.22144`; k = - a Cos[b 1/2];
- (-NIntegrate[
      (k + a Cos[b x] - 1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x])
       (D[1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2), x]),
      {x, 0.465`, 1/2}] - NIntegrate[k + a Cos[b x],
     {x, 0.43075223880624175`, 1/2}]) 2 + (-NIntegrate[
     (k + a Cos[b x] - 1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]) *
      (D[1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2), x]),
     {x, 0, 1/2}] + NIntegrate[k + a Cos[b * x], {x, 0, 1/2}]) * 2
```

```
0.26009
```

```
Remove["Global`*"]
```

1.9 Find the shortest path from the origin to B^{**} .

```
a = 25;
b = 0.22144;
k = - a Cos[b 1/2];
NMinimize[Sqrt[((1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2))^2 +
     ((k + a Cos[b x] - 1/2 (a b - 2 a b x + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]))^2)], {x,
  0, 0.5}]
{0.2666390703981955`, {x → 0.26187629499827975`}}
```

```
Remove["Global`*"]
```

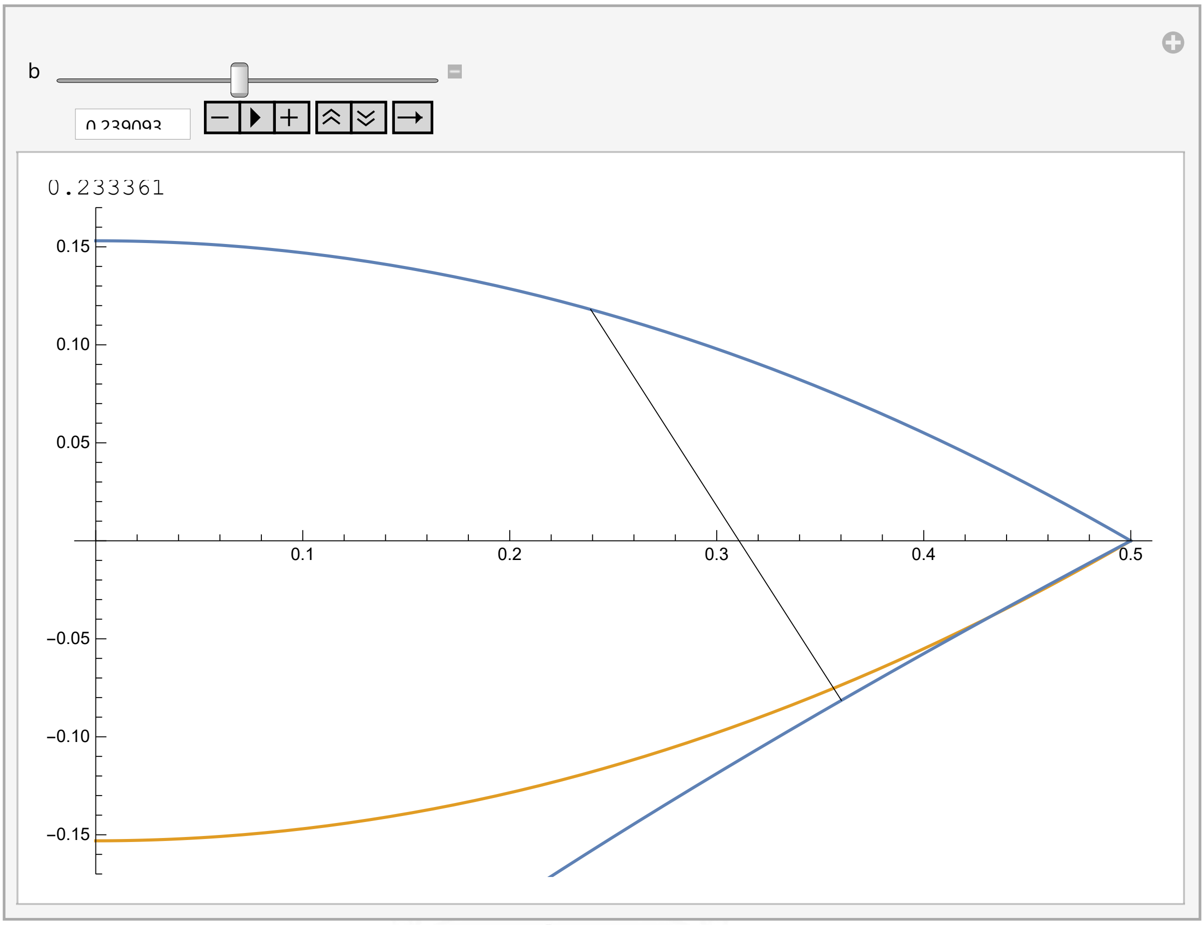
1.10 Find x_{b0}.

```
x = 0.26187629499827975`; a = 25; b = 0.22144; k = - a Cos[b 1/2];
1/2 (2 x - a b (a (b - 2 b x) + Sqrt[(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)]) Sin[b x]^2)
0.15712354663496542`
```

```
Remove["Global`*"]
```

1.11 Find x_b when $\overline{ab}$ < 0.233361.

```
Manipulate[x1 = b; nm = NMinimize[d[arc], x2];
 Column[{nm[[1]],
   Show[{Plot[{T[x], -T[x]}, {x, 0, 0.5}], ParametricPlot[Bn[x], {x, 0, 0.5}],
     Graphics[Line[arc /. nm[[2]]]]}, ImageSize → Large]}]
 , {{b, 0.2}, 0, 0.5}]
```




```
Remove["Global`*"]
```


2. The lower boundary of regions with other upper boundaries.

2.1 The lower boundary of the region with the circular upper boundary.

The upper boundary represented by the circular arc $(x - x_0)^2 + (y - y_0)^2 = r^2$ where $0 \le x \le \frac{1}{2}$

The locus of point $(\bar{x}, \bar{y})$ which is a part of the lower boundary is

$$\{\bar{x}, \bar{y}\} = \Big\{\Big(2 r^4 x - r^2 (1 + 2 x) (x - x0)^2 + (x - x0) \Big((x - x0)^3 - \sqrt{(r + x - x0) (r - x + x0)} \sqrt{r^2 (1 - 2 x)^2 (r + x - x0) (r - x + x0)}\Big)\Big) \Big/ \Big(2 (r + x - x0)^2 (r - x + x0)^2\Big), -\frac{1}{2 (r + x - x0) (r - x + x0)} \Big(\sqrt{r^2 (1 - 2 x)^2 (r + x - x0) (r - x + x0)} - 2 r^2 \Big(\sqrt{(r + x - x0) (r - x + x0)} + y0\Big) +$$

$$(x-x0)\left(\sqrt{(r+x-x0)\,(r-x+x0)}+2\,x\,y0-2\,x0\left(\sqrt{(r+x-x0)\,(r-x+x0)}+y0\right)\right)\Big)\Big\}\ \text{for}\ \left\{x,\,0,\,\frac{1}{2}\right\}.$$

The lower boundary of the region with the circular upper boundary where x0 = 0, r = 0.892, $y0 = -0.5\sqrt{-1+4\,r^2+4\,x_0-4\,x_0{}^2}$.

$$\{\bar{x},\,\bar{y}\} = \Big\{\Big(2\,r^4\,x - r^2\,(1+2\,x)\,(x-x0)^2 + (x-x0)\left((x-x0)^3-\sqrt{(r+x-x0)\,(r-x+x0)}\,\sqrt{r^2\,(1-2\,x)^2\,(r+x-x0)\,(r-x+x0)}\right)\Big)\Big/\left(2\,(r+x-x0)^2\,(r-x+x0)^2\right),\ -\frac{1}{2\,(r+x-x0)\,(r-x+x0)}\Big(\sqrt{r^2\,(1-2\,x)^2\,(r+x-x0)\,(r-x+x0)}-2\,r^2\left(\sqrt{(r+x-x0)\,(r-x+x0)}+y0\right)+(x-x0)\left(\sqrt{(r+x-x0)\,(r-x+x0)}+2\,x\,y0-2\,x0\left(\sqrt{(r+x-x0)\,(r-x+x0)}+y0\right)\right)\Big)\Big\}\ \text{for}\ \{x,\,0,\,0.433\}$$

and $y = -\sqrt{r^2-x^2+2\,x\,x0-x0^2}-y0$ for $\left\{x,\,0.3697952069716776\text{`},\,\frac{1}{2}\right\}$.

2.2 The lower boundary of the region with the elliptic upper boundary .

The upper boundary represented by the elliptic arc $\left(\frac{x-x0}{a}\right)^2+\left(\frac{y-y0}{b}\right)^2 = 1$ where $0 \le x \le \frac{1}{2}$.

The locus of point $(\bar{x},\,\bar{y})$ which is a part of the lower boundary is

$$\{\bar{x},\,\bar{y}\} = \Big\{-\frac{1}{2\,a^2\left(a^2-(x-x0)^2\right)^2}\Big(-2\,a^6\,x + a\,b\,\sqrt{a^2-(x-x0)^2}\,\sqrt{\left(\frac{1}{a^2}(1-2\,x)^2\left(a^2-(x-x0)^2\right)\left(a^4-a^2\,(x-x0)^2+b^2\,(x-x0)^2\right)\right)}\,(x-x0) + 4\,a^4\,x\,(x-x0)^2 - a^2\left(b^2\,(-1+2\,x)+2\,x\,(x-x0)^2\right)(x-x0)^2 + b^2\,(-1+2\,x)\,(x-x0)^4\Big),\ \Big(2\,a^2\,b\,\sqrt{(a+x-x0)\,(a-x+x0)} - a\sqrt{\left(\frac{1}{a^2}(1-2\,x)^2\left(a^4-a^2\,(x-x0)^2+b^2\,(x-x0)^2\right)(a+x-x0)\,(a-x+x0)\right)} + b\,(x-x0)\,\sqrt{(a+x-x0)\,(a-x+x0)}\,(-1+2\,x0)+2\,a^3\,y0-2\,a\,(x-x0)^2\,y0\Big)\Big/\left(2\,a\,(a+x-x0)\,(a-x+x0)\right)\Big\}\ \text{for}\ \left\{x,\,0,\,\frac{1}{2}\right\}.$$

The lower boundary of the region with the elliptic upper boundary where a = 1.95272`, b = 4.58588`, x0 = 0, $y0 = -\frac{b\sqrt{a^2-(0.5-x0)^2}}{a}$.

$$\{\overline{x}, \overline{y}\} = \{-\frac{1}{2\, a^2 \left(a^2 - (x - x0)^2\right)^2} \Big(-2\, a^6\, x + a\, b \sqrt{a^2 - (x - x0)^2}$$

$$\sqrt{\left(\frac{1}{a^2} (1 - 2\, x)^2 \left(a^2 - (x - x0)^2\right) \left(a^4 - a^2 (x - x0)^2 + b^2 (x - x0)^2\right)\right)} (x - x0) +$$

$$4\, a^4\, x\, (x - x0)^2 - a^2 \left(b^2 (-1 + 2\, x) + 2\, x\, (x - x0)^2\right) (x - x0)^2 +$$

$$b^2 (-1 + 2\, x)\, (x - x0)^4\Big),\ \Big(2\, a^2\, b \sqrt{(a + x - x0)\, (a - x + x0)} -$$

$$a \sqrt{\left(\frac{1}{a^2} (1 - 2\, x)^2 \left(a^4 - a^2 (x - x0)^2 + b^2 (x - x0)^2\right) (a + x - x0)\, (a - x + x0)\right)} +$$

$$b\, (x - x0) \sqrt{(a + x - x0)\, (a - x + x0)}\, (-1 + 2\, x0) + 2\, a^3\, y0 - 2\, a\, (x - x0)^2\, y0\Big) \Big/$$

$$\left(2\, a\, (a + x - x0)\, (a - x + x0)\right)\} \text{ for } \{x, 0, 0.459\}$$

and $y = -\frac{b \sqrt{a^2 - x^2 + 2\, x\, x0 - x0^2}}{a} - y0$ for $\{x, 0.41899697583328316`, \frac{1}{2}\}$.

2.3 The lower boundary of the region with the parabolic 1 upper boundary .

The upper boundary represented by the parabolic 1 arc $(x - x0)^2 = 4\, c\, (y - y0)$ where $0 \le x \le \frac{1}{2}$.

The locus of point $(\overline{x}, \overline{y})$ which is a part of the lower boundary is

$$\{\overline{x}, \overline{y}\} =$$

$$\{\frac{1}{8\, c^3} \left(8\, c^3\, x + \sqrt{(c - 2\, c\, x)^2 \left(4\, c^2 + (x - x0)^2\right)}\, (x - x0) + c\, (-1 + 2\, x)\, (x - x0)^2\right),\ \frac{1}{4\, c^2}$$

$$\left(-\sqrt{(c - 2\, c\, x)^2 \left(4\, c^2 + (x - x0)^2\right)} + c \left(x - x^2 + (-1 + x0)\, x0 + 4\, c\, y0\right)\right)\} \text{ for } \{x, 0, \frac{1}{2}\}$$

The lower boundary of the region with the parabolic 1 upper boundary where $x0 = -0.4`$, $c = -0.77$, $y0 = -\frac{(0.5 - x0)^2}{4\, c}$.

$$\{\overline{x}, \overline{y}\} =$$

$$\{\frac{1}{8\, c^3} \left(8\, c^3\, x + \sqrt{(c - 2\, c\, x)^2 \left(4\, c^2 + (x - x0)^2\right)}\, (x - x0) + c\, (-1 + 2\, x)\, (x - x0)^2\right),$$

$$\frac{1}{4\, c^2} \left(-\sqrt{(c - 2\, c\, x)^2 \left(4\, c^2 + (x - x0)^2\right)} + c \left(x - x^2 + (-1 + x0)\, x0 + 4\, c\, y0\right)\right)\} \text{ for } \{x, 0, 0.487\}$$

and $y = -\frac{x^2 - 2\, x\, x0 + x0^2 + 4\, c\, y0}{4\, c}$ for $\{x, 0.47404644105951255`, \frac{1}{2}\}$.

2.4 The lower boundary of the region with the parabolic 2 upper boundary .

The upper boundary represented by the parabolic 2 arc $(y - y0)^2 =$

$4c(x-x0)$ where $0 \le x \le \frac{1}{2}$.

The locus of point $(\bar{x}, \bar{y})$ which is a part of the lower boundary is

$$\{\bar{x}, \bar{y}\} = \{x + \frac{c(-1+2x)}{2(x-x0)} - \frac{\sqrt{c(x-x0)}\sqrt{(1-2x)^2(x-x0)(c+x-x0)}}{2(x-x0)^2},$$
$$\frac{1}{2(x-x0)}\left(\sqrt{c(x-x0)} + \sqrt{(-1+2x)^2(x-x0)(c+x-x0)} + 2x\left(\sqrt{c(x-x0)}+y0\right) - 2x0\left(2\sqrt{c(x-x0)}+y0\right)\right)\} \text{ for } \{x, 0, \frac{1}{2}\}.$$

The lower boundary of the region with the parabolic 2 upper boundary where x0 = 1, c = -0.17, $y0 = -\sqrt{2c - 4c\,x0}$.

$$\{\bar{x}, \bar{y}\} = \{x + \frac{c(-1+2x)}{2(x-x0)} - \frac{\sqrt{c(x-x0)}\sqrt{(1-2x)^2(x-x0)(c+x-x0)}}{2(x-x0)^2},$$
$$\frac{1}{2(x-x0)}\left(\sqrt{c(x-x0)} + \sqrt{(-1+2x)^2(x-x0)(c+x-x0)} + 2x\left(\sqrt{c(x-x0)}+y0\right) - 2x0\left(2\sqrt{c(x-x0)}+y0\right)\right)\} \text{ for } \{x, 0, \frac{1}{2}\}.$$

2.5 The lower boundary of the region with the hyperbolic 2 upper boundary .

The upper boundary represented by the hyperbolic 2 arc $(x-x0)^2 = 4c(y-y0)$ where $0 \le x \le \frac{1}{2}$.

The locus of point $(\bar{x}, \bar{y})$ which is a part of the lower boundary is

$$\{\bar{x}, \bar{y}\} = \{\left(2a^6x + 4a^4x(x-x0)^2 + a^2\left(b^2(-1+2x) + 2x(x-x0)^2\right)(x-x0)^2 + b^2(-1+2x)(x-x0)^4 + ab\sqrt{a^2+(x-x0)^2}\sqrt{\left(\frac{1}{a^2}(1-2x)^2\left(a^2+(x-x0)^2\right)\left(a^4+a^2(x-x0)^2+b^2(x-x0)^2\right)\right)}(-x+x0)\right)\Big/$$
$$\left(2a^2\left(a^2+(x-x0)^2\right)^2\right), \left(-2a^2b\sqrt{a^2+(x-x0)^2} - a\sqrt{\left(\frac{1}{a^2}(1-2x)^2\left(a^2+(x-x0)^2\right)\left(a^4+a^2(x-x0)^2+b^2(x-x0)^2\right)\right)} + b\sqrt{a^2+(x-x0)^2}(x-x0)(-1+2x0) + 2a^3y0 + 2a(x-x0)^2y0\right)\Big/$$
$$\left(2a\left(a^2+(x-x0)^2\right)\right)\} \text{ for } \{x, 0, \frac{1}{2}\}$$

The lower boundary of the region with the hyperbolic 2 upper boundary where x0 = 0, a = 5.35, b = 35.4`, $y0 = \frac{b\sqrt{1+4a^2-4x0+4x0^2}}{2a}$.

$$\{\bar{x}, \bar{y}\} = \Big\{\Big(2 a^6 x + 4 a^4 x (x - x0)^2 + a^2 \big(b^2 (-1 + 2 x) + 2 x (x - x0)^2\big) (x - x0)^2 + b^2 (-1 + 2 x) (x - x0)^4 + a b \sqrt{a^2 + (x - x0)^2} \sqrt{\Big(\frac{1}{a^2}(1 - 2 x)^2 \big(a^2 + (x - x0)^2\big) \big(a^4 + a^2 (x - x0)^2 + b^2 (x - x0)^2\big)\Big)} (-x + x0)\Big) \Big/ \Big(2 a^2 \big(a^2 + (x - x0)^2\big)^2\Big), \Big(-2 a^2 b \sqrt{a^2 + (x - x0)^2} - a \sqrt{\Big(\frac{1}{a^2}(1 - 2 x)^2 \big(a^2 + (x - x0)^2\big) \big(a^4 + a^2 (x - x0)^2 + b^2 (x - x0)^2\big)\Big)} + b \sqrt{a^2 + (x - x0)^2} (x - x0) (-1 + 2 x0) + 2 a^3 y0 + 2 a (x - x0)^2 y0\Big) \Big/ \Big(2 a \big(a^2 + (x - x0)^2\big)\Big)\Big\} \text{ for } \{x, 0, 0.462\}$$

and $y = -\frac{x^2 - 2 x x0 + x0^2 + 4 c y0}{4 c}$ for $\{x, 0.42479278055926334`, \frac{1}{2}\}$.

2.6 The lower boundary of the region with the cosine upper boundary .

The upper boundary represented by a graph of cosine a Cos [b x] + k where $0 \le x \le \frac{1}{2}$.

The locus of point $(\bar{x}, \bar{y})$ which is a part of the lower boundary is

$$\{\bar{x}, \bar{y}\} = \Big\{\frac{1}{2}\Big(2 x - a b \Big(a (b - 2 b x) + \sqrt{(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)}\Big) Sin[b x]^2\Big), k + a Cos[b x] - \frac{1}{2}\Big(a b - 2 a b x + \sqrt{(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)}\Big) Sin[b x]\Big\} \text{ for } \{x, 0, \frac{1}{2}\}$$

The lower boundary of the region with a graph of cosine upper boundary where a = 25, b = 0.22144, k = - a $Cos[b \frac{1}{2}]$.

$$\{\bar{x}, \bar{y}\} = \Big\{\frac{1}{2}\Big(2 x - a b \Big(a (b - 2 b x) + \sqrt{(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)}\Big) Sin[b x]^2\Big), k + a Cos[b x] - \frac{1}{2}\Big(a b - 2 a b x + \sqrt{(1 - 2 x)^2 (a^2 b^2 + Csc[b x]^2)}\Big) Sin[b x]\Big\} \text{ for } \{x, 0, 0.465\}$$

and y = -k - a Cos[b * x] for $\{x, 0.43075223880624175`, \frac{1}{2}\}$.

2.7 The lower boundary of the region with the snipping off of the parabolic 1 upper boundary .

The upper boundary represented by the snipping off of the parabolic 1 .

$(x)^2 + (y + 0.3127336317241687`)^2 = \left(\frac{1}{2}\right)^2$ where $0 \le x \le 0.1748013092940937`$ and $(x - x0)^2 = 4 c (y - y0)$ where $0 \le 0.1748013092940937` \le \frac{1}{2}$

The lower boundary of the region with the parabolic 1 upper boundary where x0 = -0.4`, c = -0.77, $y0 = -\frac{(0.5 - x0)^2}{4 c}$.

$$\{\overline{x}, \overline{y}\} = \left\{\frac{1}{8 c^3}\left(8 c^3 x + \sqrt{(c - 2 c x)^2 \left(4 c^2 + (x - x0)^2\right)} (x - x0) + c (-1 + 2 x) (x - x0)^2\right), \frac{1}{4 c^2}\left(-\sqrt{(c - 2 c x)^2 \left(4 c^2 + (x - x0)^2\right)} + c \left(x - x^2 + (-1 + x0) x0 + 4 c y0\right)\right)\right\} \text{ for } \{x, 0, 0.487\}$$

and $y = -\frac{x^2 - 2 x x0 + x0^2 + 4 c y0}{4 c}$ for $\{x, 0.47404644105951255`, \frac{1}{2}\}$.

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