CHAPTER I

PRELIMINARIES

Let S be a semigroup.

For $T \subsetneq S$, T is a <u>subsemigroup</u> of S if T forms a semigroup under the same operation on S. A subsemigroup G of S is a <u>subgroup</u> of S if G is also a group. A subsemigroup T of S is called a <u>filter</u> of S if for any $a, b \in S$, $ab \in T$ implies $a, b \in T$.

An element a of S is called an <u>idempotent</u> of S if $a^2 = a$. Let $E(S)$ denote the set of all idempotents of S, that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

For $e \in E(S)$, eSe is clearly a subsemigroup of S where $eSe = \{exe \mid x \in S\}$.

A semigroup S is a <u>band</u> if $a^2 = a$ for all a in S ; or equivalently, $E(S) = S$. A commutative band is a <u>semilattice</u>.

An element z of a semigroup S is called a <u>left</u> [right] <u>zero</u> of S if $zx = z$ [$xz = z$] for every $x \in S$. An element of a semigroup S is called a <u>zero</u> of S if it is both a left and a right zero of S. An element e of S is called a <u>left</u> [right] <u>identity</u> of S if $ex = x$ [$xe = x$] for all $x \in S$. An element of a semigroup S is called an <u>identity</u> of S if it is both a left and a right identity of S. A semigroup can have at most one zero and at most one identity. The zero and the identity of a semigroup, if exist, are usually denoted by 0 and 1, respectively.

A <u>left zero semigroup</u> is a semigroup S in which xy = x for all x, y ε S. A <u>right zero semigroup</u> is defined dually.

If S is a semigroup with zero 0 and $S \setminus \{0\}$ is a subgroup of S, then S is called a <u>group with zero</u>.

Let S be a semigroup with identity 1. An element a of S is called a <u>unit</u> of S if there exists $a' \in S$ such that $aa' = a'a = 1$. Let G be the set of all units of S, that is,

$$G = \{a \in S \mid aa' = a'a = 1 \text{ for some } a' \in S\}.$$

Then G is the greatest subgroup of S having 1 as its identity, and it is called the <u>group of units</u> or the <u>unit group</u> of the semigroup S.

Let S be a semigroup, and let 0 be a symbol not representing any element of S. Let the notation $S \cup 0$ denote the semigroup obtained by extending the binary operation on S to 0 by defining 00 = 0 and 0a = a0 = 0 for all a ε S, and then let the notation S^0 denote the following semigroup :

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero,} \\ S \cup 0 & \text{if } S \text{ has no zero.} \end{cases}$$

Similarly, let S be a semigroup and 1 a symbol not representing any element of S. Let the notation $S \cup 1$ denote the semigroup obtained by extending the binary operation on S to 1 by defining 11 = 1 and 1a = a = a1 for all a ε S, and let the notation S^1 denote the following semigroup :

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup 1 & \text{if } S \text{ has no identity.} \end{cases}$$

An element a of a semigroup S is <u>regular</u> if a = axa for some x ε S. A semigroup S is <u>regular</u> if every element of S is regular.

Let a be an element of a semigroup S. An element x of S is an <u>inverse</u> of a if a = axa and x = xax. Then a semigroup S is regular if and only if every element of S has an inverse. A semigroup S is an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a in S is denoted by a^{-1}. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute [2, Theorem 1.17]. Hence, if S is an inverse semigroup, then E(S) is a semilattice.

Let S be a semigroup and A a nonempty subset of S. Then A is called a <u>left</u> [right] ideal of S if $SA \subseteq A$ [$AS \subseteq A$]. We call A an <u>ideal</u> of S if A is both a left and a right ideal of S.

A semigroup S is a <u>left simple</u> [right simple, simple] if S is the only left ideal [right ideal, ideal] of S. Hence, a semigroup S is left simple [right simple, simple] if and only if Sa = S [aS = S, SaS = S] for all $a \in S$.

A semigroup S is <u>left cancellative</u> if for a, b, $x \in S$, xa = xb implies a = b. A <u>right cancellative semigroup</u> is defined dually. A <u>cancellative semigroup</u> is a semigroup which is both left and right cancellative.

A semigroup S is called a <u>right group</u> if it is right simple and left cancellative. A <u>left group</u> is defined dually.

Let S and T be semigroups and ψ a map from S into T. The map ψ is a <u>homomorphism</u> from S into T if $(ab)\psi = (a\psi)(b\psi)$ for all a, $b \in S$. A semigroup T is a <u>homomorphic image</u> of a semigroup S if there exists a homomorphism from S onto T. A homomorphism ψ from S into T is called an <u>epimorphism</u> if ψ is onto. A homomorphism ψ from S into T is an isomorphism if ψ is one-to-one. If there is an isomorphism from S onto T, we say that the semigroup S and T are isomorphic, and we write $S \cong T$.

A homomorphic image of a group is also a group. If a semigroup T is a homomorphic image of an inverse semigroup S by a homomorphism ψ, then T is an inverse semigroup and for each $f \in E(T)$, there is an element $e \in E(S)$ such that $e\psi = f$ [2, Lemma 7.34] and for any $a \in S$, $(a\psi)^{-1} = a^{-1}\psi$ [2, Theorem 7.36], and hence

$$E(T) = \{e\psi \mid e \in E(S)\}.$$

Let S be a semigroup. A relation ρ on S is called left [right] compatible if for a, b, c $\in$ S, $a\rho b$ implies $ca\rho cb$ [$ac\rho bc$]. An equivalence relation ρ on S is called a congruence on S if it is both left and right compatible

Let ρ be a congruence on a semigroup S. Then the set

$$S/\rho = \{a\rho \mid a \in S\}$$

of all ρ - classes of S with the operation defined by

$$(a\rho)(b\rho) = (ab)\rho$$

(a, b $\in$ S) is a semigroup and is called the quotient semigroup relative to the congruence ρ , and the map $\rho^{\natural} : S \rightarrow S/\rho$ defined by $a\rho^{\natural} = a\rho$ is an onto homomorphism and it is called the natural homomorphism of S onto S/ρ.

Let I be an ideal of a semigroup S. Then the relation ρ_I on S defined by

$$x\rho_I y \iff x, y \in I \text{ or } x = y,$$

is a congruence on S and

$$x\rho_I = \begin{cases} I & \text{if } x \in I, \\ \{x\} & \text{if } x \notin I. \end{cases}$$

The congruence ρ_I is called the Rees congruence on S induced by I or the Rees congruence on S modulo I, and S/ρ_I is called the Rees quotient semigroup relative to I which is denoted by S/I.

Let S be a semigroup. Define the relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on S as follow :

$$a\mathcal{L}b \iff S^1a = S^1b,$$
$$a\mathcal{R}b \iff aS^1 = bS^1$$

and

$$\mathcal{H} = \mathcal{L}\cap\mathcal{R}.$$

The relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are called Green's relations on S and they are equivalence relations on S. Moreover, $\mathcal{L}$ is right compatible, $\mathcal{R}$ is left compatible, $\mathcal{H}\subseteq\mathcal{L}$ and $\mathcal{H}\subseteq\mathcal{R}$. Equivalent definitions of the Green's relations $\mathcal{L}$, $\mathcal{R}$ on S are given as follow :

$$a\mathcal{L}b \iff a = xb,\ b = ya \text{ for some } x, y \in S^1$$
$$a\mathcal{R}b \iff a = bx,\ b = ay \text{ for some } x, y \in S^1.$$

For $a \in S$, let L_a, R_a and H_a denote the $\mathcal{L}$ - class of S containing a, the $\mathcal{R}$ - class of S containing a and the $\mathcal{H}$ - class of S containing a, respectively.

In any semigroup S, any $\mathcal{H}$ - class of S contains at most one idempotent [2, Lemma 2.15], any $\mathcal{H}$ - class of S containing an idempotent e of S is a subgroup of S [2, Theorem 2.16] and it is the greatest subgroup of S having e as its identity. Hence, if S is a semigroup with zero 0, then $H_o = \{0\}$ and if S is a semigroup with identity 1, then H_1 is the unit group of S.

If S is an inverse semigroup, then each $\mathcal{L}$ - class and each $\mathcal{R}$ - class of S contains exactly one idempotent [2, Corollary 2.19].

Let S be a semilattice. Then the relation $\leq$ defined on S by

$$a \leq b \iff a = ab \ (= ba)$$

is a partial order on S.

Let Y be a semilattice and let a semigroup $S = \bigcup_{\alpha \in Y} G_\alpha$ be a disjoint union of subgroups G_α of S. The semigroup S is said to be a <u>semilattice Y of groups</u> G_α if $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

Let S be a semilattice of groups. Then S is an inverse semigroup [2, Corollary 7.53], $ea = ae$ for all $e \in E(S)$, $a \in S$, and $S = \bigcup_{e \in E(S)} H_e$ which is a semilattice E(S) of groups H_e.

Let X be a set. A <u>partial transformation</u> of X is a map from a subset of X into (a subset of) X. The <u>empty transformation</u> of X is the partial transformation of X with empty domain and it is denoted by 0. For a partial transformation α of X, the domain and the range of α are denoted by $\Delta\alpha$ and $\nabla\alpha$, respectively. Let T_X be the set of all partial transformations of X (including 0). For $\alpha, \beta \in T_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \phi$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \phi$, let

$$\alpha\beta = (\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}})(\beta|_{\nabla\alpha \cap \Delta\beta})$$

(the composite map) where $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$ denote the restrictions of α and β to $(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$ and $\nabla\alpha \cap \Delta\beta$, respectively. Then T_X is a semigroup with zero 0 and identity 1 where 1 is the identity map on X and it is called the <u>partial transformation semigroup</u> on the set X. Observe that for $\alpha, \beta \in T_X$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$ and $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$.

For any set X, T_X is a regular semigroup and

$$E(T_X) = \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\alpha \text{ and } x\alpha = x \text{ for all } x \in \nabla\alpha\}.$$

Let I_X denote the set of all 1-1 partial transformations of X, that is,

$$I_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one}\}.$$

Then I_X is an inverse subsemigroup of T_X with identity 1 and zero 0, and it is called the <u>1-1 partial transformation semigroup</u> or the <u>symmetric inverse semigroup</u> on the set X. By a <u>transformation</u> of a set X we mean a map of X into itself. Then an element $\alpha \in T_X$ is a transformation of X if and only if $\Delta\alpha = X$. Let $\mathcal{T}_X$ denote the set of all transformations of X, that is,

$$\mathcal{T}_X = \{\alpha \in T_X \mid \Delta\alpha = X\}.$$

Then $\mathcal{T}_X$ is a regular subsemigroup of T_X with identity 1 and it is called the <u>full transformation semigroup</u> on the set X. The permutation group on X is denoted by G_X. Then

$$G_X = \{\alpha \in T_X \mid \Delta\alpha = \nabla\alpha = X \text{ and } \alpha \text{ is one-to-one}\}.$$

Observe that $G_X \subsetneq I_X \subsetneq T_X$ and $G_X \subsetneq \mathcal{T}_X \subsetneq T_X$.

For a partial transformation α of X, let π_α be the relation on $\Delta\alpha$ defined by

$$x\pi_\alpha y \iff x\alpha = y\alpha.$$

which is called the <u>partition</u> of α. Then the partition of any partial transformation α of X is an equivalence relation on $\Delta\alpha$. For any $\alpha \in T_X$, we have that the $\mathcal{H}$ - class of T_X containing α is

$$H_\alpha = \{\beta \in T_X \mid \Delta\beta = \Delta\alpha, \nabla\beta = \nabla\alpha \text{ and } \pi_\beta = \pi_\alpha\}.$$

The <u>shift</u> of a partial transformation α of X, $S(\alpha)$, is defined to be the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$. A partial transformation α of X is said to be <u>almost identical</u> if the shift of α is finite, that is, $|S(\alpha)| < \infty$. Let

$$U_X = \{\alpha \in T_X \mid \alpha \text{ is almost identical}\},$$

$$V_X = \{\alpha \in \mathcal{J}_X \mid \alpha \text{ is almost identical}\}$$

and

$$W_X = \{\alpha \in I_X \mid \alpha \text{ is almost identical}\}.$$

If $\alpha, \beta \in T_X$, then $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$. Hence U_X, V_X and W_X are subsemigroups of T_X, $\mathcal{J}_X$ and I_X, respectively.

By the local subsemigroups of a semigroup S we mean the subsemigroups of S in the form eSe where e is an idempotent of S. For any adjective A describing a type of semigroups, we shall say that a semigroup S is a locally A semigroup, or locally A, if each local subsemigroup of S is an A semigroup (this follows McAlister [6]).

A semigroup S is said to be factorizable if there exists a subgroup G of S such that S = GE(S) where E(S) is the set of all idempotents of S. Then a semigroup S is locally factorizable if and only if each local subsemigroup of S is factorizable.