CHAPTER IV

THE MULTIPLICATIVE SEMIGROUP OF n×n MATRICES OVER A FIELD

In this chapter, we give a significant result of matrix semigroups. It is proved that for any positive integer n and for any field F, the multiplicative semigroup of n×n matrices over the field F is locally factorizable.

Throughout this chapter, the following notation are adopted :

Let F be a field and n a positive integer. The set of all n×n matrices over the field F is denoted by $M_n(F)$, and let O_n and I_n denote the n×n zero matrix and the n×n identity matrix over F. Then under the multiplication of matrices, $M_n(F)$ is a semigroup with zero O_n and identity I_n. For the remainder of this chapter, the notation $M_n(F)$ will denote such the semigroup.

For $k \in \{0, 1, 2, \ldots, n\}$, let

$$D_n^{(k)} = (d_{ij})$$

where

$$d_{ij} = \begin{cases} 1 & \text{if } i = j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

For instance,

$$D_3^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_5^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We remark that $D_n^{(o)} = O_n$ and $D_n^{(n)} = I_n$.

First, we shall show that the semigroup $M_n(F)$ is indeed (isomorphic to) a transformation semigroup on the set F^n where $F^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in F\}$. For $A \in M_n(F)$, define the map $\alpha_A : F^n \to F^n$ by $\bar{x}\alpha_A = \bar{x}A$ where $\bar{x} \in F^n$. Then $\alpha_A \in \mathfrak{J}_{F^n} \subseteq T_{F^n}$ for all $A \in M_n(F)$, and for $A, B \in M_n(F)$, $\alpha_A\alpha_B = \alpha_{AB}$. Hence the map $A \mapsto \alpha_A$ is a homomorphism from $M_n(F)$ into T_{F^n}. To show this map is one-to-one, let $A, B \in M_n(F)$ such that $\alpha_A = \alpha_B$. Then $\bar{x}A = \bar{x}B$ for $\bar{x} \in F^n$. Let $A = (a_{ij})$ and $B = (b_{ij})$. For $i \in \{1, 2, \ldots, n\}$, we have $(a_{i1}, a_{i2}, \ldots, a_{in}) = \bar{e}_iA = \bar{e}_iB = (b_{i1}, b_{i2}, \ldots, b_{in})$ where $\bar{e}_i \in F^n$ such that the i^{th} entry is 1 and other entries are all 0, so $a_{ij} = b_{ij}$ for all $j \in \{1, 2, \ldots, n\}$. Hence $A = B$.

In this chapter, we are only interested in square matrices, therefore we shall use matrices to mean matrices in $M_n(F)$. First we recall some preliminaries concerning matrices.

An elementary row operation on a matrix A is an operation of one of the following three types :

(1) a permutation of two rows,

(2) a multiplication of a row by a nonzero scalar,

(3) an addition of one row to another.

An <u>elementary matrix</u> is any matrix which can be obtained by performing a single elementary row operation on the identity matrix. Any elementary row operation can be performed on a matrix A by multiplying A on the left by the corresponding elementary matrix [3, Theorem 6.1]. Every elementary matrix is nonsingular [3, Theorem 6.2].

Two matrices A and B are said to be <u>row equivalent</u>, written as $A \underset{R}{\sim} B$, if B is obtainable from A by a finite sequence of elementary row operations, that is, $A \underset{R}{\sim} B$ if and only if $B = E_k E_{k-1} \ldots E_1 A$ for some elementary matrices $E_1, E_2, \ldots, E_k$. Because the inverse of an elementary matrix is the product of elementary matrices [3, Theorem 6.3], then the relation $\underset{R}{\sim}$ is an equivalent relation on $M_n(F)$.

A matrix A is in <u>row - reduced echelon form</u> if

(1) the first nonzero element in each row is 1,

(2) in any column containing the first nonzero element of some row, that element is the only nonzero element in that column,

(3) the zero rows of A (if any) come last,

(4) when the leading ones in the nonzero rows are connected by a broken line, that line slopes down and to the right.

An example of a matrix A in row - reduced echelon form is

$$\begin{bmatrix} 0 & 1 & 0 & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ is some scalar in F.

Every matrix is row equivalent to a matrix in row - reduced echelon form [3, Theorem 6.5].

Let a matrix A be in row - reduced echelon form. Suppose that $A \neq 0_n$ and the first row to the p^{th} row are all the nonzero rows of A. For each $i \in \{1,2, \ldots, p\}$, let the first nonzero element of the i^{th} row be in the c_i^{th} column. Then $i \leq c_i$ for all $i \in \{1, 2, \ldots, p\}$ and

$c_1 < c_2 < \ldots < c_p$. For each i in {1, 2, 3, ..., p}, if $i = c_i$, let $E_i = I_n$, and if $i < c_i$, let E_i be the elementary matrix obtained by interchanging the i^{th} row and the c_i^{th} row in I_n. Then we have that $E_1E_2\ldots E_pA$ is a matrix having the following properties :

(a) the first nonzero element in each row is 1, and lies on the main diagonal,

(b) in any column containing 1 in the main diagonal this element 1 is the only nonzero element in the column.

For convenience, we say a matrix satisfying the conditions (a) and (b) is in the <u>special form</u>. Hence, a matrix $A = (a_{ij})$ is in the special form if and only if

(a') for each i, either $a_{ii} = 0$ or $a_{ii} = 1$,

(b') $a_{ij} = 0$ if $i > j$,

(c') for each i, $a_{ii} = 1$ implies $a_{ki} = 0$ for all $k \neq i$,

(d') for each i, $a_{ii} = 0$ implies $a_{ik} = 0$ for all k.

An example of a matrix in the special form is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where * is some scalar in F.

4.1 <u>Lemma</u>. Let $A \in M_n(F)$ be in the special form. Then $A^2 = A$, that is, $A \in E(M_n(F))$.

<u>Proof</u> : Let $A = (a_{ij}) \in M_n(F)$ be in the special form. Let

$A^2 = (x_{ij})$. Then for $i, j \in \{1, 2, \ldots, n\}$, $x_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$. To show $x_{ij} = a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$, let $i, j \in \{1, 2, \ldots, n\}$ be arbitrary fixed.

Case $a_{ii} = 0$. Then $a_{ik} = 0$ for all $k \in \{1, 2, \ldots, n\}$, so $x_{ij} = 0 = a_{ij}$.

Case $a_{ii} = 1$. For $k \in \{1, 2, \ldots, n\}$, $k \neq i$, if $a_{ik} \neq 0$, then $a_{kk} = 0$ and so $a_{kj} = 0$. Hence $a_{ik}a_{kj} = 0$ for all $k \in \{1, 2, \ldots, n\}$, $k \neq i$. Thus $x_{ij} = a_{ii}a_{ij} = a_{ij}$. #

As mentioned before, every matrix in row - reduced echelon form is row equivalent to a matrix in the special form. Hence every matrix is row equivalent to a matrix in the special form.

4.2 Lemma. For any positive integer n and for any field F, the semigroup $M_n(F)$ is factorizable.

Proof : Let $A \in M_n(F)$. Then A is row equivalent to a matrix in the special form, say B. Therefore $A = E_kE_{k-1}\ldots E_1B$ for some elementary matrices $E_1, E_2, \ldots, E_k$. Since $E_kE_{k-1}\ldots E_1$ is nonsingular, $E_kE_{k-1}\ldots E_1 \in G$ where G is the multiplicative group of nonsingular matrices in $M_n(F)$. By Lemma 4.1, $B^2 = B$, so $B \in E(M_n(F))$. Then $A = (E_kE_{k-1}\ldots E_1)B \in GE(M_n(F))$. Hence $M_n(F) = GE(M_n(F))$, so $M_n(F)$ is factorizable. #

In matrix theory, we have that if A is an idempotent in $M_n(F)$, then $A = T^{-1}D_n^{(k)}T$ for some nonsingular matrix T in $M_n(F)$ for some $k \in \{0, 1, 2, \ldots, n\}$ [4, page 226], and observe that $A = 0_n$ if and only if $k = 0$. For any nonsingular matrix T in $M_n(F)$ and for any $k \in \{0,1,\ldots, n\}$, $T^{-1}D_n^{(k)}T$ is clearly an idempotent of $M_n(F)$. Hence

$E(M_n(F)) = \{T^{-1}D_n^{(k)}T \mid T \in M_n(F)$, T is nonsingular, $k \in \{0, 1, 2, \ldots, n\}\}$.

4.3 Lemma. If A is a nonzero idempotent of $M_n(F)$, then $AM_n(F)A$ is isomorphic to $M_k(F)$ for some $k \in \{1, 2, \ldots, n\}$.

Proof : Let $A \in E(M_n(F))$ and $A \neq 0_n$. Then $A = T^{-1}D_n^{(k)}T$ for some nonsingular matrix T in $M_n(F)$ and for some $k \in \{1, 2, \ldots, n\}$. Thus

$$\begin{aligned} AM_n(F)A &= (T^{-1}D_n^{(k)}T)M_n(F)(T^{-1}D_n^{(k)}T) \\ &= T^{-1}D_n^{(k)}(TM_n(F)T^{-1})D_n^{(k)}T \\ &= T^{-1}D_n^{(k)}M_n(F)D_n^{(k)}T \end{aligned}$$

Since $D_n^{(k)}$ is the matrix (d_{ij}) where $d_{ij} = \begin{cases} 1 & \text{if } i = j \leq k, \\ 0 & \text{otherwise,} \end{cases}$

by the multiplication of matrices, we have that for any matrix $B = (b_{ij})$ in $M_n(F)$, $D_n^{(k)}BD_n^{(k)}$ is the matrix (b'_{ij})
where

$$b'_{ij} = \begin{cases} b_{ij} & \text{if } i, j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $M_n^{(k)}(F)$ denote the set $\{B = (b_{ij}) \in M_n(F) \mid b_{ij} = 0$ if $i > k$ or $j > k\}$. Then $M_n^{(k)}(F)$ is a subsemigroup of $M_n(F)$ and obviousely $D_n^{(k)}M_n(F)D_n^{(k)} = M_n^{(k)}(F)$. Hence, $AM_n(F)A = T^{-1}M_n^{(k)}(F)T$ which is isomorphic to $M_n^{(k)}(F)$ by the map $B \mapsto T^{-1}BT$ $(B \in M_n^{(k)}(F))$. Clearly,

$M_k(F)$ is isomorphic to $M_n^{(k)}(F)$ by the map $X \mapsto \bar{X}$ where if $X = (x_{ij})$ ε $M_k(F)$, then $\bar{X} = (\bar{x}_{ij})$ ε $M_n(F)$ is defined by

$$\bar{x}_{ij} = \begin{cases} x_{ij} & \text{if } i, j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $AM_n(F)A$ is isomorphic to $M_k(F)$. #

We know that $0_n M_n(F) 0_n = \{0_n\}$ which is factorizable. Hence from Lemma 4.2 and Lemma 4.3, we obtain the following theorem.

4.4 Theorem. For any positive integer n and for any field F, the multiplicative semigroup of n×n matrices over F is locally factorizable.