CHAPTER II

THE SEQUENTIAL WIENER INTEGRAL

In the first part of this chapter, we study the definition of the sequential Wiener integral with complex parameters. And in the last part of this chapter, we consider the case in which the parameter is real and positive.

In chapter I, the Wiener measure with positive real parameters is defined, but it is not defined for non-real variance parameters. For defining the kernel by (2.1.2) below, we have that

$$(*)\qquad \int_{\mathbb{R}^n} |K_\sigma(\tau,\xi)|\, d\xi = \{|\sigma|^2 \operatorname{Re}(\sigma^{-2})\}^{-n/2}$$

and this clearly approaches ∞ with n if $\operatorname{Re}(\sigma) > 0$ and $\operatorname{Im}(\sigma) \neq 0$. For each Wiener measurable set E on $C[a,b]$, we define

$$W_\sigma(E) = \int_{E^*} K_\sigma(\tau,\xi)\, d\xi$$

where E^* is the restricting set of E. We claim that W_σ is not a complex measure on $C[a,b]$. Suppose on contrary that W_σ is a complex measure on $C[a,b]$. Therefore, the total variation of W_σ is finite, i.e.,

$$(**)\qquad |W_\sigma|(C[a,b]) < \infty .$$

For each $E \subseteq \mathbb{R}^n$, we define

$$W_n(E) = \int_E K_\sigma(\tau,\xi)d\xi ,$$

thus we have

$$|W_n|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |K_\sigma(\tau,\xi)|d\xi$$

$$(***) \qquad = \sup \{ \sum_{n=1}^{\infty} | \int_{E_n} K (\tau,\xi)d\xi | : \bigcup_{n=1}^{\infty} E_n = \mathbb{R}^n \}.$$

Since $|W_\sigma|(C[a,b]) = \sup\{ \sum_{n=1}^{\infty} | \int_{E_n^*} K_\sigma(\tau,\xi)d\xi| : \bigcup_{n=1}^{\infty} E_n = C[a,b]$

and E_n is Wiener measurable$\}$,

it follows from (***) that

$$|W_\sigma|(C[a,b]) \geq \int_{\mathbb{R}^n} |K_\sigma(\tau,\xi)|d\xi$$

for all $n \in \mathbb{N}$. Hence, we have by (*) that

$$|W_\sigma|(C[a,b]) = \infty$$

which contradicts (**). Thus W_σ is not a complex measure on $C[a,b]$, so that there is no integration theory in the usual measure theoretic sense is possible for non-real variances.

Next, we shall define the integral in the other sense for complex parameters.

Let $C[a,b]$ denote the space of real continuous functions on the interval $[a,b]$ which vanish at the initial point a, and let $C_0[a,b]$

denote the set of all polygonal functions contained in $C[a,b]$. Let σ be a fixed non-vanishing complex number such that $|\arg \sigma| \leq \pi/4$.

Let $\tau = [\tau_1,\ldots,\tau_n]$ denote a variable vector of a variable number of dimensions whose components form a subdivision of $[a,b]$, so that $a < \tau_1 < \tau_2 < \ldots < \tau_n \leq b$; and let $\tau_0 = a$. We call τ a <u>subdivision vector</u> and we define the norms of subdivisions by

$$\|\tau\| = \max_{j=1,\ldots,n} (\tau_j - \tau_{j-1}).$$

Let $\xi = [\xi_1,\ldots,\xi_n]$ denote an unrestricted real vector, where n is determined by τ, and let $\xi_0 = 0$. Let $\psi_{\tau,\xi}(t)$ denote an element of $C_0[a,b]$ whose vertices have the abscissas τ_i and ordinates ξ_i, so that we have

$$\psi_{\tau,\xi}(\tau_i) = \xi_i \qquad i = 0,1,\ldots,n$$

and $\psi_{\tau,\xi}(t)$ is linear on $[\tau_{i-1},\tau_i]$ for $i = 1,\ldots,n$. Finally, let $\mathbb{R}^n$ denote an n-dimensional Euclidean space and $d\xi$ the element of volume in $\mathbb{R}^n$, i.e., $d\xi = d\xi_1 \ldots d\xi_n$.

Using the above terminology, we now define the <u>sequential Wiener integral with parameter σ</u> (or <u>sw_σ integral</u>) for functionals $F(x)$ for which the definition has meaning.

<u>Definition 2.1</u> If the limit exists, we define

$$(2.1.1) \qquad \int_{C[a,b]}^{sw_\sigma} F(x)dx = \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} K_\sigma(\tau,\xi)F(\psi_{\tau\xi})d\xi ,$$

where

$$(2.1.2)\quad K_\sigma(\tau,\xi) = \frac{1}{[\pi^n(2\sigma^2)^n(\tau_1-\tau_0)\dots(\tau_n-\tau_{n-1})]^{1/2}} \exp\left[-\sum_{i=1}^{n}\frac{(\xi_i-\xi_{i-1})^2}{2\sigma^2(\tau_i-\tau_{i-1})}\right].$$

The parameter σ will sometimes be called the <u>standard deviation parameter</u>, and σ^2 will be called the <u>variance parameter</u>.

In particular, the existence of the sw_σ integral of F implies that F is defined over all elements of $C_0[a,b]$ or at least over almost all of them in the sense that for each τ, $F(\psi_{\tau,\xi})$ is defined for almost all ξ in $\mathbb{R}^n$. It of course also implies that $K_\sigma(\tau,\xi)F(\psi_{\tau,\xi})$ is integrable in ξ over $\mathbb{R}^n$ for each τ.

<u>Example</u>. Let $F(x) = x(\tau_1)$. Then we have from (2.1.2) that

$$K_\sigma(\tau,\xi) = \frac{1}{[\pi(2\sigma^2)(\tau_1-\tau_0)]^{1/2}} \exp\left[\frac{-\xi_1^2}{2\sigma^2(\tau_1-\tau_0)}\right]$$

and $F(\psi_{\tau,\xi}) = \psi_{\tau,\xi}(\tau_1) = \xi_1$. Thus according to (2.1.1) we get

$$\int_{C[a,b]}^{sw_\sigma} F(x)dx = \lim_{\|\tau\|\to 0} \int_{\mathbb{R}} K_\sigma(\tau,\xi)F(\psi_{\tau,\xi})d\xi$$

$$= \lim_{\|\tau\|\to 0} \int_{-\infty}^{\infty} \frac{1}{[\pi(2\sigma^2)(\tau_1-\tau_0)]^{1/2}}\, \xi_1 \exp\left[\frac{-\xi_1^2}{2\sigma^2(\tau_1-\tau_0)}\right] d\xi_1 .$$

Put $\eta_1 = \dfrac{\xi_1}{\sigma\sqrt{2(\tau_1-\tau_0)}}$, so that $\xi_1 = \sigma\sqrt{2(\tau_1-\tau_0)}\,\eta_1$ and

$d\xi_1 = \sigma\sqrt{2(\tau_1-\tau_0)}\, d\eta_1$, hence

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi(\tau_1-\tau_0)}}\, \xi_1 \exp\left[\frac{-\xi_1^2}{2\sigma^2(\tau_1-\tau_0)}\right] d\xi_1$$

$$= \frac{\sigma}{\sqrt{\pi}} \sqrt{2(\tau_1-\tau_0)} \int_{-\infty}^{\infty} \eta_1 \exp(-\eta_1^2)\, d\eta_1$$

$$= 0 ,$$

since $\int_{-\infty}^{\infty} \eta_1 \exp(-\eta_1^2) d\eta_1 = 0$. Therefore,

$$\int_{C[a,b]}^{sw_\sigma} F(x)dx = 0 .$$

Definition 2.2 If σ^2 is pure imaginary, the integral (2.1.1) will be called a Feynman integral, and we use the term f_p integral to denote the $sw_{p\sqrt{i}}$ integral, where p and Re $\sqrt{i}$ are positive. Thus we define

(2.2.1) $$\int_{C[a,b]}^{f_p} F(x)dx = \int_{C[a,b]}^{sw_{p\sqrt{i}}} F(x)dx.$$

Remark 2.3

(i) When the subscript σ or p is unity, it will usually be dropped from the symbol sw_σ or f_p.

(ii) For each positive number p,

(2.3.1) $$K_{p\sigma}(\tau,\xi) = p^{-n} K_\sigma(\tau,p^{-1}\xi).$$

It follows from (2.1.2), since

$$K_{p\sigma}(\tau,\xi) = \frac{1}{[\pi^n(2p^2\sigma^2)^n(\tau_1-\tau_0)\ldots(\tau_n-\tau_{n-1})]^{1/2}} \exp\left[-\sum_{i=1}^{n} \frac{(\xi_i-\xi_{i-1})^2}{2p^2\sigma^2(\tau_i-\tau_{i-1})}\right]$$

$$= p^{-n} \frac{1}{[\pi^n(2\sigma^2)^n(\tau_1-\tau_0)\ldots(\tau_n-\tau_{n-1})]^{1/2}} \exp\left[-\sum_{i=1}^{n} \frac{(p^{-1}\xi_i-p^{-1}\xi_{i-1})^2}{2\sigma^2(\tau_i-\tau_{i-1})}\right]$$

$$= p^{-n}K_\sigma(\tau,p^{-1}\xi).$$

(iii) For each positive number p,

$$(2.3.2) \qquad \int_{C[a,b]}^{sw_{p\sigma}} F(x)dx = \int_{C[a,b]}^{sw_\sigma} F(px)dx$$

where the existence of either member implies that of the other. In particular, from (2.2.1) we have in the same sense

$$(2.3.3) \qquad \int_{C[a,b]}^{f_p} F(x)dx = \int_{C[a,b]}^{f} F(px)dx.$$

(2.3.2) follows from (2.1.1) and (2.3.1), since

$$\int_{C[a,b]}^{sw_{p\sigma}} F(x)dx = \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} K_{p\sigma}(\tau,\xi)F(\psi_{\tau,\xi})d\xi$$

$$= \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} p^{-n}K_\sigma(\tau,p^{-1}\xi)F(\psi_{\tau,\xi})d\xi$$

$$= \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} p^{-n}K_\sigma(\tau,\eta)F(\psi_{\tau,p\eta})p^n d\eta, \qquad (\eta = p^{-1}\xi)$$

$$= \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} K_\sigma(\tau,\eta)F(p\psi_{\tau,\eta})d\eta, \qquad (\psi_{\tau,p\eta} = p\psi_{\tau,\eta})$$

$$= \int_{C[a,b]}^{sw_\sigma} F(px)dx.$$

The real sequential Wiener integral. We now consider the case in which σ is real and positive, and we give a set of sufficient conditions for the existence of the limit in the right member of (2.1.1).

Definition 2.4. We say that a functional $F(x)$ is continuous in the uniform topology on $C[a,b]$ if for each $x \in C[a,b]$ and $\varepsilon > 0$, there is $\delta > 0$ such that for all $y \in C[a,b]$, $|F(x)-F(y)| < \varepsilon$ whenever $\|x-y\| < \delta$.

Lemma 2.5 Let $F(x)$ be a Borel functional defined on $C[a,b]$. Then for each $\psi_{\tau,\xi}(t) \in C_0[a,b]$, $H(\xi) = F(\psi_{\tau,\xi})$ is measurable in ξ over $\mathbb{R}^n$.

Proof : Case 1. If F is the characteristic functional χ_I of a Borel subset I of $C[a,b]$, let E be a rectangle in $\mathbb{R}^n$ defined by

$$E = \{(x(\tau_1),\ldots, x(\tau_n)) : x \in I\}.$$

Thus for each $\psi_{\tau,\xi}(t) \in C_0[a,b]$,

$$\begin{aligned} H(\xi) &= \chi_I(\psi_{\tau,\xi}) \\ &= \begin{cases} 1 & \text{if } \psi_{\tau,\xi} \in I \\ 0 & \text{if } \psi_{\tau,\xi} \notin I \end{cases} \\ &= \begin{cases} 1 & \text{if } (\psi_{\tau,\xi}(\tau_1),\ldots,\psi_{\tau,\xi}(\tau_n)) \in E \\ 0 & \text{if } (\psi_{\tau,\xi}(\tau_1),\ldots,\psi_{\tau,\xi}(\tau_n)) \notin E \end{cases} \end{aligned}$$

$$= \begin{cases} 1 & \text{if } (\xi_1,\ldots,\xi_n) \in E \\ 0 & \text{if } (\xi_1,\ldots,\xi_n) \notin E \end{cases}$$

$$= \quad \chi_E(\xi) \quad \text{is measurable in } \xi.$$

<u>Case 2</u>. If F is a simple functional; i.e., if F is any finite linear combination of characteristic functionals of disjoint Borel subsets of $C[a,b]$, then by case 1, $H(\xi)$ is measurable in ξ over $\mathbb{R}^n$.

<u>Case 3</u>. If F is an extended non-negative functional on $C[a,b]$, then there exists a non-decreasing sequence of non-negative simple functionals F_k which converges to F at each point of $C[a,b]$. Hence by case 2, the corresponding sequence of non-negative non-decreasing simple functions H_k converges to H at each point of $\mathbb{R}^n$, so that H is measurable in ξ over $\mathbb{R}^n$.

<u>Case 4</u>. Finally, if F is a Borel functional on $C[a,b]$, then $F = F^+ - F^-$, so that we have by case 3 that the corresponding functions H^+ and H^- are measurable in ξ over $\mathbb{R}^n$, and hence $H = H^+ - H^-$ is measurable in ξ over $\mathbb{R}^n$.

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<u>Corollary 2.6</u> Under the same hypothesis of Lemma 2.5, we have

$$\int_{\mathbb{R}^n} F(\psi_{\tau,\xi})K(\tau,\xi)d\xi = \int_{C[a,b]} F(x_\tau)dW(x), \tag{2.6.1}$$

where x_τ denotes the polygonal function which has the same value as x at τ_i for $i = 0,1,\ldots,n$ and is linear in between

$$x_\tau(\tau_i) = x(\tau_i) \qquad i = 0,1,\ldots, n$$

$$(\tau_i-\tau_{i-1})x_\tau(t) = (\tau_i-t)x(\tau_{i-1})+(t-\tau_{i-1})x(\tau_i)$$

$$\tau_{i-1} \le t \le \tau_i \,.$$

Proof: By Lemma 2.5, $F(\psi_{\tau,\xi}) = H(\xi)$ is measurable in ξ, hence it follows from Theorem 1 (in chapter I) that the functional $F(x_\tau) = F(x(\tau_1),\ldots,x(\tau_n))$ defined on $C[a,b]$ is Wiener measurable and (2.6.1) holds. #

Theorem 2.7 Let F(x) be Borel measurable over $C[a,b]$ and continuous in the uniform topology almost everywhere (in the Wiener sense) on the space $C[a,b]$. If F is bounded on $C[a,b]$, then the real sequential Wiener integral of F exists for $\sigma = 1$ and equals the Wiener integral

$$\text{(2.7.1)} \qquad \overset{sw}{\int_{C[a,b]}} F(x)dx = \int_{C[a,b]} F(x)dw(x).$$

Proof: It readily follows from corollary 2.6 that

$$\int_{R^n} K(\tau,\xi)F(\psi_{\tau,\xi})d\xi = \int_{C[a,b]} F(x_\tau)dW(x)$$

where x_τ denotes the polygonal function which has the same value as x at τ_i for $i = 0,1,\ldots,n$, and is linear in between

$$x_\tau(\tau_i) = x(\tau_i) \qquad i = 0,1,\ldots, n$$

$$(\tau_i-\tau_{i-1})x_\tau(t) = (\tau_i-t)x(\tau_{i-1})+(t-\tau_{i-1})x(\tau_i)$$

$$\tau_{i-1} \le t \le \tau_i \,.$$

By the continuity of F and of x, we have for almost all x in C[a,b],

$$\lim_{\|\tau\|\to 0} F(x_\tau) = F(x).$$

Since F(x) is uniformly bounded and measurable on C[a,b] and C[a,b] has a finite Wiener measure, by the Dominated Convergence Theorem

$$\begin{aligned}
\int_{C[a,b]}^{sw} F(x)dx &= \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} K(\tau,\xi)F(\psi_{\tau,\xi})\, d\xi \\
&= \lim_{\|\tau\|\to 0} \int_{C[a,b]} F(x_\tau)dW(x) \\
&= \int_{C[a,b]} (\lim_{\|\tau\|\to 0} F(x_\tau))\, dW(x) \\
&= \int_{C[a,b]} F(x)dW(x)
\end{aligned}$$

Therefore, the theorem is proved. #