CHAPTER I
INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Over the past fifty years, there have been numerous works on continued fraction expansions. In 2007, A. H. Fan, B. W. Wang and J. Wu [10] introduced a class of continued fraction expansions in the field of real numbers, called Oppenheim continued fraction expansions. In the field of p-adic numbers, there are two well-known continued fraction expansions, due respectively to Ruban [23] and Schneider [25] since 1970. In the field of formal Laurent series in x^{-1}, a Ruban continued fraction expansion was studied by T. Chaichana, V. Laohakosol and A. Harnchoowong [7] in 2006. In Chapter II, we devise an algorithm for constructing continued fraction expansions of elements in a discrete non-archimedean valued field. This algorithm embraces almost all known continued fraction expansions as special cases.

In 1987, V. Laohakosol and P. Ubolsri [15] derived some criteria for algebraic independence of elements in the field of p-adic numbers. Similar criteria in the field of formal Laurent series in x^{-1} were established by T. Chaichana and V. Laohakosol, [6], in 2007. In Chapter III, analogous criteria are derived in the function field with respect to a prime-adic valuation and we use them to obtain sufficient conditions for algebraic and linear independence of elements represented by continued fraction expansions.

In $\mathbb{F}_q((x^{-1}))$, the field of formal Laurent series in x^{-1} over a finite base field of q elements, where q is a prime power, let $[i] := x^{q^i} - x,\ d_0 := 1,\ d_i := [i]d_{i-1}^q\ (i \geq 1)$, and let

$$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i},$$

the exponential element for $\mathbb{F}_q[x]$. For brevity, put $e := e(1)$. In 1992, D. Thakur [29] showed that $e(z)$ has a continued fraction expansion with an interesting pattern. In Chapter IV, we generalize the result of Thakur by giving the explicit Ruban continued fraction expansions for elements in $\mathbb{F}_q((x^{-1}))$ of the form $\frac{e}{f(x)^m}$, where $m \in \mathbb{N}$ and $f(x)$ is a nonconstant monic polynomial over a finite field $\mathbb{F}_q$ satisfying $f(x) \mid [1]$.

To prove main results in Chapter IV, one needs the Folding Lemma. In the final chapter, Chapter V, a generalized Folding Lemma and a generalized 3-tier Folding Lemma are given and some examples are obtained by applying these lemmas.

1.2 Preliminaries

In this section, we collect basic definitions and results, given mainly without proofs, and give brief background materials needed. The first subsection deals with valuation and related concepts. Details and proofs can be found in McCarthy [17] and Bachman [2]. A principal result is Theorem 1.14, which shows how to represent elements in complete discrete non-archimedean valued fields of formal Laurent series. The second subsection deals with continued fraction expansions and their notation.

1.2.1 Valuation

Definition 1.1. A *valuation* on a field K is a map $|\cdot| : K \to \mathbb{R}$ with the following properties:

(i) for all $\alpha \in K$, $|\alpha| \geq 0$ and $|\alpha| = 0$ if and only if $\alpha = 0$,

(ii) for all $\alpha, \beta \in K$, $|\alpha\beta| = |\alpha||\beta|$,

(iii) for all $\alpha, \beta \in K$, $|\alpha + \beta| \leq |\alpha| + |\beta|$.

There is always at least one valuation on K, namely, that given by setting $|\alpha| = 1$ if $\alpha \in K \setminus \{0\}$ and $|0| = 0$. This valuation is called the *trivial valuation* on K.

Definition 1.2. A valuation $|\cdot|$ on a field K is *non-archimedean* if the condition (iii) in Definition 1.1 is replaced by a stronger condition, called the *strong triangle inequality*

$$|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}, \quad \text{for all } \alpha, \beta \in K.$$

Any other valuation on K is called *archimedean*.

A *valued field* $(K, |\cdot|)$ is a field K equipped with a valuation $|\cdot|$ on K. If the valuation is non-archimedean, then K is called a *non-archimedean valued field*.

Examples of valuations are as follows:

Example 1.3.

(1) The usual absolute value $|\cdot|$ is an archimedean valuation on $\mathbb{Q}$.

(2) Let p be a prime number. By the Fundamental Theorem of Arithmetic, each $\alpha \in \mathbb{Q} \setminus \{0\}$ can be written uniquely in the form

$$\alpha = p^{\nu_p(\alpha)} \frac{a}{b}$$

where $\nu_p(\alpha) \in \mathbb{Z}$, $a, b \in \mathbb{Z}$ $(b > 0)$, $\gcd(a, b) = 1$ and $p \nmid ab$.

Define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ by

$$|\alpha|_p = \left(\tfrac{1}{p}\right)^{\nu_p(\alpha)} \text{ if } \alpha \neq 0 \text{ and } |0|_p = 0.$$

Then $|\cdot|_p$ is a non-archimedean valuation on $\mathbb{Q}$ and called the *p-adic valuation*.

(3) Consider the field $F(x)$ of rational functions over a field F. Let $\pi(x)$ be an irreducible polynomial in $F[x]$. Any $\alpha \in F(x) \setminus \{0\}$ can be written uniquely as

$$\alpha = \pi(x)^{\nu_\pi(\alpha)} \frac{a(x)}{b(x)}$$

where $\nu_\pi(\alpha) \in \mathbb{Z}$, $a(x)$ and $b(x)$ are relatively prime elements of $F[x]$, $b(x)$ is a nonzero monic polynomial and $\pi(x) \nmid a(x)b(x)$.

Define $|\cdot|_\pi : F(x) \to \mathbb{R}$ by

$$|\alpha|_\pi = 2^{-\nu_\pi(\alpha) \deg \pi}, \text{ if } \alpha \neq 0 \text{ and } |0|_\pi = 0.$$

Then $|\cdot|_\pi$ is a non-archimedean valuation on $F(x)$ and called the *π-adic valuation*.

(4) Define $|\cdot|_\infty$ on $F(x)$ by, for all $f(x), g(x) \in F[x] \setminus \{0\}$,

$$\left|\frac{f(x)}{g(x)}\right|_\infty = 2^{\deg f - \deg g} \text{ and } |0|_\infty = 0.$$

Then $|\cdot|_\infty$ is a non-archimedean valuation on $F(x)$ and called the *degree valuation.*

Theorem 1.4. *Let $(K, |\cdot|)$ be a non-archimedean valued field and $\alpha, \beta \in K$. If $|\alpha| \neq |\beta|$, then*

$$|\alpha + \beta| = \max\{|\alpha|, |\beta|\}.$$

From a non-archimedean valuation $|\cdot|$ on a field K, we define $\nu : K \longrightarrow \mathbb{R} \cup \{\infty\}$ by

$$\nu(\alpha) = -\log_2 |\alpha| \quad \text{if } \alpha \neq 0 \quad \text{and} \quad \nu(0) = \infty.$$

With the convention $\infty + a = \infty = a + \infty$ for all $a \in \mathbb{R} \cup \{\infty\}$ and $\infty > a$ for all $a \in \mathbb{R}$, the properties of $|\cdot|$ translate to

$(i)'$ for all $\alpha \in K$, $\nu(\alpha) \in \mathbb{R} \cup \{\infty\}$ and $\nu(\alpha) = \infty$ if and only if $\alpha = 0$,

$(ii)'$ for all $\alpha, \beta \in K$, $\nu(\alpha\beta) = \nu(\alpha) + \nu(\beta)$,

$(iii)'$ for all $\alpha, \beta \in K$, $\nu(\alpha + \beta) \geq \min\{\nu(\alpha), \nu(\beta)\}$ with equality when $\nu(\alpha) \neq \nu(\beta)$.

A mapping $\nu : K \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfies $(i)' - (iii)'$ is called an *exponential valuation* of K corresponding to the valuation $|\cdot|$.

Definition 1.5. A non-archimedean valuation $|\cdot|$ is called a *discrete valuation* if $\nu(K \setminus \{0\})$ is a discrete subgroup of the additive group of real numbers, i.e., $\nu(K \setminus \{0\}) = \{0\}$ or $\nu(K \setminus \{0\})$ is an infinite cyclic subgroup of $(\mathbb{R}, +)$.

Two kinds of examples of discrete valuations are as follows:

Example 1.6.

(1) The p-adic valuation, $|\cdot|_p$, is a discrete non-archimedean valuation on $\mathbb{Q}$.

(2) The π-adic valuation, $|\cdot|_\pi$, and the degree valuation, $|\cdot|_\infty$, are discrete non-archimedean valuations on $F(x)$.

The concepts of convergence and completeness of our mentioned fields are defined in the usual ways.

Definition 1.7. Let $(K, |\cdot|)$ be a valued field. A sequence $\{a_n\}$ of elements of K is a *Cauchy sequence* in K if for any $\varepsilon > 0$ there is an integer N such that $|a_n - a_m| < \varepsilon$ whenever $m, n > N$.

Definition 1.8. Let $(K, |\cdot|)$ be a valued field. A sequence $\{a_n\}$ of elements of K *converges* to α in K if for any $\varepsilon > 0$ there is an integer N such that $|a_n - \alpha| < \varepsilon$ whenever $n > N$.

Definition 1.9. The field K is called *complete* with respect to the valuation $|\cdot|$ if every Cauchy sequence in K, with respect to $|\cdot|$, has a limit in K.

Definition 1.10. A field $\widehat{K}$ with valuation $\widehat{|\cdot|}$ is a *completion* of a field K with $|\cdot|$ if

(1) $\widehat{K}$ is an extension of K,

(2) $\widehat{K}$ is complete with respect to $\widehat{|\cdot|}$ which is a prolongation of $|\cdot|$ over K,

(3) every element of $\widehat{K}$ is a limit of some Cauchy sequence in K.

Example 1.11.

(1) In the case of $\mathbb{Q}$, with the usual absolute value, its completion is the field $\mathbb{R}$ of real numbers.

(2) In the case of $(\mathbb{Q}, |\cdot|_p)$, its completion is the *field of p-adic numbers* $(\mathbb{Q}_p, |\cdot|_p)$.

(3) In the case of $(F(x), |\cdot|_\pi)$, its completion is $(F((\pi(x))), |\cdot|_\pi)$ the *field of formal Laurent series in* $\pi(x)$ or the *function field with respect to the* π*-adic valuation.*

(4) In the case of $(F(x), |\cdot|_\infty)$, its completion is $(F((1/x)), |\cdot|_\infty)$ the *field of formal Laurent series in* $1/x$ or the *function field with respect to the degree valuation.*

Definition 1.12. Let $(K, |\cdot|)$ be a non-archimedean valued field.

(1) The set $\mathcal{O} := \{\alpha \in K : |\alpha| \leq 1\}$ is a ring, called the *valuation ring* of $(K, |\cdot|)$.

(2) The set $\mathcal{P} := \{\alpha \in K : |\alpha| < 1\}$ is the unique maximal ideal of $\mathcal{O}$.

(3) The field $\mathcal{O}/\mathcal{P}$ is called the *residue class field* of $(K, |\cdot|)$.

Example 1.13. In the case of the field of p-adic numbers, we get

$$\mathcal{O} = \left\{ \frac{a}{b} \in \mathbb{Q} : (a,b) = 1,\ p \nmid b \right\},$$
$$\mathcal{P} = \left\{ \frac{a}{b} \in \mathbb{Q} : (a,b) = 1,\ p \mid a,\ p \nmid b \right\} = p\mathcal{O},$$

and the residue class field is $\mathcal{O}/p\mathcal{O} \cong \{0, 1, \ldots, p-1\}$.

Elements in a field completed with respect to a discrete non-archimedean valuation can be uniquely represented via series expansions as stated in the next theorem, see e.g. [17].

Theorem 1.14. *Let K be a complete field with respect to a discrete non-archimedean valuation $|\cdot|$. For each integer m let τ_m be an element of K such that $\nu(\tau_m) = m$. Let $\mathcal{A}$ be a complete set of representatives in $\mathcal{O}$ of the elements of $\mathcal{O}/\mathcal{P}$, that is, $\mathcal{A}$ consists of exactly one element from each of the residue classes of $\mathcal{P}$ in $\mathcal{O}$. Then every $\alpha \in K \setminus \{0\}$ can be written uniquely in the form*

$$\alpha = \sum_{i=r}^{\infty} c_i \tau_i,$$

where $r = \nu(\alpha)$, $c_i \in \mathcal{A}$ for each i, and $c_r \neq 0$.

Example 1.15.

(1) In the case that $\tau_m = p^m$, $m \in \mathbb{Z}$, p is a prime number and $\mathcal{A} = \{0, 1, \ldots, p-1\}$, we have a unique representation of any element in the p-adic number field $\mathbb{Q}_p$ of the form

$$\sum_{i=r}^{\infty} c_i p^i,$$

where $r \in \mathbb{Z}$, $c_i \in \{0, 1, \ldots, p-1\}$ for each i and $c_r \neq 0$.

(2) An element $\tau_m = x^{-m}$ in $F((1/x))$ and the set $\mathcal{A} = F$ give a representation of an element in $F((1/x))$ of the form

$$\sum_{i=r}^{\infty} c_i x^{-i},$$

where $r \in \mathbb{Z}$, $c_i \in F$ for each i and $c_r \neq 0$.

(3) An element $\tau_m = x^m$ in $F((x))$ and the set $\mathcal{A} = F$ give a representation of an element in $F((x))$ of the form

$$\sum_{i=r}^{\infty} c_i x^i,$$

where $r \in \mathbb{Z}$, $c_i \in F$ for each i and $c_r \neq 0$.

1.2.2 Continued fraction expansions

A finite or infinite expansion of shape

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots\, a_{n-1} + \cfrac{b_n}{\ddots}}}},$$

is called a *continued fraction expansion*. The quantities a_i and b_i may be taken to be integers, real or complex numbers, functions or elements in a field, and called the *partial numerators* and *partial denominators*, respectively. When all $b_i = 1$ $(i \geq 1)$, it is usually referred to as a *regular* or *simple continued fraction expansion*.

For convenience we shall generally denote the above continued fraction expansion by

$$a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n+}\cdots, \tag{1.1}$$

which was introduced by Rogers [22] in 1907.

In addition, one also finds in this thesis the notation

$$[a_0; b_1, a_1; b_2, a_2; \ldots; b_n, a_n; \ldots].$$

A terminating or finite continued fraction expansion

$$a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n} =: \frac{C_n}{D_n}$$

is called the n^{th} *convergent* of the continued fraction expansion (1.1).

From the definition of a continued fraction expansion we have

$$\frac{C_0}{D_0} = \frac{a_0}{1}$$
$$\frac{C_1}{D_1} = a_0 + \frac{b_1}{a_1} = \frac{a_0a_1 + b_1}{a_1}$$
$$\frac{C_2}{D_2} = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2}} = a_0 + \frac{b_1a_2}{a_1a_2 + b_2} = \frac{a_0a_1a_2 + a_0b_2 + b_1a_2}{a_1a_2 + b_2} = \frac{a_2C_1 + b_2C_0}{a_2D_1 + b_2D_0}.$$

For all $n \geq 2$, assume that

$$C_n = a_nC_{n-1} + b_nC_{n-2} \quad \text{and} \quad D_n = a_nD_{n-1} + b_nD_{n-2}. \tag{1.2}$$

Then, for all $n \geq 2$,

$$\frac{C_n}{D_n} = \frac{a_nC_{n-1} + b_nC_{n-2}}{a_nD_{n-1} + b_nD_{n-2}}.$$

Relationships (1.2) were first established by Wallis [35] and were considered in detail by Euler [9]. Euler was the first person who used continued fraction expansion.

Following Euler, we put $C_{-1} = 1$ and $D_{-1} = 0$ in order to make (1.2) valid for $n = 1$.