CHAPTER III

PRISM(C_n)

In this chapter, we show that the prism of a cycle C_n, where $n \geq 3$, is an edge-odd graceful graph.

Definition 3.1 Let $n \geq 3$ and C_n be an n-cycle $u_1u_2u_3 \cdots u_nu_1$. Let $C_n' : u_1'u_2'u_3' \cdots u_n'u_1'$ be a copy of C_n. Define $\mathrm{Prism}(C_n)$, called the **prism of** C_n, by joining each corresponding vertex u_i of C_n to u_i' of C_n'. That is

$$E(\mathrm{Prism}(C_n)) = E(C_n) \cup E(C_n') \cup \{u_iu_i' \mid i \in \{1,2,3,\ldots,n\}\}.$$

Note that $\{u_iu_i' \mid i \in \{1,2,3,\ldots,n\}\}$ is the set of bridges between C_n and C_n'

$\mathrm{Prism}(C_n)$ can be viewed as $C_n \square P_2$, a cartesian product of a cycle C_n and a path P_2.

Example 3.1 From Definition 3.1, we have $\mathrm{Prism}(C_4)$ as seen in Figure 3.1.

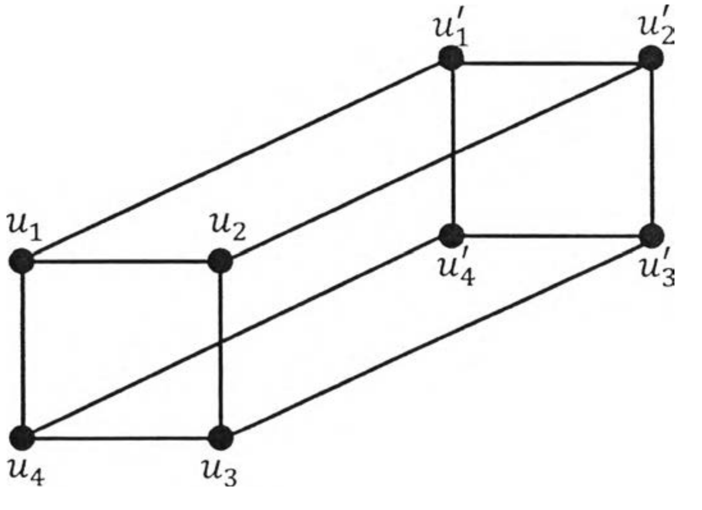

Figure 3.1 $\mathrm{Prism}(C_4)$.

First, Figure 3.2 shows one example on edge-labeling of $\text{Prism}(C_3)$.

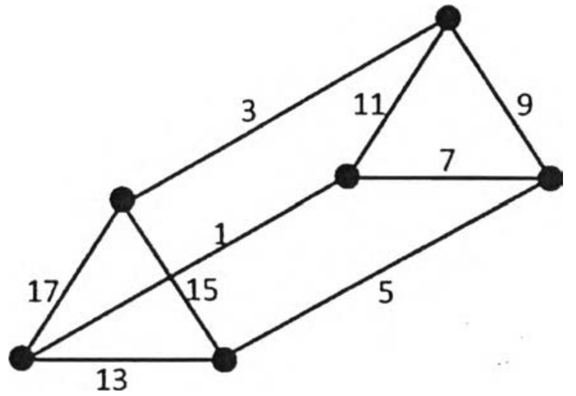

Figure 3.2 Edge-labeling for $\text{Prism}(C_3)$.

Next, for any $n \geq 4$, we can label the edges of $\text{Prism}(C_n)$ by using the following algorithm.

Algorithm 3.1

Let G denote $\text{Prism}(C_n)$, where $n \geq 4$. Then, $q = 3n$. Define $f: E(G) \to \{1, 3, 5, \ldots, 6n-1\}$ by

1.1 $f(u_{i-1}u_i) = 4n - 2i + 1$, for $i \in \{2, 3, 4, \ldots, n\}$;

1.2 $f(u_1u_n) = 4n - 1$;

1.3 $f(u'_{i-1}u'_i) = 6n - 2i + 1$, for $i \in \{2, 3, 4, \ldots, n\}$;

1.4 $f(u'_1u'_n) = 6n - 1$;

1.5 $f(u_iu'_i) = 2i - 1$, for $i \in \{1, 2, 3, \ldots, n\}$.

Example 3.2 From Algorithm 3.1, we can label each edge of $\text{Prism}(C_6)$ as shown in Figure 3.3.

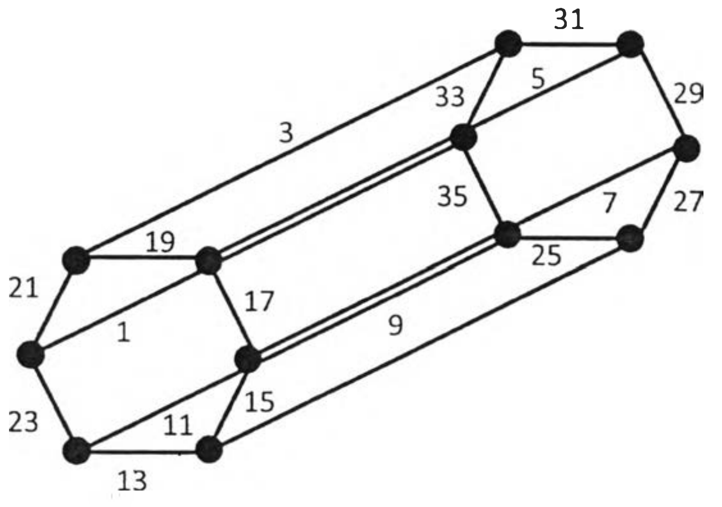

Figure 3.3 Edge-labeling for $\text{Prism}(C_6)$.

Next, we show that if $n \geq 3$, then $\text{Prism}(C_n)$ is an edge-odd graceful graph.

Theorem 3.1 *$Prism(C_n)$ is an edge-odd graceful graph whenever $n \geq 3$.*

Proof. From Figure 3.2, we can see immediately that the induced vertex-labeling is shown in Figure 3.4.

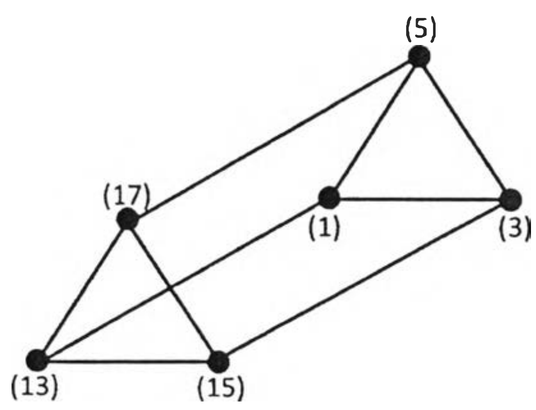

Figure 3.4 The vertex-labeling is induced from the edge-labeling in Figure 3.2.

Therefore, it is obvious from Figures 3.2 and 3.4 that $\text{Prism}(C_3)$ is an edge-odd graceful graph.

Let $n \geq 4$. We first prove that the function f defined in Algorithm 3.1 is a bijection from $E(G)$ to $\{1,3,5,\dots,6n-1\}$. From Algorithm 3.1(1.1 and 1.2), we have

$$A = \{f(u_{i-1}u_i), f(u_1u_n) \mid i \in \{2,3,4,\dots,n\}\}$$
$$= \{2n+1, 2n+3, 2n+5, \dots, 4n-3, 4n-1\}.$$

From Algorithm 3.1(1.3 and 1.4), we have

$$B = \{f(u'_{i-1}u'_i), f(u'_1u'_n) \mid i \in \{2,3,4,\dots,n\}\}$$
$$= \{4n+1, 4n+3, 4n+5, \dots, 6n-3, 6n-1\}.$$

From Algorithm 3.1(1.5), we have

$$C = \{f(u_iu'_i) \mid i \in \{1,2,3,\dots,n\}\} = \{1,3,5,\dots,2n-1\}.$$

We can see clearly that A, B and C are disjoint and

$$f\left(E\left(\text{Prism}(C_n)\right)\right) = A \cup B \cup C = \{1,3,5,\dots,6n-1\}.$$

Next, we will show that the induced vertex-labels from the edge-labels using algorithm 3.1 are in $\{0,1,2,\dots,6n-1\}$ and all distinct. From Algorithm 3.1, we have

$$f^+(u_1) = \left(f(u_1u'_1) + f(u_1u_n) + f(u_1u_2)\right) \pmod{6n}$$
$$= \left(1 + (4n-1) + (4n-3)\right) \pmod{6n}$$
$$= 2n-3;$$

$$f^+(u_n) = \left(f(u_nu'_n) + f(u_1u_n) + f(u_{n-1}u_n)\right) \pmod{6n}$$
$$= \left((2n-1) + (4n-1) + (2n+1)\right) \pmod{6n}$$
$$= 2n-1;$$

$$f^+(u_i) = \left(f(u_iu'_i) + f(u_{i-1}u_i) + f(u_iu_{i+1})\right) \pmod{6n}$$

$= \big((2i-1)+(4n-2i+1)+(4n-2(i+1)+1)\big)$

$\pmod{6n}$

$= 2n-2i-1$, for $i \in \{2,3,4,\dots,n-1\}$;

$f^+(u_1') = \big(f(u_1u_1')+f(u_1'u_n')+f(u_1'u_2')\big) \pmod{6n}$

$= \big(1+(6n-1)+(6n-3)\big) \pmod{6n}$

$= 6n-3$;

$f^+(u_n') = \big(f(u_nu_n')+f(u_1'u_n')+f(u_{n-1}'u_n')\big) \pmod{6n}$

$= \big((2n-1)+(6n-1)+(4n+1)\big) \pmod{6n}$

$= 6n-1$;

$f^+(u_i') = \big(f(u_iu_i')+f(u_{i-1}'u_i')+f(u_i'u_{i+1}')\big) \pmod{6n}$

$= \big((2i-1)+(6n-2i+1)+(6n-2(i+1)+1)\big)$

$\pmod{6n}$

$= 6n-2i-1$, for $i \in \{2,3,4,\dots,n-1\}$.

We can see that

$\{f^+(u_i) \mid i \in \{1,2,3,\dots,n\}\}$

$= \{2n-3\} \cup \{2n-1\} \cup \{2n-5, 2n-7, 2n-9, \dots, 5, 3, 1\}$

$= \{1, 3, 5, \dots, 2n-5, 2n-1, 2n-3\}$

and

$\{f^+(u_i') \mid i \in \{1,2,3,\dots,n\}\}$

$= \{6n-3\} \cup \{6n-1\} \cup \{6n-5, 6n-7, 6n-9, \dots, 4n+5, 4n+3, 4n+1\}$

$= \{4n+1, 4n+3, 4n+5, \dots, 6n-5, 6n-3, 6n-1\}$.

It is clear that if $n \geq 4$, these two sets are disjoint and both are subsets of $\{0, 1, 2, \dots, 6n-1\}$.

Therefore, the function f defined in Algorithm 3.1 is an edge-odd graceful labeling and $\mathrm{Prism}(C_n)$ is an edge-odd graceful graph for all $n \geq 3$. ■

Example 3.3 From the edge-labeling in Example 3.2, the induced vertex-labeling of $\mathrm{Prism}(C_6)$ is shown in Figure 3.5.

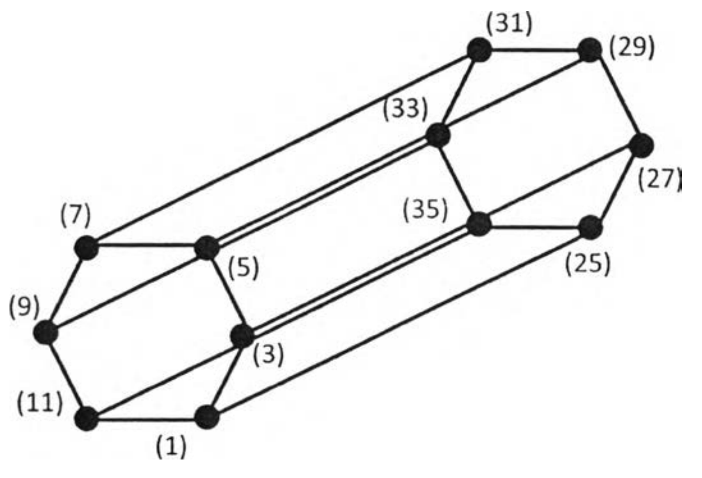

Figure 3.5 The vertex-labeling is induced from the edge-labeling in Example 3.2.