CHAPTER II
PRELIMINARIES

In this chapter, we collect notation and basic results that will be used throughout this research.

2.1 Notation

Let b be a real number and $\rho : \mathbb{R}^n \to (0, \infty)$ be defined by

$$\rho(x) := \left(\ln(e + |x|^2)\right)^b, \qquad x \in \mathbb{R}^n.$$

For $n \in \mathbb{N}$, we denote the *weighted Lebesgue space* on $\mathbb{R}^n$ with weight ρ to be the set

$$L^{\infty,\rho}(\mathbb{R}^n) := \left\{f \colon \mathbb{R}^n \to \mathbb{R} : f \text{ is measuarable and } \|f\|_{L^{\infty,\rho}} < \infty\right\},$$

where $\|f\|_{L^{\infty,\rho}}$ is the *weighted Lebesgue norm* on the $L^{\infty,\rho}(\mathbb{R}^n)$ defined by

$$\|f\|_{L^{\infty,\rho}} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \rho(x)\,|f(x)|\,.$$

For convenience, we use $\|\cdot\|_\rho$ instead of $\|\cdot\|_{L^{\infty,\rho}}$ and also denote $C(\mathbb{R}^n)$ to be the set of all continuous real-valued functions on $\mathbb{R}^n$. Finally, we denote

$$L_C^{\infty,\rho}(\mathbb{R}^n) := L^{\infty,\rho}(\mathbb{R}^n) \cap C(\mathbb{R}^n).$$

For any function f and g, the notation $f \lesssim g$ means that there exists a positive constant C such that $f \le Cg$ at every point in the domain. For any $x \in \mathbb{R}^n$, *the Japanese bracket* is $\langle x\rangle := \sqrt{1 + |x|^2}$.

2.2 Basic Analysis

In this section, we provide fundamental knowledges that will be used throughout in this work. See [5], [7], [11] for more information.

Theorem 2.1 (Hölder's inequality). *Let* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* f *and* g *are measurable functions on a measure space* (X, μ), *then*

$$\int_X |fg|\, d\mu \leq \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q\, d\mu\right)^{\frac{1}{q}}.$$

Theorem 2.2 (Minskowski's inequality). *Let* $1 \leq p < \infty$ *and* $f, g \in L^p(X, \mu)$, *then*

$$\left(\int_X |f+g|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p\, d\mu\right)^{\frac{1}{p}}.$$

Lemma 2.3. *Let* $1 \leq p < \infty$ *and* $x, y \in \mathbb{R}$. *Then*

$$\left|x\,|x|^{p-1} - y\,|y|^{p-1}\right| \leq c_p\,(|x| \vee |y|)^{p-1}\,|x-y|.$$

where $c_p = p$ *if* a *and* b *have the same sign and* $c_p = 2^p$ *if* a *and* b *have different signs.*

Theorem 2.4 (Generalized Dominated Convergence Theorem). *Let* $(X, \mathcal{A}, \mu)$ *be a measure space and* (f_n) *a sequence of measurable functions on a measurable subset* E *of* X *that converges pointwise a.e. on* E *to a function* f. *Suppose there is a sequence* g_n *of nonnegative measurable functions on* E *that converges pointwise a.e. on* E *to* g *and* $|f_n(x)| \leq g_n(x)$ *for a.e.* $x \in E$ *and all* $n \in \mathbb{N}$. *If* $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$ *then* $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Theorem 2.5 (Weierstrass M-test). *Suppose that* $(X, \|\cdot\|)$ *is a Banach space,* (Y, d) *is a metric space, for each* $n \in \mathbb{N}$, $f_n : Y \to X$, *and there exists a sequence of positive real number* $\{M_n\}_n^\infty$ *satisfying;*

$$\sup_{y \in Y} \|f_n(y)\| \leq M_n \quad \forall n \in \mathbb{N}$$

and $\sum_{n=1}^{\infty} M_n < \infty$*. Then* $S_N(y) := \sum_{n=1}^{N} f_n(y)$ *converges absolutely and uniformly to* $S(y) := \sum_{n=1}^{\infty} f_n(y)$*. Moreover, if* $f_n \in C(Y, X)$ *for all* $n \in \mathbb{N}$*, then* $S \in C(Y, X)$*.*

Theorem 2.6 (Banach Fixed Point Theorem). *Let* X *be a complete metric space with the metric* d*. Let* $A : X \to X$ *be a map. If* A *is strictly contractive on* X*, i.e. there exists a constant* $0 < k < 1$ *such that* $d(Ax, Ay) \leq kd(x, y)$*, for all* $x, y \in X$*. Then* A *has a unique fixed point.*

2.3 Uniformly Continuous Semigroups

Definition 2.7. [11] Let X be a Banach space. A family $\{\mathcal{G}(t)\}_{t\geq 0}$ of bounded linear operators from X into X is a *semigroup* if

1. $\mathcal{G}(0) = id_X$

2. $\mathcal{G}(t + s) = \mathcal{G}(t)\mathcal{G}(s)$ for every $t, s \geq 0$.

Definition 2.8. [11] A semigroup of bounded linear operators $\{\mathcal{G}(t)\}_{t\geq 0}$ on a Banach space X is *uniformly continuous* if

$$\lim_{t\to 0^+} \|\mathcal{G}(t) - id_X\| = 0.$$

where $\|\cdot\|$ is the operator norm of bounded linear operators on X.

Lemma 2.9. [11] *Let $\{\mathcal{G}(t)\}_{t\geq 0}$ be a uniformly continuous semigroup of bounded linear operators on a Banach space X. Then we have for any $s, t \geq 0$ that*

$$\lim_{t\to s} \|\mathcal{G}(t) - \mathcal{G}(s)\| = 0.$$

Proof. Let $s, t \geq 0$. Assume $s \leq t$. Then

$$\begin{aligned}
\lim_{t\to s^+} \|\mathcal{G}(t) - \mathcal{G}(s)\| &= \lim_{t\to s^+} \|\mathcal{G}((t-s)+s) - \mathcal{G}(s)\| \\
&= \lim_{t\to s^+} \|\mathcal{G}(s)\mathcal{G}(t-s) - \mathcal{G}(s)\| \\
&= \lim_{t\to s^+} \|\mathcal{G}(s)\left(\mathcal{G}(t-s) - id_X\right)\| \\
&= 0,
\end{aligned}$$

because $\|\mathcal{G}(s)\| \leq \infty$ and $\|\mathcal{G}(t-s) - id_X\| \to 0$ as $t \to s^+$. So, the proof is complete. □

Example 2.10. [11] Let X be a Banach space and $A : X \to X$ is a bounded linear operator. Then

$$\left\{\mathcal{G}(t) = e^{tA} = \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} : t \geq 0\right\}$$

is a uniformly continuous semigroup of bounded linear operators on X.