CHAPTER V

LOCAL WELLPOSEDNESS OF SEMILINEAR EQUATION

In this section we prove the local (in time) existence and uniqueness of solutions to the Cauchy problem (1.2) on weighted Lebesgue spaces by applying the Banach fixed-point theorem. Consider the Cauchy problem (1.2) and assume $u_0 \in L_C^{\infty,\rho}(\mathbb{R}^n)$.

Definition 5.1. Let $b \in \mathbb{R}, \sigma > 0, p > 1$ and $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a given function. A function $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ is a *solution* of the Cauchy problem (1.2) if $u \in C\left([0,T) \,; L_C^{\infty,\rho}(\mathbb{R}^n)\right)$ and it satisfies, for any $x \in \mathbb{R}^n$ and $t \in [0,T)$ the equation

$$u(\cdot,t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau.$$

Theorem 5.2. *Let $p > 1, \sigma > 0$ and $b \in \mathbb{R}$ be such that $b \geq \dfrac{\sigma}{p-1}$. Assume that $u_0 \in L_C^{\infty,\rho}(\mathbb{R}^n)$ and $u_0 \geq 0$. Then there exists $T_0 > 0$ such that the Cauchy problem (1.2) has a unique non-negative solution $u \in C\left([0,T_0); L_C^{\infty,\rho}(\mathbb{R}^n)\right)$.*

Proof. Let $T > 0$ be a sufficiently small number to be specified.
We define the Banach space

$$X_T := C\left([0,T) \,; L_C^{\infty,\rho}(\mathbb{R}^n)\right)$$

equipped with the norm $\|u\|_{X_T} = \sup\limits_{0\leq t<T} \|u(\cdot,t)\|_\rho$.
We define the operator $\mathcal{T}$ on X_T by

$$\mathcal{T}u(\cdot,t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau \quad t \in [0,T).$$

Let $u \in X_T$, $x \in \mathbb{R}^n$ and $t \in [0, T)$.

Let us verify that $\mathcal{T}$ is a self-map on X_T.

Step1 Verify $\mathcal{T}u(\cdot, t)$ is bounded on $\mathbb{R}^n$. We notice that if $b \geq \dfrac{\sigma}{p-1}$ then $bp \geq b + \sigma$.

$$
\begin{aligned}
&\rho(x)\,|\mathcal{T}u(x,t)| \\
&= \rho(x)\left|\left(\mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)((\ln(e+|x|^2)))^\sigma (u(x,\tau))^p d\tau\right)\right| \\
&\leq \rho(x)\,|(\mathcal{G}(t)u_0(x))| + \rho(x)\left|\left(\int_0^t \mathcal{G}(t-\tau)(\ln(e+|x|^2))^\sigma (u(x,\tau))^p d\tau\right)\right| \\
&\leq \|\mathcal{G}(t)u_0\|_\rho + \int_0^t \left|\rho(x)\mathcal{G}(t-\tau)((\ln(e+|x|^2)))^\sigma (u(x,\tau))^p\right| d\tau \\
&\lesssim (\ln(e+et^2))^{|b|}\,\|u_0\|_\rho + \int_0^t \|\mathcal{G}(t-\tau)\|_{\mathcal{G}} \sup_{x\in\mathbb{R}^n}\{(\ln(e+|x|^2))^{\sigma+b}(u(x,\tau))^p\}d\tau \\
&\lesssim (\ln(e+et^2))^{|b|}\,\|u_0\|_\rho + \int_0^t (\ln(e+e(t-\tau)^2))^{|b|} \sup_{x\in\mathbb{R}^n}\{(\ln(e+|x|^2))^{bp}(u(x,\tau))^p\}d\tau \\
&\lesssim (\ln(e+et^2))^{|b|}\,\|u_0\|_\rho + \int_0^t (\ln(e+e(t-\tau)^2))^{|b|} \sup_{x\in\mathbb{R}^n}\{(\ln(e+|x|^2))^{b}(u(x,\tau))\}^p d\tau \\
&\lesssim (\ln(e+et^2))^{|b|}\,\|u_0\|_\rho + \int_0^t (\ln(e+e(t-\tau)^2))^{|b|}\,\|u(\cdot,\tau)\|_\rho^p\, d\tau \\
&\lesssim (\ln(e+et^2))^{|b|}\,\|u_0\|_\rho + \int_0^t (\ln(e+et^2))^{|b|}\,\|u(\cdot,\tau)\|_\rho^p\, d\tau \\
&\lesssim (\ln(e+et^2))^{|b|}\,\|u_0\|_\rho + \int_0^t (\ln(e+et^2))^{|b|}\,\|u\|_{X_T}^p\, d\tau \\
&\lesssim (\ln(e+et^2))^{|b|}\,\|u_0\|_\rho + \|u\|_{X_T}^p\, T(\ln(e+eT^2))^{|b|}
\end{aligned}
$$

So we get

$$
\|\mathcal{T}u(\cdot,t)\|_\rho \leq K(\ln(e+eT^2))^{|b|}\left(\|u_0\|_\rho + T\|u\|_{X_T}^p\right),
$$

and hence, in particular, $\|\mathcal{T}u(\cdot,t)\|_\rho < \infty$. Also, $\mathcal{T}u(\cdot,t) \in C(\mathbb{R}^n)$.

Step 2 Verify that $t \mapsto \mathcal{T}u(\cdot,t)$ is continuous on $[0,T)$
Let $s,t \in [0,T]$. WLOG, $s \leq t$. Consider

$$
\begin{aligned}
&\|\mathcal{T}u(\cdot,t) - \mathcal{T}u(\cdot,s)\|_\rho \\
&\leq \|(\mathcal{G}(t)u_0 - \mathcal{G}(s)u_0)\|_\rho \\
&\quad + \left\| \int_0^t \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau - \int_0^s \mathcal{G}(s-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau \right\|_\rho \\
&\leq \|\mathcal{G}(t) - \mathcal{G}(s)\| \, \|u_0\|_\rho \\
&\quad + \left\| \int_0^t \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau - \int_0^s \mathcal{G}(s-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau \right\|_\rho \\
&= I + II.
\end{aligned}
$$

Now, We estimate II.

$$
\begin{aligned}
II &= \left\| \int_0^t \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau - \int_0^s \mathcal{G}(s-\tau)\ln(e+|\cdot|^2)^\sigma (u(\cdot,\tau))^p d\tau \right\|_\rho \\
&\leq \left\| \int_0^s \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau \right. \\
&\quad \left. + \int_s^t \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau - \int_0^s \mathcal{G}(s-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau \right\|_\rho \\
&= \left\| \int_0^s \big(\mathcal{G}(t-\tau) - \mathcal{G}(s-\tau)\big)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau \right. \\
&\quad \left. + \int_s^t \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p d\tau \right\|_\rho \\
&\leq \int_0^s \|\mathcal{G}(t-\tau) - \mathcal{G}(s-\tau)\| \left\| (\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p \right\|_\rho d\tau \\
&\quad + \int_s^t \left\| \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p \right\| d\tau \\
&\leq \|u\|_{X_T}^p \int_0^s \|\mathcal{G}(t-\tau) - \mathcal{G}(s-\tau)\| d\tau + \int_s^t \left\| \mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma (u(\cdot,\tau))^p \right\|_\rho d\tau \\
&\lesssim \|u\|_{X_T}^p \int_0^s \|\mathcal{G}(t-\tau) - \mathcal{G}(s-\tau)\| d\tau + \|u\|_{X_T}^p \left(\ln(e+eT^2)\right)^{|b|} (t-s).
\end{aligned}
$$

Taking $t \to s$, $I \to 0$ and $II \to 0$ and hence

$$\|\mathcal{T}u(\cdot, t) - \mathcal{T}u(\cdot, s)\|_\rho \to 0.$$

So $t \mapsto \mathcal{T}u(\cdot, t)$ is continuous.

Now we show that $\mathcal{T}$ is a self-map and a contraction map on some closed subspace of X_T. Let $\delta > 0$ be sufficiently large number to specified. Define

$$X_T^\delta := \{u \in X_T : \|u\|_{X_T} \le \delta\}.$$

Then X_T^δ is a closed subspace of X_T, hence it is a complete metric space.
We find a pair T, δ such that $\mathcal{T}$ is a self-map on X_T^δ. Let $u \in X_T^\delta$. It suffices to verify that $\|\mathcal{T}u\|_{X_T} \le \delta$. By **Step 1**, we have that there exists a constant K such that

$$\begin{aligned}\|\mathcal{T}u\|_{X_T} &\le K(\ln(e + eT^2))^{|b|}\left(\|u_0\|_\rho + T\|u\|_{X_T}^p\right)\\ &= K(\ln(e + eT^2))^{|b|}\|u_0\|_\rho + K(\ln(e + eT^2))^{|b|}T\delta^p.\end{aligned}$$

To obtain the result, we have to choose T and $\delta > 0$ so that

$$K(\ln(e + eT^2))^{|b|}\|u_0\|_\rho \le \frac{\delta}{2} \quad \text{and} \quad K(\ln(e + eT^2))^{|b|}T \le \frac{1}{2\delta^{p-1}} \tag{5.1}$$

For fixed $T = 1$, we have that there exists $\delta > 0$ such that

$$K(\ln(e + eT^2))^{|b|}\|u_0\|_\rho = K(\ln(2e))^{|b|}\|u_0\|_\rho \tag{5.2}$$
$$\le \frac{\delta}{2}. \tag{5.3}$$

Since $\lim_{t\to 0}(\ln(e + et^2))^{|b|}t = 0$, there exists $T_1 > 0$ such that

$$K(\ln(e + eT_1^2))^{|b|}T_1 \le \frac{1}{2\delta^{p-1}}. \tag{5.4}$$

In this step, we take $T = \min\{1, T_1\}$. Hence, (5.2) and (5.4) hold for this T. Now, we show that $\mathcal{T}$ is a contraction map on X_T^δ. Let $u, v \in X_T^\delta$. Then

$$
\begin{aligned}
&\|\mathcal{T}u - \mathcal{T}v\|_{X_T} \\
&= \sup_{0\le t\le T}\int_0^t \left\|\mathcal{G}(t-\tau)(\ln(e+|\cdot|^2))^\sigma\left((u(\cdot,\tau))^p - (v(\cdot,\tau))^p\right)\right\|_\rho d\tau \\
&\le \sup_{0\le t\le T}\int_0^t \|\mathcal{G}(t-\tau)\| \left\|(\ln(e+|\cdot|^2))^\sigma\left((u(\cdot,\tau))^p - (v(\cdot,\tau))^p\right)\right\|_\rho d\tau \\
&\le \sup_{0\le t\le T}\int_0^t \|\mathcal{G}(t-\tau)\| \sup_{x\in\mathbb{R}^n}(\ln(e+|x|^2))^{\sigma+b}\left|(u(x,\tau))^p - (v(x,\tau))^p\right| d\tau \\
&\le 2^p \sup_{0\le t\le T}\int_0^t \|\mathcal{G}(t-\tau)\| \sup_{x\in\mathbb{R}^n}(\ln(e+|x|^2))^{bp} \\
&\qquad\qquad (u(x,\tau))\vee(v(x,\tau))^{p-1}\left|u(x,\tau)-v(x,\tau)\right| d\tau \\
&= 2^p \sup_{0\le t\le T}\int_0^t \|\mathcal{G}(t-\tau)\| \sup_{x\in\mathbb{R}^n}(\ln(e+|x|^2))^{b(p-1)}(\ln(e+|x|^2))^b \\
&\qquad\qquad (u(x,\tau))\vee(v(x,\tau))^{p-1}\left|u(x,\tau)-v(x,\tau)\right| d\tau \\
&\le 2^p \sup_{0\le t\le T}\int_0^t \|\mathcal{G}(t-\tau)\| \sup_{x\in\mathbb{R}^n}\delta^{p-1}(\ln(e+|x|^2))^b\left|u(x,\tau)-v(x,\tau)\right| d\tau \\
&\le 2^p \sup_{0\le t\le T}\int_0^t \|\mathcal{G}(t-\tau)\| \,\delta^{p-1}\left\|u(\cdot,\tau)-v(\cdot,\tau)\right\|_\rho d\tau \\
&\le 2^p\delta^{p-1}K\int_0^t (\ln(e+e(t-\tau)^2))^{|b|}\|u-v\|_{X_T} d\tau \qquad \text{for some constant } K \\
&\le 2^p\delta^{p-1}K\int_0^t (\ln(e+et^2))^{|b|}\|u-v\|_{X_T} d\tau \\
&\le 2^p\delta^{p-1}K(\ln(e+eT^2))^{|b|}T\,\|u-v\|_{X_T}.
\end{aligned}
$$

The map $\mathcal{T}$ is a contraction provided

$$2^p\delta^{p-1}K(\ln(e+eT^2))^{|b|}T < 1. \tag{5.5}$$

Again, since $\lim_{t\to 0}(\ln(e+et^2))^{|b|}t = 0$, there exists $T_2 > 0$ such that

$$(\ln(e+eT_2^2))^{|b|}T_2 < \frac{1}{2^p\delta^{p-1}K}. \tag{5.6}$$

Therefore, we choose $T_0 = \min\{1, T_1, T_2\}$ and we have that

1. $K(\ln(e + eT_0^2))^{|b|}T_0 \le K(\ln(e + e(1)^2))^{|b|}(1)\frac{1}{2\delta^{p-1}}$.

2. $K(\ln(e + eT_0^2))^{|b|} \le K(\ln(e + eT_1^2))^{|b|} \le \frac{\delta}{2}$.

3. $(\ln(e + eT_0^2))^{|b|}T_0 \le (\ln(e + eT_2^2))^{|b|}T_2 < \frac{1}{2^p\delta^{p-1}K}$.

So we have already found T_0 that makes (5.1) and (5.5) holds simultaneously and hence $\mathcal{T}$ is a contraction on X_T^δ. Hence we have that $\mathcal{T}$ has a unique fixed point $u \in X_T^\delta$ by applying the Banach fixed-point theorem. □